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DEVELOPMENT OF A UNIFIED FIELD THEORY BASED ON FINSLER GEOMETRY--ETC(U)
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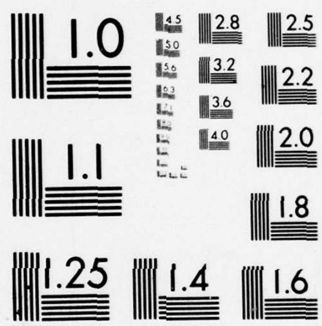

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DEVELOPMENT OF A UNIFIED FIELD THEORY BASED ON FINSLER GEOMETRY

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ABSTRACT

After we have made a differentiation between the field theories in the narrower and broader sense in the introduction and had discussed Einstein's postulations with respect to unified field theories, we formulated the fundamentals of the Finsler geometry, which we have made the basis of the developed theory, in paragraph 1. Paragraph 2 contains an overview of the universal lines of charged points, respectively of the extremals of the geometry. In paragraph 3, we have then transformed the mathematical apparatus of the theory of relativity to linear elementary spaces. Finally, in paragraph 4, the field equations were derived from a variation principle.

The theoretic field investigations which, since the inspiring Faraday idea, have provided one of the most significant and useful forms of visualization of modern physics, have reached a significant development period in the last two decades. In order to make it possible to clearly formulate the character of the problems, which led to the difficulties in the field theory, it appears to be eminently useful to us to make a differentiation which is generally not customary.

The field theory, as it appeared most consistently, first in the Faraday-Maxwell theory of electromagnetism and later in the relativistic Minkowsky electrodynamics, respectively in the Lorentz electron theory, addressed itslef to the solution of the problem to replace the elemtary laws for the forces acting momentarily at a distance by variables of state, which continuously vary in place and time, in order to make it possible to take into consideration the continuous action and the finite propagation of the physical effects. In order to solve this main problem and to obtain simple generally valid lows, differential equations had to be found, which were satisfied by the variables of state. In this respect, the field has only an abstract significance, without it being possible that any expianation, which is clear from the geometrical viewpoint, could be provided for it. The new field theories, in their classical (non-quantified) form, which play a role in the physics of atom nuclei and elementary particles, have developed through the immediate generalization of these theories, only their field equations have another form corresponding to the exchange effects, which are preassumed in various ways and their field potentials have another transformation character. The universal continuum, in which the physical phenomena, which are characterized by the field, take place is the four-dimensional flat space-time manifold, which has a pseudo-Euclidian metric, in which the field potentials, respectively the field intensities are illustrated by four-vectors, respectively tensors. Finally, we should note that the field magnitudes, from the group-theoretical viewpoint, are covariant with respect to the Lorentz transformation. In the following, we now want to designate these field theories as "field theories in the broader sense".

In contrast, we designate as "field theories in the narrower sense" those field theories which, in accordance with the original Riemann postulate, determine the physical effects of the guide field directly by the geometrical structure of the space. A well-known example of a field theory of this type is provided by the general theory of relativity of Einstein.

Since the time of Einstein's discovery, the field theory in the narrower sense has set itself the objective of illustrating all physical effects, of which the field theories in the broader sense have already been solved, by the geometrical structure of the world, in which the natural phenomena occur. These investigations, which are customarily included in the area of "the unified field theory", have even today not been concluded in a satisfactory manner. The difficulties, which, in the case of gravitation and electromagnetism, have been of considerable interest for more than 30 years, can be explained by the fact that there is no field theory in the narrower sense, which satisfies the requirements, for electrodynamics. The investigations, which addressed themselves to this problem, which thus have posed themselves the question if it might not be possible to understand the electromagnetic field, as well as the gravitational field as a change of the geometric structure of the universal continuum, have yielded the result that the four-dimensional Riemann space, which is the basis of the general theory of relativity, has no place at all for such an interpretation. If the curvature of space is used for the gravitational field, the Riemann geometry has no geometrical object, which would be suitable for the illustration of the electromagnetic effect by an electromagnetic guide field. In order to make it possible to solve the fundamental problem of the unified field theory, it is therefore necessary to perceive the universal continuum as a manifold of a more general structure than the Riemann space.

The common fundamental concept of all unified field theories is that the guide field of gravitation is illustrated by the conventional four-dimensional Riemann geometry and that the deviation from this space is established by the electromagnetic field. If, without seeking a complete coverage, we briefly indicate the course of development in accordance with the well-known pioneer work of H. Weyl, the generally known unified field theories can be essentially classified in three categories. The first category summarizes those theories, which have developed through any affinity generalization of the four-dimensional Riemann space. The second category characterizes those investigations, of which the geometric basis is the four-dimensional projective geometry. Finally, the five-dimensional theories belong into the third category.

Without now going into all of these interesting theories in greater detail, we will make a short report about the challenges which were raised by Einstein with regard to the unified theories (1).

After we have indicated the transformation group, which is, in most cases, a group of the four-dimensional (respectively five-dimensional), non-singular, continuous, affine (respectively projective) coordinate transformations, in contrast to which the physical quantities, which describe geometric objects, respectively natural laws, must be invariant, we must construct the unified field theory in such a manner that it would be co-variant in this group; *in addition*, the following two conditions must be fulfilled:

a) The theory should be covariant in a unified manner. This means that the physical quantities, which correspond to the gravitaional field, respectively the electromagnetic field, cannot be transformed independently from each other in the transformations of the mentioned group.

b) The Lagrange function of the variation problem, from which the field equations should be derived, may not decomposed into invariant components.

Not all theories can completely satisfy these two conditions, which are the natural generalizations of those which led to a unified relativistic theory of electromagnetism in the uniformalization of the electrical and the magnetic field. A difficult problem arises especially in the fulfillment of the first condition.

In the following, we will develop a unified field theory, of which the geometric basis is the Finsler space -- a metric generalization of the four-dimensional Riemann space --, and which largely fulfills the above conditions. As we will see shortly, this theory has the advantage that it has greater capability of generalization, on the one hand, by the methods, which have already shown themsleves useful in the Riemann geometry, to enable the consideration of additional fields (meson fields); on the other hand, we believe that it is capable of adapting more extensively to the original Einstein postulates.

Paragraph 1. GEOMETRICAL FUNDAMENTALS

1.1. The Fundamental Formula of the General Line Element Spaces

The generalization of the Riemann space starts (2) with an n-dimensional point space R_n, with reference to the coordinates x^i $(i = 1,2,\ldots,n)$. The expansion can be attained in such a manner that all through oriented directions $\dot{x}^i$ $(n = 1,2,\ldots,n)$ are added to each point. For the $\dot{x}^i$, of course, only their relationship is of importance. The space, which is thus expanded, of which the fundamental element is determined by $(x, \dot{x})$ (FN: In the following, the line elements $(x^i, \dot{x}^i)$ $(i = 1,2,\ldots,n)$ are understood as $(x, \dot{x})$), can be considered as a (2n-1)-dimensional line element space R^*_n.

In this space, those functional systems $A_{ii\ldots\,\ldots}^{\;\;\;\;kl}$ of the line elements $(x, \dot{x})$ are designated as tensors, which, with a coordinate transformation

$$\bar{x}^i = \bar{x}^i(x^1,\ldots,x^n),$$

$$\dot{\bar{x}}^i = \frac{\partial \bar{x}^i}{\partial x^r}\dot{x}^r$$

are transformed as follows:

$$\bar{A}_{ij\ldots}^{\;\;\;kl\ldots} = \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^r}{\partial \bar{x}^j}\cdots\frac{\partial \bar{x}^k}{\partial x^s}\frac{\partial \bar{x}^l}{\partial x^t}\cdots A_{pr\ldots}^{\;\;\;st\ldots}.$$

When $\xi^i(x, \dot{x})$ signifies a contravariant tensor of the first order, thus a contravariant vector, the transition from the line element (x, ẋ) to the adjacent line element (x + dx, ẋ + dẋ) is given by

$$D\xi^i = d\xi^i + C^i_{k,l}\xi^k d\dot{x}^l + \Gamma^i_{k,l}\xi^k dx^l \qquad (1.1)$$

where the coefficients $C^i_{k,l}$ and $\Gamma^i_{k,l}$ determine the affinity relationship of the space (FN: It is known that the $C^i_{k,l}$ are tensors, and for the transformation law of the $\Gamma^i_{k,l}$, see O. Varga (1)). The operator D is designated the invariable derivation. Formula (1.1) provides the invariant differential of the Vector ξ^i.

