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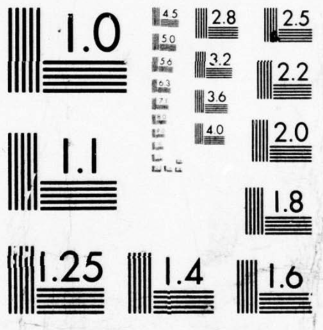

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A SET OF ORTHOGONAL POLYNOMIALS THAT GENERALIZE THE RACAH COEFFICIENTS OR 6 - j SYMBOLS

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ABSTRACT

A very general set of orthogonal polynomials with five free parameters is given explicitly, the orthogonality relation is proved and the three term recurrence relation is found.

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SIGNIFICANCE AND EXPLANATION

Orthogonal polynomials are used in numerical analysis for interpolation and quadrature, in the quantum mechanical theory of angular momentum, in statistics and many other areas. The polynomials introduced in this paper contain all the classical orthogonal polynomials as limits, and so provide a unified way of deriving some of the properties of the classical polynomials, as well as giving us a more general set of polynomials to use for applications.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A SET OF ORTHOGONAL POLYNOMIALS THAT GENERALIZE THE RACAH COEFFICIENTS OR 6 - j SYMBOLS

Richard Askey(1) and James Wilson

1. Introduction. A hypergeometric series has the form $\Sigma\, a_n$ with a_{n+1}/a_n a rational function of n. A basic hypergeometric series has a_{n+1}/a_n a rational function of q^n for a fixed q. The standard notation will be used. It is

$$(1.1)\qquad {}_rF_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{x^n}{n!},$$

where

$$(1.2)\qquad \begin{aligned}(a)_n &= a(a+1)\cdots(a+n-1), && n = 1,2,\dots\\ &= 1 && n = 0\end{aligned}$$

for hypergeometric series and

$$(1.3)\qquad {}_{r+1}\varphi_r\left(\begin{matrix}a_1,\dots,a_{r+1}\\ b_1,\dots,b_r\end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_{r+1};q)_n\, x^n}{(b_1;q)_n \cdots (b_r;q)_n\, (q;q)_n}$$

with

$$(1.4)\qquad \begin{aligned}(a;q)_n &= (1-a)\cdots(1-aq^{n-1}) && n = 1,2,\dots\\ &= 1 && n = 0\\ &= \frac{1}{(1-aq^{-n})\cdots(1-aq^{-1})} && n = -1,-2,\dots\end{aligned}$$

for basic hypergeometric series.

For readers who are unacquainted with basic hypergeometric series, observe that

$$\lim_{q\to 1} \frac{(q^\alpha;q)_n}{(q^\beta;q)_n} = \frac{(\alpha)_n}{(\beta)_n}.$$

There are reasons for using $(a;q)_n$ in (1.3) rather than $(q^\alpha;q)_n$ which go beyond a desire for a notation that is easy to set in type. There are times when we want "a" to be negative, and we can only make q^α negative by taking α complex. It is

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possible to do this but unnecessary. Also there are times when we want "a" to be independent of q. Again it is possible to take $\alpha = (\log a)/(\log q)$ so that q^α is independent of q, but it is unnecessary if we use $(a;q)_n$ rather than $(q^\alpha;q)_n$.

In [12] it was pointed out that

$$(1.5)\qquad p_n(\lambda(x)) = {}_4F_3\left(\begin{matrix} -n,\ n+\alpha+\beta+1,\ -x,\ x+\gamma+\delta+1 \\ \alpha+1,\ \beta+\delta+1,\ \gamma+1 \end{matrix}\ ;1\right)$$

