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THE DISTRIBUTION OF SUMS OF 6 EPENDENT LOG-NORMAL VARIABLES







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THE DISTRIBUTION OF SUMS OF DEPENDENT LOG-NORMAL VARIABLES

by

S. Zacks* and C. P. Tsokos** Case Western Reserve University and University of South Florida

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I. Introduction

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Distributions of sums of dependent or independent log-normal random variables appear in various fields of applied statistics. Concentrations of pollutants in the air or in water, life distributions of components in reliability systems and other applications. The available expression for the distributions -f sums of log-normal variables are complicated and inattractive. Simpler expressions are needed for practical applications. The purpose of the present paper is to provide some simple approximations and algorithms that can be easily applied for obtaining numerical results. We are concerned with two types of variables (i) $W = e^{X_1} + e^{X_2}$ and (ii) $Z = \log(e^{X_1} + e^{X_2})$, where X_1 and X_2 have a bivariate normal distribution. The dependence of the log-normal variables e^1 and e^2 is a function of the correlation ρ between X_1 and X_2 . Naus [5] derived the moment generating function of Z for the case of $\rho = 0$ and equal variances of X_1 and X_2 . Handan [2] extended Naus results to the case of arbitrary ρ and

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monants we provide a formula for a measured approximation.

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unequal variances. Lowrimore and Tsokos [4] derived the probability density function of Z and of W. Further elaboration of these distributions is given in the paper of Tsokos [6]. The problem is however, that the analytic expressions derived by Lowrimore and Tsokos are very complicated and require also numerical integration. Our aim is to develop simpler methods. In Section 2 we develop two numerical procedures for the determination of the distribution of W with arbitrary parameters. We present some numerical computations of the c.d.f. of W. These computations were performed according to a FORTRAN program given in Appendix A. In Section 3 we study the moments of W and in Section 4 we derive an approximation to the distribution of W based on the lognormal distribution having the same mean and variance. As demonstrated by numerical examples this approximation is very effective when the correlation between X1 and X2 is nonnegative. The lognormal approximation to the distribution of W, which is the normal approximation to the distribution of Z, does not provide very good results in the range of correlations close to -1. We tried therefore to correct for the pronounced skewness in the distribution of Z = log W, when p is close to -1, by employing the Edgeworth expansion. For this purpose we have to determine the moments of Z = log W. In Section 5 we discuss the problem of determining the moments of Z, in the case of X_1 and X_2 having a standard bivariate normal distribution. For the first moment an analytic expression similar to that of Hamdan [2] is given. We present however, an expression which is more suitable for numerical computations. For higher moments we provide a formula for a numerical approximation. The goodness of this approximation is also studied. Numerical computations show that the Edgeworth type of expansion mentioned earlier does not provide in the standard case any substantial improvement.

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2. The Distribution of W. a cart W mond I = 0 has go = 10 +04 = 14 (11)

Let X_1 and X_2 be random variables having a bivariate normal distribution with means μ_1 and μ_2 , variances σ_1^2 and σ_2^2 , respectively and coefficient of correlation ρ . $-\infty < \mu_1$, $\mu_2 < -$; $0 < \sigma_1$, $\sigma_2 < -$ and $-1 \le \rho \le 1$. Let $X_1 = X_2$ and let $F_{\underline{\theta}}(w)$ be the c.d.f. of W, under $\underline{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$. Obviously, $F_{\underline{\theta}}(w) = 0$ for all $w \le 0$. For w > 0 we have for all $\underline{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$

(2.1)
$$F_{\underline{\theta}}(w) = P_{\underline{\theta}}[e^{1} + e^{2} \le w]$$
$$= \frac{1}{\sigma_{1}} \int_{-\infty}^{\log w} P[e^{2} \le w - e^{y} | X_{1} = y] \phi(\frac{y - \mu_{1}}{\sigma_{1}}) dy$$

where $\phi(\mu)$ denotes the standard normal p.d.f. The conditional distribution of X_2 , given $X_1 = y$, is the normal distribution with mean $E\{X_2 | X_1 = y\} = \mu_2 + \rho \frac{\sigma_2}{\sigma_1} (y - \mu_1)$ and variance $V\{X_2 | X_1 = y\} = \sigma_2^2 (1 - \rho^2)$. Thus, if $-1 < \rho < 1$,

$$F_{\underline{\theta}}(w) = \frac{1}{\sigma_1} \int_{-\infty}^{\log w} \Phi(\frac{\log(w-e^y) - \mu_2 - \rho_2^2}{\sigma_1} \frac{(y-\mu_1)}{(y-\mu_1)}$$

(2.2)

executed. Notice that $e(-4.5) = .54 \times \frac{1}{2} - \frac{1}{2}$ Therefore, the strot constitued by angle of this character that $e_{ij} = \frac{1}{2} + \frac{1$

