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AN AUTOMATON MODELLED IN AN ENVIRONMENT. (U)
1977 J GOULD, E J WEGMAN AFOSR-75-2840
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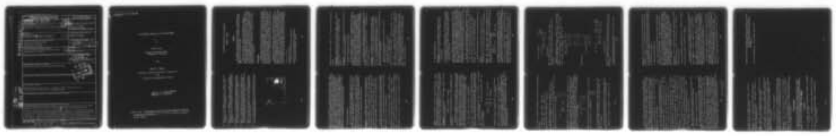

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AD A052223

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR- 78-0518
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): AN AUTOMATON MODELLED IN AN ENVIRONMENT
5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): Jerren Gould [redacted] Edward J. Wegman
8. CONTRACT OR GRANT NUMBER(s): AFOSR-75-2840
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of North Carolina, Department of Statistics, Chapel Hill, NC 27514
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304 A5
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, DC 20332
12. REPORT DATE: 1977
13. NUMBER OF PAGES: 11
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES: PROCEEDINGS OF THE FIRST INTERNATIONAL CONFERENCE ON MATHEMATICAL MODELING, v1 p501-511 1977
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

We establish the basic notion of automata in environments and solve some of the basic problems in the general theory of automata in deterministic environment (ADE).

ADE are, in general, stronger than probabilistic automata. In an effort to find when an ADE and a PA have the same capability, we introduce the concept of simulation of an ADE by a PA. Every ADE with a finite environment set can be simulated by a PA (Theorem 1). There are sets which cannot be defined by PA but can be simulated by a PA. ADE for which the environment can be reduced to a finite set can be simulated by a PA (Theorem 2).

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182 850

AFOSR-TR- 78-0518

AN AUTOMATON MODELLED IN AN ENVIRONMENT

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reprint from: Proceedings of the First International Conference on Mathematical Modeling, (X. J. R. Avula, Editor), Volume 1, 501-511 (1977).

[3] Flachs, G. M., W. E. Thompson, Y. H. U, and A. L. Gilbert, "A Real-Time Structural Algorithm," Proc. 1976 NAECON, Dayton, Ohio, May 1976, pp. 161-168.

[4] Flachs, G. M., W. E. Thompson, R. J. Black, J. M. Taylor, W. Cannon, R. Rogers, and Y. H. U, "A Pre-Prototype Real-Time Video Tracking System," Final Report to White Sands Missile Range for Contract DAAD07-76-C-0024, January 1977.

[5] Kitchell, J. W., "A General Purpose Microprogrammable Computer Architecture," Technical Report, Department of Electrical and Computer Engineering, New Mexico State University, Las Cruces, N.M., April 1977.

[6] Rogers, R., "Specifications for a Two-Pass Symbolic Assembler Generating Absolute Microcode for Arbitrary Microprogram Control Store Architectures," Technical Report, Department of Electrical and Computer Engineering, New Mexico State University, Las Cruces, N.M., April 1977.

[7] Thompson, W. E. and G. M. Flachs, "A Structure and Dynamic Mathematical Model of a Real-Time Video Tracking System," Proc. 1976 NAECON, Dayton, Ohio, May 1976, pp. 169-172.

[8] U, Y. H. and G. M. Flachs, "Structural Feature Extraction by Projection," Proc. 1976 IEEE Region V Conference, Austin, Texas, April 1976, pp. 15-19.

AN AUTOMATON MODELLED IN AN ENVIRONMENT

EDWARD J. WEGMAN

Summary

We establish the basic notion of automata in environments and solve some of the basic problems in the general theory of automata indeterministic environment (ADE).

ADE are, in general, stronger than probabilistic automata. In an effort to find when an ADE and a PA have the same capability, we introduce the concept of *simulation* of an ADE by a PA. Every ADE with a finite environment set can be simulated by a PA (Theorem 1). There are sets which cannot be defined by PA but can be defined by ADE with finite environment sets and, hence, can be simulated by a PA. ADE for which the environment can be reduced to a finite set can be simulated by a PA (Theorem 2).

Introduction

Probabilistic automata are mathematical models for systems capable of a finite number of states which admit, at discrete time intervals, certain inputs and emit certain outputs. In this paper we shall follow the notations and use the results on automata contained in the papers by Rabin (1963) and Paz (1966). In particular, the formulation given there amounts to assuming that the set of outputs contains just two elements. This is a convenient restriction which we also follow here.

