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OPTIMUM QUANTIZATION FOR MEMORYLESS DETECTION IN m-DEPENDENT NOISE

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Abstract

The problem of optimum data quantization for memoryless signal-detection systems operating in m-dependent noise environments is considered. The case where quantizer breakpoints are fixed or predetermined is considered first, and, for this case, existing results for general (unquantized) memoryless detection are modified to yield necessary and sufficient conditions for quantizer optimization in terms of asymptotic efficiency. Necessary conditions are also established for the optimum (asymptotically efficient) selection of quantizer breakpoints, and expressions are presented for the comparison of quantizer-detector performance on the basis of asymptotic relative efficiency.

1. Introduction

Signal detection systems frequently involve memoryless nonlinear operations on input data, and there are a number of practical advantages to replacing these detection nonlinearities with quantizers. Several authors have considered the problem of optimally designing quantizers for this purpose, and design criteria have been developed for quantizers to be used in a variety of signal and noise situations [1-3]. Many of these previous studies are based primarily on the assumption of independent sampling, and it is the purpose of this paper to extend some of these previous results to the case where there is dependence among the observed data samples.

To study this problem we consider the large-sample-size case of detecting known constant signals in additive noise, and model the dependence structure by assuming that the noise process is m-dependent. In general, an optimal detection procedure for this situation will require a memory of length m; however, we will restrict our study to those detectors which may be implemented without memory. The design of general (unquantized) memoryless detectors for this situation has been considered in [4] and here we apply the techniques of this earlier study to the corresponding quantizer design problem.

In Section 2 we state more precisely the problem to be considered and review previous pertinent results concerning general memoryless detection from [4] and optimum quantization from [1]. Section 3 includes the derivation of design equations for optimum quantizer design for the case where the breakpoints are fixed, as well as optimization for the general quantizer case; and expressions are presented here for the comparison of detector performance for large sample sizes. In Section 4, four-level quantization is considered in detail for two particular m-dependent noise processes, and the performance of resulting quantizer designs is compared to that of previous designs for the corresponding case where independence was assumed.

2. Preliminaries

2.1. Problem Statement and Assumptions

As noted above, we consider the detection of a known constant signal in additive m-dependent noise. Specifically, we assume that we have a sequence $\underline{x} = \{x_i;\ i=1,\ldots,n\}$ of real observations of a process $\underline{X} = \{X_i;\ i=1,\ldots,n\}$; and, on the basis of $\underline{x}$, we wish to decide between the following pair of hypotheses concerning $\underline{X}$:

$$H_0:\ X_i = N_i \quad ;\ i=1,\ldots,n$$
versus (1)
$$H_1:\ X_i = N_i + \theta \quad ;\ i=1,\ldots,n$$

where $\{N_i\}_{i=1}^{\infty}$ is a zero-mean second-order-stationary m-dependent noise process and θ is a known positive constant signal. By m-dependent, we mean that there is an integer m such that the sequences $\{N_i,\ i=1,\ldots,\zeta\}$ and $\{N_i;\ i=\xi,\ldots,\infty\}$ are independent whenever $\xi - \zeta > m$ [note that $m=0$ gives an independent process]. We will assume throughout that f, the common univariate probability density of the noise sequence, is symmetric, continuously differentiable, and strictly positive on the entire real line. Our study is restricted to the asymptotic (large-sample-size) case and, to avoid singularity, we will consider the local or small-signal limit.

2.2. Optimum Memoryless Detection

In [4], the design of optimum memoryless detectors for the problem of Eq. (1) is considered. Because techniques similar to those of [4] will be applied in this study, the results of [4] are summarized here.

Consider memoryless detectors of the form

$$\phi(g;\underline{x}) = \begin{cases} 1 ; & \sum_{i=1}^{n} g(x_i) > \tau \\ \gamma ; & \sum_{i=1}^{n} g(x_i) = \tau \\ 0 ; & \sum_{i=1}^{n} g(x_i) < \tau \end{cases} \qquad (2)$$

where g is a memoryless nonlinearity; $\phi(g;\underline{x})$ is the probability with which we accept H_1 when $\underline{x}$ is observed; and the randomization γ and threshold τ are chosen to give desired error

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probability performance. Note that the class of detectors of the form of Eq. (2) is sufficiently general to contain most memoryless detectors of interest.

