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NONLINEARITIES WITH NON-GAUSSIAN INPUTS

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Abstract

This paper is concerned with the bandlimitedness of the output of a nonlinearity with a random input. The input is taken to be a contaminated Gaussian process. It is shown that, if the nonlinearity is not a polynomial, or if the Gaussian input component is not bandlimited, then the output is not bandlimited.

I. Introduction

In statistical communication theory, as well as other areas of engineering, we are frequently interested in nonlinearities with random inputs. Also, in the context of signal transmission, the bandwidth occupied by a random signal or noise is of concern. The effect of a linear system upon the bandwidth properties of a random process is well understood; however, such is not the case for nonlinear systems. In this paper we will consider the effect of a nonlinearity upon the bandwidth of a class of random processes.

By the term nonlinearity we will refer to a time invariant zero memory nonlinearity. That is, the output at time t is a function only of the input at time t. Such nonlinearities are frequently encountered in the form of quantizers, companders, limiters, power law devices, etc.

In the next section we will precisely state our assumptions, and we will review some established results. Then in the succeeding section we will develop the principal result of the paper.

II. Preliminaries

The parameter set of the random processes will be the real line, and the parameter will be associated with time. Let N(t) be a second order random process, and assume that N(t) is second order stationary; that is

$$P\{N(t_1) \le n_1 \ , \ N(t_2) \le n_2\} =$$

$$P\{N(t_1+t_0) \le n_1 \ , \ N(t_2+t_0) \le n_2\}$$

for all t_0, t_1, t_2, n_1, and n_2. Thus the autocorrelation function $E\{N(t)\,N(s)\}$ is a function only of the time difference, $R(t-s)$. The random process is mean square continuous if, for all t,

$$\lim_{s \to t} E\{[N(s)-N(t)]^2\} = 0 \ .$$

This is equivalent to the autocorrelation function R being continuous at the origin.

If N(t) is mean square continuous, the spectral representation of N(t) is of the form

$$N(t) = \int_{-\infty}^{\infty} e^{j\omega t}\, d\tilde{N}(\omega) \ ,$$

where the integral is defined as a limit in the mean and $\tilde{N}(\omega)$ is a random process with orthogonal increments. It follows that for $\omega_2 > \omega_1$,

$$E\{[\tilde{N}(\omega_2) - \tilde{N}(\omega_1)]^2\} = F(\omega_2) - F(\omega_1)$$

and

$$R(\tau) = \int_{-\infty}^{\infty} e^{j\omega\tau}\, dF(\omega) \ .$$

The function F is called the spectral distribution function. The spectrum of the random process N(t) consists of all the numbers in whose neighborhood F is actually increasing; that is, the spectrum is the set of all numbers ω_0 such that

$$F(\omega_0+\varepsilon) - F(\omega_0-\varepsilon) > 0$$

for all $\varepsilon > 0$. These numbers are the frequencies that enter effectively in the harmonic analysis of both the autocorrelation function and the sample functions of the random process. The above properties are well known and can be found developed in [1, chapter 10] and [2, chapter 11].

Let X(t) be a second order stationary Gaussian random process with a positive variance. Assume that X(t) and N(t) are independent random processes that are mean square continuous. Let

$$Y(t) = X(t) + N(t) \ . \qquad (1)$$

Then Y(t) is a contaminated Gaussian process.

Consider the class of all Baire functions $g(\cdot)$ such that $E\{(g[Y(t)])^2\} < \infty$. All nonlinearities will be assumed to belong to this class. The nonlinearities $g_1(\cdot)$ and $g_2(\cdot)$ will be regarded as identical if $g_1[Y(t)]$ and $g_2[Y(t)]$

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are equivalent random processes. Throughout the sequel it will be assumed that $g(\cdot)$ is not a constant.

The random process $Y(t)$ will be taken as the input to the nonlinearity and the output $g[Y(t)]$ will then be a well defined second order random process that is second order stationary. We will be concerned with the spectral properties of the output, and thus we should insure that the output possesses a spectral representation. This property is established by the following two theorems.

Theorem 1: A second order random process that is second order stationary possesses a spectral representation if and only if it is mean square continuous.

This theorem is proved in [1, pp. 482-483].

Theorem 2: Let $S(t)$ be a second order random process that is first order stationary and mean square continuous. Let $g(\cdot)$ be any Baire function such that $E\{(g[S(t)])^2\} < \infty$. Then $g[S(t)]$ is mean square continuous.

This theorem is proved in [3].