Probabilistic automata are models of systems whose transition probabilistics are only related to the present state and present input and are independent of any other factors. It seems quite natural, however, to assume that a system is located within an environment which affects its properties, i.e., the transition function of the system is not only related to the present state and input, but also to the present configuration of the environment. Under previous conceptualizations, a system which performs differently under distinct environments

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would have to be regarded as a collection of distinct systems. Consideration of automata in environments allows the various external factors which may influence the behavior of the system to be incorporated in the model. Thus, systems which perform differently under distinct environments could be identical with the difference in behavior being attributed to the environments. A practical motivation for automata in environments is that even systems such as digital computers intended to be independent of external factors often do not exhibit such behavior. Furthermore, if the environment may in some way be controlled, then it may be a means of optimizing the behavior of the system.

1. Probabilistic Automata

Let Σ be the finite input set, the *alphabet*, and let Σ^*, the *set of tapes*, be the class of all finite sequences of elements of Σ. Let us also include Λ, the *empty tape*, in Σ^*. If $x = \sigma_1 \ldots \sigma_k$ is a tape, then the length $L(x)$ of x is $L(x) = k$. Note that $L(\Lambda) = 0$. If x and y are tapes, then xy will denote the tape which is the concatenation of x and y. Let M_n denote the set of all n×n stochastic matrices and V_n denote the set of all n-dimensional stochastic vectors.

Definition 1. A *probabilistic automaton* (PA) over the alphabet Σ is a system $A = (S,M,\pi_0,F)$, where $S = \{s_1,\ldots,s_n\}$ is a finite set (the *set of internal states*), M is a function $M\colon \Sigma \to M_n$ (the *matrix transition function*) such that $m_{ij}(\sigma)$ is the probability of changing to state s_j under input σ given that the system is in state s_i, $\pi_0 \in V_n$ (the *initial state distribution*), and $F \subset S$ (the *set of acceptance states*).

The function M can be extended to define the transition probabilities for going from state s_i to state s_j by a sequence $x \in \Sigma^*$ of inputs. For $x = \sigma_1\sigma_2\cdots\sigma_k$, we obtain $M(x)$ by the rule $M(x) = M(\sigma_1)M(\sigma_2)\ldots M(\sigma_k)$. Note that $M(\Lambda) = I_n$, the n×n identity matrix.

Let $A = (S,M,\pi_0,F)$ be a PA over alphabet Σ. We define the state distribution of A after input tape x as $\pi(x) = \pi_0 M(x)$. A tape $x \in \Sigma^*$ is said to be accepted by A if a state in F is obtained after tape x is input. Let η^F be an n-dimensional column vector such that $\eta_i^F = \begin{cases} 1 & s_i \in F \\ 0 & s_i \notin F \end{cases}$. The probability that tape x is accepted by A is defined as $p(x)$ and is calculated $p(x) = \pi(x)\eta^F = \pi_0 M(x)\eta^F$. A PA A may be used to define sets of tapes.

Definition 2. Let A be a PA and λ be a real number, $\lambda \in [0,1)$. The *set of all tapes accepted* by A with cut-point λ is defined as $T(A,\lambda) = \{x \mid x \in \Sigma^*, \lambda < p(x)\}$. If $x \in T(A,\lambda)$, we say x is accepted by A with *cut-point* λ.

Definition 3. Let $U \in \Sigma^*$. U is a *probabilistic cut-point event* (PCE) if $U = T(A,\lambda)$ for some PA A and some $\lambda \in [0,1)$.

2. Automata in Deterministic Environments.

We shall consider the aggregate of all configurations of the environment as an abstract set denoted by E. Presently, we do not make any stipulations as to the origin, form, structure, or cardinality of E. As each new symbol is input to the system, the configuration of the environment is observed and the probabilistic transition ensues as a function of the input symbol and the configuration of the environment. We also view the initial distribution as a function of the environment.

Let E^∞ denote the cartesian product of a countable number of copies of the environment set E. The *total environment*, or *environment sequence*, is a mapping $\underline{e}(k) = e_k \in E$ for $k = 0,1,\ldots$. We also denote $\underline{e}$ by the sequence $(e_0,e_1,\ldots)$.

Definition 4. An *automaton in a deterministic environment* (ADE) is a system $A = (\Sigma,S,G,\pi_0,F,E)$, where Σ is a finite *input alphabet*, $S = \{s_1,\ldots,s_n\}$ is the finite *internal state set*, G is a mapping $G\colon \Sigma \times E \to M_n$ (the *basic matrix transition function*), $\pi_0\colon E \to V_n$ (the *initial distribution function*), $F \subset S$ (the *set of acceptance states*), and E is the *set of environments*.