$$(1.6)\qquad \lambda(x) = x(x+\gamma+\delta+1)$$

is a polynomial of degree n in $\lambda(x)$ which is orthogonal on $x = 0,1,\ldots,N$ when $\alpha+1$, $\beta+\delta+1$ or $\gamma+1$ is $-N$. This orthogonality relation is equivalent to Racah's orthogonality for functions that are usually called Racah coefficients or $6-j$ symbols. These polynomials contain as limiting cases the classical polynomials of Jacobi, Laguerre and Hermite and their discrete analogues which go under the names of Hahn, Meixner, Krawtchouk and Charlier polynomials. All of these polynomials can be given as hypergeometric series. Since basic hypergeometric extensions of the classical polynomials have been found [8], [2] it is natural to look for a basic hypergeometric extension of (1.5). The right polynomials to consider are balanced ${}_4\varphi_3$'s

$$(1.7)\qquad p_n(\mu(x);a,b,c,d;q) = p_n(\mu(x)) = {}_4\varphi_3\left(\begin{matrix} q^{-n},\ q^{n+1}ab,\ q^{-x},\ q^{x+1}cd \\ aq,\ bdq,\ cq \end{matrix}\ ;q,q\right)$$

where

$$(1.8)\qquad \mu(x) = q^{-x} + q^{x+1}cd\,.$$

Since

$$p_n(\mu(x)) = 1 + \sum_{k=1}^{n} \frac{(q^{-n};q)_k(q^{n+1}ab;q)_k q^k}{(aq;q)_k(bdq;q)_k(cq;q)_k(q;q)_k} \prod_{j=0}^{k-1} [1 + q^{2j+1}cd - q^j(\mu(x))]$$

it is clear that $p_n(\mu(x))$ is a polynomial of degree n in the variable $\mu(x)$.

The adjective "balanced" refers to a condition put on the parameters. For basic hypergeometric series it means that the product of the numerator parameters times q is the product of the denominator parameters. In this case $q^{-n+n+1}abq^{-x+x+1}cdq = abcdq^3$.

2. <u>Orthogonality</u>. Assume that one of aq, cq or bdq is q^{-N}. Then the orthogonality relation is

$$\sum_{x=0}^{N} p_n(\mu(x);a,b,c,d;q)p_m(\mu(x);a,b,c,d;q)w(x) = 0, \quad m \neq n, \quad 0 \leq m,n \leq N, \tag{2.1}$$

where

$$w(x) = \frac{(cdq;q)_x(1 - cdq^{2x+1})(aq;q)_x(bdq;q)_x(cq;q)_x(abq)^{-x}}{(q;q)_x(1 - cdq)(cda^{-1}q;q)_x(b^{-1}cq;q)_x(dq;q)_x}.$$

Observe that

$$(q^{-x}d^{-1};q)_m(q^{x+1}c;q)_m = \prod_{j=0}^{m-1}(1 - d^{-1}q^{-x+j})(1 - cq^{x+j+1})$$

$$= \prod_{j=0}^{m-1}(1 + cq^{2j+1}d^{-1} - d^{-1}q^j\mu(x))$$

is a polynomial of degree m in $\mu(x)$. To prove (2.1) for $m \neq n$ it will suffice to show that

$$I = \sum_{x=0}^{N} p_n(\mu(x);a,b,c,d;q)(q^{-x}d^{-1};q)_m(q^{x+1}c;q)_m w(x) = 0; \quad m < n. \tag{2.2}$$

The advantage of (2.2) over (2.1) is that the polynomial of degree m can be attached to the weight function. Using the definition of $p_n(\mu(x))$ in (2.2) gives

$$I = \sum_{x=0}^{N}\sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^{n+1}ab;q)_k(q^{-x};q)_k(q^{x+1}cd;q)_k q^k}{(aq;q)_k(bdq;q)_k(cq;q)_k(q;q)_k}$$

$$\frac{(q^{-x}d^{-1};q)_m(q^{x+1}c;q)_m(cdq;q)_x(1 - cdq^{2x+1})(aq;q)_x(bdq;q)_x(cq;q)_x}{(q;q)_x(1 - cdq)(a^{-1}cdq;q)_x(b^{-1}cq;q)_x(dq;q)_x(abq)^x}$$