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where $\Phi(u)$ is the standard normal c.d.f. The following are some special cases: (i) If $\mu_1 = \mu_2 = \mu$ and $\sigma_1 = \sigma_2 = \sigma$ then the expressions are slightly simplified since e^{μ} is a scale parameter of the distribution and

$$\mathbf{F}_{(\mu,\sigma,\mu,\sigma,\rho)}(\mathbf{w}) = \mathbf{F}_{(0,\sigma,0,\sigma,\rho)} \left(\frac{\mathbf{w}}{\mathbf{e}^{\mu}}\right), \ 0 \leq \mathbf{w} \leq \infty.$$

(ii) If $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$ and $\rho = 1$ then W has a lognormal distribution. Indeed, in this case $X_2 = X_1$ with probability 1 and

(2.3) The P[W \leq w] = P[e^X] $\leq \frac{w}{2}$] = $\Phi(\frac{\log w - \mu'}{\sigma})$, is the interval of the product of the produ

where $\mu' = \mu_1 + \log 2$. (iii) When $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$ and $\rho = -1$ then $X_2 = -X_1$ with probability 1 and the distribution of W is given by,

(2.4)
$$F_{\underline{\theta}}^{(-1)}(w) = \begin{cases} 0 , \text{ if } w \leq 2, \\ \phi(\frac{\xi_2(w) - \mu_1}{\sigma_1}) - \phi(\frac{\xi_1(w) - \mu_1}{\sigma_1}), \text{ if } w > 2, \end{cases}$$

where $\xi_1(w) = \frac{1}{2}(w - \sqrt{w^2 - 4})$ and $\xi_2(w) = \frac{1}{2}(w + \sqrt{w^2 - 4})$. Notice that $e^x + e^{-x} \ge 2$ for all x.

2.1 Numerical Determination of the Distribution of W.

The integrand of (2.2) can be easily computed for each y value. A numerical integration of (2.2) over the range $(\mu_1 - 4.5 \sigma_1, \log w)$ can then be readily executed. Notice that $\Phi(-4.5) = .34 \times 10^{-5}$. Therefore, the error committed by neglecting the range of $y < \mu_1 - 4.5 \sigma_1$ is smaller than $.34 \times 10^{-5}$. For this reason, for values of w smaller than e , we approximate the value of $F_{\underline{\theta}}(w)$ by 0.

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An m-point approximation to (2.2) is given by

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 $\frac{\log(w-e^{n'j}) - (\mu_2 - \rho \frac{\sigma_2}{\sigma_1} \mu_1) - \rho \frac{\sigma_2}{\sigma_1} n'_j}{\sigma_2 \sqrt{1 - \delta^2}}).$

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 $\left[\begin{array}{c} \bullet(\frac{h_{j}-\mu_{1}}{\sigma_{1}}) - \bullet(\frac{h_{j-1}-\mu_{1}}{\sigma_{1}})\right],$

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where 000.4 0.81938 A 1983 1994 0.86402 $n_j = \mu_1 - 4.5 \sigma_1 + j \Delta(w)$, (2.6) · · · · · 10.91382 $\Delta(w) = (\log w - \mu_1 + 4.5 \sigma_1)/\mu_1$.2283.2.0

 $n_{j} = n_{j} - \Delta(w)/2.$

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In Tables 1 we provide the results of computing $F_A(w)$ according to (2.5) with $\mu = 20$ and subintervals for the case of $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\rho = -.99(.33).99.$

Theoretically, the limit of (2.5) as $m \rightarrow \infty$ is the integral (2.2). We see in Table 1 that the differences between the results of computations with **m** = 20 and **m** = 80 are in most cases in the third decimal place. It is not difficult, however, to determine the distribution functions very accurately by applying the method with a large m value. The computation of the seven distributions of Table 1, with m = 80 required about 60 seconds (double precision) on a relatively slow computer (Honeywell GE-400). A Fortran program according to which these computations were performed is available upon request. To evaluate the goodness of the approximation with m = 80 we provide in Table 2 a comparison of the results obtained for the case of $\rho = 1$ from (2.5) against the exact log-normal distribution, given by formula (2.3).

Table 1.		Distributio	n of W for	$\mu_1 = \mu_2 =$	• 0, $\sigma_1 =$	$\sigma_2 = 1.$	Determined	according
		to (2.5) fo	r m = 80 a	and m = 20.		int .	(M) 1	(5-2)
1	n W/p	99	66	33	0	33	.66	.99
	1.000	0.00000	0.02006	0.06698	0.11346	0.15739	0.19994	0.24281
	2.000	0.12205	0.29188	0.35051	0.39414	0.48153	0.46590	0.49900
	3.000	0.66113	0.59716	0.59660	0.60781	0.62269	0.63926	0.65685
	4.000	0.81100	0.77229	0.75013	0.74334	0.74393	0.74856	0.75560
	5.000	0.88036	0.86294	0.84076	0.82774	0.82141	0.81938	0.82013
	6.000	0.92414	0.91197	0.89489	0.88122	0.87221	0.86683	0.86402
80	7.000	0.94697	0.94042	0.92823	0.91600	0.90645	0.89961	0.89490
	8.000	0.96162	0.95799	0.94947	0.93922	0.93014	0.92287	0.91726
	9.000	0.97150	0.96940	0.96345	0.95512	0.94690	0.93976	0.93382
	10.000	0.97838	0.97712	0.97293	0.95901	0.95901	0.95228	0.94635
	1.000	0.00000	0.02004	0.06696	0.11353	0.15755	0.20028	0.24416
	2.000	0.12185	0.29124	0.35052	0.39417	0.43135	0.46579	0.49262
	3.000	0.66129	0.59864	0.59759	0.60813	0.62229	0.63864	0.65854
	4.000	0.82205	0.77607	0.75199	0.74399	0.74351	0.74765	0.76265
	5.000	0.88671	0.86693	0.84293	0.82859	0.82105	0.81835	0.82802
	6.000	0.92364	0.91506	0.89697	0.88215	0.87193	0.86578	0.87066
20	7.000	0.94654	0.94247	0.93004	0.91692	0.90625	0.89860	0.89974
	8.000	0.96123	0.95924	0.95095	0.94008	0.93000	0.92193	0.92038
	9.000	0.97115	0.97012	0.96463	0.95589	0.94681	0.93890	0.93555
	10.000	0.97804	0.97751	0.97385	0.96693	0.95897	0.95150	0.94701
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Theoretically, the Hailt of (2.5) as 2 - - is the integral (2.2). He