The matrix transition function M is defined on $\Sigma^* \times E^\infty$ by:

1. $M(\Lambda,\underline{e}) = I_n$ for all $\underline{e} \in E^\infty$
2. If $x \in \Sigma^*$ such that $L(x) = k$ and $\underline{e} \in E^\infty$, then $M(x\sigma,\underline{e}) = M(x,\underline{e})G(\sigma,e_{k+1})$.

Hence, for all $x \in E^*$ and $\underline{e} \in E^\infty$, $M(x,\underline{e}) = \prod_{i=1}^{k} G(\sigma_i,e_i)$, where $x = \sigma_1 \cdots \sigma_k$ and $\underline{e} = (e_0,e_1,\ldots)$. We denote the state distribution of A after input tape x in total environment $\underline{e}$ by $\pi(x,\underline{e}) = \pi_0(e_0)M(x,\underline{e})$. Similarly, we let $p(x,\underline{e}) = \pi(x,\underline{e})\eta^F$ be the probability x is accepted by A in total environment $\underline{e}$. It is clear from their definitions that these functions do not depend on the environments beyond the input symbols. Let $Q(e_0,\ldots,e_k) = \{\underline{e} \in E^\infty \mid \underline{e}(i) = e_i \text{ for } i=0,1,\ldots,k\}$. The lemma follows easily. We omit the proof.

Lemma 1. Let A be an ADE. For any $k \geq 0$ and any fixed $e_0,\ldots,e_k \in E$, the following functions are constant in $Q(e_0,\ldots,e_k)$: $M(x,\cdot)$, $\pi(x,\cdot)$, and $p(x,\cdot)$ for each $x \in \Sigma^*$ such that $L(x) \leq k$. Moreover, $M(x,\cdot)$ is constant in $Q'(e_1,\ldots,e_k) = \{\underline{e} \in E^\infty | \underline{e}(i) = e_i$ for $i=1,\ldots,k\}$ for $L(x) \leq k$.

Definition 5. Let A be an ADE and λ a real number, $\lambda \in [0,1)$. The set of tapes $T(A,\underline{e},\lambda)$ defined by

$$T(A,\underline{e},\lambda) = \{x | x \in \Sigma^*, \lambda < p(x,\underline{e})\}$$

is called the *set of tapes accepted by A in total environment* $\underline{e}$ *with cut-point* λ.

Probabilistic automata can be considered as a special case of ADE. For any PA, $A = (S,M,\pi_0,F)$, we can define an ADE A' with any nonempty environment set E such that $\pi_0(e) = \pi_0$ and $G(\sigma,e) = M(\sigma)$ $\forall \sigma \in \Sigma, \forall e \in E$. Such an ADE is not influenced by the configuration of the environment. Hence for any $\lambda \in [0,1)$, $T(A,\lambda) = T(A',\underline{e},\lambda)$ $\forall \underline{e} \in E^\infty$. Let $\Sigma=\{\sigma\}$. Paz (1970) demonstrated that there exists $U \subset \Sigma^*$ such that U is not a PCE. Let us now consider a two-state ADE over alphabet Σ, where

$$E = \{0,1\},\ \pi_0(e) = (1-e,e), G(\sigma,e) = \begin{pmatrix} 1-e & e \\ 1-e & e \end{pmatrix}\ \forall e \in E\ ,$$

and $\eta^F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let us denote the tape of k repetitions of the symbol σ by $\sigma^k = \sigma\sigma\ldots\sigma$. Note that $\sigma^0 = \Lambda$, the empty tape. Thus, for our particular situation all of the elements of Σ^* are of the form σ^k for some $k \geq 0$. Let V be any subset of Σ^*; then $V = \{\sigma^{k_1},\sigma^{k_2},\ldots\}$. We construct $\underline{e}(k) = \begin{cases} 1 & \text{if } k = k_i \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$ Such an $\underline{e}$ is an element of E^∞. Now for $k \geq 1$, we have

$$M(\sigma^k,\underline{e}) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} & \text{if } k = k_i \text{ for some } i \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} & \text{otherwise} \end{cases} .$$

Therefore, $p(\sigma^k,\underline{e}) = \pi_0(e_0)M(\sigma^k,\underline{e})\eta^F = \begin{cases} 1 & \text{if } k = k_i \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$

When $k = 0$, $p(\sigma^0,\underline{e}) = p(\Lambda,\underline{e}) = \pi_0(e_0)\eta^F = \begin{cases} 1 & \text{if } k_i = 0 \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$
Let $\lambda = 0$, then $V = T(A,\underline{e},\lambda)$. Since V is arbitrary, every subset of our particular Σ^* is definable by A in some total environment. But there exists $U \subset \Sigma^*$ which is not definable by a PA. Thus, the class of automata in deterministic environments gives us a strictly larger class of definable sets.