The quantities $C^i_{k,l}$ and $\Gamma^i_{k,l}$ determine the essential characterizing objects of the space, which are of importance for a field theory. While a point space can be characterized by the Riemann curvature tensor, the R^*_n, just introduced, for the characterization of the space has, in addition to the curvature tensor

$$\Omega^i_{k.rs} = \frac{\partial \Gamma^{*i}_{k.r}}{\partial x^s} - \frac{\partial \Gamma^{*i}_{k.s}}{\partial x^r} - \frac{\partial \Gamma^{*i}_{k.r}}{\partial \dot{x}^\rho} G^\rho_s + \frac{\partial \Gamma^{*i}_{k.s}}{\partial \dot{x}^\rho} G^\rho_r + \Gamma^{*\rho}_{k.r}\Gamma^{*i}_{\rho.s} - \Gamma^{*\rho}_{k.s}\Gamma^{*i}_{\rho.r} + C^i_{k.\rho}\Omega^{\rho}_{*.rs} \qquad (1.2)$$

also the two tensors

$$C^i_{k.l}, \qquad \Omega^i_{k.l} = \frac{1}{2}(\Gamma^{*i}_{k.l} - \Gamma^{*i}_{l.k}). \qquad (1.3)$$

which are called the torsion tensors of the R^*_n. The quantities introduced in equations (1.2) and (1.3) can be expressed by $C^i_{k,l}$, respectively $\Gamma^{*i}_{k,l}$ in the following manner:

$$G^i = \Gamma^i_{k} \dot{x}^k \ \text{[illegible]} \qquad (1.4a)$$

$$G^i_k = \frac{\partial G^i}{\partial \dot{x}^k}, \qquad G^i_{k,l} = \frac{\partial G^i_k}{\partial \dot{x}^l}. \qquad (1.4b)$$

$$\Gamma^{*i}_{k} = \Gamma^i_{k} \ \text{[illegible]} \ C^i_{k} G \ \text{[illegible]} \qquad (1.4c)$$

$$\Omega^{*i}_{\ \ rs} = \Omega^i_{\ \ rs}\dot{x}\ \text{[illegible]}\ G \ \text{[illegible]}\ G^i_{r\rho}G^\rho_s - G^i_{s\rho}G^\rho_r \qquad (1.4d)$$

Depending on the geometrical conditions, which satisfy the quantities $C^i_{k,l}$, $\Gamma^i_{k,l}$ and $\Gamma^{*i}_{k,l}$, R^*_n has various geometric structures. Among these spaces are also those which are the direct generalization of the R_n covered by E. Schroedinger (3).

In view of the fact that, in R^*_n, the $\Gamma^i_{k,l}$ with the $C^i_{k,l}$ together are the functions which determine the affinity relationship, still other $C^i_{k,l}$, respectively $\Gamma^{*i}_{k,l}$ occur in the symmetry conditions of $\Gamma^i_{k,l}$, which permit a greater multiplicity of the possibilities in the structure of the space.

In the following, we will investigate only those fields in greater d-tail, of which the quantities $\Gamma^i_{k,l}$ and $C^i_{k,l}$ of the corresponding R^*_n can be derived from a metric fundamental tensor.

1.2. Fundamental Tensors of the Metric R^*_n

It is known that the metric spaces under the R^*_n are the so-called Finsler spaces (3). The square of the element of arc is specified by

$$ds^2 = g_{ik}(x, \dot{x})\, dx^i\, dx^k \tag{1.5}$$

where the metric fundamental tensor $g_{ik}(x,\dot{x})$ is generated from a fundamental function $\mathfrak{L}(x, \dot{x})$ in the form (FN: In the following, we will consistently use the designation introduced in equation (1.4b). For example: $F_{r_1 \ldots r_n; s_1 \ldots ; s_m} \overset{\mathrm{def}}{=} \frac{\partial^{n+m} F}{\partial x^{r_1} \ldots \partial x^{r_n} \partial \dot{x}^{s_1} \ldots \partial \dot{x}^{s_m}}$)

$$g_{rs} = \tfrac{1}{2} (\mathfrak{L}^2)_{;r;s} \tag{1.6}$$

The function $\mathcal{L}(x, \dot{x})$ should be homogeneous in the first dimension in the $\dot{x}^i$, but $g_{r,s}$ homogeneous in the zero dimension because of (1.6). Because of the Euler relationship

$$\mathfrak{L}^2 = g_{rs}\, \dot{x}^r \dot{x}^s. \tag{1.7}$$

The fundamental quantities of the space are:

a) The unit vector $\vec{l}$, which possesses the same direction as its line element, of which the contravariant, respectively covariant components are

$$l^r = \frac{\dot{x}^r}{\mathfrak{L}}, \qquad l_r = \mathfrak{L}_{;r} \tag{1.8}$$

b) The torsion tensor

$$A_{ilm} \overset{\mathrm{def}}{=} \mathfrak{L} \cdot C_{ilm}. \tag{1.9}$$

where

$$C_{klm} = \tfrac{1}{2} g_{kl\,m} = \tfrac{1}{4} (\mathfrak{L}^2)_{klm} \qquad (1.10)$$

c) The functions, which determine the extremals of the space, are:

$$G_k \overset{\text{def}}{=} \tfrac{1}{4}\{(\mathfrak{L}^2)_{kr}\, \dot{x}^r - (\mathfrak{L}^2)_{k}\}. \qquad (1.11)$$

$$G^i = g^{ik} G_k. \qquad (1.12)$$

The quantities G^i_k, which are introduced in equation (1.4a), are, in the metric case,

$$G^i_k = G^i_{\;k}. \qquad (1.13)$$

as can be easily demonstrated (5).

d) In addition to the $C^{\,l}_{k,m}$, the transformation parameters are the

$$\Gamma_k{}^i{}_m = g^{il}\,\Gamma_{klm}. \qquad (1.14)$$

$$\Gamma_{klm} = \tfrac{1}{2}\{g_{kl\,m} - g_{km\,l} + g_{lm\,k}\} - C_{kmr} G^r_l - C_{lmr} G^r_k \qquad (1.15)$$

In *addition* to the transformation parameters, it is also customary to use the quantities

$$\Gamma^*_{klm} = \Gamma_{klm} - C_{klr} G^r_m \qquad (1.16)$$

which are symmetrical in the indexes k and m, as results from (1.15).

e) Among the curvature tensors of the Finsler space, especially the Riemann curvature tensor $R^{\,i}_{k,rs}$ has an importance for us. $R^{\,i}_{k,rs}$ agrees with the tensor $\varrho^{\,i}_{k,rs}$ given in equation (1.2) formally, only that, instead of G^i_k of the affine space, there is G^i_k given in (1.11) and (1.12) and, instead of $C_{k\cdot p}^{\;i}\,\varrho_{*\cdot rs}^{\;p}$ the tensor $A_{k\cdot p}^{\;i}\,R_{*\cdot rs}^{\;p}$, where

$$R_{*\cdot rs}^{\;p} \overset{\text{def}}{=} R_{k\cdot rs}^{\;p}\, l^k \qquad (1.17)$$

Additional torsion tensors of the space are

$$S_{ijkh} = A_{jk\cdot}^{\;\;s} A_{sih} - A_{jh\cdot}^{\;\;s} A_{sik} \qquad (1.18)$$

and

$$P_{i\,kh}^{\;j} = 2\,\Gamma^{*}{}_{i\,[k|h]}^{\;j} - \Gamma_k A_{i\cdot h}^{\;j} - A_{i\cdot s}^{\;j}\, \Gamma_r A_{k\cdot h}^{\;s}\, l^r \qquad (1.19)$$

(for $\bar{\nabla}_k$ see (1.24).

f) As is customary in the general theory of relativity, we also define, in the Finsler space, the tensor:

$$\bar{R}_{ik} = R_{i.ks}^{\;\;s},$$

from which the invariant

$$\bar{R} = g^{ik}\bar{R}_{ik}$$

is derived.