Table 2. The Distrib	oution of	W for P1	= µ2 = 0, σ	= o, = 1,	point al sa	
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cleionty siduol; shaces (2.000	0.5000	0.4916		encipudinter	5
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log-normal discribution, given by formula (2.2).

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2.2 Gauss-Legendre Quadrature.

The numerical integration method prescribed in Section 2.1 is quite simple in the sense that it is based on subintervals of equal length. The results obtained seems to be quite stable over all the range of $-1 \le \rho \le 1$. However, as seen in Table 2, there is some difference although small, between the numerical results obtained and what should be obtained in the case of $\rho = 1$. We therefore investigate here what numerical results can be obtained by applying the Gauss-Legendre quadrature formula, with 80 cut-points, for integrating (2.2) numerically. An m-point Gauss-Legendre quadrature formula is

(2.7)
$$\int_{a}^{b} f(x)dx = (b-a) \sum_{i=1}^{m} p_{i} f(x_{i}) + R_{m}$$

where $x_i = \frac{b-a}{2} \xi_i + \frac{a+b}{2}$, i=1,..., $m \xi_i$ is the i-th zero of the Legendre polynomial $P_m(\xi)$ over $-1 \le \xi \le 1$, and $P_i = 1/(1 - \xi_i^2) [P_m'(\xi_i)]^2$ is a weight assigned to ξ_i (i=1,...,m). R_m is a proper remainder term (see Abramowitz and Segun [8; pp. 888]). The values of ξ_i and $2p_i$ for m = 80 are tabulated in Abramowitz and Segun [8; pp. 918]. For the case under consideration, let

(2.8)
$$f(x;\alpha,\beta, s,w) = \frac{1}{s} [\log(w-e^{x}) - \alpha - \beta x], -\infty \le x \le \log w$$

where $\beta = \rho \sigma_2 / \sigma_1$, $\alpha = \mu_2 - \beta \mu_1$ and $s^2 = \sigma_2^2 (1 - \rho^2)$. By simple change of variables, we can write the c.d.f. of W in the form

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Hence, the Gauss-Legendre approximation is, according to (2.7)

(2.10) $F_{\underline{\theta}}(w) \stackrel{\sim}{=} \phi(\frac{\log w - \mu_1}{\sigma_1}) \int_{\underline{i}=1}^{\underline{m}} p_i \phi(f(\mu_1 + \sigma_1 \phi^{-1}(y_i); \alpha, \beta, s, w))$ where $y_i = \frac{1}{2} \phi(\frac{\log w - \mu_1}{\sigma_1})(1 + \xi_i)$, $i=1, \dots, m$.
In Table 3 we present the results of the numerical determination of the distributions corresponding to those of Table 1, according to the Gauss-Legendre method with m = 80.

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The comparisons of Table 1 and 3 show that the two methods yield very close results. In Table 4 we provide further comparisons of the two methods in non-standard cases.

Further simplication of the calculations without sacrificing much accuracy can be achieved by applying formula (2.10) for a small value of m. We have seen in Table 1 that (2.5) provides highly accurate results with m = 20. For small values of m formula (2.5) may not yield sufficiently accurate results, as shown in Table 5, since it is based on a partition to equal size subintervals.

On the other hand, formula (2.10) with m = 6 yields accurate results when $|\rho|$ is not too close to 1. This is seen in Table 6. For $\rho = 1$ and $\rho = -1$ we can compute the distributions exactly by other formulae.

For m = 6 formula (2.6) should be used with the following constants (see Abramowitz and Segun [8, pp. 921]).

1	$\frac{1+\xi_1}{2}$	P.	blas, we can write th
	03376	-1	
2	.16939	.18038	
1.0.0 (3) ⁽¹⁾ 4.	.38069	.23395	* (v) .*
4	.61930	.23395	
5	.83060	.18038	
6	.96623	.08566	

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Table 3.	The Distr	ibution of	W for H.	= µ_ =	0, σ. = σ	. = 1.	Determined
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According to The Gauss-Legendre Quadrature; m = 80.