For large-sample-size situations, the performance of detectors is commonly compared on the basis of asymptotic relative efficiency (ARE). For the situation of Eq. (1), the asymptotic efficiency of a detector ϕ_1 relative to another detector ϕ_2 is defined as [5]

$$\mathrm{ARE}(1,2) = \lim_{\substack{n\to\infty \\ \theta\to 0}} e(\alpha,\theta,n) \tag{3}$$

where $e(\alpha,\theta,n)$ is the relative number of samples ϕ_2 requires to achieve the same power (i.e., probability of correct detection) that ϕ_1 achieves for sample size n when both are operating at false-alarm probability α and the signal strength is θ. In [4], the optimum memoryless detector is considered to be that detector of the form of Eq. (2) which is the most efficient asymptotically in the sense of Eq. (3). By considering a sequence of signal strengths $\theta_n = K/\sqrt{n}$ where $K > 0$, it can be shown under mild restrictions that the asymptotic efficiency of a detector $\phi(g_1;\cdot)$ of the form of Eq. (2) relative to another detector $\phi(g_2;\cdot)$ of the form of Eq. (2) is given by

$$\mathrm{ARE}(g_1,g_2) = \eta(g_1)/\eta(g_2) \tag{4}$$

where

$$\eta(g) = \left(\int gf'\right)^2/\sigma_0^2(g) \tag{5}$$

is the efficacy of $\phi(g;\cdot)$; the function f is the noise density; and the quantity $\sigma_0^2(g)$ is defined as

$$\sigma_0^2(g) = \mathrm{Var}_0[g(X_1)] + 2\sum_{j=1}^{m} \mathrm{Cov}_0[g(X_1),g(X_{j+1})] \ . \tag{6}$$

Note that the subscript 0 denotes quantities computed under the signal-absent hypothesis H_0.

We see from Eq. (4) that the most efficient memoryless detector of the form of Eq. (2) is based on a nonlinearity g_0 solving

$$g_0 = \arg\{\max_g \eta(g)\} \tag{7}$$

In [4] it is shown under mild restrictions that g_0 solves Eq. (7) if, and only if, g_0 (or some constant multiple of g_0) satisfies the integral equation

$$g_{f0}(x) - \int_{-\infty}^{\infty} K(x,y)g_0(y)dy = g_0(x) \tag{8}$$

for all $x \in (-\infty,\infty)$. Here $g_{f0} \triangleq -f'/f$ is the optimum choice of nonlinearity for the problem in Eq. (1) when the noise process is an independent sequence (m=0); and the kernel K(x,y) is given by

$$K(x,y) = \sum_{j=1}^{m} [f_{j,1}(y|x) + f_{1,j}(y|x)] \tag{9}$$

where $f_{j,1}$ is the conditional density of N_{j+1} given N_1 and vice-versa for $f_{1,j}$.

Equation (8) is a Fredholm equation of the second kind, and under further mild conditions, the solution is given by

$$g_0(x) = g_{f0}(x) - \sum_{\nu=0}^{\infty} (c_\nu\lambda_\nu)/(1+\lambda_\nu)\varphi_\nu(x) \tag{10}$$

where equality is in the sense of uniform convergence. Here $\{\varphi_\nu;\nu=0,\ldots,\infty\}$ is a sequence of functions, orthogonal with respect to f, satisfying

$$\lim_{N\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| K(x,y)/f(y) - \sum_{\nu=0}^{N} \lambda_\nu\varphi_\nu(x)\varphi_\nu(y)\right|^2 f(x)f(y)dxdy = 0 \tag{11}$$

where

$$\lambda_\nu = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_\nu(x)\varphi_\nu(y)K(x,y)f(x)dxdy; \ \nu=0,1,\ldots \tag{12}$$

and the coefficients c_ν are given by

$$c_\nu = \int g_{f0}\varphi_\nu f \ ; \ \nu=0,1,\ldots \ . \tag{13}$$

If g is another nonlinearity in the span of $\{\varphi_\nu;\nu=0,\ldots,\infty\}$, then the ARE of $\phi(g_0;\cdot)$ to $\phi(g;\cdot)$ is given by

$$\mathrm{ARE}(g_0,g) = \left[\sum_{\nu=0}^{\infty} c_\nu^2/(1+\lambda_\nu)\right]\cdot\left[\sum_{\nu=0}^{\infty} \beta_\nu^2(1+\lambda_\nu)\right]/\left(\sum_{\nu=0}^{\infty} c_\nu\beta_\nu\right)^2 \tag{14}$$

where $\beta_\nu = \int g\varphi_\nu f \ ; \ \nu=0,1,\ldots$. Further details of these results can be found in [4].