3. Finite Environment Sets

Definition 6. A PA, A', over alphabet Σ' *simulates* the ADE, $A = (\Sigma,S,G,\pi_0,F,E)$, if there is a relation R between Σ'^* and $\Sigma^* \times E^\infty$ such that

1. for any $(x,\underline{e}) \in \Sigma^* \times E^\infty$ there exists $x' \in \Sigma'^*$ such that $[x',(x,\underline{e})] \in R$;
2. if $[x',(x,\underline{e})] \in R$, then for each $\lambda \in [0,1)$, $x' \in T(A',\lambda)$ iff $x \in T(A,\underline{e},\lambda)$;
3. if for some $\lambda \in [0,1)$, $x' \in T(A',\lambda)$, then there exists $(x,\underline{e}) \in \Sigma^* \times E^\infty$ such that $[x',(x,\underline{e})] \in R$.

If a PA A' simulates an ADE A, given any input tape x for A in total environment $\underline{e}$, we have a rule R to find an input tape x' for A' such that x' has the same acceptance properties as x for A in total environment $\underline{e}$. We say a tape $x' \in \Sigma'^*$ is *admissible* if and only if there is an input tape x for A in some total environment $\underline{e}$ from which the rule R will yield x'. By condition 3 above we see that tapes which are not admissible are not accepted by A' for any $\lambda \in [0,1)$; that is, if x' is not admissible, then the probability of acceptance of x' by A' is zero.

By lemma 1, we see that the probability of acceptance of any tape $x \in \Sigma^*$ does not depend on any environments other than $e_0,e_1,\ldots,e_k$, where $k = L(x)$. Hence, the relation R need only depend on x and $e_0,e_1,\ldots,e_k$, where $k = L(x)$. Thus all pairs of the form $(x,\underline{e})$, where $\underline{e} \in Q(e_0,\ldots,e_k)$ and $k = L(x)$, can be considered equivalent.

We shall use the notation $\#(A)$ to denote the cardinality of the set A.

Theorem 1: Let $A = (\Sigma,S,G,\pi_0,F,E)$ be an ADE such that $\#(E) < \infty$. There exists a PA A' over some finite alphabet Σ' which simulates A.

Proof: Let $m = \#(\Sigma)$ and $\mu = \#(E)$. Without loss of generality we shall let $\Sigma = \{1,2,\ldots,m\}$ and $E = \{0,1,\ldots,\mu-1\}$. Let us consider $\Sigma' = \{0,1,\ldots,\mu(m+1)-1\}$. Let ϕ be any symbol such that $\phi \notin \Sigma$. We now define a mapping g: $(\Sigma \cup \{\phi\}) \times E \rightarrow \Sigma'$ as follows:

$$g(\phi,e) = e \qquad \forall e \in E$$
$$g(\sigma,e) = e + \sigma\mu \qquad \forall e \in E \quad \text{and} \quad \forall \sigma \in E$$

This mapping is a one-to-one correspondence. We extend g to $((\Sigma \cup \{\phi\}) \times E)^*$, the set of all finite sequences of elements of $(\Sigma \cup \{\phi\}) \times E$, by component-wise application of the mapping and concatenation of the results. That is, $a_1,a_2,\ldots,a_k \in (\Sigma \cup \{\phi\}) \times E$, then $a_1a_2\ldots a_k \in ((\Sigma \cup \{\phi\}) \times E)^*$ and $g(a_1a_2\ldots a_k) = g(a_1)g(a_2)\ldots g(a_k)$. The extension of g is a one-to-one correspondence between $((\Sigma \cup \{\phi\}) \times E)^*$ and Σ'^*. Let b be any nonempty element of $((\Sigma \cup \{\phi\}) \times E)^*$, then b is isomorphic to $(y,(e_0,\ldots,e_k))$, where for some $k \geq 0$, $y \in (\Sigma \cup \{\phi\})^*$, $L(y) = k+1$, and

$e_0,\ldots,e_k \in E$. We define the relation R to be the set of all elements of the form:

$$[g(\phi x,(e_0,\ldots,e_k)),\ (x,\underline{e})], \quad \text{where } x \in \Sigma^*,$$

$$L(x) = k, \underline{e} = (e_0,\ldots,e_k,\ldots) \quad \text{and } k = 0,1,\ldots\ .$$

Note that when $x = \Lambda \in \Sigma^*$, then $L(\Lambda) = 0$ and $\phi\Lambda = \phi$.