					A 4	120 J. M. J.	0	
	₩:/ρ	99	66	33	0	.33	.66	.99
	1.000	0.00000	0.02006	0.06698	0.11345	0.15737	0.19992	0.24278
ik.0	2.000	0.12207	0.29191	0.35054	0.39416	0.43153	0.46590	0.49900
12.0	3.000	0.66176	0.59728	0.59668	0.60785	0.62272	0.63928	0.65689
3.0	4.000	0.81144	0.77245	0.75022	0.74340	0.74398	0.74860	0.75565
1.16	5.000	0.88254	0.86301	0.84081	0.82780	0.82147	0.81943	0.82019
5.0	6.000	0.92191	0.91186	0.89489	0.88127	0.87227	0.86689	0.86409
\$.0	7.000	0.94567	0.94010	0.92817	0.91603	0.90651	0.89967	0.89497
6.7	8.000	0.96088	0.95759	0.94935	0.93923	0.93019	0.92292	0.91732
	9.000	0.97105	0.96910	0.96330	0.95510	0.94695	0.93981	0.93388
8.2	10.000	0.97810	0.97699	0.97281	0.96621	0.95905	0.95233	0.94641
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We remark that for m = 6 formula (2.10) can be used also with hand calculators

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	Table 4.	The Distr:	ibution of	W Compute	d According	; to (2.5)	and Accor	ding
•	10042313	to Gauss-]	Legendre Q	uadrature.			na nana ser	
		Case I: µ	$= 0, \mu_2$	= 0, σ ₁ =	1, σ ₂ = 3;	Case II:	μ ₁ = 0, μ	2 = 3,
R.	88.	$\sigma_1 = 1, \sigma_2$	2 = 4.	- Th	33	Q2	0157	
ase I	W/p	99	66	33	0	.33	.66	.99
0.495	1.000	0.00000	0.06995	0.13479	0.18889	0.23925	0.29053	0.34915
	3.222	0.49500	0.47368	0.49128	0.51462	0. 53951	0.56491	0 58860
	5.444	0.65530	0.64135	0.63729	0.64234	0.65173	0.66301	0.6738
	7.667	0.72296	0 71551	0 70896	0.70810	0.71110	0.71640	0.7220
12 86	9 880	0 76153	0.75604	0.75124	0.76855	0.7/1119	0.71040	0.7220
	12 111	0 78723	0.79604	0.73124	0.74635	0.74690	0.73123	0.7344
_T	16 222	0.76723	0.70404	0.77948	0.77630	0.77540	0.77630	0.7761
CD R	14.333	0.80002	0.00303	0.79999	0.79093	0.79347	0.79340	0.7965.
	10.330	0.82063	0.818/4	0.815/9	0.81295	0.81125	0.810/9	0.81134
	10.//8	0.83245	0.83091	0.82847	0.82589	0.82413	0.82339	0.82360
	21.000	0.84232	0.84103	0.8389/	0.83665	0.83489	0.83398	0.83390
	1.000	0.00000	0.07007	0.13503	0.18916	0.23950	0.29067	0.3491
	3.223	0.49724	0.47431	0.49169	0.51490	0.53965	0.56490	0.5886
	5.444	0.65536	0.64165	0.63751	0.64249	0.65179	0.66297	0.6738
	7.667	0.72293	0.71559	0.70907	0.70819	0.71122	0 71638	0.7220
	9.889	0.76147	0.75693	0.75126	0.74858	0.74891	0.75119	0.7543
2.5)	12,111	0.78715	0.78400	0.77945	0.77635	0.77545	0.77625	0.7780
,	14.333	0.80594	0 80359	0 79994	0 79690	0.79545	0 79544	0 7964
	16.556	0.82055	0.81870	0 81574	0 81291	0 81123	0 81075	0.8112
	18 778	0 83237	0.83097	0.828/2	0.82585	0 82409	0.82335	0 8235
	21,000	0.84223	0.84099	0.83892	0.83660	0.83485	0.83395	0.8339
T	т	0104225	0.04077	0.03072	0105000	0105105	0105575	
ase 1.	0 135	0 00000	0 00027	0 00367	0 00002	0 01659	0 021/3	0 0227
	0.155	0.00000	0.00027	0.00307	0.00992	0.01038	0.02143	0.0227
	1 000	0.00000	0.02109	0.03/13	0.00922	0.11/91	0.14273	0.13/94
	2 701	0.15/85	0.24605	0.29959	0.34449	0.30031	0.43029	0.4011
	2.701	0.0/301	0.65902	0.00000	0.0/334	0.09230	0.71097	0.7490
-L	7.839	0.869/2	0.86/39	0.86341	0.86091	0.86120	0.86402	0.00/0
	20.086	0.9314/	0.93104	0.93015	0.928/8	0.92/35	0.92612	0.9252
	54.598	0.95984	0.95977	0.95957	0.95922	0.958/5	0.95816	0.95/5
	148.413	0.97724	0.97725	0.97719	0.97710	0.97698	0.97682	0.9765
	403,429	0.98777	0.98781	0.98777	0.98774	0.98772	0.98770	0.98768
	0.135	0.00000	0.00027	0.00367	0.00995	0.01662	0.02149	0.0227
	0.368	0.00000	0.02176	0.05734	0.08949	0.11825	0.14321	0.1586
	1.000	0.15748	0.24682	0.30027	0.34516	0.38752	0.43113	0.4803
	2.781	0.68028	0.66717	0.66809	0.68012	0.69846	0.72228	0.7538
2.5)	7.389	0.86966	0.86737	0.86346	0.86100	0.86128	0.86403	0.8677
	20.086	0.93140	0.93099	0.93012	0.92876	0.92730	0.92607	0.9251
	54.598	0.95977	0.95971	0.95954	0,95921	0.95870	0.95810	0.9574
	148,413	0.97717	0.97719	0.97717	0.97709	0.97695	0.97675	0.9765
	403.429	0.98772	0.98775	0.98775	0.98774	0.98770	0.98763	0.9875

