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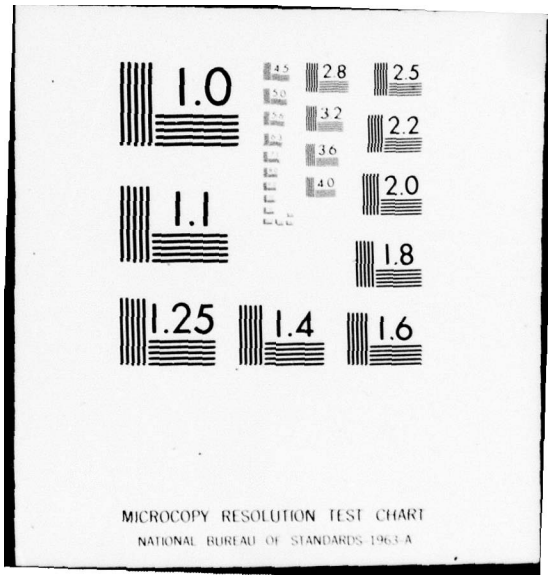
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ON THE STRUCTURE OF A LIAPUNOV FUNCTIONAL FOR A DIFFERENCE-DIFFERENTIAL EQUATION WITH ONE DELAY.

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ON THE STRUCTURE OF A LIAPUNOV FUNCTIONAL FOR A DIFFERENCE-DIFFERENTIAL EQUATION WITH ONE DELAY

Abstract: A quadratic positive definite Liapunov functional that yields necessary and sufficient conditions for the asymptotic stability of the solutions of the matrix difference-differential equation $\dot{x}(t) = Ax(t) + Bx(t-\tau)$ is constructed and its structure is analyzed. The method of construction is a natural generalization of the same problem for an ordinary differential equation, and this relationship is emphasized.

1. Introduction.

In this note we construct a Liapunov functional for the determination of the asymptotic behavior of the solutions of the linear autonomous matrix difference-differential equation with one delay

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \quad t > 0,$$

where $x(t)$ is an n-vector function of time, A and B are constant $n \times n$ matrices and $\tau \geq 0$.

In this note, we generalize results presented in [7] for a scalar equation; essential to this generalization was the study, presented in [2], of the existence, uniqueness and structure of the solutions of a special functional differential equa-

tion which is intimately connected with the construction of the Liapunov functional presented here.

The problem of constructing Liapunov functionals for equations of this type has been previously considered by Repin [8], Datko [3] and Hale [5,6]. This study is in the spirit of the previous ones, but is more specific.

2. The Difference-Differential Equation.

Denote by $L_2([a,b],\mathscr{R}^n)$ the space of Lebesgue square integrable functions defined on $[a,b]$ with values in $\mathscr{R}^n$, and for a fixed $\tau \geq 0$ consider the Hilbert space $\mathscr{H} = \mathscr{R}^n \times L_2([-\tau,0],\mathscr{R}^n)$ with inner product $\langle u_1,u_2\rangle = v_1^T v_2 + \int_{-\tau}^{0} \phi_1^T(\theta)\phi_2(\theta)d\theta$, where $u_i = (v_i,\phi_i) \in \mathscr{H}$, and the naturally induced norm is $||(v,\phi)||_{\mathscr{H}}^2 = v^T v + \int_{-\tau}^{0} \phi^T(\theta)\phi(\theta)d\theta$. With $x\colon [-\tau,\infty) \to \mathscr{R}^n$, for $t \geq 0$ we denote by x_t the function $x_t\colon [-\tau,0] \to \mathscr{R}^n$, where $x_t(\theta) = x(t+\theta)$.

Consider the matrix difference-differential equation

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \qquad t > 0, \tag{2.1}$$

where A,B are $n \times n$ matrices, $x(t)$ is an n-vector and $\tau \geq 0$, together with the initial conditions

$$x_0(0) = \xi, \quad x_0 = \phi, \tag{2.2}$$

where $(\xi,\phi) \in \mathscr{H}$.

A solution of this initial value problem is for each $t > 0$, a function $x \in L_2([-\tau,t],\mathscr{R}^n)$ such that x is absolutely continuous for $t \geq 0$, satisfies (2.1) a.e. on $[0,t]$ and $x(0) = \xi$, $x(\theta) = \phi(\theta)$ a.e. for $\theta \in [-\tau,0]$. It is known [1, 6] that (2.1)-(2.2) has a unique solution, defined on $[-\tau,\infty)$, which depends continuously on the initial data in the norm of $\mathscr{M}$.