We shall now construct a PA A' to simulate A. Let $A' = (S', M', \pi_0', F')$ be a PA over Σ'. Let $S' = S \cup \{s_{n+1}, s_{n+2}\}$, where $S = \{s_1,\ldots,s_n\}$ is the state set of A. For any $\sigma' \in E \subset \Sigma'$, then $0 \le \sigma' \le \mu-1$. For any σ' such that $0 \le \sigma' \le \mu-1$, we define

$$M'(\sigma') = \begin{pmatrix} & 0 & & \begin{matrix} 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{matrix} \\ \pi_0^{(1)}(\sigma') & \cdots & \pi_0^{(n)}(\sigma') & \begin{matrix} 0 & 0 \end{matrix} \\ 0 & \cdots & 0 & \begin{matrix} 0 & 1 \end{matrix} \end{pmatrix}$$

where $\pi_0^{(i)}(\sigma')$ is the i-th component of $\pi_0(\sigma')$. Note that $0 \le \sigma' \le \mu-1$ implies $\sigma' \in E$. These values of σ' are used to incorporate the initial distribution function of A into the PA model A'. The values $\sigma' \in E'$ such that $\mu \le \sigma' \le \mu(m+1)-1$ are used in order that A' can imitate the basic matrix transition function A. The set $\{\sigma' \mid \sigma' \in \Sigma', \mu \le \sigma' \le \mu(m+1)-1\}$ is in one-to-one correspondence with the set $\Sigma \times E$ under the mapping g. So for $\mu \le \sigma' \le \mu(m+1)-1$, define

$$M'(\sigma') = \begin{pmatrix} G(g^{-1}(\sigma_i')) & \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \\ 0 & \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

where G is the basic matrix transition function of A. Also, let $\pi_0' = (0,\ldots,0,1,0)$ and $F' = F$. Therefore, $\eta^{F'} = \begin{pmatrix} \eta^F \\ \cdots \\ 0 \\ 0 \end{pmatrix}$.

It is clear from the definition of R that for any $(x,\underline{e}) \in \Sigma^* \times E^\infty$ there exists a unique $x' = g(\phi x,(e_0,\ldots,e_k))$, where $k = L(x)$, such that $[x',(x,\underline{e})] \in R$.

Let $p'(x')$ denote the probability that A' accepts x' and $p(x,\underline{e})$ denote the probability that A accepts x in total environment $\underline{e}$.

Suppose $[x',(\Lambda,\underline{e})] \in R$. Therefore, $x' = g(\phi,e_0) = e_0$. So

$$p'(x') = p'(e_0) = \pi_0' M'(e_0)\eta^{F'}$$

$$= (0,\ldots,0,1,0) \begin{pmatrix} & 0 & & \begin{matrix} 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{matrix} \\ \pi_0^{(1)}(e_0) & \cdots & \pi_0^{(n)}(e_0) & \begin{matrix} 0 & 1 \end{matrix} \\ 0 & \cdots & 0 & \begin{matrix} 0 & 1 \end{matrix} \end{pmatrix} \begin{pmatrix} \eta^F \\ \cdots \\ 0 \\ 0 \end{pmatrix}$$

$$= \pi_0(e_0)\eta^F = p(\Lambda,\underline{e}) .$$

Thus, for each $\lambda \in [0,1)$, $p'(x') = p'(e_0) > \lambda$ iff $p(\Lambda,\underline{e}) > \lambda$.

That is, if $[x',(\Lambda,\underline{e})] \in R$, then for each $\lambda \in [0,1)$, $x' \in T(A',\lambda)$ iff $\Lambda \in T(A,\underline{e},\lambda)$.

Suppose $[x',(x,\underline{e})] \in R$, where $x \ne \Lambda$. Hence, $L(x) > 0$. By the definition of R we have that $x' = g(\phi x,(e_0,\ldots,e_k))$, where $x = \sigma_1\ldots\sigma_k$ has length k. So $x' = g(\phi,e_0)g(\sigma_1,e_1)\ldots g(\sigma_k,e_k) = e_0\sigma_1'\ldots\sigma_k'$. Thus, $p'(x') = p'(e_0\sigma_1'\ldots\sigma_k') = \pi_0' M(e_0\sigma_1'\ldots\sigma_k')\eta^{F'}$. But

$$\prod_{i=1}^{k} M'(\sigma_i') = \prod_{i=1}^{k} \begin{pmatrix} G(g^{-1}(\sigma_i')) & \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