$$= \sum_{k=0}^{n}\sum_{x=k}^{N} \frac{(q^{-n};q)_k(q^{n+1}ab;q)_k(aq^{k+1};q)_{x-k}(bdq^{k+1};q)_{x-k}}{(q;q)_k(a^{-1}cdq;q)_k(b^{-1}cq;q)_x(q;q)_{x-k}}$$

$$\frac{(cq^{k+1};q)_{x-k+m}(cdq;q)_{x+k}(1 - cdq^{2x+1})(-1)^{k+m}q^{k-x(k+m+1)+\binom{k}{2}+\binom{m}{2}}}{(dq;q)_{x-k}(1 - cdq)}$$

$$= \frac{q^{\binom{m}{2}}(-1)^m}{d^m} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{n+1}ab;q)_k q^{k(k+1)/2}}{(q;q)_k (-1)^k q^{k(k+m+1)} (ab)^k} \cdot \sum_{x=0}^{N-k} \frac{(aq^{k+1};q)_x (bdq^{k+1};q)_x}{(a^{-1}cdq;q)_{x+k}}$$

$$\cdot \frac{(cq^{k+1};q)_{x+m} (cdq;q)_{x+2k} (1 - cdq^{2x+2k+1})}{(b^{-1}cq;q)_{x+k} (q;q)_x (dq;q)_{x+k-m} (1 - cdq)} \left(\frac{1}{abq^{k+m+1}}\right)^x$$

$$= \frac{q^{\binom{m}{2}}(-1)^m}{d^m} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{n+1}ab;q)_k (cdq;q)_{2k} (cq^{k+1};q)_m (1 - cdq^{2k+1}) (-1)^k q^{-mk-k}}{(q;q)_k (a^{-1}cdq;q)_k (b^{-1}cq;q)_k (dq;q)_{k-m} (1 - cdq) q^{\binom{k}{2}} (ab)^k}$$

$$\cdot \sum_{x=0}^{N-k} \frac{(cdq^{2k+1};q)_x (1 - cdq^{2k+2x+1}) (aq^{k+1};q)_x (bdq^{k+1};q)_x (cq^{k+m+1};q)_x}{(q;q)_x (1 - cdq^{2k+1}) (a^{-1}cdq^{k+1};q)_x (b^{-1}cq^{k+1};q)_x (dq^{k-m+1};q)_x} \left(\frac{q^{-m-k-1}}{ab}\right)^x .$$

The sum on x can be evaluated since it is a very well poised ${}_6\varphi_5$. The required sum is

$$(2.3) \qquad {}_6\varphi_5\left(\begin{matrix} a^2, aq, -aq, b, c, q^{-N} \\ a, -a, a^2b^{-1}q, a^2c^{-1}q, a^2q^{N+1} \end{matrix}; q, \frac{a^2q^{N+1}}{bc}\right) = \frac{(a^2q;q)_N (a^2qb^{-1}c^{-1};q)_N}{(a^2qb^{-1};q)_N (a^2qc^{-1};q)_N}$$

$$= \frac{(a^2q;q)_\infty (a^2qb^{-1}c^{-1};q)_\infty (a^2q^{N+1}b^{-1};q)_\infty (a^2q^{N+1}c^{-1};q)_\infty}{(a^2q^{N+1};q)_\infty (a^2q^{N+1}b^{-1}c^{-1};q)_\infty (a^2qb^{-1};q)_\infty (a^2qc^{-1};q)_\infty} .$$

Up until now no assumptions on q have been made other than the implicit assumption that there are no zeros in the denominator. To make the calculations that follow a little easier we will assume $|q| < 1$ and define $(a;q)_\infty$ by

$$(2.4) \qquad (a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n) .$$

Since we are only dealing with polynomials it is easy to remove this restriction on q. A proof of (2.3) using orthogonal polynomials is given in [1]. One needs to set $bc = aq$ in the formula in Theorem 12. A proof is also given in [10, (3.3, 1.4)]. However in the appendix in [10] this formula is given with some misprints.