The initial value problem (2.1)-(2.2) can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} x_t(0) \\ x_t \end{pmatrix} = \mathscr{A}\begin{pmatrix} x_t(0) \\ x_t \end{pmatrix}, \tag{2.3}$$

$$(x_0(0),x_0) = (\xi,\phi) \in \mathscr{M}, \tag{2.4}$$

where

$$\mathscr{A}\begin{pmatrix} x_t(0) \\ x_t \end{pmatrix} = \begin{pmatrix} Ax_t(0) + Bx_t(-\tau) \\ \frac{\partial x_t(\theta)}{\partial \theta}, -\tau \leq \theta \leq 0 \end{pmatrix}, \tag{2.5}$$

and this operator has a domain, dense in $\mathscr{M}$, defined by

$$\mathscr{D}(\mathscr{A}) = \{(\xi,\phi) \in \mathscr{M} \mid \phi \text{ is A.C. in } [-\tau,0],$$

$$\phi' \in L_2[-\tau,0],\ \phi(0) = \xi\}.$$

The operator $\mathscr{A}$ is the generator of the C_0-semigroup $\mathscr{S}(t)$, where $\mathscr{S}(t): \mathscr{M} \to \mathscr{M}$ is given by $\mathscr{S}(t)(\xi,\phi) = (x(t),x_t)$, the solution pair of (2.3), (2.4).

It is known [1,6] that there is a constant γ such that the spectrum of $\mathscr{A}$ lies in the left-half plane $Re(\lambda) \leq \gamma$, and that for every $\in > 0$ there exists a constant $K \geq 1$ such that

$$||\mathscr{S}(t)||_{(\mathscr{M},\mathscr{M})} \leq Ke^{(\gamma+\in)t}. \tag{2.6}$$

The spectrum of $\mathscr{A}$ consists of those complex λ which satisfy the characteristic equation

$$\det[\lambda I - A - Be^{-\lambda\tau}] = 0. \tag{2.7}$$

Finally, [5,6], a useful representation of the solutions of (2.1) is given for every $t,\ u \geq 0$, by

$$x_{t+u}(0) = S(u)x_t(0) + \int_{-\tau}^{0} S(u-\alpha-\tau)Bx_t(\alpha)d\alpha, \tag{2.8}$$

where the matrix S is the solution of the matrix initial value problem

$$\frac{d}{dt}S(t) = S(t)A + S(t-\tau)B,$$
$$S(0) = I,\ S(t) = 0 \quad \text{for} \quad t < 0. \tag{2.9}$$

We now turn to the construction of a Liapunov functional.

3. A Quadratic Functional.

Associated with the functional differential equation (2.1)-(2.2), or (2.3)-(2.4), and motivated by the results in [7], we wish to consider the real symmetric quadratic form on $\mathscr{M}$,

$$V(\xi,\phi) = \xi^T M\xi + e^{\delta\tau}\int_{-\tau}^{0}\phi^T(\theta)Re^{2\delta\theta}\phi(\theta)d\theta + \\ + \xi^T Q(0)\xi + 2\xi^T\int_{-\tau}^{0}Q(\alpha+\tau)e^{\delta(\alpha+\tau)}B\phi(\alpha)d\alpha \qquad (3.1) \\ + 2\int_{-\tau}^{0}\int_{\alpha}^{0}\phi^T(\alpha)B^TQ(\beta-\alpha)e^{\delta(\alpha+\beta+2\tau)}B\phi(\beta)d\beta d\alpha .$$

Here δ is a real number, M,R are constant $n \times n$ real positive definite matrices and $Q(\alpha)$ is a continuously differentiable matrix that is assumed to satisfy the initial value problem for the functional differential equation

$$Q'(\alpha) = (A^T+\delta I)Q(\alpha) + e^{\delta\tau}B^TQ^T(\tau-\alpha), \quad 0 \le \alpha \le \tau \qquad (3.2)$$

$$Q(0) = Q(0)^T = Q_0, \qquad (3.3)$$

where the superscript T denotes transpose and Q_0 is a symmetric but otherwise arbitrary matrix.