$$= \begin{pmatrix} M(x,\underline{e}) & \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Hence we have

$$p'(x') = \pi_0' M'(e_0) \begin{pmatrix} M(x,\underline{e}) & \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0\ldots0 \\ 0\ldots0 \end{matrix} & \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} \end{pmatrix} \eta^{F'}$$

$$= (0,\ldots,0,1,0) \begin{pmatrix} & 0 & & \begin{matrix} 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{matrix} \\ \pi_0^{(1)}(e_0) & \cdots & \pi_0^{(n)}(e_0) & \begin{matrix} 0 & 1 \end{matrix} \\ 0 & \cdots & 0 & \begin{matrix} 0 & 1 \end{matrix} \end{pmatrix} \begin{pmatrix} M(x,\underline{e}) & \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0\ldots0 \\ 0\ldots0 \end{matrix} & \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} \end{pmatrix} \begin{pmatrix} \eta^F \\ \cdots \\ 0 \\ 0 \end{pmatrix}$$

$$= \pi_0(e_0)M(x,\underline{e})\eta^F = p(x,\underline{e}) .$$

So, if $[x',(x,\underline{e})] \in R$ and $x \ne \Lambda$, then for each $\lambda \in [0,1)$ $p'(x') > \lambda$ iff $p(x,\underline{e}) > \lambda$. Thus, in this case, we have $x' \in T(A',\lambda)$ iff $x \in T(A,\underline{e},\lambda)$ for each $\lambda \in [0,1)$.

We have verified that $\forall x \in \Sigma^*$ and $\forall \underline{e} \in E^\infty$ if $[x',(x,\underline{e})] \in R$, then for each $\lambda \in [0,1)$, $x' \in T(A',\lambda)$ iff $x \in T(A,\underline{e},\lambda)$.

Let $x' \epsilon \Sigma^*$ such that there does not exist $x \epsilon \Sigma^*$ and $\underline{e} \epsilon E^\infty$ so that $[x',(x,\underline{e})] \epsilon R$. Hence, there does not exist $x \epsilon \Sigma^*$ and $\underline{e} \epsilon E^\infty$ such that $x' = g(\phi x, (e_0, \ldots, e_k))$, where $k = L(x)$. Therefore, such an x' is not admissible. However, $x' = g(b)$ for some $b \epsilon ((\Sigma \cup \{\phi\}) \times E)^*$ since g is a one-to-one correspondence between $((\Sigma \cup \{\phi\}) \times E)^*$ and Σ'^*. Λ', the empty tape for A', is the image under g of the empty tape in $((\Sigma \cup \{\phi\}) \times E)^*$. Clearly, Λ' is not admissible. $p'(\Lambda') = \pi'_0 \eta^{F'} = (0,\ldots,0,1,0) \begin{pmatrix} \eta^F \\ \vdots \\ 0 \\ 0 \end{pmatrix} = 0$. Hence $\Lambda' \notin T(A',\lambda)$ for any $\lambda \epsilon [0,1)$.

Let $x' \epsilon \Sigma'^*$ be any tape which is not admissible and $x' \neq \Lambda'$. $x' = \sigma'_0 \ldots \sigma'_k$ is the image under g of some element b $((\Sigma \cup \{\phi\}) \times E)^*$, where b has the form $(y,(e_0,\ldots,e_k))$ and $L(x') = L(y) = k+1$. Note that $y \epsilon (\Sigma \cup \{\phi\})^*$. Let $y = \tau_0 \tau_1 \ldots \tau_k$, where $\tau_i \epsilon \Sigma \cup \{\phi\}$ for $i = 0,1,\ldots,k$. x' is not admissible iff $\tau_0 \neq \phi$ or $\tau_i \notin \Sigma$ for some $i = 1,2,\ldots,k$. If $\tau_0 \neq \phi$, then $\tau_0 \epsilon \Sigma$. So $\sigma'_0 = g(\tau_0, e_0)$ and, hence, $\mu \le \sigma'_0 \le \mu(m+1)-1$. By our definition of M', we see that with input $\sigma'_0 \sigma'_1 \ldots \sigma'_k$, A' enters state s_{n+2} at the first transition. But s_{n+2} is an absorbing state and $s_{n+2} \notin F = F'$. Hence, $p'(x') = 0$. So $x' \epsilon T(A',\lambda)$ for any $\lambda \epsilon [0,1)$. If $\tau_i \notin \Sigma$ for some $i = 1,2,\ldots,k$; say $\tau_j \notin \Sigma$. Then $\tau_j = \phi$. So $\sigma'_j = g(\tau_j, e_j) = g(\phi, e_j) = e_j$ and hence, $0 \le \sigma'_j \le \mu-1$. By our definition of M' we see that with input $\sigma'_0 \sigma'_1 \ldots \sigma'_k$, A' enters state s_{n+2} at the (j+1)-th transition. Again, s_{n+2} is an absorbing state and $s_{n+2} \notin F = F'$. Hence, $p'(x') = 0$ and $x' \notin T(A',\lambda)$ for any $\lambda \epsilon [0,1)$.