erente a	a Canada and	. J. material	and ano	an Anglanan	Survey out to	made olde	water arms
W/p	99	66	33	0	.33	.66	.99
1.000	0.00000	0.02050	0.07004	0.11845	0.16275	0.20154	0.22662
2.000	0.09668	0.30755	0.36506	0.40738	0.44136	0.46498	0.43157
3.000	0.65238	0.61829	0.61385	0.62212	0.63209	0.63592	0.56577
4.000	0.84762	0.78664	0.76445	0.75557	0.75147	0.74380	0.65754
5.000	0.87245	0.87047	0.85124	0.83737	0.82709	0.81399	0.72333
6.000	0.89576	0.91525	0.90219	0.88858	0.87638	0.86130	0.77234
7.000	0.94787	0.94138	0.93322	0.92157	0.90948	0.89421	0.80998
8.000	0.96941	0.95779	0.95285	0.94344	0.93232	0.91775	0.83961
9.000	0.97429	0.96867	0.96574	0.95832	0.94847	0.93500	0.86339
10.000	0.97656	0.87618	0.97449	0.96871	0.96014	0.94789	0.88276

Table 5. The Distribution of W for $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$. According

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	table 6.	The Dist:	ribution of	f W Determ	ined by For	rmula (2.1)	0).	
	a train	Case I:	$\mu_1 = 0, \mu_2$	= 0, ₀ =	1, $\sigma_2 = 1$; Case II	$= \mu_1 = 0,$	μ ₂ = 3,
		σ ₁ = 1,	σ ₂ = 4; m ·	- 6.			936 _a) 8 .3	wice \$5M
	abred di		5 bris (0)*	N VU Bades	tab ana a s		terdest a	
ase I	W/p	0.99	66	33	0	.33	.66	.99
	1.000	0.00000	0.02007	0.06714	0.11344	0.15720	0.19994	0.24896
	2.000	0.11675	0.29190	0.35060	0.39388	0.43139	0.46605	0.53030
	3.000	0.63077	0.59662	0.59623	0.60685	0.62180	0.63946	0.63481
	4.000	0.83510	0.77805	0.75230	0.74339	0.74276	0.74764	0.74155
	5.000	0.86792	0.87131	0.84572	0.83008	0.82143	0.81729	0.85772
	6.000	0.90840	0.91886	0.90063	0.88528	0.87397	0.86476	0.88071
	7.000	0.95439	0.94530	0.93345	0.92085	0.90982	0.89886	0.89062
	.8.000	0.97747	0.96143	0.95368	0.94406	0.93451	0.92393	0.89707
	9.000	0.98548	0.97190	0.96666	0.95854	0.95177	0.94264	0.90196
	10.000	0.98922	0.97902	0.97531	0.97008	0.96394	0.95656	0.91261
								and the second se

0.00369

0.05731

0.30052

0.66231

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0.92898

0.95916

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0.43104

0.71796

0.86519

0.92802

0.96007

0.97850

0.98906

0.01662

0.11806

0.38741

0.69346

0.86259

0.92835

0.95924

0.97725

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0.76278

0.88895

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0.93661

0.99119

0.99982

at to (2.5) with m = 6. Other is solded subtray all of any as

Case II

0.135

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2.718

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0.94280

0.99226

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0.66079

0.86865

0.93262

0.96146

0.97885

0.98913

C

3. The Moments of W and Some Characteristics of Its Distribution

As seen in the various tables of Section 2, the distribution of W is considerably skewed in non-standard cases. We develop here formulae for the moments of W and measures of skewness, kurtosis and other characteristics. The r-th moment of W is given by,

0.65239

000.2

(3.1)
$$M_{\mathbf{r}}(\underline{\theta}) = E_{\underline{\theta}} \{W^{\mathbf{r}}\} = E_{\underline{\theta}} \{\sum_{j=0}^{\mathbf{r}} (\mathbf{j}^{\mathbf{r}}) \exp \{(\mathbf{r}-\mathbf{j})X_{1} + \mathbf{j}X_{2}\}\}$$
$$= \sum_{j=0}^{\mathbf{r}} (\mathbf{j}^{\mathbf{r}}) \exp \{\mathbf{j}\mu_{2} + (\mathbf{r}-\mathbf{j})\mu_{1} + \frac{1}{2}((\mathbf{r}-\mathbf{j})^{2}\sigma_{1}^{2} + 2\sigma_{1}\sigma_{2}\rho_{1}(\mathbf{r}-\mathbf{j}) + \sigma_{2}^{2}\mathbf{j}^{2}\}\}.$$