Evaluation of the Fréchet differentiable Liapunov functional (3.1) along the solutions of (2.3)-(2.4) with initial conditions in $\mathscr{D}(\mathscr{A})$ yields a function of time, which we denote by $V(t) = V(x_t(0),x_t)$. This function of time is differentiable along

such solutions, and a laborious but straightforward computation, which makes use of (3.2) and (3.3), shows that the derivative of this function along these solutions is given by

$$\dot{V}(t) = \frac{d}{dt} V(x_t(0),x_t) = -2\delta V(x_t(0),x_t) + \tag{3.4}$$

$$+ x_t^T(0)[(A^T+\delta I)Q(0) + Q(0)(A+\delta I) + (A^T+\delta I)M + M(A+\delta I) +$$

$$+ e^{\delta\tau}B^TQ^T(\tau) + e^{\delta\tau}Q(\tau)B + 2e^{\delta\tau}R]x_t(0)$$

$$- e^{\delta\tau}h(x_t(0),x_t(\tau)) \equiv U(x_t(0),x_t),$$

where

$$h(x_t(0),x_t(-\tau)) = [x_t^T(0),-e^{-\delta\tau}x_t^T(-\tau)]\begin{bmatrix} R & MB \\ B^TM & R \end{bmatrix}\begin{bmatrix} x_t(0) \\ -e^{-\delta\tau}x_t(-\tau) \end{bmatrix}. \tag{3.5}$$

A direct application of Theorem 3.9 of [9] shows that for any solution with initial condition on $\mathscr{K}$, $\dot{V}(t) \leq U(x_t(0),x_t)$.

It is our purpose to show that, through the functional (3.1) and its derivative (3.4) it is possible to estimate the rate of growth or decay of the solutions for our original functional differential equation (2.3)-(2.4). For this purpose appropriate choices must be made for the positive definite matrices M and R, of the constant δ and of the matrix $Q(\alpha)$ determined by (3.2) and (3.3).

To be more specific, denoting by $\gamma = \max\{\mathrm{Re}\ \lambda \mid \det[\lambda I - A - Be^{-\lambda\tau}] = 0\}$, we wish to show that for every $\epsilon > 0$ and $-\delta = \gamma + 2\epsilon$ it is possible to choose matrices M, R and $Q(\alpha)$ satisfying (3.2) and (3.3) for which there exist positive constants c_1, c_2 such that

$$c_1 ||(\xi,\phi)||^2_{\mathscr{M}} \leq V(\xi,\phi) \leq c_2 ||(\xi,\phi)||^2_{\mathscr{M}}, \tag{3.6}$$

and

$$\dot{V}(\xi,\phi) \leq -2\delta V(\xi,\phi). \tag{3.7}$$

These last relationships imply that $\{V(\xi,\phi)\}^{1/2}$ $\tilde{|||}(\xi,\phi)\tilde{|||}_{\mathscr{M}}$ is a *norm equivalent to the original norm on* $\mathscr{M}$ *and that, in this norm*

$$\tilde{|||}(x_t(0),x_t)\tilde{|||}_{\mathscr{M}} \leq \tilde{|||}x_0(0),x_0\tilde{|||}_{\mathscr{M}} e^{-\delta t}, \tag{3.8}$$

whereas in the original norm on $\mathscr{M}$

$$||(x_t(0),x_t)||_{\mathscr{M}} \leq \left(\frac{c_2}{c_1}\right)^{1/2} ||(x_0(0),x_0)||_{\mathscr{M}} e^{-\delta t}. \tag{3.9}$$

These estimates are precisely those stated in (2.6) and are the best possible. It should be noted that the norm induced by the square root of the Liapunov functional is the best possible one in the sense that it yields (2.6) with $K = 1$. Moreover, if

$\gamma < 0$, then the Liapunov functional (3.1) shows that (2.1)-(2.2) is uniformly exponentially stable.

First of all, consider an appropriate choice for the matrix $Q(\alpha)$, a solution of (3.2), (3.3) for $0 \leq \alpha \leq \tau$. In [2] it was shown that equation (3.2) with initial conditions (3.3) has a unique solution; moreover, it was shown that the linear vector space of all solutions of (3.2) has dimension n^2, and the structure of these solutions as well as simple methods of computation which take advantage of the particular structure of the equation were presented.

