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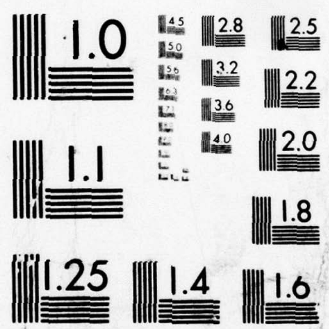

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Research Report CCS 295

GRADIENT STATES FOR SOME DUALITIES WITH THE C^2 EXTREMAL PRINCIPLE

by

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ABSTRACT

Gradient characterizations of some convex function infima are derived which apply to extension of the Charnes-Cooper duality state characterizations to more general classes of convex programming problems via the Charnes-Cooper extremal principle for optimization dualities.

The C^2 extremal principle for dualities which was originally presented in Charnes, Cooper and Seiford [1] is an approach to deriving dual optimization problems with proper duality inequality which simplifies and generalizes the Fenchel-Rockafellar scheme [2, 3]. The derivation is accomplished in two stages. The first is the achievement of the duality inequality. The second is the decoupling of the primal and dual variables.

The C^2 Extremal Principle

Let $K(\delta,x)$ be a real valued function which is concave in δ for

$$(\delta,x)\varepsilon\Delta\oplus X\subseteq R^m\oplus R^s$$

and for which

$$g(\delta)\equiv\inf_{x\varepsilon X} K(\delta,x)$$

exists for each $\delta\varepsilon\Delta$. Let T be a map from the convex set $Z\subseteq R^n$ into X. If $K(\delta,T(z)) = f(z)$, a convex function for $z\varepsilon Z$, $\delta\varepsilon\Gamma$, then $\Gamma\oplus T[Z]$ is the decoupling set for (δ,x). If we further require $\Delta\cap\Gamma$ to be a convex set, the problems

$$\sup g(\delta),\quad \delta\varepsilon\Delta\cap\Gamma \tag{1}$$

and

$$\inf f(z),\quad z\varepsilon Z \tag{2}$$

are dual convex programming problems.

As an example in the use of the use of the C^2 extremal principle, we derive the dual problems for linear programming. Let

$$K(y,x) = y^T x$$

for

$$y \varepsilon \Delta = \{y : y \geq 0\}$$

and

$$x \varepsilon X = \{x : x \geq b\}.$$

Then

$$g(y) = \inf_{x} K(y,x) = y^T b.$$

If we let $x = Az$, then

$$y^T b \leq y^T Az \quad \forall y \varepsilon \Delta, \ \forall z \varepsilon Z = \{z : Az \geq b\}.$$

Defining

$$\Gamma = \{y : y^T A = c^T\}$$

we have

$$\sup_{y \varepsilon \Delta \cap \Gamma} y^T b \leq \inf_{z \varepsilon Z} c^T z$$

or equivalently

$$\begin{array}{lcl} \sup b^T y & \leq & \inf c^T z \\ \text{subject to} & & \text{subject to} \\ A^T y = c & & Az \geq b \\ y \geq 0 & & \end{array} \qquad (3)$$

It is well known that a duality gap cannot occur in linear programming. In the general case the existence or non-existence of duality gaps is dependent on the choices of $\Delta \cap \Gamma$ and Z.

Gradient Characterization of Some Convex Function Infima

The extension of our characterization to duality states for more general cases of dual convex problems of the form given in Charnes, Cooper and Seiford [1] depends on developing properties characterizing the existence or non-existence of infima for special classes of convex functions. In the following theorem we adduce some such properties.

Theorem 1: Let $f: X \to R$ be convex and differentiable on an open convex set $X \subseteq R^m$. For the linear function $A: Z \to X$ with Z convex, consider

$$C(z) = f(Az) - b^T z.$$

If we define

$$\Gamma = \{\delta : \delta^T A = b^T\}$$

$$\Delta_x = \nabla f[X]$$

$$\Delta_z = \nabla f[A(Z)]$$

then

(i) a) C(z) is bounded below implies

$\overline{\Delta}_z \cap \Gamma \neq \phi$

b) $\Delta_x \cap \Gamma \neq \phi$ implies C(z) is bounded below

(ii) a) C(z) has an infimum implies

$$\overline{\Delta}_z \cap \Gamma \neq \phi, \; \Delta_z \cap \Gamma = \phi$$

b) $\Delta_x \cap \Gamma \neq \phi$, $\Delta_z \cap \Gamma = \phi$ implies C(z) has an infimum

(iii) C(z) has a minimum if and only if $\Delta_z \cap \Gamma \neq \phi$.