Using (2.3) in I gives

$$I = \frac{q^{\binom{m}{2}}(-1)^m (cq;q)_m}{d^m} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{n+1}ab;q)_k (cdq;q)_{2k} (1 - cdq^{2k+1})}{(q;q)_k (a^{-1}cdq;q)_k (b^{-1}cq;q)_k (1 - cdq)}$$

$$\frac{(cq^{m+1};q)_k (cdq^{2k+2};q)_\infty (a^{-1}b^{-1}c;q)_\infty (a^{-1}dq^{-m};q)_\infty (b^{-1}q^{-m};q)_\infty (-1)^k q^{-\frac{k^2}{2}-\frac{k}{2}-mk} (ab)^{-k}}{(dq;q)_{k-m} (cq;q)_k (a^{-1}cdq^{k+1};q)_\infty (b^{-1}cq^{k+1};q)_\infty (dq^{k-m+1};q)_\infty (a^{-1}b^{-1}q^{-k-m-1};q)_\infty}$$

$$= \frac{q^{\binom{m}{2}}(-1)^m (a^{-1}b^{-1}c;q)_\infty (a^{-1}dq^{-m};q)_\infty (b^{-1}q^{-m};q)_\infty (cdq^2;q)_\infty (cq;q)_m}{d^m (a^{-1}cdq;q)_\infty (b^{-1}cq;q)_\infty (dq;q)_\infty (a^{-1}b^{-1}q^{-m-1};q)_\infty}$$

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{n+1}ab;q)_k (cq^{m+1};q)_k q^k}{(q;q)_k (abq^{m+2};q)_k (cq;q)_k} .$$

This sum is a balanced ${}_3\varphi_2$, and so can be summed using

(2.5) $${}_3\varphi_2\left(\begin{matrix} q^{-n}, aq^n, b \\ c, d \end{matrix}; q, q\right) = \frac{(c/b;q)_n (d/b;q)_n b^n}{(c;q)_n (d;q)_n}$$

when $abq = cd$. The final result is

$$I = \frac{A(abc^{-1}q;q)_n (q^{-m};q)_n (cq^{m+1})^n}{(abq^{m+2};q)_n (cq;q)_n}$$

where A is the coefficient of the sum above. So $I = 0$ for $m = 0,1,\ldots,n-1$.

The value of the sum in (2.1) when $m = n$ can be found from this sum. However it is easier to obtain it from results in the next section.

3. Recurrence relation. If $p_n(x)$ are orthogonal with respect to a positive measure then

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x); \quad p_{-1}(x) \equiv 0 . \tag{3.1}$$

If the measure has infinitely many points of support then (3.1) holds for $n = 0,1,\ldots$. When there are only finitely many point masses, say $N + 1$, then (3.1) holds for $n = 0,1,\ldots,N - 1$, and when $n = N$ the zeros of $p_{N+1}(x)$ determine the location of the point masses. For a proof of this old fact see [3]. It is implicit in some of Tchebychef's work on continued fractions. We have shown that $\{p_n(\mu(x))\}$ is orthogonal, so (3.1) becomes

$$\mu(x)p_n(\mu(x)) = A_n p_{n+1}(\mu(x)) + B_n p_n(\mu(x)) + C_n p_{n-1}(\mu(x)) . \tag{3.2}$$

When $x = 0$, $p_n(\mu(0)) = 1$, so (3.2) can be written as

$$[\mu(x) - \mu(0)]p_n(\mu(x)) = A_n[p_{n+1}(\mu(x)) - p_n(\mu(x))] - C_n[p_n(\mu(x)) - p_{n-1}(\mu(x))] . \tag{3.3}$$