Associated with equation (3.2)-(3.3) is an integral, whose structure is motivated by a similar integral for ordinary differential equation. Indeed, let W be a symmetric matrix and let $S(t)$ be the solution of equation (2.9). Consider the expression

$$\tilde{Q}(\alpha) = \int_0^\infty S^T(u) e^{\delta u} W S(u-\alpha) e^{\delta(u-\alpha)} du. \tag{3.10}$$

Since, for every $\in > 0$, $||S(t)|| \leq \tilde{K} e^{(\gamma+\in)t}$ for some $\tilde{K} \geq 1$ and since $\delta = -\gamma - 2\in$, it follows that this integral converges. Moreover, it immediately follows from the definition of $\tilde{Q}(\alpha)$ that $\tilde{Q}(\alpha) = \tilde{Q}^T(-\alpha)$, $\tilde{Q}(0) = \tilde{Q}^T(0)$ and, since $S(t)$ satisfies (2.9), that $\tilde{Q}(\alpha)$ satisfies

$$\tilde{Q}'(\alpha) = (A^T+\delta I)\tilde{Q}(\alpha) + B^T e^{\delta\tau} Q^T(\tau-\alpha), \quad 0 \leq \alpha \leq \tau$$

$$\tilde{Q}^T(0) = \tilde{Q}(0) = \int_0^\infty S^T(u) e^{\delta u} W S(u) e^{\delta u} du \tag{3.11}$$

and, moreover, given the continuity of $Q(\alpha)$, that

$$\tilde{Q}'(0) + \tilde{Q}'^T(0) = (A^T+\delta I)\tilde{Q}(0) + \tilde{Q}(0)(A+\delta I) + B^T e^{\delta T} Q^T(\tau) + Q(\tau) e^{\delta T} B$$
$$= -S^T(0)W = -W. \tag{3.12}$$

These observations, and the uniqueness of the solutions of (3.2)-(3.3), show that $\tilde{Q}(\alpha)$, defined by (3.10), is the unique solution of (3.2)-(3.3) with the initial conditions prescribed by the second equation of (3.11).

It is easily seen that the map $W \to \tilde{Q}(0)$, defined by

$$\tilde{Q}(0) = \int_0^\infty S^T(u) e^{\delta u} W S(u) e^{\delta u} du,$$

as a map on the space of $n \times n$ symmetric matrices is one-to-one, onto and it maps positive definite matrices W into positive definite matrices $\tilde{Q}(0)$.

With this particular characterization of the matrix function $\tilde{Q}(\alpha)$ it is possible to bring into evidence the particular structure of the Liapunov functional (3.1). Indeed, substitution of (3.10) into

(3.1) for $Q(\alpha)$ yields, after some rearrangements and interchanges of integrals, that our functional is of the form

$$V(x_t(0),x_t) = x_t^T(0)Mx_t(0) + e^{\delta\tau}\int_{-\tau}^{0} x_t^T(\theta)Re^{2\delta\theta}x_t(\theta)d\theta + \int_0^{\infty} x_{t+u}(0)We^{2\delta u}x_{t+u}(0)du. \tag{3.13}$$

Similarly, using this notation, equation (3.4) becomes

$$U(x_t(0),x_t) = -2\delta V(x_t(0),x_t) + x_t^T(0)[-W + (A^T+\delta I)M + M(A+\delta I) + 2e^{\delta\tau}R]x_t(0) - e^{\delta\tau}[x_t^T(0),-e^{-\delta\tau}x_t^T(-\tau)]\begin{bmatrix} R & MB \\ B^TM & R \end{bmatrix}\begin{bmatrix} x_t(0) \\ -e^{-\delta\tau}x_t(-\tau) \end{bmatrix}. \tag{3.14}$$

Given the nonnegative nature of the last term in (3.13) it follows that, letting $\lambda_{min}(A)$ denote the smallest eigenvalue of a symmetric matrix A,

$$\min(\lambda_{min}(M),e^{-|\delta|\tau}\lambda_{min}(R))||(\xi,\phi)||_{\mathcal{W}}^2 \leq V(\xi,\phi),$$

yielding a value of c_1, for equation (3.6), given by

$$c_1 = \min(\lambda_{min}(M),e^{-|\delta|\tau}\lambda_{min}(R)). \tag{3.15}$$

Similarly, since from (2.6),

$$||x_u(0)||_{\mathscr{H}^n} \leq ||(x_u(0),x_u)||_{\mathscr{H}} \leq Ke^{(\gamma+\epsilon)u}||(x_0(0),x_0)||_{\mathscr{H}},$$

it follows that

$$V(\xi,\phi) \leq [\max\{\lambda_{max}(M), e^{|\delta|\tau}\lambda_{max}(R)\} + \frac{\lambda_{max}(W)}{2\epsilon} K^2]||(\xi,\phi)||^2_{\mathscr{H}}.$$