O. Varga (6) has introduced the tensor

$$T_{k.rs}^{\;i} \overset{\text{def}}{=} \varrho_{k.rs}^{\;i} - C_{k.p}^{\;i}\,\varrho_{*.rs}^{\;p}$$

as the principal curvature tensor of the affine continuous multiplicity in general affine continuous line element spaces. In the following, we will see immediately that the principal curvature tensor

$$H_{k.rs}^{\;i} \overset{\text{def}}{=} R_{k.rs}^{\;i} - A_{k.p}^{\;i}\,R_{*.rs}^{\;p} \tag{1.20}$$

can also be used in the Finsler space in the formation of the curvature scalar, instead of the Riemann curvature tensor $R_{*.rs}^{\;i}$.

Because of the skew-symmetry of $R_{*.rs}^{\;i}$, in the indexes r and s

$$g^{ik}A_{i.s}^{\;r}R_{*.kr}^{\;s} = A^{kr}_{\;\;s}R_{*.kr}^{\;s} = 0,$$

thus because of (1.20), the invariant

$$R = g^{ik}H_{i.ks}^{\;s} \tag{1.21}$$

which we will designate in the following as the principal curvature invariant of the space, is identical with $\bar{R}$.

In the formation of R, the reduced principal curvature tensor $H_{k,rs}^{\;i}$ can also be used instead of the tensor $R_{k,rs}^{\;i}$, therefore we want to designate

$$R_{ik} \overset{\text{def}}{=} H_{i.ks}^{\;s}, \tag{1.22}$$

or in explicit form

$$\left.\begin{aligned} R_{ik} = {} & (\Gamma^{*\,s}_{i.k})_{(s)} - (\Gamma^{*\,s}_{i.k})_{;r}\,G^r_s - (\Gamma^{*\,s}_{i.s})_{(k)} + (\Gamma^{*\,s}_{i.s})_{(;r)}\,G^r_k + \\ & + \Gamma^{*\,s}_{i.k}\,\Gamma^{*\,r}_{s.r} - \Gamma^{*\,r}_{i.s}\,\Gamma^{*\,s}_{r.k}, \end{aligned}\right\} \tag{1.23}$$

as Einstein-Ricci curvature tensor of our space (FN: It should be noted that R_{ik} is symmetrical in its indexes i and k only in the Riemann geometry (see Cartan, l.c. equation XXIII.))

g) We now also have to still give the covariant derivation of the tensors. For example, let $T_{i.h}^{k}$ be a mixed tensor of the third order, then

$$\nabla T^{k}_{i.h} = (T^{k}_{i.h})_{(l)} - (T^{k}_{i.h})_{(s)} G^{s}_{l} - \Gamma^{*s}_{i.l} T^{k}_{s.h} + \Gamma^{*k}_{s.l} T^{s}_{i.h} - \Gamma^{*s}_{h.l} T^{k}_{i.s}. \qquad (1.24)$$

In the general metric line element spaces, there also exists the well-known identity:

$$\nabla_s g^{ik} = 0. \qquad (1.25)$$

Paragraph 2. ON THE UNIVERSAL LINES OF THE GIVEN GENERAL FIELD THEORY

2.1. Geodetic Lines

If a charged mass point is in any condition of motion in the four-dimensional space-time continuum, which, in our case, is a Finsler space, its path can be designated as a universal line in the customary manner. However, we now consider the space as a four-dimensional metric point multiplicity, of which the metric fundamental tensor g_{ik} is however not only a function of the locus, but also of the direction (7). The equations of the universal lines can be given in the form

$$x^i = x^i(t) \qquad (2.1)$$

When the arc length is selected as the parameter (t≡s), then

$$l^i = \dot{x}^i \quad \text{and} \quad F^2 = g_{ik} l^i l^k = 1.$$

"The actual motion of a body comes about by the competition of two influences, the guide field, which transmits the universal direction of the body from moment to moment, and the force, which deflects the body from this natural motion" (8). In the general theory of relativity, it is customary to determine the gravitational forces, which determine the motion of the body, by the structure of the guide field. If only gravitational forces act on a mass point, its path is a geodetic line of the space, which is a Riemann space in this case.

We now also want to take into consideration the effect of the electromagnetic field by the influence of the structure of the space. When, in addition to the gravitational forces, only electromagnetic forces act on the charged mass point, its path, in the guide field, will be an extremal of the Finsler space. The metric field is the potential of the guide field, which, in the prevailing case, is the combination of the electromagnetic and the gravitational field. The structural deviation of this space from the Riemann geometry, thus the space torsion, will be determined by the electromagnetic potentials.

The extremals of the Finsley space, which, in our case, are thus the universal lines of a mass point carrying out inertial motion, are derived by the variation of the integral

$$\int \mathcal{L}(x, dx)$$

The arc length on a universal line (2.1) between the parameter values t_1 and t_2 is given by the formula

$$s = \int_{t_1}^{t_2} \mathcal{L}\left(x, \frac{dx}{dt}\right) dt \qquad (2.2)$$

For physical reasons, it is appropriate, instead of the arc length s, to introduce the proper time τ by the equation

$$\tau = \frac{1}{c} s$$

which is identical to the arc length if, as we will always assume in the following, the propagation velocity of light is selected as the unit.

For the extremals

$$\delta \int ds = \delta \int \mathcal{L}(x, \dot{x})\, dt = 0$$

and the differential equations of the extremals are:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0. \qquad (2.3)$$

2.2. The Metric Fundamental Form and the Null Sphere

As in th- general theory of relativity, we also have quite analogous differences between the egometry of the positive definite and the indefinite arc elements of the Finsler space (9). For the geodetic lines emanating from a point $P_0(x_0^i)$, the following conditions can exist:

$$g_{ik}(x, \dot{x}) \frac{dx^i}{dt} \frac{dx^k}{dt} > 0 \qquad (2.4a)$$

$$g_{ik}(x, \dot{x}) \frac{dx^i}{dt} \frac{dx^k}{dt} < 0 \qquad (2.4b)$$

$$g_{ik}(x, \dot{x}) \frac{dx^i}{dt} \frac{dx^k}{dt} = 0 \qquad (2.4c)$$

The directions $\dot{x}^i = dx^i/dt$, which satisfy equations (2.4a), (2.4b), respectively (2.4c), are called time-like, space-like, respectively null directions. In the limiting case, when there is no electromagnetic and gravitational field, our space is transformed into a pseudoeuclidic space with the arc element

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

The permissible point transformations of the Finsler space are constrained by this characterization of the space-time world (10). The permissible coordinate transformations should transform the space-like, respectively time-like directions always into the same.

In order to make it possible to transform the arc element (1.5) into the arc element of the pseudoeuclidic space, the following relationships must be fulfilled (11):

$$g_{00} > 0, \qquad g_{11} < 0, \qquad \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} < 0.$$

The entirety of the geodetic null lines, which thus satisfy (2.4c), lie on a mantle of a sphere, which is designated as null sphere. For the lines of the null sphere, the arc length (2.2) cannot be selected as parameter.

Obviously, equations (2.3) are unusable for the null lines, which are characterized by (2.4c), because of the disappearance of the parameter (FN: see equations (2.2), (1.7) and (2.4c)). In order to be able to also define the geodetic lines on the null sphere, we will use their well-known other definition, in accordance with which the geodetic line is a curve, which always retains its direction. This means analytically, because of

$$C^{\,i}_{k.l}\,\dot{x}^k = 0, \tag{2.4}$$

that (FN: Varga, O.:(1))

$$\frac{D}{ds}\left(\frac{dx^i}{ds}\right) = \frac{d^2x^i}{ds^2} + \Gamma^{\,i}_{k.l}\,\frac{dx^k}{ds}\,\frac{dx^l}{ds} = 0. \tag{2.5}$$

It can easily be verified that equations (2.5) are identical to (2.3) (FN: see equations (2.4), (1.11), (1.14) and (1.15). In accordance with Pauli (FN: see Pauli: l.c.), we define the geodetic null line as follows: A geodetic null line is a curve for which a curve parameter t exists in such a manner that the differential equations

$$\frac{d^2x^i}{dt^2} + \Gamma^{\,i}_{k.l}(x,\dot{x})\,\frac{dx^k}{dt}\,\frac{dx^l}{dt} = 0 \tag{2.6a}$$

are fulfilled and

$$\mathfrak{L}(x,\dot{x}) = 0 \tag{2.6b}$$

If there is no electromagnetic field, our conditions (2.6) and (2.4c) are transformed into those of Pauli, which are given in the Riemann geometry.