A_n is determined by equating the highest powers of $\mu(x)$. It is

$$A_n = \frac{(1 - abq^{n+1})(1 - aq^{n+1})(1 - bdq^{n+1})(1 - cq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})} \tag{3.4}$$

since

$$p_n(\mu(x)) = \frac{(q^{-n};q)_n (q^{n+1}ab;q)_n q^n (-1)^n q^{\binom{n}{2}}}{(aq;q)_n (bdq;q)_n (cq;q)_n (q;q)_n} [\mu(x)]^n + \text{lower terms} . \tag{3.5}$$

The easiest way to find C_n is to first simplify (3.3). A routine calculation gives

$$p_{n+1}(\mu(x)) - p_n(\mu(x)) = \frac{-q^{-n}(1 - q^{2m+2}ab)(1 - q^{-x})(1 - q^{x+1}cd)}{(1 - aq)(1 - bdq)(1 - cq)} \tag{3.6}$$

$$\cdot \, {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{n+2}ab, q^{-x+1}, q^{x+2}cd \\ aq^2, bdq^2, cq^2 \end{matrix} ; q,q\right) .$$

So (3.3) can be rewritten as

$$(3.7)\qquad -{}_4\varphi_3\left(\begin{matrix} q^{-n},\ q^{n+1}ab,\ q^{-x},\ q^{x+1}cd \\ aq,\ bdq,\ cq \end{matrix};\ q,q\right)$$

$$= \frac{-A_n q^{-n}(1 - q^{2n+2}ab)}{(1 - aq)(1 - bdq)(1 - cq)}\ {}_4\varphi_3\left(\begin{matrix} q^{-n},\ q^{n+2}ab,\ q^{-x+1},\ q^{x+2}cd \\ aq^2,\ bdq^2,\ cq^2 \end{matrix};\ q,q\right)$$

$$+ \frac{C_n q^{-n+1}(1 - q^{2n}ab)}{(1 - aq)(1 - bdq)(1 - cq)}\ {}_4\varphi_3\left(\begin{matrix} q^{-n+1},\ q^{n+1}ab,\ q^{-x+1},\ q^{x+2}cd \\ aq^2,\ bdq^2,\ cq^2 \end{matrix};\ q,q\right).$$

Now there are a couple of ways to proceed. If $x = 1$ then all the ${}_4\varphi_3$'s can be evaluated, but the reduction is more complicated than it has to be. Another way is to set $q^{-x} = aq$ and use (2.5) on all the series. This calculation gives

$$(3.8)\qquad C_n = \frac{cdq(1 - q^n)(1 - bq^n)(1 - abc^{-1}q^n)(1 - ad^{-1}q^n)}{(1 - abq^{2n+1})(1 - abq^{2n})}.$$

Formula (3.2) is an analogue for balanced ${}_4\varphi_3$'s of one of the contiguous relations of Gauss. Set

$$\varphi = \varphi\left(\begin{matrix} a,b,c,d \\ e,f,g \end{matrix};\ q,q\right)$$

where $abcdq = efg$ and one of a, b, c or d is q^{-n}. Also set

$$\varphi(a+,b-) = \varphi\left(\begin{matrix} aq,bq^{-1},c,d \\ e,f,g \end{matrix};\ q,q\right).$$

Then (3.2) becomes

$$(3.9)\qquad efg(1 - b)(a - e)(a - f)(a - g)(aq - b)\varphi(a-,b+)$$

$$+ \left[\begin{matrix} a^2 efg(1 - e)(1 - d)(aq - b)(a - bq) \\ - efg(1 - b)(a - e)(a - f)(a - g)(aq - b) \\ + a^2 cdq(1 - a)(e - b)(f - e)(g - b)(a - bq) \end{matrix}\right]\varphi$$

$$- a^2 cdq(1 - a)(e - b)(f - e)(g - b)(a - bq)\varphi(a+,b-) = 0,$$

or

$$\begin{aligned}(3.10)\qquad & efg(1-b)(a-e)(a-f)(a-g)(aq-b)[\varphi(a-,b+)-\varphi] \\ & + a^2cdq(1-a)(e-b)(f-e)(g-b)(a-bq)[\varphi-\varphi(a+,b-)] \\ & + a^2efg(1-e)(1-d)(aq-b)(a-bq)\varphi = 0 .\end{aligned}$$

