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Numerical Determination of Minimum Mass Structures with Specified Natural Frequencies[1]

by

A. MIELE[2], A. MANGIAVACCHI[3], B.P. MOHANTY[4], and A.K. WU[5]

Abstract. The problem of the axial vibration of a cantilever beam is investigated both analytically and numerically. The mass distribution that minimizes the total mass for a given value of the frequency parameter β is determined using both

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the sequential ordinary gradient-restoration algorithm (SOGRA) and the modified quasilinearization algorithm (MQA). Concerning the minimum value of the mass, SOGRA leads to a solution precise to at least 4 significant digits and MQA leads to a solution precise to at least 6 significant digits.

Comparison of the optimal beam (a variable-section beam) with a reference beam (a constant-section beam) shows that the weight reduction depends strongly on the frequency parameter β. This weight reduction is negligible for $\beta \to 0$, is 11.3% for $\beta = 1$, is 55.3% for $\beta = 1.4$, and approaches 100% for $\beta \to \pi/2$.

Key Words. Structural optimization, dynamic optimization, axial vibrations, frequency constraint, fundamental frequency constraint, optimal structures, cantilever beams, bars, sequential gradient-restoration algorithm, modified quasilinearization algorithm, numerical methods, computing methods.

1. Introduction

Under the sponsorship of the Office of Scientific Research, Office of Aerospace Research, United States Air Force, the Aero-Astronautics Group of Rice University has developed in recent years several algorithms dealing with the numerical solution of optimal control problems on a digital computer.

Noteworthy among those algorithms are the sequential ordinary gradient-restoration algorithm (SOGRA, Refs. 3-4) and the modified quasilinearization algorithm (MQA, Refs. 5-6). For a survey of these algorithms, see Ref. 7.

The algorithms discussed in Refs. 3-7 are general, all-purpose algorithms. As far as aerospace engineering is concerned, these algorithms can be applied to a variety of fields, for example, optimum trajectories, optimum aerodynamic shapes, and optimum structures. With reference to optimum structures, two categories of problems can be identified: (i) static optimization problems, for example, minimization of the weight of a structure for a given load distribution; and (ii) dynamic optimization problems, for example, minimization of the weight of a structure for given frequency constraints.

This paper is the first of a series dealing with dynamic optimization problems and is being written at the suggestion of Dr. V.B. Venkayya, Flight Dynamics Laboratory, Wright Patterson AFB, Ohio. The questions posed by Dr. Venkayya were as follows: can SOGRA and MQA be usefully applied to the solution of dynamic optimization problems? How reliable and precise are the solutions obtained with these algorithms under realistic engineering conditions?

At Dr. Venkayya's urging, the problem of the axial vibration of a cantilever beam was investigated, with the following objective in mind: to find the mass distribution that minimizes the total mass for a prescribed fundamental frequency. This problem was studied first, because a closed-form solution, due to Turner (Ref. 8), is available. Hence, it constitutes an ideally suited test problem for checking the accuracy and reliability of SOGRA and MQA in the dynamic optimization of structures.

Outline. This paper is divided in three parts: physical problem (Sections 2-4), optimization problem (Sections 5-6), and numerical results (Sections 7-9). The first part contains a description of the physical problem, both in dimensional form (Section 2) and in dimensionless form (Section 3); a reference solution, that pertaining to a constant-section beam, is given in Section 4. The second part contains a description of the mathematical problem (Section 5); Section 6 presents several alternative formulations and an essential simplification: the elimination of a redundant constraint. The third part contains the description of the algorithms (Section 7), the experimental conditions (Section 8), the numerical results (Section 9), and the conclusions (Section 10).

Notation. Throughout the paper, the following notation is employed.

Physical System

A	Cross-sectional area, ft^2
E	Modulus of elasticity, lb ft^{-2}
F_e	Elastic force, lb
F_i	Inertia force, lb
L	Length of the beam, ft
M	Mass per unit length, lb ft^{-2} sec^2
M_o	Tip mass, lb ft^{-1} sec^2
M_*	Total mass of the beam, lb ft^{-1} sec^2
t	Time, sec
U	Axial displacement, ft
x	Axial coordinate, ft
$Y(x)$	Displacement function, ft
$Z(t)$	Temporal function, dimensionless
ε	Normal strain, dimensionless
ρ	Density, lb ft^{-4} sec^2
σ	Normal stress, lb ft^{-2}
ω	Natural frequency, sec^{-1}

Subscripts for the Physical System

x	Derivative with respect to the axial coordinate
t	Derivative with respect to time

Normalized System

I	Total mass of the beam, $I = M_*/M_o$
m	Mass per unit length, $m = ML/M_o$
t	Axial coordinate, $t = x/L$
u	First derivative of the displacement, $u = \dot{y}$
w	Second derivative of the displacement, $w = \ddot{y}$
y	Axial displacement, $y = Y(x)/Y(L)$
z	Auxiliary variable, $z = mu$
β	Frequency parameter, $\beta = \omega L \sqrt{(\rho/E)}$

Superscript for the Normalized System

Derivative with respect to the axial coordinate t (for example, $\dot{y} = dy/dt$)

2. Physical System

We consider the problem of the axial vibration of a cantilever beam or bar, that is, a beam having a fixed end and a free end. Attached to the free end is the concentrated mass M_o. The beam has variable cross section; hence, it has variable mass per unit length M. The total mass of the beam is given by[6]

$$M_* = \int_0^L M dx \ , \tag{1}$$

where x denotes an axial coordinate, measured from the fixed end of the beam, and L is the length of the beam. Note that $x = 0$ at the fixed end and $x = L$ at the free end.

Assume that the system composed by the masses M_o and M_* is excited by some external axial disturbance. Then, the beam vibrates axially under the combined effect of the elastic forces and the inertia forces. These forces are in equilibrium.

Equilibrium Equation. Let F_e denote the elastic force at station x; let $\tilde{F}_e$ denote the elastic force at station $\tilde{x} = x + dx$; let $\tilde{F}_e - F_e$ denote the net elastic force acting on the beam element having length dx; and let dF_i denote the inertia force associated with the mass element Mdx. The equilibrium condition between elastic forces and inertia forces requires that

$$\tilde{F}_e - F_e = dF_i \ , \tag{2}$$

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[6]Note that $M = M(x)$.

where [7]

$$F_e = \sigma A \, , \tag{3-1}$$

$$\tilde{F}_e = \sigma A + [\partial(\sigma A)/\partial x] dx \, , \tag{3-2}$$

$$dF_i = M U_{tt} \, dx \, . \tag{3-3}$$

In the above equations, σ denotes the normal stress, U the axial displacement, U_t the axial velocity, U_{tt} the axial acceleration, and t is the time. From (2)-(3), the equilibrium condition takes the form

$$\partial(\sigma A)/\partial x = MU_{tt} \, . \tag{4}$$

The left-hand side of Eq. (4) can be expressed in terms of M and U if two observations are made. First, the mass per unit length M, the cross-sectional area A, and the density of the material ρ satisfy the relation

$$M = \rho A \, . \tag{5}$$

Second, on account of Hooke's law, the normal stress σ, the normal strain $\varepsilon = U_x$, and the modulus of elasticity E satisfy the relation[8]

$$\sigma = E\varepsilon = EU_x \, . \tag{6}$$

Therefore, upon combining Eqs. (4)-(6), the equilibrium condition

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[7]Note that $A = A(x)$.