2.3. The Osculatory Riemann Space

We now want to draw attention to a concept, which is important in the following, and which has the relationship that it is impossible to "transform away" the entire electromagnetic and gravitational field of a charged mass point in the entire universal continuum all at once by constant transformations, but, in sufficiently small environments of a universal point of the space-time continuum, we can always specify a system, in which the guide field disappears. It is well-known how this can be accomplished in the case of the Riemann space. We still want to refer to the fact that a similar relationship also exists between the Riemann and the Finsler geometry.

A so-called osculatory Riemann space with the following perperties can be associated with any continuously differentiable series of line elements

$$x^i = x^i(t), \qquad (2.7a)$$

$$\dot{x}^i = \dot{x}^i(t) \qquad (2.7b)$$

a) Let the metric fundamental tensor $\gamma_{ik}(x)$ of the osculatory space be identical to that of the Finsler space.

b) The geodetic lines of the two spaces should osculate each other.

c) Let the invariant differential of a vector $\xi^i(x, \dot{x})$ be identical in both spaces.

For the purpose of the construction of the osculatory Riemann space, we expand the field of the line elements (2.7) beyond the curve (2.7a), which can be carried out in the following manner: Through each element $(x, \dot{x})$ of the series (2.7), we lay an extremal of the Finsler space, which then form, as an entirety, a single parametric system. We embed these in a system of extremals, which covers a certain point area B smoothly. It is now possible to clearly associate a direction, namely the direction of the tangent vector of the extremal

$$r^i = r^i(x) \qquad (2.8)$$

with each point x^i of B. We also want to assume that the line elements are so selected that they agree with the derivations of the extremals in accordance with the arc length:

$$\dot{x}^i(t) = r^i(x(t)). \qquad (2.9)$$

By inserting r^i in $g_{ik}(x, \dot{x})$, we obtain a tensor

$$\gamma_{ik}(x) = g_{ik}(x, r(x)), \qquad (2.10)$$

which we select as the metric fundamental tensor of the osculatory Riemann space.

It can be easily proven that, with this construction, the osculatory Riemann space has the characteristics specified in a), b), respectively c) (12).

We wish to still note two additional properties of the osculatory space:

d) The covariant derivations of the tensors are identical along (2.7) in both spaces.

e) The Riemann curvature tensor of the osculating space agrees, along (2.7) with the principal curvature tensor of the Finsler space.

In order to prove d) and e), we still should note that the vectors of the field (2.8), of which the points of engagement are on (2.7a), are parallel in the osculating space; thus

$$r^i_{,k} - \dot{\Gamma}^{\,i}_{k\,l}\, r^l = 0. \qquad (2.11a)$$

where the $\dot{\Gamma}^{\,i}_{k\,l}$ signify the Christoffel symbols of the .
A more extensive calculation shows that (2.11a) can be brought into the form

$$r^i_{,k} = -G^i_k \qquad (2.11b)$$

(FN: Varga, O.: (3)).

If we now calculate the covariant derivation of a vector $\xi^i(x,\dot{x})$ in the osculatory Riemann space along (2.7):

$$\overset{0}{\nabla}_k \xi^i = [\xi^i(x, r(x))]_{(k)} + \overset{0}{\Gamma}{}^{\;i}_{k.s}\, \xi^s. \qquad (2.12)$$

It is now seen immediately that, because of (2.10)

$$\left.\begin{aligned} \overset{0}{\Gamma}_{kpl} &= \frac{1}{2}\{g_{kp(l)} + g_{pl(k)} - g_{kl(p)}\} + \\ &\quad + \frac{1}{2}\frac{\partial g_{kp}}{\partial r^s} r^s_{(l)} + \frac{1}{2}\frac{\partial g_{pl}}{\partial r^s} r^s_{(k)} - \frac{1}{2}\frac{\partial g_{kl}}{\partial r^s} r^s_{(p)} \\ &= \frac{1}{2}\{g_{kp(l)} + g_{pl(k)} - g_{kl(p)}\} - C_{kps} G^s_l - C_{pls} G^s_k + C_{kls} G^s_p \\ &= \Gamma^*_{kpl} \end{aligned}\right\} \qquad (2.13)$$

and

$$[\xi^i(x, r(x))]_{(k)} = \xi^i_{(k)} - \xi^i_{(r)} G^r_k \qquad (2.14)$$

By inserting (2.13) and (2.14) in (2.12),

$$\overset{0}{\nabla}_k \xi^i = \nabla_k \xi^i,$$

along (2.7), QED.

In order to also prove e), we note that the Riemann curvature tensor is

$$\overset{0}{R}{}_{j.kl}^{\;i} \overset{\mathrm{def}}{=} \overset{0}{\Gamma}{}^{\;i}_{j.k(l)} - \overset{0}{\Gamma}{}^{\;i}_{j.l(k)} + \overset{0}{\Gamma}{}^{\;s}_{j.k} \overset{0}{\Gamma}{}^{\;i}_{s.l} - \overset{0}{\Gamma}{}^{\;s}_{j.l} \overset{0}{\Gamma}{}^{\;i}_{s.k}$$

In accordance with (2.9), (2.11) and (2.13),

$$\overset{0}{R}{}_{j.kl}^{\;i} = H_{j.kl}^{\;i}. \qquad (2.15)$$

QED.

We now want to add the final comment that the geodetic null lines of the two spaces also osculate each other, which follows directly from identity (2.13) and from the equations of the geodetic null lines (2.8) of the two geomertries.

2.4. The Fundamental Postulate of the Specified Theory

Following these general preparations, we want to heuristically summarize the fundamental postulate of our investigations as follows.

If a charged mass point is under the influence of a guide field in any state of motion, its path in the four-dimensional space-time continuum is given by its universal line. If, in accordance with the general field theory in the narrower sense, we want to express the guide field by the geometrical structure of the space, and the guide field is a pure gravitational field, a suitable Riemann space can always be constructed in such a manner that the specified universal line of the mass point would be a geodetic line of the geometry. If, however, in addition to the gravitational field, an electromagnetic field is also present, the Riemann space will not suffice to express the physical properties of the space. However, it is possible to use, in this case, a more general geometry -- namely the Finsler geometry --, which is suitable for the description of this combined guide field.

If the universal line of the charged mass point is specified in the Riemann space, then we can select, from the multiplicities of the possible Finsler spaces, those of which one of the geodetic lines agrees with the specified universal line. In this manner, the charged mass point carries out an inertial motion in these spaces.

Paragraph 3. INFINITESIMAL COORDINATE TRANSFORMATIONS AND VARIATION THEOREMS IN THE FINSLER SPACE

After we have seen how the guide field is related with the metric of the universal continuum, we must investigate the question how the metric is influenced by the masses and by the charges, which produce the guide field. It must of course primarily be expected of the equations, which specify the relationship between the guide field and the structure of the space, that they should be generally covariant. However, in order to obtain a clear definition of these field equations, we want to derive these from a variation principle, which is customary in the field theories. In order to realize this program, some mathematical preparations, such as the concept of the measure of capacity, the analysis of the tensor densities, furthermore, infinitesimal transformations, variation methods and various identities applied in the theory shall be discussed in this paragraph.

3.1. The Concept of the Measure of Capacity in the Finsler Space

Because, in accordance with the Cartan theory, the Finsler geometry can be considered as a line element multiplicity, the measure of capacity of a partial area of the space has no importance in the usual sense. This is related to the fact that the measure of capacity in the usual sense is a concept which is defined for point multiplicities. In the Finsler space, an integrable area -- consisting of points -- has a measure of capacity only with respect to a defined directional field (13).