This yields for equation (3.6) a value of c_2 given by

$$c_2 = \max[\lambda_{max}(M), e^{|\delta|\tau}\lambda_{max}(R)] + \frac{\lambda_{max}(W)}{2\epsilon} K^2. \qquad (3.16)$$

It remains to be shown that equation (3.7) holds. For this purpose inspection of equation (3.14) indicates that it is necessary and sufficient to show, by appropriate choices of positive definite matrices W,R and M, to have the matrix

$$G = \begin{bmatrix} W - (A+\delta I)^T M - M(A+\delta I) - Re^{\delta\tau} & MBe^{\delta\tau} \\ B^T Me^{\delta\tau} & Re^{\delta\tau} \end{bmatrix} \qquad (3.17)$$

positive semidefinite. But this is always possible; indeed, a particularly simple choice of such matrices is to let W,R and M to be nonnegative multiples of the identity matrix, i.e. $M = I$, $R = k_R I$ and $W = k_W I$. Letting $k_R = \sqrt{\lambda_{max}(BB^T)}$ and

$$k_W = \max\{0, \lambda_{max}(A+A^T) + 2\delta + 2e^{\delta\tau}\sqrt{\lambda_{max}(BB^T)}\},$$

then (3.17) is positive semidefinite and therefore, for these choices of M,R and W,

$$\dot{V}(x_t(0),x_t) \leq -2\delta V(x_t(0),x_t) \tag{3.18}$$

for every $(x_t(0),x_t) \in \mathcal{C}$.

It is to be noted that, with these choices, the Liapunov functional is particularly simple and reduces to

$$V(x_t(0),x_t) = x_t^T(0)x_t(0) + e^{\delta\tau}\sqrt{\lambda_{max}(BB^T)}\int_{-\tau}^{0} x_t^T(\theta)x_t(\theta)e^{2\delta\theta}d\theta +$$

$$+ \max\{0, \lambda_{max}(A+A^T) + 2\delta + 2e^{\delta\tau}\sqrt{\lambda_{max}(BB^T)}\}. \tag{3.19}$$

$$\cdot \int_0^{\infty} x_{t+s}^T(0)x_{t+s}(0)e^{2\delta s}ds.$$

This functional is a direct generalization of that used in [7], and generalizes those used by Hale [5,6] and Datko [3]. We recapitulate the above results in the form of a

Theorem: Consider the retarded equation $\dot{x}(t) = Ax(t) + Bx(t-\tau)$ and the Liapunov functional V given by equation (3.1). If $\gamma = \max\{\text{Re } \lambda \mid \det[\lambda I - A - Be^{-\lambda\tau}] = 0\}$ and $\in > 0$, then there exist constant positive definite matrices M and R and a differentiable matrix $Q(\alpha)$, $0 \leq \alpha \leq \tau$ with $Q(0) = Q(0)^T$ such that the functional V is positive definite, bounded above, and $\dot{V} \leq 2(\gamma+\in)V$.

Of course, if $\gamma < 0$, the above result implies exponential asymptotic stability; moreover, the rate of decay is precisely the expected one.

4. A Comparison with Ordinary Differential Equations.

In this section, it is our object to point out the intimate relationship between the results obtained for our functional equation and the classical results on the construction of Liapunov functions for ordinary differential equations.

Recall, [4], that given the system of ordinary differential equations

$$\dot{x}(t) = Cx(t), \tag{4.1}$$

where C is an $n \times n$ matrix, a Liapunov function for this

system can be always taken as the quadratic form

$$V(x) = x^T P x, \tag{4.2}$$

where P is a positive definite matrix. Moreover, if $\gamma = \max\{\text{Re } \lambda \mid \det[\lambda I - C] = 0\}$ and if, for $\in > 0$, $-\delta = \gamma + 2\in$, then given an arbitrary positive definite matrix W, the algebraic equation

$$(C+\delta I)^T P + P(C+\delta I) = -W, \tag{4.3}$$

has a unique solution P which is positive definite. This matrix P, if used in (4.2) along the solutions of the differential equation (4.1) yields, upon differentiation,

$$\dot{V}(x(t)) = -2\delta V(x(t)) - x^T W x \leq -2\delta V(x(t)). \tag{4.4}$$

Furthermore, the unique positive definite solution of (4.3) can be obtained as the integral

$$P = \int_0^\infty e^{C^T u} e^{\delta u} W e^{C u} e^{\delta u} du. \tag{4.5}$$