[8]In Eq. (6), a one-dimensional state of stress is assumed.

takes the form

$$(E/\rho)(\partial/\partial x)(MU_x) - MU_{tt} = 0 . \tag{7}$$

<u>Boundary Conditions</u>. The partial differential equation (7) must be solved subject to appropriate boundary conditions.

(i) At the fixed end of the beam, the displacement must vanish. Therefore,

$$U = 0 \quad \text{at} \quad x = 0 . \tag{8}$$

(ii) At the free end of the beam, the elastic force acting on the cross section A must balance the inertia force associated with the concentrated mass M_o. Clearly,

$$F_e + F_i = 0 \quad \text{at} \quad x = L , \tag{9}$$

where

$$F_e = \sigma A = (E/\rho)MU_x , \tag{10}$$

$$F_i = M_o U_{tt} . \tag{11}$$

Therefore, upon combining Eqs. (9)-(11), we see that

$$(E/\rho)MU_x + M_o U_{tt} = 0 \quad \text{at} \quad x = L . \tag{12}$$

(iii) The spatial boundary conditions (8) and (12) must be completed by appropriate temporal boundary conditions.

However, these temporal boundary conditions are not specified here: we are only interested in the spatial behavior of the system.

Separation of Variables. We assume that the nature of the temporal boundary conditions is such that the technique of separation of variables is applicable to the system composed of Eqs. (7), (8), (12). Therefore, we write the unknown function $U(x,t)$ in the form

$$U(x,t) = Y(x)Z(t). \tag{13}$$

Upon substitution, we see that the previous differential system splits into a spatial system and a temporal system, each described by ordinary differential equations.

(i) For the spatial system, Eqs. (7), (8), (12) can be rewritten as

$$(E/\rho)(d/dx)(MY_x) + \omega^2 MY = 0 , \tag{14}$$

$$Y = 0 \quad \text{at} \quad x = 0 , \tag{15}$$

$$(E/\rho)MY_x - \omega^2 M_o Y = 0 \quad \text{at} \quad x = L , \tag{16}$$

where ω denotes any of the natural frequencies of the system under consideration.[9] Clearly, the solution $Y(x)$ of (14)-(16) depends on the mass distribution $M(x)$ as well as on several constants characteristic of the system $(L, E, \rho, \omega, M_o)$.

[9] Note that ω^2 is the separation constant, that is, the constant which allows the separation of the spatial system from the temporal system.

(ii) For the temporal system, Eq. (7) implies that

$$Z_{tt} + \omega^2 Z = 0 \ . \tag{17}$$

Of course, this equation must be completed by appropriate and consistent temporal boundary conditions.

(iii) In the remainder of this paper, we focus our attention on the solution of the spatial system (14)-(16). For problems where the function $M(x)$ is prescribed a priori (for instance, the study of the axial vibration of a constant section beam), the system (14)-(16) is linear. For problems where the function $M(x)$ is to be determined (for instance, the optimization problem described later on in this paper), the system (14)-(16) is nonlinear. In either case, the system (14)-(16) is homogeneous in the unknown function $Y(x)$, a circumstance to be exploited later on in the paper.

3. Normalized System

The equations describing the spatial system can be rewritten in a simpler form if the following dimensionless quantities are introduced:

$$t = x/L\ , \qquad m = ML/M_o\ , \qquad y = Y(x)/Y(L), \tag{18}$$

$$\beta = \omega L\sqrt{(\rho/E)}\ , \qquad I = M_*/M_o\ . \tag{19}$$

With these definitions, the mass integral (1) becomes

$$I = \int_0^1 m dt \tag{20}$$

and Eqs. (14)-(16) become[10]

$$(d/dt)(m\dot{y}) + \beta^2 my = 0\ , \tag{21}$$

$$y(0) = 0\ , \tag{22}$$

$$m(1)\dot{y}(1) = \beta^2\ , \qquad y(1) = 1\ . \tag{23}$$

The quantity β, which is proportional to the natural frequency ω, is called the frequency parameter. The additional boundary condition (23-2) is due to the fact that the displacement function $Y(x)$ has been normalized in terms of $Y(L)$, the value assumed by the displacement function at the free end of the beam. This is possible, owing to the fact that the spatial system (14)-(16) is homogeneous in the displacement function $Y(x)$.

Note that, for problems where the function m(t) is prescribed a priori (for instance, the study of the axial vibration of a constant section beam), the system (21)-(23) is linear. On the other hand, for problems where the function m(t) is to be determined (for instance, the minimum mass problem described in Section 5), the system (21)-(23) is nonlinear.

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[10] Equation (23-2) is a normalization condition for the displacement function y(t).

4. Constant-Section Beam

In this section, we consider the solution of the system (21)-(23) for the particular case of a constant-section structure. For

$$m = \text{const}\,, \tag{24}$$

the mass integral (20) reduces to

$$I = m \tag{25}$$

and Eqs. (21)-(23) become

$$\ddot{y} + \beta^2 y = 0\,, \tag{26}$$

$$y(0) = 0\,, \tag{27}$$

$$m\dot{y}(1) = \beta^2, \qquad y(1) = 1\,. \tag{28}$$

The solution of (26), (27), and (28-2), valid for any value of the frequency parameter β, is given by

$$y = \sin(\beta t)/\sin\beta\,. \tag{29}$$

The mass per unit length m corresponding to the frequency parameter β can be computed from (28-1):

$$m = \beta\tan\beta\,. \tag{30}$$

Therefore, the mass integral is given by

$$I = \beta \tan\beta \, . \tag{31}$$

In order to understand the significance of (31), recall that I is the ratio of the beam mass M_* to the tip mass M_o, and consider two limiting cases: (i) negligible mass ratio and (ii) infinite mass ratio.

If $I = 0$, then the solution of (31) is

$$\beta = n\pi \, , \qquad n = 0,1,2,\ldots\ldots \, . \tag{32}$$

On the other hand, if $I = \infty$, then the solution of (31) is

$$\beta = (2n+1)\pi/2 \, , \qquad n = 0,1,2,\ldots\ldots \, . \tag{33}$$

Since the first natural frequency or fundamental frequency corresponds to $n = 0$, we conclude that, for any mass ratio in the range

$$0 \leq I \leq \infty \, , \tag{34}$$

the smallest value of the frequency parameter β consistent with the transcendental equation (31) lies in the range

$$0 \leq \beta \leq \pi/2 \, . \tag{35}$$

We shall recall this result in the optimization problem formulated in Section 5. Specifically, we shall pose the following problem. For a given value of the frequency parameter

β in the range (35), is there a beam yielding a smaller value of the mass integral (20) than the value (31) associated with the constant-section beam? In particular, is there a variable-section design yielding the smallest value of the mass integral (20) among all the designs satisfying the feasibility equations (21)-(23)?