A random process is said to be bandlimited if it has a bounded spectrum; that is, if there exists a finite number M such that the absolute value of any point in the spectrum is less than M. If the Gaussian process $X(t)$ possesses a spectral density function (i.e. the spectral distribution function is absolutely continuous), the following theorem completely characterizes the bandlimitedness of the output when the input is $X(t)$. The nonlinearity in the following theorem is assumed to belong to the class described earlier when $N(t)$ is identically zero.

Theorem 3: Assume that the Gaussian process $X(t)$ possesses a spectral density function. Then $g[X(t)]$ is bandlimited if and only if

A. $X(t)$ is bandlimited,

and

B. $g(\cdot)$ is a polynomial.

This theorem is proved in [4].

Notice that Theorem 3 does not hold in general if we drop the Gaussian assumption. For example, let $G(t)$ be a bandlimited Gaussian process possessing a spectral density function. Then let arc tan $[G(t)]$ be the input. By Theorem 3 this input is not bandlimited; and if $g(\cdot)$ is a polynomial, the output also will not be bandlimited. However, if $g(x) = \tan x$, then the output will be bandlimited, even though the nonlinearity is not a polynomial.

III. Development

In this section we will establish necessary conditions for the output to be bandlimited when the input is the contaminated Gaussian process $Y(t)$ given by (1).

For the moment, consider two zero mean mutually Gaussian random variables X and Y, with correlation coefficient ρ, and each with variance $\sigma^2 > 0$. If $|\rho| < 1$, then a bivariate density function $f(x,y)$ exists. Using the Mehler formula, we can write

$$f(x,y) =$$

$$\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2\sigma^2(1-\rho^2)}(x^2-2\rho xy+y^2)\right] =$$

$$p(x)\,p(y) \sum_{n=0}^{\infty} \rho^n\,\theta_n(x)\,\theta_n(y)$$

where

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

$$\theta_n(x) = \frac{H_n\left(\frac{x}{\sigma}\right)}{\sqrt{n!}},$$

$H_n(\cdot)$ is the n-th Hermite polynomial given by

$$H_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right),$$

and the series is convergent pointwise as well as in an L_2 sense [5]. Let $g_1(\cdot)$ and $g_2(\cdot)$ be any Baire functions such that

$$E\{[g_i(X)]^2\} < \infty, \quad i = 1,2.$$

Then $g_1(\cdot)$ admits the expansion

$$g_1(x) = \sum_{n=0}^{\infty} b_n\,\theta_n(x)$$

where

$$b_n = \int_{-\infty}^{\infty} g_1(x)\ \theta_n(x)\ p(x)\ dx$$

and the convergence is in the sense that

$$\lim_{K\to\infty} \int_{-\infty}^{\infty} \left[g_1(x) - \sum_{n=0}^{K} b_n\ \theta_n(x)\right]^2 p(x)\ dx = 0.$$

Similarly,

$$g_2(x) = \sum_{n=0}^{\infty} c_n\ \theta_n(x)$$

where

$$c_n = \int_{-\infty}^{\infty} g_2(x)\ \theta_n(x)\ p(x)\ dx$$

and the convergence is in the same sense. It then follows in a straightforward fashion that

$$E\{g_1(X)\ g_2(Y)\} = \sum_{n=0}^{\infty} b_n\ c_n\ \rho^n.$$

Now assume that $\rho=1$. Using the Schwarz inequality, it follows that $X=Y$ with probability one. Then we get that

$$E\{g_1(X)\ g_2(Y)\} = E\{g_1(X)\ g_2(X)\}$$

$$= \sum_{n=0}^{\infty} b_n\ c_n\ .$$

Now assume that $\rho=-1$. It then follows from the Schwarz inequality that $X=-Y$ with probability one. Then $E\{g_1(X)\ g_2(Y)\} = E\{g_1(X)\ g_2(-X)\}$. However, since $\theta_n(x)$ is even for even n and odd for odd n, we get that

$$\int_{-\infty}^{\infty} g_2(-x)\theta_n(x)p(x)dx=(-1)^n \int_{-\infty}^{\infty} g_2(x)\theta_n(x)p(x)dx.$$