It is therefore a directional field defined by the equations

$$\dot{x}^i = r^i(x) \qquad (3.1)$$

in this manner, the measure of capacity of an area V can be defined, with reference to (3.1), by the integral

$$\int_V \sqrt{-g(x, r(x))}\, dx \qquad (3.2)$$

where dx is the abbreviation of

$$dx = dx^1 dx^2 dx^3 dx^4$$

and

$$g \overset{\mathrm{def}}{=} \det g_{ik}(x, \dot{x})$$

Because of the transformation formula of $\sqrt{-g}$, (3.2) is an invariant, which characterizes the measure of the area V. (FN: In order to make it possible to define the measure of capacity in a generally affine cohesive line element space, a function q with the transformation character of $\sqrt{-g}$ must be constructed. (see Varga, O.: (4)))

3.2 Tensor Densities and Their Analysis

If the integral (3.1)

$$\int \mathfrak{W}^{ik}_{..}\, dx$$

is a tensor of the second order along (3.1), then, with

WEYL[3] $\mathfrak{W}^{ik}_{\cdot\cdot}$ is called a tensor density of the second order. This is developed by multiplying a conventional tensor with $\sqrt{-g}$. The tensor densities of teh zero order, which will play an outstanding role in the following, are the scalar densities.

The algebra of the tensor densities also does not introduce anything new in the Finsler space, although there are certain differences in their analysis, because the components of the tensors and tensor densities are now a function of the line element $(x, \dot{x})$.

In order to be able to define the divergence of a tensor density, we will start, in accordance with Weil (15), with a static vector field

$$\eta^i = \eta^i(x, \dot{x}), \qquad (3.3)$$

which thus satisfies the condition

$$\Gamma_k \eta^i \equiv \eta^i_{(k)} - \eta^i_{(r)} G^r_k + \Gamma^{*\,i}_{r\,k}\,\eta^r = 0 \qquad (3.4)$$

by definition. If $\mathfrak{A}_{i\cdot}$ is a mixed tensor density (FN: $\mathfrak{A}_{ik}$ shall be symmetrical in i and k), then

$$\mathfrak{a}^k = \mathfrak{A}_{i\cdot}^{\;k}\,\eta^i$$

will be a vector density, of which the divergence can be defined as follows:

$$\mathfrak{Div}\,\mathfrak{a} \overset{\text{def}}{=} \mathfrak{a}^k_{\cdot(k)}. \qquad (3.5)$$

In accordance with (3.4), (3.5) can explicitly be brought into the form

$$\mathfrak{a}^k_{(k)} = \{\mathfrak{A}_{i\cdot(k)}^{\;k} - \Gamma^{*\,r}_{i\,k}\,\mathfrak{A}_{r\cdot}^{\;k}\}\,\eta^i + \mathfrak{A}_{i\cdot}^{\;k}\,G^r_k\,\eta^i_{(r)} \qquad (3.6)$$

(FN: This formula also applies in the affine cohesive line element space).

For special applications, it appears appropriate to restate our formula (3.6) in a certain manner.

Through a simple calculation, the result is

$$\Gamma^{*\,r}_{i\,k}\,\eta^i\,\mathfrak{A}_{r\cdot}^{\;k} = \{\tfrac{1}{2} g_{kr(i)} - C_{krl}\,G^l_i\}\,\eta^i\,\mathfrak{A}^{kr}$$

and

$$\mathfrak{a}^k_{(k)} = \{\mathfrak{A}_{i\cdot(k)}^{\;k} - \tfrac{1}{2} g_{kr(i)}\,\mathfrak{A}^{kr} + C_{krl}\,G^l_i\,\mathfrak{A}^{kr}\}\,\eta^i + \eta^i_{(r)}\,G^r_k\,\mathfrak{A}_{i\cdot}^{\;k}. \qquad (3.7)$$

After transvection of equation (3.4) with $\dot{x}^k$, the identity

$$G^i_l \eta^l = -\eta^i_{,k}\dot{x}^k - \eta^i_{,l} G^l_k \dot{x}^k \qquad (3.8)$$

can be written because of (FN: See also (1.4a), 1.13), and (1.4c) and observe the symmetry of Γ^*_{ikj} in the indexes i and j)

$$\Gamma^{*i}_{l\,k}\dot{x}^k = G^i_l$$

In accordance with (3.7), (3.8) and (1.10), in view of the homogeneity of the second order of G'

$$\mathfrak{d}^k_{|k} = \{\mathfrak{A}^{ik}_{|k} - [illegible]\}\eta_i - [illegible] \qquad (3.9)$$

Through analogous calculations, the result is that, for a

$$\mathfrak{d}^k = \mathfrak{A}^{ik}\eta_i$$

defined with the aid of a covariant static vector field

$$\eta_i = \eta_i(x, \dot{x})$$

the divergence can be defined in the following manner:

$$\mathfrak{d}^k_{|k} = \{\mathfrak{A}^{ik}_{|k} - \Gamma^{*r}_{i\,k}\mathfrak{A}^{ik}\}\eta_r - G^r_k\mathfrak{A}^{ik}\eta_{i\cdot r} \qquad (3.10)$$

or

$$\mathfrak{d}^k_{|k} = \{\mathfrak{A}^{ik}_{|k} - [illegible]\}\eta_r - G^r_k\mathfrak{A}^{ik}\eta_{i\cdot r} \qquad (3.11)$$

In the Riemann geometry, our formulas (3.9), (3.10) and (3.11) are transformed into the well-known expressions of the divergence (FN: See Pauli, l.c., formulas (149) and (150)), but it will be more useful for the applications to consider those static vector fields, which are a function of the locus only, in order to make the divergence of the considered tensor density independent of the vector field η^i (or η_i). For this purpose, we use a static vector field

$$\xi^i = \xi^i(x) \qquad (3.12)$$

of which the analytical definition in accordance with (3.4) is given by $\nabla_k \xi^i = \xi^i_{(k)} - \Gamma^{*i}_{r\,k}\xi^r = 0$

For the construction of such a vector field (3.12), that process can be used, for example, which leads to the formation of the osculatory Riemann space (FN: Varga, O.: ($\underline{3}$)).

With the aid of the static vector field (3.12), the formulas of the divergence are given, instead of (3.6), respectively

(3.7) and (3.10), respectively (3.11), by

$$\mathfrak{Div}\,\mathfrak{a} = \mathfrak{a}^{k}_{(k)} \overset{\text{def}}{=} \mathfrak{Div}_i\,\mathfrak{A}\cdot\xi^{i} \overset{\text{def}}{=} \mathfrak{Div}^{i}\,\mathfrak{A}\cdot\xi_i \qquad (3.13)$$

with (FN: See Pauli: l.c., p 609)

$$\mathfrak{Div}_i\,\mathfrak{A} \overset{\text{def}}{=} \mathfrak{A}^{k}_{.i(k)} - \Gamma^{*}{}^{r}_{i.k}\,\mathfrak{A}^{k}_{.r} = \mathfrak{A}^{k}_{.i(k)} - \tfrac{1}{2} g_{sk(i)}\,g^{ks} - C_{skj}\,G^{j}_{i}\,\mathfrak{A}^{ks}_{..} \qquad (3.14)$$

respectively with

$$\mathfrak{Div}^{i}\,\mathfrak{A} \overset{\text{def}}{=} \mathfrak{A}^{ik}_{..(k)} - \Gamma^{*}{}^{i}_{r.k}\,\mathfrak{A}^{rk}_{..} \qquad (3.15)$$

and formula (3.13) can also be written (FN: See equation (3.9)) with (3.12) in the important form

$$\mathfrak{a}^{k}_{(k)} = \{\mathfrak{A}^{k}_{i.(k)} - \tfrac{1}{2} g_{sk(i)}\,\mathfrak{A}^{ks}_{..}\}\,\xi^{i} - \tfrac{1}{2} g_{sk(r)}\,\mathfrak{A}^{ks}_{..}\,\xi^{r}_{(t)}\,\dot{x}^{t}, \qquad (3.16)$$

or

$$\mathfrak{a}^{k}_{(k)} = \{\mathfrak{A}^{k}_{i.(k)} + \tfrac{1}{2} \mathfrak{A}_{ks}\,g^{ks}_{(i)}\}\,\xi^{i} + \tfrac{1}{2} \mathfrak{A}_{ks}\,g^{ks}_{(r)}\,\xi^{r}_{(t)}\,\dot{x}^{t} \qquad (3.17)$$

We can add the same physical significance to the concept which was just given concerning the divergence of the tensor densities as Weyl did in the case of the general theory of relativity (16).