5. Optimization Problem

In the previous section, the solution valid for a constant-section beam was given. Here, we consider a variable-section beam and formulate the following optimization problem: For a given value of the frequency parameter β, find the mass distribution $m(t)$ and the displacement distribution $y(t)$ such that the mass integral is minimized, while the feasibility equations are satisfied. This is a problem of the calculus of variations or a problem of optimal control, depending on the formulation being employed (Refs. 9-10).

Formulation (F1). In this formulation, the optimization problem has the following format:[11]

$$I = \int_0^1 m\,dt \ , \tag{36}$$

$$m\ddot{y} + \dot{m}\dot{y} + \beta^2 my = 0 \ , \tag{37}$$

$$y(0) = 0 \ , \tag{38}$$

$$y(1) = 1 \ , \qquad m(1)\dot{y}(1) = \beta^2 \ . \tag{39}$$

Clearly, the unknowns are the functions $m(t)$ and $y(t)$.

Formulation (F2). Let the following auxiliary variables be introduced:

$$u = \dot{y} \ , \qquad w = \ddot{y} \ . \tag{40}$$

[11] The dot denotes derivative with respect to the independent variable t.

With the aid of these variables, the problem (36)-(39) can be rewritten in optimal control format as follows:

$$I = \int_0^1 m\,dt\ , \tag{41}$$

$$\dot{y} = u\ , \qquad \dot{m} = -(m/u)(w + \beta^2 y)\ , \qquad \dot{u} = w\ , \tag{42}$$

$$y(0) = 0\ , \tag{43}$$

$$y(1) = 1\ , \qquad m(1)u(1) = \beta^2\ . \tag{44}$$

In this formulation, the unknowns are the state variables $y(t)$, $m(t)$, $u(t)$ and the control variable $w(t)$. Note that the control appears linearly in the differential equations. Hence, the Hamiltonian is linear in the control $w(t)$, and the problem is singular. Also note that the system (41)-(44) is autonomous, since the independent variable t does not appear explicitly on the right-hand side of Eqs. (41)-(42). As a consequence, the Hamiltonian is constant along the interval of integration.

6. Alternative Formulations of the Optimization Problem

In this section, we present some alternative formulations of the optimization problem, having important computational implications.

Formulation (F3). Let the following auxiliary variable be introduced:

$$z = mu \ . \tag{45}$$

Let the variable $m(t)$ of Formulation (F2) be eliminated and replaced by $z(t)$. With this understanding, the problem (41)-(44) can be reformulated as follows:

$$I = \int_0^1 (z/u)dt \ , \tag{46}$$

$$\dot{y} = u \ , \qquad \dot{z} = -\beta^2 yz/u \ , \qquad \dot{u} = w \ , \tag{47}$$

$$y(0) = 0 \ , \tag{48}$$

$$y(1) = 1 \ , \qquad z(1) = \beta^2 \ . \tag{49}$$

In this formulation, the unknowns are the state variables $y(t)$, $z(t)$, $u(t)$ and the control variable $w(t)$. After these functions are determined, the function $m(t)$ can be computed a posteriori with the relation

$$m = z/u \ . \tag{50}$$

A characteristic of this formulation is that all the boundary conditions are linear. The Hamiltonian is linear in the control $w(t)$, and hence the problem is singular. Once more, the system is autonomous, and the Hamiltonian is constant along the interval of integration.

Formulation (F4). Inspection of (46)-(49) shows that the control $w(t)$ appears only in Eq. (47-3). Also, both $u(0)$ and $u(1)$ are free. Hence, one surmises that the differential constraint (47-3) might be redundant. This can be shown as follows. Assume that the problem (46)-(49) is solved bypassing Eq. (47-3); with the solution $y(t)$, $z(t)$, $u(t)$ known, one can always determine a posteriori the function $w(t)$ in such a way that the feasibility equation (47-3) is satisfied. This simple but important observation leads to a new formulation of the optimal control problem:

$$I = \int_0^1 (z/u)\,dt\,, \tag{51}$$

$$\dot{y} = u\,, \qquad \dot{z} = -\beta^2 yz/u\,, \tag{52}$$

$$y(0) = 0\,, \tag{53}$$

$$y(1) = 1\,, \qquad z(1) = \beta^2\,. \tag{54}$$

In this formulation, the unknowns are the state variables $y(t)$, $z(t)$ and the control variable $u(t)$. After these functions are determined, the functions $m(t)$ and $w(t)$ are computed a posteriori with the relations

$$m = z/u , \qquad w = \dot{u} . \tag{55}$$

A characteristic of this formulation is that all the boundary conditions are linear. In addition, the state variable $u(t)$ of Formulation (F3) becomes the control variable $u(t)$ of Formulation (F4). The Hamiltonian is nonlinear in the control $u(t)$, and hence the problem is nonsingular. Once more, the system is autonomous, and the Hamiltonian is constant along the interval of integration.

Formulation (F5). This formulation is a slight modification of the previous one and is obtained by eliminating the control variable $u(t)$ and replacing it with the new control variable $m(t)$. On account of (55-1), Eqs. (51)-(54) become

$$I = \int_0^1 m dt , \tag{56}$$

$$\dot{y} = z/m , \qquad \dot{z} = -\beta^2 ym , \tag{57}$$

$$y(0) = 0 , \tag{58}$$

$$y(1) = 1 , \qquad z(1) = \beta^2 . \tag{59}$$

In this formulation, the unknowns are the state variables $y(t)$, $z(t)$ and the control variable $m(t)$. After these functions are determined, the functions $u(t)$ and $w(t)$ are computed a posteriori with the relations

$$u = z/m\,, \qquad w = \dot{u}\,. \tag{60}$$

A characteristic of this formulation is that all the boundary conditions are linear. In addition, the Hamiltonian is nonlinear in the control $m(t)$, and hence the problem is nonsingular. Once more, the system is autonomous, and the Hamiltonian is constant along the interval of integration. Note that this formulation can be obtained directly from (20)-(23) after consideration of the relation

$$z = m\dot{y}\,. \tag{61}$$

<u>Remark</u>. The analytical justification for the passage from Formulation (F3) to Formulation (F4) is as follows. With reference to Formulation (F3), let $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ denote variable Lagrange multipliers associated with the differential constraints (47). Let

$$H = z/u - \lambda_1 u + \lambda_2 \beta^2 yz/u - \lambda_3 w \tag{62}$$

denote the Hamiltonian of the system (46)-(49). The condition optimizing the control $w(t)$ of Formulation (F3) takes the form

$$H_w = 0 , \qquad 0 \leq t \leq 1 , \tag{63}$$

which implies that

$$\lambda_3(t) = 0 , \qquad 0 \leq t \leq 1 . \tag{64}$$

Note that both u(0) and u(1) are free; hence, the transversality condition requires that

$$\lambda_3(0) = \lambda_3(1) = 0 , \tag{65}$$

and these relations are consistent with (64). From (64)-(65), it is natural to infer that the differential constraint (47-3) is redundant, a circumstance which has important analytical and computational implications.