Thus, if for some $\lambda \epsilon [0,1)$, $x' \epsilon T(A',\lambda)$, then x' is admissible; that is, there exists $(x,\underline{e}) \epsilon \Sigma^* \times E^\infty$ such that $[x',(x,\underline{e})] \epsilon R$.

Consequently, A', as constructed, simulates A. □

The results of the previous section and Theorem 1 seem to present a paradox. We have found for a particular Σ such that there exists $U \subset \Sigma^*$ which is not a PCE; that is, for any $\lambda \epsilon [0,1)$ there is no PA A_p such that $U = T(A_p,\lambda)$. However, we showed that there exists an ADE A with finite environment set over Σ such that $U = T(A,\underline{e},\lambda)$ for some set $\underline{e} \epsilon E^\infty$ and $\lambda \epsilon [0,1)$. By Theorem 1 there exists a PA A' over an expanded alphabet Σ' such that A' simulates A. Let $U' = \{g(\phi x, \underline{e}) | x \epsilon U\}$, where $\underline{e}$ is the total environment whereby $U = T(A,\underline{e},\lambda)$. $U' = T(A',\lambda)$ and, hence, is a PCE. Thus we not only have expanded Σ to Σ' but also enriched the alphabet with a structure that had been included in the environment. The problem for $U \subset \Sigma^*$ remains because U' has a different structure within Σ'^*. In fact, we can find a subset of Σ'^* which is not a PCE.

page complete

<u>Corollary 1</u>: Let $A = \{\Sigma,S,G,\pi_0,F,E\}$ be an ADE. For any fixed $\underline{e} \epsilon E^\infty$ such that $E = \{e_i | i=0,1,\ldots\}$ is a finite set, then there exists a PA A' over some alphabet Σ' such that for any $x \epsilon \Sigma^*$ and $\lambda \epsilon [0,1)$, $x \epsilon T(A,\underline{e},\lambda)$ iff $g_E(x,\underline{e}) \epsilon T(A',\lambda)$, where g_E is defined for the ADE $A_E = (\Sigma,S,G,\pi_0,F,E)$ as the function g in theorem 1.

4. Reducing the Environment Set

Let $A = (\Sigma,S,G,\pi_0,F,E)$ be an ADE and let e and e' be two distinct configurations of the environment. If $\pi_0(e) = \pi_0(e')$ and $G(\sigma,e) = G(\sigma,e')$ for all $\sigma \epsilon \Sigma$, then e and e' have identical effects on the system. That is, the configurations e and e' are equivalent in the sense that for any input tape $x \epsilon \Sigma^*$ and total environment $\underline{e} \epsilon E^\infty$, $\pi(x,\underline{e})$ remains constant if we substitute e for e' whenever e' occurs in $\underline{e}$. Thus, we do not need to include e' in the environment since it does not augment the structure of A. Moreover, we can reduce E so that no two distinct environments are equivalent in the above sense.

<u>Definition 7</u>: Let $A = (\Sigma,S,G,\pi_0,F,E)$ be an ADE, then we say E is *irreducible* if for every $e,e' \epsilon E$, $e = e'$ iff $\pi_0(e) = \pi_0(e')$ and $G(\sigma,e) = G(\sigma,e')$ for every $\sigma \epsilon E$.

Given any arbitrary n-dimensional vector ξ, we define $|\xi| = \max_i |\xi_i|$. Also, for any n×n matrix Q we define $|Q| = \max_{i,j} |q_{ij}|$.