Indeed, for each j=0,...,r,

$$(r-j)X_{1} + jX_{2} = N((r-j)\mu_{1} + j\mu_{2}, (r-j)\sigma_{1}^{2} + 2j(r-j)\rho\sigma_{1}\sigma_{2} + j^{2}\sigma_{2}^{2}).$$

Moreover, $E \{e^{N(\xi, \tau^{2})}\} = exp\{\xi + \tau^{2}/2\}.$

The central moments of W are denoted by $M_r^*(\theta)$ and are given in terms of $M_r(\theta)$ by the formula

(3.2)
$$M_{\mathbf{r}}^{*}(\theta) = \sum_{j=0}^{\mathbf{r}} {r \choose j} (-1)^{j} M_{\mathbf{r}-j}(\theta) (M_{1}(\theta))^{j}.$$

When X_1 and X_2 have the same marginal distributions, i.e., $\mu_1 = \mu_2 = \mu$ and $\sigma_1 = \sigma_2 = \sigma$, then e^{μ} is a scale parameter of the distribution of W and we have

(3.3)

$$M_{1}(\theta) = e^{\mu} \cdot 2 e^{\sigma^{2}/2}$$

$$s.d._{w}(\theta) = e^{\mu}\sqrt{2} e^{\sigma^{2}/2} (e^{\sigma^{2}} + e^{\rho\sigma^{2}} - 2)^{1/2},$$
where s.d._w(θ) is the standard deviation of W, under θ .

In Figure 1 we illustrate the standard deviation of W, as a function of ρ , for $\sigma = 1$, 1.5 and 2, and $\mu = 0$, on a logarithmic scale.

We see in Figure 1 that the standard deviation of W changes very slowly when $\rho \leq 0$ and increases faster over the range of positive ρ values. Also, the relative rate of increase grows fast with σ . In other words, for $\sigma = 2 \text{ s.d.}_{W}(\Theta)$ is relatively constant over $\rho \leq 0$, compared to the case with $\sigma = 1$.

Other parameters of interest are the coefficients of skewness, $\gamma_1(\underline{\theta}) = (M_3^*(\theta))^2 / (M_3^*(\theta))^3$ and of kurtosis $\gamma_2(\theta) = M_4^*(\theta) / (M_2^*(\theta))^2$. Again, when $\mu_1 = \mu_2 = \mu$ these parameters do not depend on μ (or on the scale parameter e^{μ}). For $\sigma_1 = \sigma_2 = \sigma$ we obtain that

(3.4)
$$M_3^*(\theta) = 2e^{\frac{3}{2}\sigma^2}[e^{3\sigma^2} + 3e^{\sigma^2}(e^{2\sigma^2\rho} - 2) - 6e^{\sigma^2\rho} + 8]$$

and

(3.5)
$$M_{4}^{*}(\underline{\theta}) = 2e^{2\sigma^{2}}[e^{2\sigma^{2}}(e^{4\sigma^{2}} + 4e^{\sigma^{2}+3\sigma^{2}\rho} + 3e^{4\sigma^{2}\rho}) - 8e^{\sigma^{2}}(e^{2\sigma^{2}} + 3e^{2\sigma^{2}\rho}) + 24e^{\sigma^{2}}(e^{\sigma^{2}} + e^{\sigma^{2}\rho}) - 24].$$

The coefficients of skewness and kurtosis, $\gamma_1(\underline{0})$ and $\gamma_2(\underline{0})$ are plotted in Figure 2 as functions of ρ for cases of $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2 = 1$, 1.5, 2.0. These plots show that the distribution of W becomes extremely skewed and flat when σ becomes large.

COEFFICIENT OF CORRELATION, O

In Figure 1 we illustrate the standard deviation of U, as a function of P,

Figure 1. The Standard Deviation of W for $\mu = 0$. For $\sigma = 1, 1.5, 2.0$. We see in Figure 1 that the standard deviation of W changes very slowly when 2 < 0 and increases faster over the range of positive P values. Also, the relative rate of increase grows fair with a. In other words, for a = 2 a.d., (b) 1 2= 0 1 1 100 T , sancerer $\frac{1}{2} \left(\frac{1}{2} \right)^{2} = \left(\frac{1}{2} \right)^{2} \left(\frac{1}{2} \right)^{2}$ detrash. 5 When D, * Do = D these parameters do not dopend on D (or of the scale parameter STANDARD DEVIATION W (P e^{μ}). For $\sigma_1 = \sigma_2 = \sigma$ we obtain that (**(a) = 2a² o² (-3a² + 3e² (e²a² a - 2) - 6e⁶²a J=1.5 10 N. (E) = 20²⁰ 10² 10⁴ + 40² + 40² + 30² + 30⁴ - 30² N 19 5 · J=1 int bedroit ore The coefficients of skewness and kurtcais, 2.0 Figure 2 as functions of a for cases of $u_1 = u_2$ and $u_1 = c_1 = 1$, 1.5, 2.0. These plots show that the distribution of W becomes extremely showed and flat >0 -1.0 -.75 -.50 -.25 0 .25 .50 .75 1.0 COEFFICIENT OF CORRELATION, P

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4. Approximating the Distribution of W by a Lognormal Distribution.