2.3. Optimum Quantization for the Independent (m=0) Case

In this paper we will consider the subclass of the class of memoryless detectors of Eq. (2) for which the nonlinearity g is an M-level quantizer, that is, we consider the class of all detectors of the form

$$\phi(Q;\underline{x}) = \begin{cases} 1 \ ; & \\ \gamma \ ; & \sum_{i=0}^{n} Q(x_i) = \tau \\ 0 \ ; & \end{cases} \tag{15}$$

where Q is an M-level quantizer; and we wish to choose Q in an optimal way for the detection problem of Eq. (1). Note that an M-level quantizer can be parameterized by a pair $Q \equiv (\underline{t},\underline{q})$ where $\underline{t}=(t_0,t_1,\ldots,t_M)$ is an ordered breakpoints vector $(-\infty = t_0 < t_1 < \ldots < t_{M-1} < t_M = +\infty)$; $\underline{q} \in R^M$ is a levels vector; and we take

$$Q(x) = q_k \text{ when } x \in (t_{k-1},t_k] \ ; \ k=1,\ldots,M \tag{16}$$

Thus the problem of optimal quantizer design is one of performance optimation over a set of (2M-1) real parameters.

The problem of optimal quantizer-detector design for the independent noise (m=0) case of Eq. (1) has been considered by Kassam in [1]. Specifically it is shown in [1] that, for this case, if we fix the breakpoints $\underline{t}$ then an optimum (asymptotically-most-efficient) choice of the levels vector $\underline{q}$ is given by

$$q_k^{\ell o} = [f(t_{k-1}) - f(t_k)]/\int_{t_{k-1}}^{t_k} f(x)dx \; ; \; k=1,\ldots,M \quad (17)$$

where f is the univariate noise density. Thus, for each $\underline{t}$, the optimum choice of levels $\underline{q}^{\ell o}$ can be written in terms of $\underline{t}$ and the problem of choosing an optimal quantizer Q is reduced to that of choosing an optimal breakpoints vector $\underline{t}^{\ell o}$. It is shown in [ ] that the optimum $\underline{t}$ for this case (m=0) must satisfy

$$g_{\ell o}(t_k^{\ell o}) = [q_k^{\ell o} + q_{k+1}^{\ell o}]/2 \; ; \; k=1,\ldots,(M-1) \quad (18)$$

where $\underline{q}^{\ell o}$ is from Eq. (17) and the function $g_{\ell o} = f'/f$. Thus a simultaneous solution to Eqs. (17) and (18) will yield the optimum quantizer parameters for the independent noise case of Eq. (1). Further details may be found in [1].

3. Optimum Quantization for m-Dependent Noise

3.1. The Efficacy of Quantizer-Detectors

We see from the results summarized in Section 2 that the optimum memoryless nonlinearity for detecting a constant signal in m-dependent noise [Eq. (1)] is given by Eq. (10). Similarly, the optimum quantizer-detector parameters for the independent noise case (m=0) of Eq. (1) are seen to be solutions to Eqs. (17) and (18). In this section we derive quantizer design equations analogous to those Eqs. (17) and (18) for the general case of m-dependent noise using analysis similar to that used to derive Eq. (10).

We consider the class of memoryless quantizer-detectors $\phi(Q;\cdot)$ of the form of Eq. (15). As in Eq. (16), the quantizer Q is parameterized by identifying $Q \equiv (\underline{t},\underline{q})$ where $\underline{t}$ and $\underline{q}$ are the breakpoints and levels of Q, respectively. Adopting the optimality criterion of maximum ARE, we wish to choose vectors $\underline{t}^0$ and $\underline{q}^0$ so that the quantizer $Q^0 = (\underline{t}^0,\underline{q}^0)$ is optimum for the problem of Eq. (1). Within minor restrictions (as in [4]), the relative efficiency of one quantizer-detector $\phi(Q^1;\cdot)$ of the form of Eq. (15) relative to another quantizer-detector $\phi(Q^2;\cdot)$ of the form of Eq. (15) is given by

$$\text{ARE}(Q^1,Q^2) = \eta(Q^1)/\eta(Q^2) \quad (19)$$

where, as in Eq. (15), the efficacy of $\phi(Q;\cdot)$ is given by

$$\eta(Q) = (\int Qf')^2/\sigma_0^2(Q), \quad (20)$$

and $\sigma_0^2(Q)$ is defined by Eq. (6). Thus the optimum M-level quantizer Q^0 will be given by

$$Q^0 = \arg\{\max_{Q\in\mathcal{P}_M} \eta(Q)\} \quad (21)$$

where $\mathcal{P}_M$ is the class of all M-level quantizers (Here we must introduce the restriction to those Q for which $\sigma_0^2(Q) > 0$. Without this property, Eq. (19) does not hold. Note that this is not a very restrictive limitation since we always have $\sigma_0^2(Q) \geq 0$, and the case $\sigma_0^2(Q)=0$ is of limited interest.)