7. Description of the Algorithms

With the sequential ordinary gradient-restoration algorithm (SOGRA, Refs. 3-4), numerical solutions can be obtained using any of the optimal control formulations of Sections 5-6. This is because SOGRA can be applied regardless of whether the problem is singular or nonsingular. On the other hand, with the modified quasilinearization algorithm (MQA, Refs. 5-6), numerical solutions can be obtained using Formulations (F4) and (F5), but not Formulations (F2) and (F3). This is because the optimal control problem is singular with the latter formulations, while it is nonsingular with the former. For these reasons, the experiments reported here refer to Formulations (F4) and (F5).

Note that both SOGRA and MQA require the state vector to be known at the initial point, so that the trajectories in the state space can be anchored at the initial point. With reference to the fourth and fifth formulations (state vector given at the final point), this situation can be achieved by replacing the independent variable t with its complement

$$\tau = 1-t, \tag{66}$$

that is, by reversing the sense of integration and interchanging the final point with the initial point. Even though this replacement of independent variable is necessary computationally, it is not formally executed here, for two reasons: for the sake of brevity and in order to avoid complicating the description of the problems and the interpretation of the results. Therefore, in the following sections, we shall retain the independent variable t, that is, we shall imagine the trajectories in the state space to be anchored at the final point at every iteration.

Formulation (F4). With the formulation represented by Eqs. (51)-(54), the Hamiltonian is given by[12]

$$H = z/u - \lambda_1 u + \lambda_2 \beta^2 yz/u \, , \tag{67}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ denote variable Lagrange multipliers associated with the differential constraints (52). Hence, the optimality conditions take the form[12]

$$\lambda_1 + (z/u^2)(1 + \lambda_2 \beta^2 y) = 0 \, , \tag{68}$$

$$\dot{\lambda}_1 = \lambda_2 \beta^2 z/u \, , \qquad \dot{\lambda}_2 = (1/u)(1 + \lambda_2 \beta^2 y) \, , \tag{69}$$

$$\lambda_1(0) + \mu = 0 \, , \qquad \lambda_2(0) = 0 \, . \tag{70}$$

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[12] Equation (68) optimizes the control distribution u(t). In Eq. (70-1), the symbol μ denotes a constant Lagrange multiplier associated with the initial condition (53).

Formulation (F5). With the formulation represented by Eqs. (56)-(59), the Hamiltonian is given by

$$H = m - \lambda_1 z/m + \lambda_2 \beta^2 ym \,, \tag{71}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ denote variable Lagrange multipliers associated with the differential constraints (57). Hence, the optimality conditions take the form[13]

$$\lambda_1 z/m^2 + 1 + \lambda_2 \beta^2 y = 0 \,, \tag{72}$$

$$\dot{\lambda}_1 = \lambda_2 \beta^2 m \,, \qquad \dot{\lambda}_2 = -\lambda_1/m \,, \tag{73}$$

$$\lambda_1(0) + \mu = 0 \,, \qquad \lambda_2(0) = 0 \,. \tag{74}$$

Auxiliary Functionals. In addition to the functional I, several auxiliary functionals are of interest in the implementation of SOGRA and MQA. These auxiliary functionals are denoted by the symbols J, P, Q, R and have the following meaning: J is the augmented functional, that is, the functional I

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[13] Equation (72) optimizes the control distribution $m(t)$. In Eq. (74-1), the symbol μ denotes a constant Lagrange multiplier associated with the initial condition (58).

augmented linearly by the constraints through Lagrange multipliers; P is the constraint error, that is, the norm squared of the error in the feasibility equations; Q is the optimality condition error, that is, the norm squared of the error in the optimality conditions; and $R = P + Q$ is the total error in the system composed of the feasibility equations and the optimality conditions. In the definitions of J, P, Q, R, it is tacitly assumed that the final conditions (54) or (59) are satisfied at every iteration of SOGRA and MQA.

Convergence Conditions. The auxiliary functionals P, Q, R are defined in such a way that

$$P = Q = R = 0 \tag{75}$$

for the optimal solution. For an approximation to the optimal solution, Eqs. (75) are replaced by the inequalities

$$P \leq \varepsilon_1, \qquad Q \leq \varepsilon_2, \tag{76}$$

or

$$R \leq \varepsilon_3, \tag{77}$$

where ε_1, ε_2, ε_3 are small, preselected numbers. Inequalities (76) are of interest for SOGRA, and Ineq. (77) is of interest for MQA.

Sequential Ordinary Gradient-Restoration Algorithm. The sequential ordinary gradient-restoration algorithm (SOGRA, Refs. 3-4) is an iterative technique which includes a sequence of cycles having the following properties: (i) the functions available both at the beginning and at the end of each cycle are feasible; that is, they are consistent with the feasibility equations within the preselected accuracy (76-1); and (ii) the functions produced at the end of each cycle are characterized by a value of the functional I which is smaller than that associated with the functions available at the beginning of the cycle.

To achieve the above properties, each cycle is made of two phases, a gradient phase and a restoration phase. These phases are now described.

The gradient phase is started only when the constraint error P satisfies Ineq. (76-1). It involves a single iteration, which is designed to decrease the value of the functional I or the augmented functional J, while satisfying the constraints to first order. During this iteration, the first variation of the functional I is minimized, subject to the linearized constraints and a quadratic constraint on the variations of the control.

The restoration phase is started only when the constraint error P violates Ineq. (76-1). The restoration phase involves one or more iterations. In each restorative iteration, the objective is to reduce the constraint error P, while the constraints are satisfied to first order and the norm of the variations of the control is minimized. The restoration phase is terminated whenever Ineq. (76-1) is satisfied.

In summary, the main properties of the sequential ordinary gradient-restoration algorithm can be written as follows:

$$\tilde{J} < J\,, \qquad P \leq \varepsilon_1\,, \tag{78}$$

$$\tilde{P} < P\,, \qquad P > \varepsilon_1\,, \tag{79}$$

$$\tilde{I} < I\,, \qquad P \leq \varepsilon_1\,, \qquad \tilde{P} \leq \varepsilon_1\,, \tag{80}$$

where (78) hold for a gradient iteration, (79) hold for a restorative iteration, and (80) hold for a complete gradient-restoration cycle. In the above relations, I, J, P denote quantities evaluated at the beginning of an iteration or a cycle, while $\tilde{I}$, $\tilde{J}$, $\tilde{P}$ denote quantities evaluated at the end of an iteration or a cycle.

At each iteration of the gradient phase or the restoration phase, a linear, two-point boundary-value problem must be solved. If the method of particular solutions is employed (Refs. 11-12), one must execute $q+1$ independent sweeps of the linearized system governing the variations associated with the gradient phase or the restoration phase, where q is the number of initial conditions. Since $q = 1$ for the present problem, each gradient iteration or restorative iteration requires 2 sweeps of the linearized system. For the details, see Refs. 3-4 and 11-12.