3.3. Variation Methods

In order to make it possible to formulate our variation principles, it is appropriate to use infinitesimal coordinate transformations. This method, which is carried out by Pauli (17) in the Riemann geometry, has the advantage that certain characteristic differential equations, which are fulfilled by the quantities which characterize the geometric structure of the guide field, can be obtained directly.

Such an infinitesimal coordinate transformation is also specified in the Finsler space by

$$\bar{x}^{i} = x^{i} + \varepsilon\,\xi^{i}(x) \qquad (3.18)$$

where ε designates an infinitesimal parameter. This transformation can be plainly interpreted as a generalized Galilei transformation with the velocity vector ξ^{i}.

We will develop the differences of the overlined and non-overlined quantities in accordance with powers of ε; the term in ε of the first order is designated as the variation of the particular quantity.

Analytically, this variation of the tensors can be specified in two different ways by the following formulas (FN: Pauli designates the variation $\bar{\delta}$ as δ)

$$\bar{\delta} A^{i\ldots}_{\ldots k\ldots} = \bar{A}^{i\ldots}_{\ldots k\ldots}(\bar{x}, \dot{\bar{x}}) - A^{i\ldots}_{\ldots k\ldots}(x, \dot{x}) \qquad (3.19a)$$

respectively

$$\delta^* A^{i\ldots}_{\ldots k\ldots} = \bar{A}^{i\ldots}_{\ldots k\ldots}(x, \dot{x}) - A^{i\ldots}_{\ldots k\ldots}(x, \dot{x}) \qquad (3.19b)$$

As in the Riemann space, there is also, for the variation $\bar{\delta}$, in the Finsler case, for example,

$$\bar{\delta} A^i = \varepsilon\, \xi^i_{,r} A^r, \qquad \bar{\delta} A_i = -\varepsilon\, \xi^r_{,i} A_r \qquad (3.20)$$

respectively

$$\bar{\delta} A_{ik} = -\varepsilon \{\xi^r_{,i} A_{rk} + \xi^r_{,k} A_{ir}\}. \qquad (3.21)$$

Because of the dependence of the quantities of $\dot{x}^i$, the variation δ^* has the following explicit form in the Finsler space

$$\delta^* A^i = -\varepsilon \{A^i_{|r} \xi^r - A^r \xi^i_{,r} + A^i_{;r} \xi^r_{,s} \dot{x}^s\} \qquad (3.22)$$

respectively

$$\delta^* A_{ik} = -\varepsilon \{A_{ik|r} \xi^r + A_{rk} \xi^r_{,i} + A_{ir} \xi^r_{,k} + A_{ik;r} \xi^r_{,s} \dot{x}^s\}. \qquad (3.23)$$

The formulas for the other components can be easily calculated.

In the variation calculation, it is also customary to introduce the variation

$$\delta A^{i\ldots}_{\ldots k\ldots} = \bar{A}^{i\ldots}_{\ldots k\ldots}(\bar{x}, \dot{\bar{x}}) - A^{i\ldots}_{\ldots k\ldots}(x, \dot{x}) \qquad (3.24)$$

which, as can be easily verified. is, for example,

$$\delta A_{ik} = \varepsilon \{A_{ik|r} \xi^r + A_{ik;s} \xi^s_{,r} \dot{x}^r\}$$

for tensors of the second order.

Between the δ symbols, which are thus introduced, there is the following important relationship

$$\delta = \bar{\delta} - \delta^* \qquad (3.25)$$

For scalars, $\mathfrak{J}$ will be $-\mathfrak{J}^*$.

We now want to also give the following formula, which is calculated in an elementary manner:

$$\delta\sqrt{-g} = -\tfrac{1}{2}\sqrt{-g}\, g_{ik}\, \delta g^{ik}$$

respectively

$$\delta\mathfrak{g}^{ik} = \sqrt{-g}\,\{\delta g^{ik} - \tfrac{1}{2} g_{rs}\, g^{ik}\, \delta g^{rs}\}. \qquad (3.26)$$

3.4. <u>The Generalizing Palatini Identity</u>

It is well known that the identity of Palatini plays an important role in the derivation of the field equations of the general theory of relativity from a variation principle. For us, also, the generalization of this identity makes a simplification possible.

In order to make it possible to derive the generalized Palatini identity, we form the variation of the Einstein-Ricci tensor (1.23):

$$\left.\begin{aligned}\delta R_{ik} = {} & (\delta\Gamma^{*}{}_{i\cdot k}^{s})_{(s)} - (\delta\Gamma^{*}{}_{i\cdot k}^{s})_{(r)}\, G^{r}_{s} + (\delta\Gamma^{*}{}_{i\cdot k}^{r})\,\Gamma^{*}{}_{r\cdot s}^{s} - (\delta\Gamma^{*}{}_{r\cdot k}^{s})\,\Gamma^{*}{}_{i\cdot s}^{r} - \\ & - (\delta\Gamma^{*}{}_{i\cdot r}^{s})\,\Gamma^{*}{}_{s\cdot k}^{r} - (\delta\Gamma^{*}{}_{i\cdot s}^{s})_{(k)} + (\delta\Gamma^{*}{}_{i\cdot s}^{s})_{(r)}\, G^{r}_{k} + \\ & + (\delta\Gamma^{*}{}_{r\cdot s}^{s})\,\Gamma^{*}{}_{i\cdot k}^{r} - (\Gamma^{*}{}_{i\cdot k}^{s})_{(r)}\,(\delta G^{r})_{(s)} + (\Gamma^{*}{}_{i\cdot s}^{s})_{(r)}\,(\delta G^{r})_{(k)}.\end{aligned}\right\}(3.27)$$

For the transformation law of $\Gamma^{*}{}_{b\cdot c}^{a}$, there is (FN: Varga, O.: (<u>I</u>), equation (1.17))

$$\bar\Gamma^{*}{}_{b\cdot c}^{a} = \frac{\partial x^{k}}{\partial \bar x^{b}}\,\frac{\partial \bar x^{a}}{\partial x^{l}}\,\frac{\partial x^{j}}{\partial \bar x^{c}}\,\Gamma^{*}{}_{k\cdot j}^{l} + \frac{\partial^{2} x^{l}}{\partial \bar x^{b}\,\partial \bar x^{c}}\,\frac{\partial \bar x^{a}}{\partial x^{l}}. \qquad (3.28)$$

$\Gamma^{*}{}_{b\cdot c}^{a}$ is thus not a tensor, but yet, as can be easily verified, $\delta\Gamma^{*}{}_{b\cdot c}^{a}$ will be a mixed tensor of the third order. Therefore, equation (3.27) can be written, in accordance with definition (1.24) of the covariant derivation, in the form

$$\left.\begin{aligned}\delta R_{ik} = {} & \nabla_{s}(\delta\Gamma^{*}{}_{i\cdot k}^{s}) - \nabla_{k}(\delta\Gamma^{*}{}_{i\cdot s}^{s}) - (\Gamma^{*}{}_{i\cdot k}^{s})_{(r)}\,(\delta G^{r})_{(s)} + \\ & + (\Gamma^{*}{}_{i\cdot s}^{s})_{(r)}\,(\delta G^{r})_{(k)}\end{aligned}\right\} \qquad (3.29)$$

Equation (3.29) is the desired generalization of the identity of Palatini (18).

The two last members in equation (3.29) make the usual application of the Palatini identity difficult. Added to this is the fact that the transformation of the first members, in accordance with the Gauss theorem, into a surface integral provides additional interfering terms, which contain derivations in $\dot{x}^i$, because of the special form of the covariant derivation in the Finsler spaces.