Let us now bring into evidence the relationship between the results obtained in the previous section and these results. The

functional differential equation under consideration is

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \quad t > 0 . \tag{4.1'}$$

Once again, we assume that $\gamma = \max\{\mathrm{Re}\ \lambda \mid \det[\lambda I - A - Be^{-\lambda\tau}\} = 0\}$ and, for $\in > 0$, let $-\delta = \gamma + 2\in$. The Liapunov functional is then of the form

$$V(x_t(0),x_t) = x_t^T(0)Mx_t(0) + e^{\delta\tau}\int_{-\tau}^{0} x_t^T(\theta)Re^{2\delta\theta}x_t(\theta)d\theta \tag{4.2'}$$

$$+ x_t^T(0)Q(0)x_t(0) + 2x_t^T(0)\int_{-\tau}^{0} Q(\alpha+\tau)e^{\delta(\alpha+\tau)}Bx_t(\tau)d\theta +$$

$$+ 2\int_{-\tau}^{0}\int_{\alpha}^{0} x_t^T(\alpha)B^TQ(\beta-\alpha)e^{\delta(\alpha+\beta+2\tau)}Bx_t(\beta)d\beta d\alpha.$$

The choice of the positive definite symmetric matrices M and R in this expression is rather arbitrary. Their purpose is to insure the strict positive definiteness of the functional on the Hilbert space $\mathscr{H}$; it should be noted that if $M = R = 0$, the functional (4.2') is positive, but does not satisfy a relationship of the type $V(x_t(0),x_t) \geq c_1||(x_t(0),x_t)||^2_{\mathscr{H}}$ for $c_1 > 0$. The requirement that the matrix G of equation (3.17) be positive semidefinite is always satisfied for $M = R = 0$ and W positive semidefinite. Given an arbitrary positive definite matrix W, it is always possible to select positive definite matrices M and R so that G is positive semidefinite.

The choice of the continuously differentiable matrix $Q(\alpha)$, $0 \leq \alpha \leq \tau$ is critical. Given an arbitrary positive definite matrix W it must satisfy the functional equation

$$\begin{aligned} Q'(\alpha) &= (A^T+\delta I)Q(\alpha) + B^T e^{\delta\tau} Q^T(\tau-\alpha), \quad 0 \leq \alpha \leq \tau, \\ Q(0) &= Q^T(0), \end{aligned} \tag{4.3'}$$

with the condition

$$(A^T+\delta I)Q(0) + Q(0)(A+\delta I) + B^T e^{\delta\tau} Q^T(\tau) + Q(\tau)e^{\delta\tau}B = -W. \tag{4.3"}$$

In the previous section it was shown that such a $Q(\alpha)$ always exists and is unique.

With such a choice of $Q(\alpha)$, one then obtains that, along all solutions of the functional differential equation (4.1')

$$\dot{V}(x_t(0),x_t) \leq -2\delta V(x_t(0),x_t). \tag{4.4'}$$

Moreover, such a matrix $Q(\alpha)$ exists, is unique and a representation of it is given by the integral

$$Q(\alpha) = \int_0^\infty S^T(u)e^{\delta u}WS(u-\alpha)e^{\delta(u-\alpha)}du, \tag{4.5'}$$

where $S(t)$ is the solution of equation (2.9).

The strong relationship between the unprimed and primed equations is now clear. Indeed, note that for $\tau = 0$, the matrix C in (4.1) becomes $A + B$ and the matrix P in (4.3) becomes $Q(0) + M$ and all the primed equations become the unprimed ones.

Equations (4.3') and (4.3") are of a much more complex nature than the familiar algebraic equation (4.3). However, in spite of its appearance, the linear vector space of the solutions of (4.3') is not infinite dimensional but, as was pointed out in the previous section, has dimension n^2. Hence, although the problem of construction of the Liapunov functional for the functional differential equation does not reduce to the solution of an algebraic equation such as (4.3), it reduces to the solution of a linear differential equation, equation (3.10), which can be accomplished without difficulty.

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