Modified Quasilinearization Algorithm. The modified quasilinearization algorithm (MQA, Refs. 5-6) is an iterative technique which includes a sequence of iterations having the following property: the functions produced at the end of each iteration are characterized by a value of the functional R which is smaller than that associated with the functions available at the beginning of the iteration. Therefore, the main property of the modified quasilinearization algorithm can be written as follows:

$$\tilde{R} < R , \tag{81}$$

with the implication that

$$\tilde{P} < P \quad \text{and/or} \quad \tilde{Q} < Q . \tag{82}$$

At each iteration, both the feasibility equations and the optimality conditions are satisfied to first order. To achieve this, a linear, two-point boundary-value problem must be solved. If the method of particular solutions is employed (Refs. 11-12), one must execute $n + 1$ independent sweeps of the linearized system governing the variations associated with MQA, where n is the number of state variables. Since $n = 2$ for the present problem, each iteration requires 3 sweeps of the linearized system. For the details, see Refs. 5-6 and 11-12.

8. Experimental Conditions

The problem of determining the mass distribution that minimizes the total mass for a given value of the frequency parameter β was solved using both Formulation (F4) and Formulation (F5). The following values were given to the frequency parameter:

$$\beta = \pi/6 , \qquad \beta = \pi/5 , \qquad \beta = \pi/4 , \qquad \beta = \pi/3 , \qquad \beta = \pi/2 . \qquad (83)$$

Computations were performed on the IBM 370/155 computer of Rice University, Houston, Texas, using both the sequential ordinary gradient-restoration algorithm (SOGRA) and the modified quasilinearization algorithm (MQA). Both algorithms were programmed in FORTRAN IV, and the numerical results were obtained in double-precision arithmetic.

The interval of integration was divided into 100 steps for SOGRA and 50 steps for MQA. The differential equations were integrated using Hamming's modified predictor-corrector method, with a special Runge-Kutta starting procedure. The definite integrals I, J, P, Q were computed using a modified Simpson's rule.

Nominal Functions. For each formulation, two sets of nominal functions were employed, more specifically:

$$\text{(N1)} \quad y(t) = t , \qquad z(t) = \beta^2, \qquad u(t) = 1 , \qquad (84)$$

$$\text{(N2)} \quad y(t) = t^2, \qquad z(t) = \beta^2, \qquad u(t) = 1, \tag{85}$$

and

$$\text{(N3)} \quad y(t) = t, \qquad z(t) = \beta^2, \qquad m(t) = 1, \tag{86}$$

$$\text{(N4)} \quad y(t) = t^2, \qquad z(t) = \beta^2, \qquad m(t) = 1. \tag{87}$$

The nominal functions (84) and (85) were employed in connection with Formulation (F4), and the nominal functions (86) and (87) were employed in connection with Formulation (F5). Note that these nominal functions satisfy the boundary conditions, but violate the differential constraints.

Stopping Conditions. The sequential ordinary gradient-restoration algorithm was programmed to stop whenever a solution consistent with the following inequalities was obtained:[12]

$$P \leq E\text{-}08, \qquad Q \leq E\text{-}04. \tag{88}$$

The modified quasilinearization algorithm was programmed to stop whenever the following inequality was satisfied:

$$R \leq E\text{-}10, \tag{89}$$

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[12]The symbol $E \pm ab$ stands for $10^{\pm ab}$.

with the implication that

$$P \leq E\text{-}10, \qquad Q \leq E\text{-}10. \tag{90}$$

Remark 8.1. For the sake of brevity, the computational details pertaining to SOGRA and MQA (for instance, solution of the linear, two-point boundary value problem, computation of the stepsize, safeguards, and nonconvergence conditions) are omitted, and the reader is referred to Refs. 3-4 and Refs. 5-6.

Remark 8.2. Ordinarily, algorithms of the second-order type require the specification of nominal functions not only for the state and the control, but also for the multipliers. This is not the case with MQA. Once the nominal state and the nominal control are specified, the nominal multipliers $\lambda_1(t)$, $\lambda_2(t)$, μ are determined automatically through a subroutine which supplies the solution of the following auxiliary minimization problem: minimize the error in the optimality conditions Q with respect to multipliers $\lambda_1(t)$, $\lambda_2(t)$, μ. This is an important feature of MQA, certainly very useful to engineers, since it circumvents the need for an arbitrary guess of the multiplier functions.

9. Numerical Results

Under the experimental conditions outlined in the previous section, 40 computer runs were executed: the minimization problem was solved for five different values of the frequency parameter, using two algorithms (SOGRA and MQA), two formulations [(F4) and (F5)], and two sets of nominal functions for each formulation [(N1) and (N2) for Formulation (F4), and (N3) and (N4) for Formulation (F5)]. Convergence to the desired stopping condition was achieved in 37 runs. The nonconvergence occurring in the remaining 3 runs was due to poor choice of the nominal functions. Incidentally, all the nonconvergence cases occurred for $\beta = \pi/2$, which is the upper limit to the range of values of the frequency parameter investigated. The numerical results are presented in Tables 1-24.

Tables 1-4 present summary results pertaining to SOGRA at convergence. For each value of the frequency parameter β, the tables show the number of iterations for convergence N, the computed value of the functional I, the exact value of the functional I_e, the number of correct significant digits M (determined by comparing I with I_e), and the error in the optimality conditions Q. Clearly, as far as the minimum mass is concerned, the solutions obtained are precise to at least 4 significant digits; in some cases, they are precise to 6 significant digits.

Tables 5-8 present summary results pertaining to MQA at convergence. For each value of the frequency parameter β, the tables show the number of iterations for convergence N, the computed value of the functional I, the exact value of the functional I_e, the number of correct significant digits M (determined by comparing I with I_e), and the total error in the system R. Clearly, as far as the minimum mass is concerned, the solutions obtained are precise to at least 6 significant digits; in many cases, they are precise to 7 significant digits.

Due to the large number of runs performed, it is impractical to present the convergence history for every value of the frequency parameter β. However, for a particular value of the frequency parameter (namely, $\beta = \pi/4$), the convergence history of SOGRA is given in Tables 9-12 and the convergence history of MQA is given in Tables 13-16. The descent property in I, characteristic of SOGRA, is apparent from Tables 9-12. Analogously, the descent property in R, characterisitc of MQA, is apparent from Tables 13-16.

Due to the large number of runs performed, it is impractical to present the converged solutions y(t), z(t), m(t), u(t), w(t) for every case. Therefore, Tables 17-21 present only one set of converged solutions: these are the solutions corresponding to the lowest value of R achieved computationally for any given value of the frequency parameter β. Even though the tolerance level set for MQA is $R \leq E\text{-}10$, values of the total error much lower that E-10 were achieved during computation, more precisely, $R \leq E\text{-}16$ for $\beta = \pi/5$ and $R \leq E\text{-}18$ for the remaining values of β.