However, when we use the Palatini identity along one line element sequence (3.1) with the conditions (2.9) and (2.11), the Palatini identity is simplified. In equation (3.27), in accordance with (2.9) and (2.11), there is

$$(\delta\Gamma^{*}{}_{i\cdot k}^{\cdot s})_{(s)} - (\delta\Gamma^{*}{}_{i\cdot k}^{\cdot s})_{(\dot{r})} G^r_s - (\Gamma^{*}{}_{i\cdot k}^{\cdot s})_{(\dot{r})}\, \delta G^r_s = [\delta\Gamma^{*}{}_{i\cdot k}^{\cdot s}(x, r(x))]_{(s)} = (\delta\overset{0}{\Gamma}{}_{i\cdot k}^{\cdot s})_{(s)}$$

and, in a similar manner,

$$(\delta\Gamma^{*}{}_{i\cdot s}^{\cdot s})_{(k)} - (\delta\Gamma^{*}{}_{i\cdot s}^{\cdot s})_{(\dot{r})} G^r_k - (\delta\Gamma^{*}{}_{i\cdot s}^{\cdot s})_{(\dot{r})}\, \delta G^r_k = (\delta\overset{0}{\Gamma}{}_{i\cdot s}^{\cdot s})_{(k)} .$$

We therefore have along (3.1), with the mentioned condition, instead of (3.29):

$$\delta\overset{0}{R}_{ik} = \nabla_s(\delta\overset{0}{\Gamma}{}_{i\cdot k}^{\cdot s}) - \nabla_k(\delta\overset{0}{\Gamma}{}_{i\cdot s}^{\cdot s}) . \qquad (3.30)$$

We have thus given the relationship of the Palatini identity of the Finsler space with that of the osculatory Riemann space.

3.5 Identities

It is well-known that the Bianchi identities play a significant role in the general theory of relativity and in the modern unified field theories. They can be used to derive those identities, which express the conservation laws of mass. In the Finsler space also, important formulas can be derived from the generalized Biachi identities, which are the direct generalizations of the relativistic conservation laws.

There exist the identities (FN: Cartan, E.: l.c., p. 37):

$$\nabla_l R_{ij,kh} - \nabla_k R_{ij,hl} - \nabla_h R_{ij,lk} = M_{ij,khl} . \qquad (3.31)$$

where $$M_{ijkhl} \overset{def}{=} P_{ijkm} R_*{}^m{}_{hl} - P_{ijhm} R_*{}^m{}_{lk} - P_{ijlm} R_*{}^m{}_{kh} \tag{3.32}$$

which illustrate the generalized Bianchi identities.

Because of the skew symmetry of the tensor R_{ijkh} in the indexes (i, j) and (k, h),

$$\nabla_l R_{ijkh} - \nabla_k R_{ijlh} - \nabla_h R_{ijkl} = M_{ijkhl}$$

develops from (3.31), and, after transvection with $g^{jh}g^{ik}$,

$$\nabla_l R - 2\nabla_k R^k_l = g^{jh} g^{ik} M_{ijkhl} = M_l$$

or

$$\nabla_k (R^k_l - \tfrac{1}{2}\delta^k_l R) = -\tfrac{1}{2} M_l. \tag{3.33}$$

M_l, as can be determined immediately, disappears identically in the Riemann space. The required generalization is obtained in this manner. We will return elsewhere to the physical content of this identity.

Paragraph 4, DERIVATION OF THE FIELD EQUATIONS

4.1. General Considerations

In order to make it possible to derive the field equations, which are the immediate generalizations of the Einstein gravitation equations, we now want to formulate the variation principle in the following manner, with consideration of paragraph 2.3 and paragraph 3.1. If we form along a field of direction

$$\dot{x}^i = v^i(x) \tag{4.1}$$

the invariant integral

$$I = \int \mathfrak{R}\, dx, \tag{4.2}$$

where R is the principal curvature invariant density, which is defined by equation (1.21).

In the general theory of relativity, the field equations are derived by the disappearance of the variation of (4.2) and, in this manner, the field equations for the entire space are obtained. A different situation obviously prevails in the Finsler space. The field equations, which could be directly derived by the disappearance of δI, would be defined only with respect to the directional field (4.1).

In order to expand the field equations to the entire space, we require that

a) an absolute parallelism of the line elements will exist in the space (in this manner, equations (2.11) can be fulfilled along each extremal associated with the line element series),

b) that the tensor relations, which are obtained as field equations, exist along each line element series (3.1) of the Finsler space,

c) that, with respect to each line element series (3.1), the field equations of the Finsler space agree with those of the Riemann space, which osculates along (3.1).*

For the determination of the effect of the electromagnetic field on the geometrical structure of the space, we still must

d) specify a tensor relation, which gives the Finsler space a special character

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* It can be shown in a simple manner that this continuation is independent of the selection of the osculatory Finsler space. On this comment, see O. Varga ([3]).

4.2. First Group of Field Equations

Following these general preparations, we now form the variation of the integral (see (2.10)) (4.2):

$$\delta I = \int R_{ik}\, \delta g^{ik}\, dx - \int g^{ik}\, \delta R_{ik}\, dx. \tag{4.3}$$

With the aid of the generalizing Palatini identity, we now prove that the second integral disappears. In accordance with equations (3.27), (2.15) and (3.29), in view of requirement b), we now have:

$$\delta R_{ik} = \delta \overset{0}{R}_{ik}. \qquad (4.4)$$

Following transvection with g^{ik}, we obtain, because of (1.25), with the abbreviation

$$\overset{0}{T}{}^{s} \overset{\mathrm{def}}{=} g^{ik}\,\delta \overset{0}{\Gamma}{}^{s}_{ik} - g^{is}\,\delta \overset{0}{\Gamma}{}^{k}_{ik}, \qquad (4.5)$$

that

$$g^{ik}\,\delta R_{ik} = \nabla_s T^s \qquad (4.6)$$

Because, in the osculatory Riemann space, it is well knwon that

$$\nabla_s T^s = \frac{1}{\sqrt{\gamma}}\, T^s_{,s} \qquad (4.7)$$

$(\gamma = \det \gamma_{ik})$, we now have

$$g^{ik}\,\delta R_{ik} = T^s_{,s}, \qquad (4.8)$$

in this manner, the second integral disappears -- in accordance with the Gauss law, when we assume, as is customary, that the variation of g_{ik} and its derivations will be equal to zero at the boundary of the area of integration.

Then, in accordance with (3.26),

$$\delta I = \int \{\overset{0}{\mathfrak{R}}_{ik} - \tfrac{1}{2} g_{ik} \overset{0}{\mathfrak{R}}\}\,\delta g^{ik}\,dx = 0.$$

Our field equations are thus in the space osculating along (4.1) for the vacuum

$$\overset{0}{R}_{ik} - \tfrac{1}{2} g_{ik} \overset{0}{R} = 0 \qquad (4.9)$$

and, in accordance with the requirements b) and c), the field equations will be in the Finsler space

$$R_{ik} - \tfrac{1}{2} g_{ik} R = 0 \qquad (4.10)$$

In order to also fulfill condition a), we must also require that

$$R_{*} = 0 \qquad (4.11)$$

will be in the space (FN: Varga, O.: ($\underline{2}$), paragraph 4).

4.3. The Second Group of Field Equations

In order to make it possible to also fulfill condition d), we must specify those tensor relations, which determine the special character of the Finsler space. We want to characterize this characteristic of the space by the potentials of the electromagnetic field. It is obvious that we cannot derive this relation from a variation principle, because the electromagnetic field influences the dependence of the fundamental quatities of the space of the $\dot{x}^i$, which cannot be established through a variation of an invariant space integral (in the usual sense).

It is fortunate that we must consider the torsion of the space -- if we want to establish this group of the field equations in accordance with our fundamental hypothesis formulated in 2.3 --; it is therefore necessary only to give a relation for the torsion tensor A_{ikh}. Because A_{ikh} is now a tensor which is symmetrical in its indexes i, k, h, we must form a tensor of the third order, which is symmetrical in its indexes, from the known electromagnetic field tensors, in order to be able to identify this with A_{ikh}. The energy-tension tensor of the electromagnetic field S_{ik} is a symmetrical tensor, from which we derive the tensor of the third order

$$-\tfrac{1}{2}\{\Gamma_h S_{ik} + \Gamma_i S_{kh} + \Gamma_k S_{hi}\} \tag{4.12}$$

which has the desired character of symmetry; this tensor can thus be identified with A_{ikh} (19).