To find the sum of (2.1) when $m = n$, call it h_n. Then

$$A_n h_{n+1} = \sum_x \mu(x) p_n(\mu(x)) p_{n+1}(\mu(x)) w(x)$$

and

$$\begin{aligned} C_n h_{n-1} &= \sum_x \mu(x) p_n(\mu(x)) p_{n-1}(\mu(x)) w(x) \\ &= A_{n-1} h_n \end{aligned}$$

so

$$(3.11)\qquad h_n = \frac{C_n}{A_{n-1}} h_{n-1} = \cdots = \frac{C_n \cdots C_1}{A_{n-1} \cdots A_0} h_0 .$$

The ${}_6\varphi_5$ sum (2.3) gives

$$(3.12)\qquad h_0 = \frac{(cdq^2;q)_\infty \left(\frac{c}{ab};q\right)_\infty \left(\frac{d}{a};q\right)_\infty \left(\frac{1}{b};q\right)_\infty}{\left(\frac{cdq}{a};q\right)_\infty \left(\frac{cq}{b};q\right)_\infty (dq;q)_\infty \left(\frac{1}{abq};q\right)_\infty}$$

and so

$$h_n = \frac{(q;q)_n (1-abq)(bq;q)_n (ad^{-1}q;q)_n (abc^{-1}q;q)_n (cdq)^n}{(abq;q)_n (1-abq^{2n+1})(aq;q)_n (bdq;q)_n (cq;q)_n} h_0 .$$

Once this formula has been found some of the mystery of Section 2 can be removed. It is natural to ask where the weight function came from. Observe that

$${}_4\varphi_3\left(\begin{matrix} q^{-n},\ q^{n+1}ab,\ q^{-x},\ q^{x+1}cd \\ aq,\ bdq,\ cq \end{matrix} ;\ q,q\right)$$

is symmetric in n and x when (a,b) is changed into (c,d). This symmetry carries over to $w(x)$ and h_n, that is $w(n)$ is just h_0/h_n with (a,b) interchanged with (c,d). The reason for this is that a matrix that is orthogonal by rows is also orthogonal by columns. The usefulness of this remark was mentioned by Karlin and McGregor [9] in connection with the Hahn and dual Hahn polynomials. Also see Eagleson [6]. In fact this is how we found the weight function. However we could not give a proof by this method without first proving the recurrence relation (3.3) directly. This would be very tedious, so it is preferable to prove the orthogonality directly.

To show from the recurrence relation that the masses must be located at $x = 0,1,\ldots,N$, observe first that $A_N = 0$, since one of aq, bdq and cq was assumed to be q^{-N}. For definiteness take $bdq = q^{-N}$. The other cases are handled in a similar fashion. Formula (3.3) will hold when $n = N$ if we can show that

$$(1 - q^{-x})(1 - q^{x+1}cd)p_N(\mu(x)) = C_N[p_N(\mu(x)) - p_{N-1}(\mu(x))] .$$