Table 22 is obtained from Tables 17-21 by normalizing the mass distribution $m(t)$ with respect to the mass per unit length at $t = 0$. Therefore, Table 22 shows the ratio $m(t)/m(0)$ for several values of the frequency parameter β. Clearly, the mass distribution of the optimal beam approaches that of the constant-section beam for $\beta \to 0$ and deviates considerably from it for $\beta \to \pi/2$. In particular, the ratio $m(1)/m(0)$ of tip mass per unit length to root mass per unit length approaches 1 for $\beta \to 0$ and 1/6 for $\beta \to \pi/2$.

Finally, Tables 23-24 show the mass integral for the optimal beam I_o, the mass integral for the constant-section beam I_c, as well as the quantities I_o/I_c and $(I_c - I_o)/I_c$ for several values of the frequency parameter β. For $\beta \to 0$, the optimal beam exhibits no weight advantage over the constant-section beam. For $\beta = 0.5$, the weight advantage is only 0.6%. For $\beta = 1.0$, the weight advantage is 11.3%. For $\beta = 1.4$, the weight advantage is 55.3%. Finally, for $\beta \to \pi/2$, the weight advantage approaches 100%. The explanation for this limiting result is simple: for $\beta \to \pi/2$, the mass of the optimal beam is finite, while the mass of the constant-section beam is infinitely large.

Table 1. Summary results, SOGRA, Formulation (F4), Nominal (N1).

β	N	I	I_e	M	Q
π/6	3	0.3001435E+00	0.3001434E+00	6	0.85E-07
π/5	3	0.4495492E+00	0.4495488E+00	5	0.85E-06
π/4	3	0.7545937E+00	0.7545892E+00	4	0.15E-04
π/3	4	0.1560903E+01	0.1560918E+01	5	0.28E-04
π/2	10	0.5295951E+01	0.5295977E+01	5	0.88E-03 (*)

(*) Algorithm unable to reach desired stopping condition; loss of descent property on I due to numerical inaccuracy.

Table 2. Summary results, SOGRA, Formulation (F4), Nominal (N2).

β	N	I	I_e	M	Q
π/6	4	0.3001435E+00	0.3001434E+00	6	0.85E-07
π/5	4	0.4495492E+00	0.4495488E+00	5	0.85E-06
π/4	4	0.7545937E+00	0.7545892E+00	4	0.15E-04
π/3	5	0.1560903E+01	0.1560918E+01	5	0.28E-04
π/2	11	0.5295951E+01	0.5295977E+01	5	0.88E-03(*)

(*) Algorithm unable to reach desired stopping condition; loss of descent property on I due to numerical inaccuracy.

Table 3. Summary results, SOGRA, Formulation (F5), Nominal (N3).

β	N	I	I_e	M	Q
π/6	19	0.3001453E+00	0.3001434E+00	5	0.32E-04
π/5	14	0.4495573E+00	0.4495488E+00	4	0.84E-04
π/4	10	0.7545631E+00	0.7545892E+00	4	0.36E-04
π/3	12	0.1560911E+01	0.1560918E+01	6	0.51E-04
π/2	(*)	(*)	0.5295977E+01	(*)	(*)

(*) Nonconvergence, failure of initial restoration phase; algorithm unable to find a feasible solution in less than 10 restorative iterations.

Table 4. Summary results, SOGRA, Formulation (F5), Nominal(N4).

β	N	I	I_e	M	Q
π/6	21	0.3001449E+00	0.3001434E+00	5	0.25E-04
π/5	14	0.4495525E+00	0.4495488E+00	4	0.38E-04
π/4	10	0.7545679E+00	0.7545892E+00	4	0.26E-04
π/3	15	0.1560913E+01	0.1560918E+01	6	0.54E-04
π/2	(*)	(*)	0.5295977E+01	(*)	(*)

(*) Nonconvergence, failure of initial restoration phase; algorithm unable to find a feasible solution in less than 10 restorative iterations.

Table 5. Summary results, MQA, Formulation (F4), Nominal (N1).

β	N	I	I_e	M	R
$\pi/6$	3	0.3001434E+00	0.3001434E+00	7	0.20E-18
$\pi/5$	3	0.4495488E+00	0.4495488E+00	7	0.74E-16
$\pi/4$	3	0.7545892E+00	0.7545892E+00	7	0.12E-12
$\pi/3$	4	0.1560918E+01	0.1560918E+01	7	0.32E-18
$\pi/2$	(*)	(*)	0.5295977E+01	(*)	(*)

(*) Nonconvergence, stepsize bisection limit reached.

Table 6. Summary results, MQA, Formulation (F4), Nominal (N2).

β	N	I	I_e	M	R
$\pi/6$	3	0.3001434E+00	0.3001434E+00	7	0.17E-18
$\pi/5$	3	0.4495488E+00	0.4495488E+00	7	0.70E-16
$\pi/4$	3	0.7545892E+00	0.7545892E+00	7	0.97E-13
$\pi/3$	4	0.1560918E+01	0.1560918E+01	7	0.11E-18
$\pi/2$	6	0.5295977E+01	0.5295977E+01	7	0.15E-15

Table 7. Summary results, MQA, Formulation (F5), Nominal (N3).

β	N	I	I_e	M	R
$\pi/6$	9	0.3001433E+00	0.3001434E+00	6	0.79E-10
$\pi/5$	6	0.4495488E+00	0.4495488E+00	7	0.15E-14
$\pi/4$	5	0.7545892E+00	0.7545892E+00	7	0.41E-18
$\pi/3$	4	0.1560917E+01	0.1560918E+01	6	0.20E-10
$\pi/2$	7	0.5295976E+01	0.5295977E+01	6	0.38E-17

Table 8. Summary results, MQA, Formulation (F5), Nominal (N4).

β	N	I	I_e	M	R
$\pi/6$	9	0.3001433E+00	0.3001434E+00	6	0.59E-10
$\pi/5$	7	0.4495487E+00	0.4495488E+00	6	0.55E-10
$\pi/4$	4	0.7545890E+00	0.7545892E+00	6	0.88E-11
$\pi/3$	4	0.1560917E+01	0.1560918E+01	6	0.83E-10
$\pi/2$	7	0.5295976E+01	0.5295977E+01	6	0.78E-18

Table 9. Convergence history, $\beta = \pi/4$, SOGRA, Formulation (F4), Nominal (N1).

N	Phase	P	Q	I
0		0.48E-01		
1	REST	0.10E-33	0.21E-01	0.76081
2	GRAD	0.13E-04		
3	REST	0.82E-33	0.15E-04	0.75459

Table 10. Convergence history, $\beta = \pi/4$, SOGRA, Formulation (F4), Nominal (N2).