Since Q is parameterized by the vectors $\underline{t}$ and $\underline{q}$, the efficacy $\eta(Q)$ is a function of (2M-1) real parameters (note that t_0 and t_M are fixed) so we may write explicitly $\eta(Q) \equiv \eta(\underline{t},\underline{q})$. Thus Eq. (21) is equivalent to finding $Q^0 = (\underline{t}^0,\underline{q}^0)$ with

$$(\underline{t}^0,\underline{q}^0) = \arg\{\max_{(\underline{t},\underline{q})} \eta(\underline{t},\underline{q})\} \quad (22)$$

Note from Eq. (20) that $\eta(Q)$ is invariant to additive constants, that is $\eta(Q+\beta) = \eta(Q)$ for any constant β. Note also that f is assumed to be symmetric; thus, we may, without loss of generality, restrict our study to those quantizers Q which have zero mean under H_0; i.e., we restrict Q to satisfy $E_0[Q(X_j)] = \int Qf = 0$.

It can be shown that $\eta(\underline{t},\underline{q})$ is given by

$$\eta(\underline{t},\underline{q}) = (\underline{q}^T\underline{\Delta f})^2/[\underline{q}^T(\underline{F} + \underline{P})\underline{q}] \quad (23)$$

where the superscript T denotes transposition. Here the vector $\underline{\Delta f}$ has components

$$(\Delta f)_k = [f(t_k) - f(t_{k-1})]; \; k=1,\ldots,M; \quad (24)$$

the matrix $\underline{F} = \text{diag}\{f_1,\ldots,f_M\}$ where

$$f_k = \int_{t_{k-1}}^{t_k} f(x)dx \; ; \; k=1,\ldots,M; \quad (25)$$

and the M × M matrix $\underline{P}$ has entries

$$P_{k,\ell} = \sum_{j=1}^{m} [P_0[X_1\in(t_{k-1},t_k],X_{j+1}\in(t_{\ell-1},t_\ell]] + P_0[X_1\in(t_{\ell-1},t_\ell],X_{j+1}\in(t_{k-1},t_k]]]; \quad (26)$$

$$k,\ell=1,\ldots,M$$

where P_0 denotes probability under the signal-absent hypothesis H_0. Recall that f is the univariate density, m is the dependence parameter, and M is the dimension (i.e., number of levels) of the quantizer.

3.2. Results for Fixed Breakpoints - Optimum Choice of Levels

As in the independent noise case of [1], it is convenient to consider first the situation where the breakpoints vector $\underline{t}$ is predetermined or fixed. Here we are free to choose only the levels, and in this case we look for optimum levels $\underline{q}$ by searching for a solution $\underline{q}^0$ to satisfy

$$\underline{q}^0 = \arg\{\max_{\underline{q}\in R^M} \eta(\underline{t},\underline{q})\} \quad (27)$$

To do so, we assume that the matrix $(\underline{F} + \underline{P})$ is

positive definite, where $\underset{\sim}{F}$ and $\underset{\sim}{P}$ are from Eqs. (25 and (26), respectively*. Within this restriction, a necessary condition for a solution to Eq. (27) is that

$$\operatorname{grad}_{\underline{q}} \eta(\underline{t},\underline{q})\Big|_{\underline{q}=\underline{q}^0} = 0 \qquad (28)$$

This leads to the condition

$$(\underline{\Delta f}) + (\underset{\sim}{F}+\underset{\sim}{P})\underline{q}^0 = 0 \qquad (29)$$

where $\underline{\Delta f}$, $\underset{\sim}{F}$, and $\underset{\sim}{P}$ are defined by Eqs. (24) through (26). [Note that the dependence on $\underline{t}$ of $\underline{\Delta f}$, $\underset{\sim}{F}$, $\underset{\sim}{P}$ and $\underline{q}^0$ is implicit in Eq. (29).] Equation (29) can also be shown to be a sufficient condition for $\underline{q}^0$ to solve Eq. (27).

Since the noise density f is assumed to have support $(-\infty,\infty)$, the terms $(f_1,\ldots,f_M)$ on the diagonal of the matrix $\underset{\sim}{F}$ are all positive. Thus, $\underset{\sim}{F}$ is invertible and Eq. (29) can be rewritten in a more intuitively appealing form:

$$\underline{q}^{\ell o} - \underset{\sim}{K}\,\underline{q}^0 = \underline{q}^0 \qquad (30)$$

where the M x M matrix $\underset{\sim}{K} = \underset{\sim}{F}^{-1}\underset{\sim}{P}$ has entries

$$K_{k,\ell} = P_{k,\ell}/f_k \;;\quad k,\ell=1,\ldots,M \qquad (31)$$

and $P_{k,\ell}$ and f_k are from Eqs. (25) and (26), and the vector $\underline{q}^{\ell o} = \underset{\sim}{F}^{-1}(\underline{\Delta f})$ has components

$$q_k^{\ell o} = -(\Delta f)_k/f_k = [f(t_{k-1})-f(t_k)]/\int_{t_{k-1}}^{t_k} f(x)dx;\quad k=1,\ldots,M \qquad (32)$$

is the optimum choise of levels for the independent noise (m=0) case from Eq. (17). Comparing Eq. (30) to the integral equation [Eq. (8)] of Section 2, we see the analogy between the quantized (fixed-breakpoints) case and the continuous nonlinearity case.