The electromagnetic field tensors, which were provided by the known field theories, are of course a function of only the x^i, because these series are based on point spaces. We should, therefore, first transform these tensors into the Finsler space.

If the guide field of the electromagnetic field is already determined by its potentials

$$\Phi_i = \Phi_i(x)$$

then we form therefrom -- as is customarily done -- the tensors

$$F_{ik} \overset{\text{def}}{=} \Gamma_i \Phi_k - \Gamma_k \Phi_i \tag{4.13}$$

with $\quad \Gamma_i \Phi^i = 0 \qquad (4.14)$

and

$$S_{ik} \overset{\text{def}}{=} \frac{1}{4\pi}\left\{F_{ir}F^{r}_{\cdot k} - \frac{1}{4} g_{ik} F_{rs} F^{rs}_{\cdot\cdot}\right\}, \tag{4.15}$$

where, as $\overline{\nabla}$, the covariant differential operator of the Finsler geometry is of course understood (FN: It should be observed that F_{ik} is also a function of $\dot{x}^i$ because of definition (1.24) of the covariant derivation).

The tensors F_{ik} and S_{ik} should satisfy the relations

$$\nabla_k F^{k}_{i\cdot} = 4\pi s_i \tag{4.16a}$$

and

$$S^{i}_{i\cdot} = 0 \tag{4.16b}$$

which are the direct generalizations of the corresponding relations of the relativistic Maxwell-Lorentz theories, where $s_i = s_i(x)$ signifies the vector of the current density. Equations (4.16) can of course also be considered the generalizing field equations of the electromagnetic field.

Following these preparations, it is now possible for us to identify the torsion tensor with (4.12) in order to completely determine the geometrical structure of the space through the electromagnetic guide field:

$$A_{ikh} = -\tfrac{1}{2}\{\nabla_h S_{ik} + \nabla_i S_{kh} + \nabla_k S_{hi}\}. \tag{4.17}$$

Equation (4.17), together with (4.14) and (4.16) form the second group of our field equations.

4.4. The Complete System of the Field Equations for Vacuum and theMetric Fundamental Form of the Space

We now want to summarize the field equations defined in the previous sections:

$$R_{ik} - \tfrac{1}{2} g_{ik} R = 0, \tag{4.18a}$$

$$R_{*\cdot rs}^{\;\,p} = 0, \tag{4.18b}$$

$$\nabla_k F^{k}_{i\cdot} = 0, \tag{4.18c}$$

$$\nabla_i \Phi^i = 0, \tag{4.18d}$$

$$S^{i}_{i\cdot} = 0, \tag{4.18e}$$

$$A_{ikh} = -\tfrac{1}{2}\{\nabla_h S_{ik} + \nabla_i S_{kh} + \nabla_k S_{hi}\}. \tag{4.18f}$$

Equations (4.18a) resolve into two groups:

$$R_{\underline{ik}} - \tfrac{1}{2} g_{ik} R = 0$$

and

$$R_{\check{ik}} = 0,$$

where $R_{\underline{ik}}$ is the symmetrical and $R_{\check{ik}}$ the skew-symmetrical portion of the tensor R_{ik}.

Because of the properties of symmetry of the tensor A_{ikh} and the skew symmetry of the tensor $R^{\;p}_{*.rs}$ in the indexes r and s, field equations (4.18) provide 66 differential equations.

If we now assume the fundamental form of the Finsley geometry in the form (20)

$$\mathfrak{L}(x, \dot{x}) = \sqrt[4]{\frac{(a_{ijklm}\, \dot{x}^i \dot{x}^j \dot{x}^k \dot{x}^l \dot{x}^m)^2}{(b_{rs}\, \dot{x}^r \dot{x}^s)^3}} \qquad (4.19)$$

where the tensors a_{ijklm}, respectively b_{rs}, which are symmetrical in all their indexes, are a function of only x^i. We must of course *assume that* L *is not infinite for any direction* $\dot{x}^i$.

If the denominator is the sixth power of a linear form

$$f \stackrel{\text{def}}{=} \alpha_i \dot{x}^i$$

and the numerator has the form

$$f^6 (c_{ik}\, \dot{x}^i \dot{x}^k)^2$$

then the fundamental form (4.19) will determine a Riemann geometry with the metric fundamental tensor c_{ik}.

In other cases, we want to assume that

$$b_{rs}\, \dot{x}^r \dot{x}^s$$

is a positive-definite quadratic form.

The field equations provide differential equations for the components of the tensors a_{ijklm} and b_{rs}. The tensor a_{ijklm} has 56 components and the tensor b_{rs} 10, the number of equations (4.18) thus agress with the unknown.

4.5. Comments on the Field Equations

a) Equations (4.18) experss in explicit form that the two essential quantities of the geometry are determined by the combined electromagnetic and gravitational field. In this case, it is noteworthy that, while R_{ik} is a function of the gravitational field alone in the Riemann space, the electromagnetic field also provides a contribution to R_{ik} in the Finsler space. This is shown, on the one hand, by the occurrence of $R_{\check{ik}}$ (which identically disappears in the Riemann space), and, on the other hand, by the dependence of the generalized Einstein-Ricci tensors on the $\dot{x}i$; which means that the field does not resolve into its components in any manner.

b) The charged mass points, which move under the influence of the electromagnetic and gravitational field, carry out an inertial motion in the Finsler space, as we have noted in 2.3. Thus, their universal lines are geodetic lines (2.3) of the space, of which the differential equations, thus the equations of motion of the mass point, determine, which can be derived through the variation of (4.17) in accordance with 2.1.
(FN: We wave the consideration of the effect of the electromagnetic characteristic field, which is excited by the moving charged mass point. This consideration is a much more significant problem in the derivation of the field equations for nonempty space than for the gravitational fields.)

c) The tensor $A_{i.k}^{\ \ k}$, which, in a way, determines the geometrical structure of the space (21), has a concrete physical meaning in our case. This result shows immediately when we transvect equation (4.18f) with g^{kh}. In accordance with (4.18e), the result is

$$A_i = A_{i.k}^{\ \ k} = -\Gamma_k S_{i.}^{\ k}, \tag{4.20}$$

where A_i obviously indicates the Lorentz power density (22).

d) If, in expression (4.19) of the curve element, the numerator can be divided by the denominator, L is reduced, as we have already shown, to a form

$$\mathfrak{L}(x, \dot{x}) = \sqrt{c_{ik} \dot{x}^i \dot{x}^k}\,.$$

which obviously determines a Riemann geometry, of which the metric fundamental tensor is precisely $c_{ik}(x)$. This happens only in the case

$$A_{ikh} = 0$$

i.e. because of (4.17) in the absence of the electromagnetic field. It cannot be determined with certainty whether this occurs only in the absence of the electromagnetic field. In accordance with (4.17) it is possible that $A_{ikh} = 0$ without the identical disappearance of S_{ik}. However, in this case, also, the Finsler geometry is transformed into the Riemann, and the given unified theory completely adapts to the Einstein general theory of relativity. The discussion of this problem will be of special interest in the case of the matter and charge-filled space.

e) Finally, without going into details, we want to note that, when current density $s_i(x)$ exists in the space, equation (4.16a) indicates the possible generalization of the field equations given for vacuum. In order to also take the presence of matter into consideration, we can start with identity (3.32), but we want to come back to that another time.

CONCLUSIONS

In conclusion, we want to briefly discuss in how far the developed theory satisfies the postulates of Einstein, which were mentioned in the introduction, and the problems which we set as our objectives. It can be immediately determined that field equations (4.18) completely adapt to the first Einstein postulate. It is obvious that the invariant (1.21) does not resolve into several invariant components in any manner. The second postulate is also fulfilled and the fact that it was necessary for us to consider facts which are foreign to the variation principle in the development of the second group of field equations is related, on the one hand, with the inner character of the Finsler geometry, which differs from the Riemann space and, on the other hand, with our postulated requirement that we sought the direct generalization of the Einstein theory of gravitation. If we had not taken this last requirement into consideration and if we had not sought to develop a generalization containing the unchanged Einstein theory, then it would have been possible for us to base the variation principle on another invariant, which would have summarized the electromagnetic and the gravitational field still closer. However, we do not wish to cover this possibility at this time.

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