Both sides vanish when $x = 0$, since $p_n(\mu(0)) = 1$. Use (3.6) on $p_N(\mu(x)) - p_{N-1}(\mu(x))$ and the value of C_N given in (3.8), to see that this is equivalent to

$$(3.13) \quad {}_3\varphi_2\left(\begin{matrix} q^{N+1}ab,\ q^{-x},\ q^{x+1}cd \\ aq,\ cq \end{matrix} ;\ q,q\right) = \frac{cdq(1 - q^N)(1 - bq^N)(1 - abc^{-1}q^N)(1 - ad^{-1}q^N)}{(1 - abq^{2N})(1 - abq^{2N+1})}$$

$$\cdot \frac{(-q^{1-N})(1 - abq^{2N})}{(1 - aq)(1 - bdq)(1 - cq)}\ {}_3\varphi_2\left(\begin{matrix} q^{N+1}ab,\ q^{1-x},\ q^{x+2}cd \\ aq^2,\ cq^2 \end{matrix} ;\ q,q\right),$$

when $x = 1,2,\ldots,N$. For the series is terminated by q^{1-x}, so it is correct to replace the factors $(q^{-N};q)_k/(q^{-N};q)_k$ by 1, since they do not vanish. Since $x = 1,2,\ldots,N$ the series in (3.13) can be summed using (2.5). Again a simple calculation shows that (3.13) holds for $x = 1,2,\ldots,N$. Thus the recurrence relation (3.2) holds when $n = N$ and $x = 0,1,\ldots,N$. Therefore the point masses must be located at $x = 0,1,\ldots,N$.

4. Summary and miscellaneous results. For ease of reference we state the two main results again:

$$\sum_{x=0}^{N} p_n(\mu(x);a,b,c,d;q)p_m(\mu(x);a,b,c,d;q)w(x) \tag{4.1}$$

$$= \delta_{m,n}h_n, \quad aq,\ bdq \ \text{ or } \ cq = q^{-N},$$

where

$$p_n(\mu(x);a,b,c,d;q) = {}_4\varphi_3\left(\begin{matrix} q^{-n},\ q^{n+1}ab,\ q^{-x},\ q^{x+1}cd \\ aq,\ bdq,\ cq \end{matrix}\ ;\ q,q\right) \tag{4.2}$$

$$\mu(x) = q^{-x} + q^{x+1}cd \tag{4.3}$$

$$w(x) = \frac{(cdq;q)_x(1-cdq^{2x+1})(aq;q)_x(bdq;q)_x(cq;q)_x}{(q;q)_x(1-cdq)(a^{-1}cdq;q)_x(b^{-1}cq;q)_x(dq;q)_x(abq)^x} \tag{4.4}$$

$$h_n = \frac{(q;q)_n(1-abq)(bq;q)_n(ad^{-1}q;q)_n(abc^{-1}q;q)_n(cdq)^n}{(abq;q)_n(1-abq^{2n+1})(aq;q)_n(bdq;q)_n(cq;q)_n} \tag{4.5}$$

$$\cdot \frac{(cdq^2;q)_\infty(a^{-1}b^{-1}c;q)_\infty(a^{-1}d;q)_\infty(b^{-1};q)_\infty}{(a^{-1}cdq;q)_\infty(b^{-1}cq;q)_\infty(dq;q)_\infty(a^{-1}b^{-1}q^{-1};q)_\infty}.$$

These infinite products look like they must have $|q| < 1$ before they make sense. However, since one of aq, bdq or cq is q^{-N} these products all reduce to finite products. For example, when $aq = q^{-N}$, then

$$\frac{(cdq^2;q)_\infty(a^{-1}b^{-1}c;q)_\infty(a^{-1}d;q)_\infty(b^{-1};q)_\infty}{(a^{-1}cdq;q)_\infty(b^{-1}cq;q)_\infty(dq;q)_\infty(a^{-1}b^{-1}q^{-1};q)_\infty} = \frac{(cdq^2;q)_\infty(b^{-1}cq^{N+1};q)_\infty(dq^{N+1};q)_\infty(b^{-1};q)_\infty}{(cdq^{2+N};q)_\infty(b^{-1}cq;q)_\infty(dq;q)_\infty(b^{-1}q^N;q)_\infty}$$