Let us assume that E is an irreducible environment set for the ADE $A = (\Sigma,S,G,\pi_0,F,E)$. Consider the function $d(e,e') = \max(|\pi_0(e) - \pi_0(e')|, \max_{\sigma \epsilon \Sigma} |G(\sigma,e) - G(\sigma,e')|)$, where $e,e' \epsilon E$. Clearly, $d(e,e') = d(e',e)$ and $d(e,e') \ge 0$ for all $e,e' \epsilon E$. When $e = e'$, then $d(e,e') = 0$. Also, for any $e,e',e'' \epsilon E$, $d(e,e') \le d(e,e'') + d(e'',e')$. Since E is irreducible, if $e \neq e'$, then by definition $d(e,e') > 0$. So the function d, as defined above, is a metric on E. Since irreducibility is needed only to show that if $e \neq e'$, then $d(e,e') \neq 0$, the function d is a pseudo-metric on any arbitrary environment set.

Let us now consider an arbitrary ADE $A = (\Sigma,S,G,\pi_0,F,E)$ for which the environment set is reducible to a set $E^\dagger$. That is, there is an onto mapping $h: E \to E^\dagger \subset E$, where if $h(e) = e^\dagger$, then $d(e,e^\dagger) = 0$. The mapping h is unique only when $E^\dagger$ is irreducible. h can be extended to $h: E^\infty \to (E^\dagger)^\infty$ by component-wise application. That is, $h(\underline{e}) = (h(e_0),h(e_1),\ldots)$. So for all $e \epsilon E$, $d(e,h(e)) = 0$ and, hence, we know that $\pi_0(e) = \pi_0(h(e))$ and $G(\sigma,e) = G(\sigma,h(e))$ for all $\sigma \epsilon \Sigma$. So by extension, $M(x,\underline{e}) = M(x,h(\underline{e}))$, $\pi(x,\underline{e}) = \pi(x,h(\underline{e}))$, and $p(x,\underline{e}) = p(x,h(\underline{e}))$ for all $x \in \Sigma^*$ and for all $\underline{e} \in E^\infty$.

<u>Theorem 2</u>: Let $A = (\Sigma,S,G,\pi_0,F,E)$ be any ADE, then $T(A,\underline{e},\lambda) = T(A,h(\underline{e}),\lambda)$ for all $\underline{e} \in E^\infty$ and $\lambda \in [0,1)$. Furthermore, if $\#(h(E)) < \infty$, then A can be simulated by a PA.

<u>Proof</u>. Let $\underline{e} \in E^\infty$. We have seen that $p(x,\underline{e}) = p(x,h(\underline{e}))$ for all $x \in \Sigma^*$. Then, clearly $T(A,\underline{e},\lambda) = T(A,h(\underline{e}),\lambda)$ for any $\lambda \in [0,1)$.

Let $E\dagger = h(E)$. So if $\#(E\dagger) < \infty$, then by theorem 1 there exists PA A' over alphabet Σ' which simulates the ADE $A\dagger = (\Sigma,S,G,\pi_0,F,E\dagger)$. Thus there exists a relation $R\dagger$ between Σ'^* and $((\Sigma \cup \{\phi\}) \times E\dagger)^*$ satisfying definition 6. We now define a relation R between Σ'^* and $((\Sigma \cup \{\phi\}) \times E)^*$ by the rule:

$$[x',(x,\underline{e})] \in R \quad \text{iff} \quad [x',(x,h(\underline{e}))] \in R\dagger .$$

It is easily verified that R is the relation to justify that A' simulates A. Note that since he is an onto mapping, x' is not admissible under the relation R iff x' is not admissible under the relation $R\dagger$. □

<u>Corollary 2</u>: Let $A = (\Sigma,S,G,\pi_0,F,E)$ be an ADE. For any fixed $\underline{e} \in E^\infty$ such that $E = \{e_i | i=0,1,\dots\}$ is reducible to a finite set, then there exists a PA A' over some alphabet Σ' such that for any $x \in \Sigma^*$ and $\lambda \in [0,1)$, $x \in T(A,\underline{e},\lambda)$ iff $g(x,h(\underline{e})) \in T(A',\lambda)$, where g is defined in theorem 1 and h is defined in theorem 3.

Acknowledgement

We would like to acknowledge the influence of Gerard P. Weeg who initiated and spurred our interest in the area of automata theory.

This research was supported by the Air Force Office of Scientific Research under Contract AFOSR-~~75-2706~~. 75-2140

References

1. Paz, A. (1966). Some aspects of probabilistic automata. *Information and Control*, *9*, 26-60.

2. Paz, A. (1970). *Probabilistic Automata*. Academic Press, New York.

3. Rabin, M.O. (1963). Probabilistic automata. *Information and Control*, *6*, 230-245.

Keywords and Phrases: Automaton; probabilistic automaton; automaton in environments; probabilistic tables; cut-point; probabilistic cut-point event; environment.

A.M.S. Classification No.: 94A30; 94A35; 68A25; 60J20.