The positive definiteness of $(\underset{\sim}{F}+\underset{\sim}{P})$ implies that $(\underset{\sim}{I}+\underset{\sim}{K})$ is invertible ($\underset{\sim}{I}$ is the M x M identity matrix). Thus we have the solution

$$\underline{q}^0 = (\underset{\sim}{I} + \underset{\sim}{K})^{-1}\underline{q}^{\ell o} \qquad (33)$$

Using the identity

$$(\underset{\sim}{I} + \underset{\sim}{K})^{-1} = \underset{\sim}{I} - \underset{\sim}{K}(\underset{\sim}{I} + \underset{\sim}{K})^{-1} \qquad (34)$$

Eq. (33) becomes

$$\underline{q}^0 = \underline{q}^{\ell o} - \underset{\sim}{K}(\underset{\sim}{I} + \underset{\sim}{K})^{-1}\underline{q}^{\ell o} \qquad (35)$$

Note that Eq. (35) is directly analogous to the continuous nonlinearity solution given by Eq. (10) of Subsection 2.2. Note also that Eq. (35) could be derived by applying the Schwarz inequality to the efficacy $\eta(Q)$. The approach here is used because of the analogy to that for the continuous nonlinearity case.

Even if $(\underset{\sim}{F}+\underset{\sim}{P})$ is not positive definite, Eq. (30) still has a solution given by

$$\underline{q}^0 = \underline{q}^{\ell o} - \underset{\sim}{K}(\underset{\sim}{I} + \underset{\sim}{K})^{+}\underline{q}^{\ell o} \qquad (36)$$

where the superscript + denotes the generalized inverse. Equation (36) can be verified by direct substitution into Eq. (30) if we note that the matrix $(\underset{\sim}{I}+\underset{\sim}{K})(\underset{\sim}{I}+\underset{\sim}{K})^{+}$ is the projection onto the range space of $(\underset{\sim}{I}+\underset{\sim}{K})$. Note however that, if $(\underset{\sim}{F}+\underset{\sim}{P})$ is not positive definite, then the necessity of Eq. (28) is not assured and the expression of Eq. (19) cannot be used for all quantizers Q. [Since $\sigma_0^2(Q)$ will equal zero for some Q.]

3.3. Results for Fixed Breakpoints-Efficacy Expressions

The efficacy $\eta(Q) \equiv \eta(\underline{t},\underline{q})$ of a quantizer-detector $\phi(Q;\cdot)$ can be computed from Eq. (23). For fixed $\underline{t}$, $\eta(\underline{t},\underline{q})$ is maximized when $\underline{q} = \underline{q}^0$, the solution from Eq. (33). Combining Eqs. (23) and (33) we have that, since $(\underset{\sim}{F}+\underset{\sim}{P})$ is invertible,

$$\max_{\underline{q}} \eta(\underline{t},\underline{q}) = \eta(\underline{t},\underline{q}^0) = (\underline{\Delta f})^T[\underset{\sim}{F}+\underset{\sim}{P}]^{-1}(\underline{\Delta f}) \qquad (37)$$

For the independent noise (m=0) case, we have $\underset{\sim}{P} \equiv 0$, and Eq. (36) reduces to

$$\eta(\underline{t},\underline{q}^0) = (\underline{\Delta f})^T\underset{\sim}{F}^{-1}(\underline{\Delta f}) = \sum_{k=1}^{M} \{[f(t_{k-1})-f(t_k)]^2/\int_{t_{k-1}}^{t_k} f(x)dx\}$$

which agrees with previously established results for this case [1].