$$= \frac{(cdq^2;q)_N(b^{-1};q)_N}{(b^{-1}cq;q)_N(dq;q)_N}.$$

$$-(1-q^{-x})(1-q^{x+1}cd)p_n(\mu(x);a,b,c,d;q) \tag{4.6}$$

$$= A_np_{n+1}(\mu(x);a,b,c,d;q) - (A_n + C_n)p_n(\mu(x);a,b,c,d;q)$$

$$+ C_np_{n-1}(\mu(x);a,b,c,d;q),$$

where $p_{-1}(\mu(x);a,b,c,d;q) \equiv 0$ and

$$A_n = \frac{(1 - abq^{n+1})(1 - aq^{n+1})(1 - bdq^{n+1})(1 - cq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})}, \tag{4.7}$$

$$C_n = \frac{(1 - q^n)(1 - bq^n)(c - abq^n)(d - aq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})}. \tag{4.8}$$

When aq, bdq or cq is q^{-N} then (4.6) holds for $n = 0,1,\ldots,N-1$ and all x if the basic hypergeometric series that define $p_n(\mu(x))$ are assumed to terminate so that $p_n(\mu(x))$ is a polynomial of degree n, and to hold when $n = N$ when $x = 0,1,\ldots,N$. When none of aq, bdq or cq is equal to q^{-N} then (4.6) holds for all x for $n = 0,1,\ldots$.

For the polynomials $p_n(\mu(x))$ to be orthogonal with respect to a positive measure it is necessary and sufficient that

$$A_{n-1}C_n > 0 \tag{4.9}$$

See [7, Chapter II, Theorem 1.5]. If (4.9) holds for $n = 1,2,\ldots,$ then the measure has infinitely many points of support; when it holds for $n = 1,2,\ldots,N$ then the measure can be taken to have support on $N + 1$ points. In this paper we have only considered some cases when the measure is purely discrete and is supported on a finite set of points. In a later paper we will treat the general cases where the measure has both an absolutely continuous part and a discrete part.

There are many special cases of the orthogonality relation (4.1) which are interesting. When $d = 0$ and $cq = q^{-N}$ the polynomials were discovered by Hahn, and their weight function was found a couple of years ago by Andrews and Askey. Delsarte [4], Dunkl [5] and Stanton [11] have considered special cases of these polynomials. When $b = 0$ the polynomials are called dual Hahn polynomials. The orthogonality when $aq = q^{-N}$ was also found by Andrews and Askey.

Another interesting special case is Stanton's q-analogue of the Krawtchouk polynomials. These are

$$K_n(q^{-x};c,q^{-N-1};q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{-x}, -cq^{k+1} \\ 0, q^{-N} \end{matrix}; q,q\right) .$$

To obtain these from the q-Racah polynomials (1.7) first set $c = d = 0$ and $aq = q^{-N}$, then set $b = -cq^{N+1}$ so $ab = -c$. The weight function is then

$$w(x) = \frac{(q^{-N};q)_x}{(q;q)_x}\left(-\frac{1}{cq}\right)^x .$$

When $q \to 1$ this converges to $\frac{(-N)_x}{x!}\left(\frac{1}{c}\right)^x(-1)^x = \binom{N}{x}\left(\frac{1}{c}\right)^x$, which is the weight function for the Krawtchouk polynomials.

A word of caution about characterization theorems needs to be said. There are many theorems that say "the classical polynomials are the only *polynomials* to have a given property". Such theorems are often misleading. For example, Eagleson [6] showed that the Charlier, Krawtchouk and Meixner polynomials are the only polynomials that are self dual. He is able to prove this theorem and yet miss the polynomials $p_n(\mu(x);a,b,a,b)$, which are clearly symmetric in n and x because his definition of self dual or *symmetrizable is too restricted*. A characterization theorem that leads to new orthogonal polynomials is usually interesting, one that says the classical polynomials are the only polynomials with a given property are usually much less interesting and if they keep people from looking for new polynomials they are harmful.

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A very general set of orthogonal polynomials with five free parameters is given explicitly, the orthogonality relation is proved and the three term recurrence relation is found.

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