N	Phase	P	Q	I
0		0.36E+00		
1	REST	0.29E-03		
2	REST	0.90E-33	0.21E-01	0.76081
3	GRAD	0.13E-04		
4	REST	0.27E-32	0.15E-04	0.75459

Table 11. Convervenge history, $\beta = \pi/4$, SOGRA, Formulation (F5), Nominal (N3).

N	Phase	P	Q	I
0		0.27E+00		
1	REST	0.13E+00		
2	REST	0.11E-01		
3	REST	0.58E-04		
4	REST	0.37E-08	0.31E+00	0.83009
5	GRAD	0.16E-01		
6	REST	0.80E-04		
7	REST	0.41E-08	0.46E-02	0.75504
8	GRAD	0.25E-05		
9	REST	0.26E-12	0.24E-03	0.75462
10	GRAD	0.95E-08	0.36E-04	0.75456

Table 12. Convergence history, $\beta = \pi/4$, SOGRA, Formulation (F5), Nominal (N4).

N	Phase	P	Q	I
0		0.55E+00		
1	REST	0.81E-01		
2	REST	0.38E-02		
3	REST	0.92E-05		
4	REST	0.10E-09	0.28E+00	0.82212
5	GRAD	0.13E-01		
6	REST	0.48E-04		
7	REST	0.15E-08	0.46E-02	0.75503
8	GRAD	0.27E-05		
9	REST	0.20E-12	0.19E-03	0.75461
10	GRAD	0.56E-08	0.26E-04	0.75456

Table 13. Convergence history, $\beta = \pi/4$, MQA, Formulation (F4), Nominal (N1).

N	P	Q	R	I
0	0.48E-01	0.32E-01	0.80E-01	0.61685
1	0.84E-05	0.23E-02	0.23E-02	0.75463
2	0.48E-08	0.96E-06	0.96E-06	0.75455
3	0.44E-15	0.12E-12	0.12E-12	0.75458

Table 14. Convergence history, $\beta = \pi/4$, MQA, Formulation (F4), Nominal (N2).

N	P	Q	R	I
0	0.36E+00	0.27E-01	0.39E+00	0.61685
1	0.27E-03	0.30E-02	0.33E-02	0.74916
2	0.72E-08	0.75E-06	0.76E-06	0.75454
3	0.10E-14	0.96E-13	0.97E-13	0.75458

Table 15. Convergence history, $\beta = \pi/4$, MQA, Formulation (F5), Nominal (N3).

N	P	Q	R	I
0	0.27E+00	0.20E+00	0.47E+00	1.00000
1	0.48E-01	0.94E-01	0.14E+00	0.75124
2	0.30E-03	0.71E-02	0.74E-02	0.74933
3	0.69E-06	0.26E-04	0.27E-04	0.75419
4	0.21E-10	0.58E-09	0.61E-09	0.75458
5	0.15E-19	0.39E-18	0.41E-18	0.75458

Table 16. Convergence history, $\beta = \pi/4$, MQA, Formulation (F5), Nominal (N4).

N	P	Q	R	I
0	0.55E+00	0.16E+00	0.72E+00	1.00000
1	0.98E-01	0.45E-01	0.14E+00	0.77369
2	0.21E-03	0.19E-02	0.22E-02	0.74914
3	0.14E-06	0.29E-05	0.30E-05	0.75442
4	0.33E-12	0.84E-11	0.88E-11	0.75458

Table 17. Converged solution, $\beta = \pi/6$, MQA.

t	y	z	m	u	w
0.0	0.0000	0.3126	0.3270	0.9557	0.0000
0.1	0.0956	0.3121	0.3261	0.9570	0.0262
0.2	0.1914	0.3108	0.3235	0.9609	0.0524
0.3	0.2878	0.3087	0.3191	0.9675	0.0789
0.4	0.3850	0.3058	0.3131	0.9767	0.1055
0.5	0.4833	0.3021	0.3056	0.9886	0.1325
0.6	0.5829	0.2977	0.2968	1.0032	0.1598
0.7	0.6840	0.2927	0.2868	1.0206	0.1875
0.8	0.7871	0.2870	0.2757	1.0408	0.2157
0.9	0.8923	0.2808	0.2639	1.0638	0.2446
1.0	1.0000	0.2741	0.2515	1.0897	0.2741

Table 18. Converged solution, $\beta = \pi/5$, MQA.

t	y	z	m	u	w
0.0	0.0000	0.4753	0.5072	0.9371	0.0000
0.1	0.0937	0.4743	0.5052	0.9389	0.0370
0.2	0.1879	0.4715	0.4992	0.9445	0.0741
0.3	0.2828	0.4669	0.4896	0.9538	0.1116
0.4	0.3788	0.4606	0.4764	0.9668	0.1495
0.5	0.4763	0.4527	0.4602	0.9837	0.1880
0.6	0.5756	0.4434	0.4414	1.0044	0.2272
0.7	0.6773	0.4327	0.4204	1.0292	0.2674
0.8	0.7816	0.4209	0.3979	1.0580	0.3085
0.9	0.8890	0.4082	0.3742	1.0909	0.3509
1.0	1.0000	0.3947	0.3499	1.1282	0.3947

Table 19. Converged solution, $\beta = \pi/4$, MQA.

t	y	z	m	u	w
0.0	0.0000	0.8170	0.9037	0.9041	0.0000
0.1	0.0905	0.8145	0.8981	0.9069	0.0558
0.2	0.1815	0.8071	0.8817	0.9153	0.1120
0.3	0.2737	0.7949	0.8553	0.9293	0.1688
0.4	0.3676	0.7783	0.8200	0.9491	0.2267
0.5	0.4637	0.7578	0.7775	0.9747	0.2860
0.6	0.5627	0.7340	0.7293	1.0063	0.3471
0.7	0.6652	0.7074	0.6774	1.0442	0.4103
0.8	0.7718	0.6786	0.6234	1.0885	0.4761
0.9	0.8832	0.6482	0.5688	1.1395	0.5448
1.0	1.0000	0.6168	0.5150	1.1976	0.6168

Table 20. Converged solution, $\beta = \pi/3$, MQA.

t	y	z	m	u	w
0.0	0.0000	1.7549	2.0937	0.8381	0.0000
0.1	0.0839	1.7453	2.0709	0.8427	0.0920
0.2	0.1688	1.7171	2.0044	0.8566	0.1851
0.3	0.2556	1.6717	1.8999	0.8798	0.2803
0.4	0.3451	1.6114	1.7654	0.9127	0.3785
0.5	0.4385	1.5390	1.6103	0.9557	0.4808
0.6	0.5366	1.4576	1.4443	1.0091	0.5885
0.7	0.6407	1.3700	1.2760	1.0736	0.7026
0.8	0.7517	1.2791	1.1123	1.1499	0.8244
0.9	0.8711	1.1873	0.9584	1.2388	0.9552
1.0	1.0000	1.0966	0.8175	1.3413	1.0966

Table 21. Converged solution, $\beta = \pi/2$, MQA.