Let $LN(n, \tau^2)$ denote a lognormal distribution corresponding to the normal distribution $N(n, \tau^2)$. We consider a lognormal approximation to the distribution of W, with parameters n and σ^2 determined so that the first two moments of $LN(n, \tau^2)$ and of W coincide. In Figure 3 the distribution of W (in the standard case) is plotted on a normal probability paper versus log W, for $\rho = -.2(.2).8$. We see that in the standard case the distribution $F_{\underline{\theta}}(w)$ for nonnegative ρ values is very close to a lognormal distribution. The lognormal approximation is very good for $\rho = -.20$.

We consider now the lognormal approximation to $F_{\underline{\theta}}(w)$. By the methods of moment equations we determine η and τ^2 by equating the first two moments of W to those of LN(η , τ^2). The equations to be solved are:

$$exp\{n + \tau^2/2\} = exp\{\mu_1 + \frac{\sigma_1^2}{2}\} + exp\{\mu_2 + \frac{\sigma_2^2}{2}\},$$

(4.1) and

 $\exp \{2n + 2\tau^2\} = \exp\{2\mu_1 + 2\sigma_1^2\} + \exp\{2\mu_2 + 2\sigma_2^2\} + 2\exp\{\mu_1 + \mu_2 + \frac{1}{2}(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)\}.$

These equations yield the solutions:

(4.2)
$$\tau^{2} = \log \frac{e^{2\mu_{1}+2\sigma_{1}^{2}} + e^{2\mu_{2}+2\sigma_{2}^{2}} + 2e^{\mu_{1}+\mu_{2}+\frac{1}{2}(\sigma_{1}^{2}+2\rho\sigma_{1}\sigma_{2}+\sigma_{2}^{2})}}{e^{2\mu_{1}+\sigma_{1}^{2}} + e^{2\mu_{2}+\sigma_{2}^{2}} + 2e^{\mu_{1}+\mu_{2}+\frac{1}{2}(\sigma_{1}^{2}+\sigma_{2}^{2})}}$$

and

(4.3)
$$\eta = \log(e^{\mu_1 + \sigma_1^2/2} + e^{\mu_2 + \sigma_2^2/2}) - \tau^2/2.$$

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The approximation to the distribution of W is then

(4.4)
$$F_{\underline{\theta}}(w) = \phi(\frac{\log w - \eta}{\tau}).$$

In Table 7 we compare the exact distribution of W in the standard case with the lognormal approximation for $\rho = -.75(.25).75$. Table 7 confirms the earlier conclusion from Figure 3, that the lognormal approximation (4.4) is very good for nonnegative values of p. In Table 8 we provide a comparison between the exact distribution and the lognormal approximation in the case of $\mu_1 = \mu_2 = 1$ and $\sigma_1 = \sigma_2 = 1$. Here also the approximation is also good ρ nonnegative values of p. The extent to which the lognormal deviates from the exact in the case of $\rho = -.75$ is shown in Figure 4. Due to the asymmetry of the distribution, the lognormal distribution provides a better approximation at the right hand tail of the distribution than at the left hand tail. Between the 10th and 90th percentiles the lognormal distribution is good even in the case of p = -.75. siglans + 12 th + .uslass = 1 rs + rsi qua

In order to improve the approximation, especially for negative values of p, we consider the Edgeworth expansion (see Johnson and Kotz [3; pp. 17]). A These equations y taid the solutional 2-term approximation formula is

n = 10258 + 12 + 14 - 2/2 - 2/2 ·

(4.5)
$$F_{\underline{\theta}}(w) = \phi(\frac{\log w - \eta}{\tau}) - \frac{\gamma_1^*(\underline{\theta})}{6} \left[\left(\frac{\log w - \eta}{\tau}\right)^2 - 1 \right]$$
$$\phi(\frac{\log w - \eta}{\tau});$$

normal distribution M(n. "); the co

(4.6) $\mathbf{F}_{\underline{\theta}}(\mathbf{w}) = \mathbf{G}_{\underline{\theta}}^{(2)}(\mathbf{w}) - \frac{1}{24}(\gamma_{2}^{*}(\underline{\theta}) - 3) \cdot \left[\left(\frac{\log w - n}{\tau} \right)^{3} - 3 \cdot \left(\frac{\log w - n}{\tau} \right) \right] \\ + \left(\frac{\log w - n}{\tau} \right) - \frac{1}{72} \gamma_{1}^{*}(\underline{\theta}) \left[\left(\frac{\log w - n}{\tau} \right)^{5} - 10 \left(\frac{\log w - n}{\tau} \right)^{3} + 15 \left(\frac{\log w - n}{\tau} \right) \right] + \left(\frac{\log w - n}{\tau} \right),$

where $\gamma_1^{*}(\underline{\theta})$ and $\gamma_2^{*}(\underline{\theta})$ are the coefficients of skewness and kurtosis of $Z = \log W$, and $\phi(u)$ is the standard normal p.d.f. $G_{\underline{\theta}}^{(2)}(w)$ is the R.H.S. of (4.5). In order to apply these approximations we have to discuss the problem of computing the moments of $Z = \log W$, which are required for $\gamma_1^{*}(\underline{\theta})$ and $\gamma_2^{*}(\underline{\theta})$. This problem is discussed in Section 5.