Using this, we then get that

$$E\{g_1(X)\ g_2(Y)\} = \sum_{n=0}^{\infty} (-1)^n\ b_n\ c_n.$$

Now we return to the problem at hand. We will first calculate the autocorrelation function of the output of the nonlinearity. Thus we are interested in the quantity

$$E\{g[Y(t)]\ g[Y(s)]\} =$$

$$E\{g[X(t)+N(t)]\ g[X(s)+N(s)]\}.$$

Associate the mean of $X(t)$ with the random process $N(t)$, and therefore, without loss of generality, assume that $X(t)$ has zero mean. Let σ^2 denote the variance of $X(t)$, and let $\rho(t-s)$ denote the correlation coefficient of $X(t)$ and $X(s)$. Using the results outlined above, we have that the conditional expectation is almost surely given by

$$E\{g[Y(t)]\ g[Y(s)]\,|\,N(t),N(s)\} =$$

$$\sum_{n=0}^{\infty} [\rho(t-s)]^n\ b_n[N(t)]\ b_n[N(s)]$$

where

$$b_n(y) = \int_{-\infty}^{\infty} \theta_n(x)\ g(x+y)\ p(x)\ dx.$$

Thus we see that the output autocorrelation function is given by

$$E\left\{g[Y(t)]\ g[Y(s)]\right\} =$$

$$E\left\{\sum_{n=0}^{\infty} [\rho(t-s)]^n\ b_n[N(t)]\,b_n[N(s)]\right\}.$$

Notice that

$$\left|[\rho(t-s)]^n\ b_n[N(t)]\ b_n[N(s)]\right| \leq$$

$$(b_n[N(t)])^2 + (b_n[N(s)])^2$$

and

$$E\left\{\sum_{n=0}^{\infty} \left[\left(b_n[N(t)]\right)^2 + \left(b_n[N(s)]\right)^2\right]\right\} =$$

$$2\ E\{(g[Y(t)])^2\} < \infty.$$

Thus the Dominated Convergence Theorem justifies exchanging the expectation and the infinite summation. Therefore, we have that

$$E\{g[Y(t)]\ g[Y(s)]\} =$$

$$\sum_{n=0}^{\infty} [\rho(t-s)]^n\ E\{b_n[N(t)]\,b_n[N(s)]\}.$$

Notice that $b_n(\cdot)$ can be interpreted as a nonlinearity, and thus $b_n[N(t)]$ is a second order random process that is second order stationary. Thus, $E\{b_n[N(t)]\ b_n[N(s)]\}$ is a nonnegative definite function of $(t-s)$, say $R_n(t-s)$, and we see from Theorem 2 that $R_n(\cdot)$ is continuous. Therefore,

$$E\{g[Y(t)]\ g[Y(s)]\} = \sum_{n=0}^{\infty} [\rho(t-s)]^n R_n(t-s).$$

Letting $r(\cdot)$ denote the autocorrelation function of the output, we get that

$$r(\tau) = \sum_{n=0}^{\infty} [\rho(\tau)]^n R_n(\tau) . \qquad (2)$$

Since

$$r(0) = \sum_{n=0}^{\infty} R_n(0) < \infty,$$

we see that the convergence in (2) is uniform.

Let $G(\omega)$ be the spectral distribution function corresponding to $r(\tau)$. That is,

$$r(\tau) = \int_{-\infty}^{\infty} e^{j\omega\tau}\, dG(\omega).$$

If we require $G(-\infty) = 0$ and $G(\cdot)$ to be right continuous, then $G(\cdot)$ is uniquely determined by $r(\cdot)$ [2, pp. 519-522]. In the sequel all spectral distribution functions will be assumed to be right continuous and to vanish at minus infinity. Therefore, we can also uniquely determine the spectral distribution functions corresponding to $\rho(\cdot)$ and $R_n(\cdot)$, that is,

$$\rho(\tau) = \int_{-\infty}^{\infty} e^{j\omega\tau}\, d\, D(\omega)$$

$$R_n(\tau) = \int_{-\infty}^{\infty} e^{j\omega\tau}\, d\, D_n(\omega).$$

Recall that the spectral distribution function of the product of autocorrelation functions is given by the convolution of the spectral distribution functions corresponding to each of the autocorrelation functions. This can be easily seen if one forms the analogy between autocorrelation functions and characteristic functions and between spectral distribution functions and probability distribution functions.

Due to uniform convergence, we can transform (2) term by term to get

$$G(\omega) = D_0(\omega) + D(\omega) * D_1(\omega) + \sum_{n=2}^{\infty} ([D(\omega)]^{*n}) * D_n(\omega) , \qquad (3)$$

where * denotes convolution, that is,

$$F_1(\omega) * F_2(\omega) = \int_{-\infty}^{\infty} F_1(\omega-\lambda)\, dF_2(\lambda) ,$$

and $[\cdot]^{*n}$ denotes multifold convolution.

Recall that to any spectral distribution function there corresponds a symmetric nonnegative definite function [2, pp. 519-522], and to any such symmetric nonnegative definite function there corresponds a stationary second order random process [2, p. 72]. Thus, for any spectral distribution function F, define B(F) as the supremum of the spectrum of the associated random process. Notice that -B(F) is then the infimum of the spectrum of the associated random process. Thus a random process having a spectral distribution function F is bandlimited if and only if $B(F) < \infty$. For two spectral distribution functions F_1 and F_2, it follows in a straightforward fashion that $B(F_1 * F_2) = B(F_1) + B(F_2)$.