t	y	z	m	u	w
0.0	0.0000	6.1911	9.0703	0.6825	0.0000
0.1	0.0685	6.1155	8.8501	0.6910	0.1691
0.2	0.1387	5.8977	8.2309	0.7165	0.3424
0.3	0.2124	5.5620	7.3207	0.7597	0.5241
0.4	0.2913	5.1422	6.2573	0.8217	0.7188
0.5	0.3774	4.6739	5.1695	0.9041	0.9313
0.6	0.4729	4.1888	4.1521	1.0088	1.1668
0.7	0.5800	3.7118	3.2603	1.1384	1.4312
0.8	0.7015	3.2600	2.5149	1.2962	1.7310
0.9	0.8403	2.8436	1.9134	1.4861	2.0735
1.0	1.0000	2.4674	1.4406	1.7126	2.4674

Table 22. Optimal mass distribution $m(t)/m(0)$ for several values of the frequency parameter β .

t	$\beta = 0$	$\beta = \pi/6$	$\beta = \pi/5$	$\beta = \pi/4$	$\beta = \pi/3$	$\beta = \pi/2$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0000	0.9972	0.9960	0.9938	0.9891	0.9757
0.2	1.0000	0.9891	0.9843	0.9757	0.9573	0.9074
0.3	1.0000	0.9757	0.9652	0.9464	0.9074	0.8071
0.4	1.0000	0.9573	0.9394	0.9074	0.8431	0.6898
0.5	1.0000	0.9344	0.9074	0.8603	0.7691	0.5699
0.6	1.0000	0.9074	0.8703	0.8071	0.6898	0.4577
0.7	1.0000	0.8768	0.8290	0.7496	0.6094	0.3594
0.8	1.0000	0.8431	0.7845	0.6898	0.5312	0.2772
0.9	1.0000	0.8071	0.7378	0.6294	0.4577	0.2109
1.0	1.0000	0.7691	0.6898	0.5699	0.3904	0.1588

Table 23. Comparison of the optimal beam (subscript o) with the constant section beam (subscript c).

β	I_o	I_c	I_o/I_c	$(I_c-I_o)/I_c$
0	0.00000	0.00000	1.0000	0.0000
$\pi/6$	0.30014	0.30229	0.9929	0.0071
$\pi/5$	0.44954	0.45650	0.9848	0.0152
$\pi/4$	0.75458	0.78539	0.9608	0.0392
$\pi/3$	1.56091	1.81379	0.8606	0.1394
$\pi/2$	5.29597	∞	0.0000	1.0000

Table 24. Comparison of the optimal beam (subscript o) with the constant section beam (subscript c).

β	I_o	I_c	I_o/I_c	$(I_c-I_o)/I_c$
0.0	0.0000	0.0000	1.0000	0.0000
0.1	0.0100	0.0100	1.0000	0.0000
0.2	0.0405	0.0405	0.9999	0.0001
0.3	0.0927	0.0928	0.9992	0.0008
0.4	0.1687	0.1691	0.9976	0.0024
0.5	0.2715	0.2731	0.9941	0.0059
0.6	0.4053	0.4104	0.9874	0.0126
0.7	0.5754	0.5896	0.9760	0.0240
0.8	0.7887	0.8237	0.9575	0.0425
0.9	1.0537	1.1341	0.9291	0.0709
1.0	1.3810	1.5574	0.8868	0.1132
1.1	1.7839	2.1612	0.8254	0.1746
1.2	2.2784	3.0865	0.7382	0.2618
1.3	2.8845	4.6827	0.6160	0.3840
1.4	3.6263	8.1170	0.4468	0.5532
1.5	4.5338	21.1521	0.2143	0.7857
$\pi/2$	5.2959	∞	0.0000	1.0000

10. Conclusions

In this report, we have investigated the axial vibration of a cantilever beam both analytically and numerically. We have determined the mass distribution that minimizes the total mass for a given value of the fundamental frequency using both the sequential ordinary gradient-restoration algorithm (SOGRA) and the modified-quasilinearization algorithm (MQA). Concerning the minimum value of the mass, SOGRA leads to a solution precise to at least 4 significant digits and MQA leads to a solution precise to at least 6 significant digits.

Comparison of the optimal beam (a variable-section beam) with a reference beam (a constant-section beam) shows that the weight reduction depends strongly on the frequency parameter β. This weight reduction is negligible for $\beta \to 0$, is 0.6% for $\beta = 0.5$, is 11.3% for $\beta = 1$, is 55.3% for $\beta = 1.4$, and approaches 100% for $\beta \to \pi/2$.

11. Appendix: Analytical Solutions

In this appendix, we summarize the analytical solutions available in two limiting cases: the constant-section beam and the optimal beam.

Constant-Section Beam. For this structure, the functions $y(t)$, $z(t)$, $m(t)$, $u(t)$, $w(t)$ are given by

$$y = \sin(\beta t)/\sin\beta \,, \tag{91-1}$$

$$z = \beta^2 \cos(\beta t)/\cos\beta \,, \tag{91-2}$$

$$m = \beta \tan\beta \,, \tag{91-3}$$

$$u = \beta \cos(\beta t)/\sin\beta \,, \tag{91-4}$$

$$w = -\beta^2 \sin(\beta t)/\sin\beta \,. \tag{91-5}$$

The associated mass integral is given by

$$I = \beta \tan\beta \,. \tag{92}$$

Optimal Beam. For this structure, the functions $y(t)$, $z(t)$, $m(t)$, $u(t)$, $w(t)$ are given by (Ref. 8)

$$y = \sinh(\beta t)/\sinh(\beta) \,, \tag{93-1}$$

$$z = \beta^2 \cosh(\beta)/\cosh(\beta t) \,, \tag{93-2}$$

$$m = \beta \sinh(\beta)\cosh(\beta)/\cosh^2(\beta t) \,, \tag{93-3}$$

$$u = \beta\cosh(\beta t)/\sinh(\beta) , \tag{93-4}$$

$$w = \beta^2\sinh(\beta t)/\sinh(\beta) . \tag{93-5}$$

The associated mass integral is given by

$$I = \sinh^2(\beta) . \tag{94}$$

Limiting Case. For $\beta = 0$, Eqs. (91) and (93) reduce to

$$y = t , \quad z = 0 , \quad m = 0 , \quad u = 1 , \quad w = 0 , \tag{95}$$

and the mass integrals (92) and (94) yield

$$I = 0 . \tag{96}$$

Therefore, for $\beta = 0$, the constant-section beam and the optimal beam are identical.

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The problem of the axial vibration of a cantilever beam is investigated both analytically and numerically. The mass distribution that minimizes the total mass for a given value of the frequency parameter β is determined using both the sequential ordinary (OVER)

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20. Abstract

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Comparison of the optimal beam (a variable-section beam) with a reference beam (a constant-section beam) shows that the weight reduction depends strongly on the frequency parameter β. This weight reduction is negligible for $\beta \to 0$, is 11.3% for $\beta = 1$, is 55.3% for $\beta = 1.4$, and approaches 100% for $\beta \to \pi/2$.

beta beta approaches beta

beta approaches pi/2