The asymptotic efficiency of the quantizer-detector using $Q^0=(\underline{t},\underline{q}^0)$ relative to a quantizer-detector using $Q=(\underline{t},\underline{q})$ is given by the ratio of their efficacies

$$\mathrm{ARE}(Q^0,Q) = \eta(\underline{t},\underline{q}^0)/\eta(\underline{t},\underline{q}) \qquad (38)$$

Again since $\underline{t}$ is such that $(\underset{\sim}{F}+\underset{\sim}{P})$ is invertible, Eqs. (23), (37) and (38) imply that

$$\mathrm{ARE}(Q^0,Q) = \frac{[(\underline{\Delta f})^T(\underset{\sim}{F}+\underset{\sim}{P})^{-1}(\underline{\Delta f})][\underline{q}^T(\underset{\sim}{F}+\underset{\sim}{P})\underline{q}]}{[(\underline{\Delta f})^T\underline{q}]^2} \qquad (39)$$

Specifically, the improvement gained by using $\underline{q}^0$ over the independent-noise levels $\underline{q}^{\ell o} = -\underset{\sim}{F}^{-1}(\underline{\Delta f})$ is given by

$$\mathrm{ARE}(Q^0,Q^{\ell o}) = \frac{[(\underline{\Delta f})^T(\underset{\sim}{F}+\underset{\sim}{P})^{-1}(\underline{\Delta f})][(\underline{\Delta f})^T\underset{\sim}{F}^{-1}(\underset{\sim}{F}+\underset{\sim}{P})\underset{\sim}{F}^{-1}(\underline{\Delta f})]}{[(\underline{\Delta f})^T\underset{\sim}{F}^{-1}(\underline{\Delta f})]^2} \qquad (40)$$

If the breakpoints $\underline{t}$ are chosen so that they partition the real line into equiprobability segments (under H_0), then $\underset{\sim}{F}=\frac{1}{M}\underset{\sim}{I}$, and we have

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*It is easily shown that, with $Q=(\underline{t},\underline{q})$ and $\int Qf=0$, we have $\sigma_0^2(Q) = \underline{q}^T(\underset{\sim}{F}+\underset{\sim}{P})\underline{q} \geq 0$ where $\underset{\sim}{F}$ and $\underset{\sim}{P}$ are from Eqs. (25) and (26) respectively. Thus the matrix $(\underset{\sim}{F}+\underset{\sim}{P})$ is at least non-negative definite. Note further that if the kernel K(x,y) from Eq. (7) is a positive definite kernel, then $(\underset{\sim}{F}+\underset{\sim}{P})$ will be positive definite for all $\underline{t}$. This latter case holds, for example, for Gaussian noise processes.

$$\mathrm{ARE}(Q^0,Q^{\ell o})\Big|_{\underline{t}=\underline{t}_{equi}} = \frac{[(\underline{\Delta f})^T(\underline{I}+\underline{K})^{-1}(\underline{\Delta f})][(\Delta f)^T(\underline{I}+\underline{K})(\underline{\Delta f})]}{[(\underline{\Delta f})^T(\underline{\Delta f})]^2} \tag{41}$$

Note that Eq. (41) implies that, for $\underline{t}=\underline{t}_{equi}$,

$$\frac{(1+\lambda_{max})}{(1+\lambda_{min})} \geq \mathrm{ARE}(Q^0,Q^{\ell o})\Big|_{\underline{t}=\underline{t}_{equi}} \geq \frac{(1+\lambda_{min})}{(1+\lambda_{max})} \tag{42}$$

where λ_{max} is the maximum eigenvalue of $\underline{K}$ and λ_{min} is the minimum eigenvalue of $\underline{K}$. [Note that the right-hand side of Eq. (42) is trivially true since we have by definition of Q^0 that

$$\mathrm{ARE}(Q^0,Q^{\ell o}) \geq 1 \geq (1+\lambda_{min})/(1+\lambda_{max})].$$

3.4. Choosing Optimum Breakpoints

We turn now to the problem of choosing an optimal set of breakpoints $\underline{t}$. An overall optimum choice of quantizer $Q^0 \equiv (\underline{t}^0,\underline{q}^0)$ will be one for which $\eta(\underline{t},\underline{q})$ is a maximum over all possible choices of $\underline{t}$ and $\underline{q}$. From the results of Subsection 3.2 we have that, for fixed $\underline{t}$, the optimal choice of levels is made by choosing $\underline{q}^0$ from Eq. (33) corresponding to $\underline{t}$. Writing explicitly $\underline{q}^0=\underline{q}^0(\underline{t})$ to emphasize the dependence on $\underline{t}$ of the solution from Eq. (33), we have

$$\max_{\underline{q}} \eta(\underline{t},\underline{q}) = \eta(\underline{t},\underline{q}^0(\underline{t})). \tag{43}$$

Thus, as in the independent noise case of Subsection 2.3, the problem of maximizing $\eta(\underline{t},\underline{q})$ over $(\underline{t},\underline{q})$ can be reduced to that of maximizing $\eta(\underline{t},\underline{q}^0(\underline{t}))$ over $\underline{t}$; that is, we look for $\underline{t}^0$ to solve

$$\underline{t}^0 = \arg\{\max_{\underline{t}} \eta(\underline{t},\underline{q}^0(\underline{t}))\} \tag{44}$$

and the overall optimal quantizer will be given by $Q^0 \equiv (\underline{t}^0,\underline{q}^0(\underline{t}^0))$.