Notice that if $R_n(0) = 0$, then $b_n[N(t)] = 0$ with probability one. If all but a finite number of the $R_n(0)$ were zero, then all but a finite number of the $b_n[N(t)]$ would be zero with probability one; and in this case $g[x+N(t)]$ would be a polynomial in x (with random coefficients). Therefore, if $g(\cdot)$ is not a polynomial, then an infinite number of the $R_n(0)$ are nonzero. Notice that if $R_n(0)$ is nonzero, then $D_n(\omega)$ is not identically zero.

Assume that $g(\cdot)$ is not a polynomial. Notice that for $K \geq 2$,

$$B\left[\sum_{n=2}^{K} ([D(\omega)]^{*n}) * D_n(\omega)\right] \leq B[G(\omega)],$$

and as K approaches infinity, the left hand side of the above inequality increases to infinity. This result is summarized in the following theorem.

<u>Theorem 4</u>: Let the random process Y(t) be given by (1) and assume that $g(\cdot)$ is not a polynomial. Then $g[Y(t)]$ is not bandlimited.

Since we are assuming that the nonlinearity is not constant, then for at least one value of $n \geq 1$, $R_n(0) > 0$, and thus $D_n(\omega)$ is not identically zero. Thus we see from (3) that if the Gaussian component X(t) is not bandlimited, then the output will also not be bandlimited. This observation is summarized in the following theorem.

Theorem 5: Let the random process Y(t) be given by (1) and assume that the Gaussian component is not bandlimited. Then g[Y(t)] is not bandlimited.

IV. Comments

Notice that Theorems 4 and 5 hold for any contamination component N(t) which satisfies the mild conditions imposed earlier. From the development of these theorems we see that the Gaussian component plays a major role in causing the spectrum of the random process to be spread by the nonlinearity. Also, it is impossible for the independent contamination component to "undo" this effect of the Gaussian component.

Let G(t) be a stationary, mean square continuous, bandlimited Gaussian process, and let N(t) = arc tan[G(t)]. Assume that the Gaussian process G(t) is independent of the Gaussian process X(t). Let the nonlinearity be given by g(x) = tan(x). Then g[N(t)] is bandlimited. However, on the basis of Theorem 4, g[X(t) + N(t)] is not bandlimited, even though the positive variance of X(t) can be made very small.

Now assume N(t) is identically zero. Then (2) becomes

$$r(\tau) = \sum_{n=0}^{\infty} b_n^2 [\rho(\tau)]^n$$

where

$$b_n = \int_{-\infty}^{\infty} g(x)\, \theta_n(x)\, p(x)\, dx ,$$

and (3) becomes

$$G(\omega) = b_0^2 I\{\omega \geq 0\} + b_1^2 D(\omega) + \sum_{n=2}^{\infty} b_n^2 [D(\omega)]^{*n}.$$

Then we see that if g(·) is a polynomial, only a finite number of the b_n are nonzero. In this case, we see that g[X(t)] is bandlimited if and only if X(t) is bandlimited. In the case where g(·) is not a polynomial, an infinite number of the b_n are nonzero, and thus g[X(t)] cannot be bandlimited. Thus we see that Theorem 3 holds without the assumption of the existence of a spectral density function.

Using the characterization of spherically invariant random processes given in [6], it can be shown in a straightforward fashion that the results of this paper hold if the term "Gaussian process" is replaced with the term "spherically invariant random process."

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References

1. M. Loève, Probability Theory. New York: Van Nostrand, 1963.

2. J. L. Doob, Stochastic Processes. New York: John Wiley, 1953.

3. G. L. Wise and J. B. Thomas, "On a Completeness Property of Series Expansions of Bivariate Densities," Journal of Multivariate Analysis, Vol. 5, pp. 243-247, June 1975.

4. G. L. Wise, A. P. Traganitis, and J. B. Thomas, "The Effect of a Memoryless Nonlinearity on the Spectrum of a Random Process," IEEE Transactions on Information Theory, Vol. IT-23, pp. 84-89, January 1977.

5. R. Leipnik, "Integral Equations, Biorthonormal Expansions, and Noise," SIAM Journal of Applied Mathematics, Vol. 7, pp. 6-30, March 1959.

6. G. L. Wise and N. C. Gallagher, Jr., "On Spherically Invariant Random Processes," IEEE Transactions on Information Theory, Vol. IT-24, pp. 118-120, January 1978.

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