<u>Proof</u>: (i) a) Suppose $\overline{\Delta}_z \cap \Gamma = \phi$. Then

$$\nabla C(z) = \nabla f(Az)^T A - b^T$$

is bounded away from zero, i.e.,

$$\| \nabla C(z) \| \geq \varepsilon > 0.$$

Consider the differential equation system.

$$\dot{z}(t) = \frac{-\nabla C(z(t))}{\| \nabla C(z(t)) \|}.$$

The function $-\nabla C(\cdot)/\| \nabla C(\cdot) \|$ is continuous [2] and bounded, since ∇C is the gradient of a <u>convex</u> function. Hence there exists a solution, z(t). For $F(t) \equiv C(z(t))$

$$F'(t) = \nabla C(z(t)) \cdot \dot{z}(t)$$

$$= \nabla C(z(t)) \cdot \left(\frac{-\nabla C(z(t))}{\| \nabla C(z(t)) \|} \right)$$

$$= -\| \nabla C(z(t)) \| \leq -\varepsilon < 0.$$

Thus as $t \to +\infty$

$$F(t) = C(z(t)) \downarrow -\infty$$

and C is unbounded below.

(i) b) If $\Delta_x \cap \Gamma \neq \phi$, let $\bar{\delta} \varepsilon \Delta_x \cap \Gamma$.

Then $C(z) = f(Az) - b^T z = f(Az) - \bar{\delta}^T Az$.

Hence $\inf_z C(z) \geq \inf_x f(x) - \bar{\delta}^T x \geq f(x_o) - \bar{\delta}^T x_o$

by the differentiable convexity of f, where x_o satisfies $\nabla f(x_o) = \bar{\delta}$.

(iii) Suppose C(z) attains its minimum at z_o. Then

$$\nabla C(z_o)^T = \nabla f(Az_o)^T A - b^T = 0.$$

Setting $\delta = \nabla f(Az_o)$ we have

$$\delta \varepsilon \Delta_z \cap \Gamma.$$

Conversely, if $\delta_o \varepsilon \Delta_z \cap \Gamma$, then $\delta_o \varepsilon \Gamma$

so $C(z) = f(Az) - \delta_o^T Az$

and $\nabla C(z)^T = \nabla f(Az)^T \cdot A - \delta_o^T A$

$= (\nabla f(Az)^T - \delta_o^T A.$

Since $\delta_o \varepsilon \Delta_z$, $\exists z_o$ and that $\nabla f(Az_o) = \delta_o$.

Hence $\nabla C(z_o) = 0$ and C(z) attains its minimum at z_o.

(ii) a) and b) now follow by exhaustion.

<u>Corollary 1</u> $\Gamma = \phi \Rightarrow C(z)$ is unbounded below.

<u>Proof</u>: Consider the dual linear programming problems

(I)	(II)
max $b^T z$	min $\delta^T 0$
s.t. $Az = 0$	s.t. $\delta^T A = b^T$

If $\Gamma = \phi$, then II is infeasible. Since $z = 0$ satisfies (I), there exists a sequence z_n such that

$$Az_n = 0 \ (\forall n) \text{ and } b^T z_n \uparrow +\infty .$$

Thus $C(z_n) = f(Az_n) - b^T z_n$

$$= f(0) - b^T z_n \to -\infty .$$

That the characterization given by Theorem 1 is a best possible is shown by the following examples.

<u>Example 1</u>: To show that (in i,b) we need $\Delta_x \cap \Gamma \neq \phi$ (rather than $\overline{\Delta}_x \cap \Gamma \neq \phi$) to insure C(z) is bounded below, consider

$$f(x) = \begin{cases} -\ell n(-x) & \text{if } x \leq -1 \\ x + 1 & \text{if } x > -1 \end{cases}$$

Then $b = 0 \ \varepsilon \ \overline{\Delta}_x$ but $\lim_{x \to -\infty} f(x) - 0 \cdot x = -\infty$.

<u>Example 2</u>: To show (in i, a) that C(z) bounded below only *guarantees* $\overline{\Delta}_z \cap \Gamma \neq \phi$ and not $\Delta_z \cap \Gamma \neq \emptyset$, consider

$$f(z) = e^z .$$

Then $f(z) = e^z - 0 \cdot z > 0 \qquad (\forall z)$

but $0 \notin \Delta_z$, $0 \ \varepsilon \ \overline{\Delta}_z$.

<u>Conclusion</u>

In other work now in progress we employ these results to obtain duality state characterizations of dual convex programs derived from the C^2 principle. We also make applications to two-person zero-sum games whose payoff function is of the form $K(\delta,x) = f(x) - \delta^T x + g(\delta)$ where f(x) is convex and $g(\delta)$ is concave. Such games have arisen in contexts where the x-player corresponds to a government agency and the δ-player is the totality of enterprise groups whose activities are being regulated.

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