To look for solutions to Eq. (44), we restrict the noise process to be such that the matrix $(\underline{F}+\underline{P})$ is positive definite for every choice of $\underline{t}$. For this case, we have from Eq. (37) that

$$\eta(\underline{t},\underline{q}^0(\underline{t})) = (\Delta f)^T(\underline{F}+\underline{P})^{-1}(\Delta f) \tag{45}$$

A necessary condition for $\underline{t}^0$ to maximize (45) is that

$$\underset{\underline{t}}{\mathrm{grad}}\ \eta(\underline{t},\underline{q}^0(\underline{t}))\Big|_{\underline{t}=\underline{t}^0} = 0 \tag{46}$$

Using the identity

$$\partial(\underline{F}+\underline{P})^{-1}/\partial t_k = (\underline{F}+\underline{P})^{-1}[\partial(\underline{F}+\underline{P})/\partial t_k](\underline{F}+\underline{P})^{-1},$$

Eq. (49) can be shown to hold if

$$\underline{G}_{\ell o}(\underline{t}^0) = \underline{L}\,\underline{q}^0(\underline{t}^0) \tag{47}$$

where the vector $\underline{G}_{\ell o}(\underline{t}^0)=(g_{\ell o}(t_1^0),\ldots,g_{\ell o}(t_{M-1}^0))^T$; the function $g_{\ell o}=-f'/f$; and the (M-1)XM matrix $\underline{L}$ has entries

$$L_{k,\ell}=[\delta_{k,\ell}+\delta_{k+1,\ell}]/2+\left(\sum_{p=0}^{M} q_p^0(\partial P_{p,\ell}/\partial t_k)\Big|_{\underline{t}=\underline{t}^0}\right)\Big/[2(q_k^0-q_{k+1}^0)f(t_k^0)] \tag{48}$$

for $k=1,\ldots,(m-1)$ and $\ell=1,\ldots,M$. Here $\delta_{k,\ell}$ is the Kronecker delta and the matrix $\underline{P}$ is from Eq. (26).

Thus, combining Eqs. (33) and (47), an overall optimal quantizer $Q^0 \equiv (\underline{t}^0,\underline{q}^0)$ will be a solution to the equations

$$\underline{G}_{\ell o}(\underline{t}^0) = \underline{L}\,\underline{q}^0 \tag{49}$$

and

$$\underline{q}^0 = (\underline{I}+\underline{K})^{-1}\underline{q}^{\ell o} \tag{50}$$

where the quantities $\underline{q}^{\ell o}$, $\underline{K}$ and $\underline{L}$ are computed for $\underline{t}=\underline{t}^0$ from Eqs. (31) - (33) and (48). That is, under the above assumptions, Eqs. (49) and (50) are necessary conditions that must be satisfied by an optimum (most-efficient) quantizer for a memoryless decision between the hypothesis pair of Eq. (1). Given that $\underline{t}^0$ is optimal and $(\underline{F}+\underline{P})$ is positive definite, Eq. (50) is sufficient for $\underline{q}^0$ to be optimal; however, the sufficiency of Eq. (49) can be determined analytically only for special cases, and a numerical test must be used in the general case.

Note that for the independent-noise (m=0) case, the entries of the matrix $\underline{L}$ become

$$L_{k,\ell} = [\delta_{k,\ell} + \delta_{k+1,\ell}]/2 \qquad \begin{matrix} k=1,\ldots,(M-1 \\ \ell=1,\ldots,M \end{matrix}$$

and the matrix $\underline{P}$ is identically zero. Thus, for this case, Eqs. (49) and (50) reduce to Eqs. (17) and (18) so that Eqs. (49) and (50) agree with the previously established results for this case.

4. Example - 4-Level Quantization

Example 1 - Gaussian Noise

To illustrate the results of Section 3, we consider the particular problem of designing 4-level quantizers (M=4) for use in detecting known signals in m-dependent Gaussian noise.

Specifically, we assume that the noise process $\{N_i;i=1,\ldots,n\}$ of Eq. (1) arises from uniform sampling of a stationary Gaussian process N(t). We assume that N(t) has zero mean unit variance, and autocorrelation $\rho(\tau)=E\{N(t)N(t+\tau)\}$ given by

$$\rho(\tau) = \begin{cases} (1-|\tau|/D) & ;\ |\tau| \leq D \\ 0 & ;\ |\tau| > D \end{cases} \tag{51}$$

where $D > 0$. The samples are taken at intervals of length $D/(m+1)$, and, at this sampling rate, $\{N_i;i=1,\ldots,\infty\}$ is m-dependent. For this case, the elements of the matrix $\underline{K}$ of Eq. (31) are given by

$$K_{k,\ell}=[2\int_{t_{k-1}}^{t_k}\int_{t_{\ell-1}}^{t_\ell}[\sum_{j=1}^{m} w(x,y;\rho_j)]dxdy]/[\Phi(t_k)-\Phi(t_{k-1})] \quad (52)$$

where $w(x,y;\rho)$ is the bivariate density function of two standard jointly Gaussian random variables with correlation coefficient ρ; Φ is the Gaussian distribution function; and

$$\rho_j=\rho(jD/(m+1))=(1-j/(m+1)) \; ; \; j=1,\ldots,m \quad (53)$$

Here, the optimum quantizer $Q^0=(\underline{t}^0,\underline{q}^0)$ for the case M=4 (i.e., 4-level quantization) is characterized by

$$-q_1^0=+q_4^0; \; -q_2^0=+q_3^0>0$$
$$\text{and} \quad (54)$$
$$-t_0^0=t_4^0=\infty; \; t_2^0=0; \; t_1^0=t_3^0>0$$

Since $\eta(Q)$ is independent of scale (i.e., $\eta(\alpha Q)=\eta(Q)$ for $|\alpha|\neq 0$), we may thus characterize Q^0 completely by specifying the upper breakpoint t_3^0 and the levels ratio (q_4^0/q_3^0). Table I gives values of these parameters for several values of the dependence parameter m. Note that m=0 gives the independent case. Also given in Table I are values of $ARE(Q^0,Q^{\ell o})$ and $ARE(Q^0,\ell.d.)$ where $Q^{\ell o}$ denotes the m=0 quantizer and 'ℓ.d.' denotes the linear detector (i.e., Eq. (2) with $g(x)\equiv x$) which is the optimum memoryless detector for this case [4] $ARE(Q^0,Q\ell o)$ is a measure of the improvement gained by these techniques over that system designed by merely ignoring the dependence, and $ARE(Q^0,\ell.d.)$ is a measure of the degradation in performance due to quantization.

Example 2 - Cauchy Noise

As a second example consider the noise process $\{N_i'\}_{i=1}^{\infty}=\{T(Z_i)\}_{i=1}^{\infty}$ where $T(z)=\tan(\pi erf(z/\sqrt{2})/2)$ and $\{Z_i\}_{i=1}^{\infty}$ is a Gaussian process identical to the process $\{N_i\}_{i=1}^{\infty}$ of Example 1. This will yield a Cauchy noise process with univariate density $f(x)=[\pi(1+x^2)]^{-1}$. The optimum 4-level quantizer for $\{N_i'\}_{i=1}^{\infty}$ will also satisfy Eq. (54) and its parameters are given in Table II. Note that, here, there are two equivalent optimal quantizers for the m=0 case. These are denoted by $Q_1^{\ell o}$ and $Q_2^{\ell o}$, respectively, and performance of Q^0 relative to each of these is given in Table II.

Note that $q_4^0<0$ for some values of m, this is consistent with results for the unquantized memoryless case for this example as discussed in [4].

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Table I. 4-Level Quantizer Parameters for Gaussian Noise

m	t_3^0	(q_4^0/q_3^0)	$ARE(Q^0,Q^{\ell o})$	$ARE(Q^0,\ell.d.)$
0	0.982	3.34	1.00	.882
1	1.03	3.40	1.000	.935
2	1.09	3.46	1.002	.953
5	1.19	3.44	1.005	.972
10	1.24	3.40	1.006	.981
15	1.26	3.38	1.007	.983
25	1.27	3.38	1.007	.986

Table II. 4-Level Quantizer Parameters for a Cauchy Noise Process

m	t_3^0	(q_4^0/q_3^0)	$ARE(Q^0,Q_1^{\ell o})$	$ARE(Q^0,Q_2^{\ell o})$
0_1	0.254	2.92	1.00	1.00
0_2	3.94	.342	1.00	1.00
1	3.94	-.238	2.53	2.14
2	2.13	-.0802	2.90	2.31
5	2.12	-.237	3.50	2.63
10	1.88	-.227	3.81	2.80

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The problem of optimum data quantization for memoryless signal-detection systems operating in m-dependent noise environments is considered. The case where quantizer breakpoints are fixed or predetermined is considered first, and, for this case, existing results for general (unquantized) memoryless detection are modified to yield necessary and sufficient conditions for

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quantizer optimization in terms of asymptotic efficiency. Necessary conditions are also established for the optimum (asymptotically efficient) selection of quantizer breakpoints, and expressions are presented for the comparison of quantizer-detector performance on the basis of asymptotic relative efficiency.