In Table 9 we provide the results of a 2-term Edgeworth expansion, for the case presented in Table 8.

A 4-term approximation is given in Table 10. The comparison of these Tables with Table 9 showing sometimes certain improvements but not substantial ones.

Other types of approximations that we attempted did not yield better results.

	(lower);	μ ₁ = μ ₂ =	ο, σ ₁ = σ	2 - 1	ocijertvo	iggs alot-	ê si brin
W/p	75 C	50	(T.25)*	0_ (*	.25	.50	.75
1.000	0.00946	0.04229	0.07849	0.11346	0.14692	0.17938	0.21151
2.000	0.26969	0.32330	0.36195	0.39414	0.42283	0.44948	0.47499
3.000	0.60336	0.59423	0.59875	0.60781	0.61889	0.63107	0.64397
4.000	0.78277	0.75881	0.74750	0.74334	0.74333	0.74594	0.75029
5.000	0.86991	0.85105	0.83682	0.82774	0.82248	0.81994	0.81936
6.000	0.91602	0.90363	0.89113	0.88122	0.87402	0.86906	0.86585
7.000	0.94279	0.93491	0.92509	0.91600	0.90850	0.90262	0.89814
8.000	0.95943	0.95437	0.94698	0.93922	0.93218	0.92617	0.92119
9.000	0.97031	0.96701	0.96153	0.95512	0.94881	0.94307	0.93803
10.000	0.97771	0.97551	0.97147	0.96625	0.96074	0.95545	0.95058
1.000	0.0801	0.0939	0.1108	0.1311	0.1548	0.1819	0.2118
2.000	0.3483	0.3651	0.3840	0.4047	0.4271	0.4507	0.4751
3.000	0.5806	0.5886	0.5977	0.6078	0.6190	0.6311	0.6440
4.000	0.7338	0.7348	0.7364	0.7386	0.7416	0.7455	0.7503
5.000	0.8292	0.8265	0.8240	0.8218	0.8202	0.8194	0.8193
6.000	0.8883	0.8842	0.8799	0.8757	0.8718	0.8685	0.8658
7.000	0.9255	0.9211	0.9163	0.9114	0.9066	0.9021	0.8981
8.000	0.9493	0.9452	0.9406	0.9356	0.9306	0.9257	0.9212
9.000	0.9649	0.9612	0.9570	0.9524	0.9476	0.9427	0.9380
10.000	0.9753	0.9721	0.9684	0.9643	0.9598	0.9552	0.9506

Tables with Table 9 showing amorthes cartain improvements but not substantial

Other types of approximations that we attempted did not plaid better besuics.

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Table 7. The Exact Distribution of W (upper) and the Lognormal Approximation

Table 8. The Exact Distribution of W (upper) and Its Lognormal Approximation

		(lower) for μ_1	= µ ₂ = 1,	°1 - °2	- 1.			
W/p	-1.00	. .75	50	25	0	.25	.50	.75	1.00
1	0.0000	0.0000	0.0001	0.0014	0.0049	0.0110	0.0197	0.0310	0.0452
2	0.0000	0.0008	0.0106	0.0292	0.0518	0.0765	0.1025	0.1299	0.1587
3	0.4602	0.0183	0.0616	0.1034	0.1415	0.1768	0.2105	0.2435	0.2761
4	0.6141	0.0873	0.1560	0.2061	0.2473	0.2835	0.3168	0.3486	0.3795
5	0.7094	0.2080	0.2715	0.3156	0.3514	0.3829	0.4119	0.4397	0.4666
6	0.7743	0.3481	0.3876	0.4188	0.4458	0.4705	0.4939	0.5168	0.5393
7	0.8208	0.4795	0.4927	0.5102	0.5280	0.5458	0.5635	0.5816	0.5998
8	0.8551	0.5893	0.5826	0.5883	0.5981	0.6096	0.6223	0.6360	0.6504
9	0.8812	0.6758	0.6570	0.6540	0.6573	0.6636	0.6719	0.6818	0.6929
10	0.9014	0.7421	0.7176	0.7087	0.7070	0.7091	0.7138	0.7206	0.7289
11	0.9173	0.7924	0.7666	0.7540	0.7487	0.7476	0.7494	0.7536	0.7595
12	0.9300	0.8307	0.8060	0.7914	0.7837	0.7802	0.7797	0.7817	0.7857
13	0.9403	0.8600	0.8378	0.8225	0.8132	0.8079	0.8056	0.8059	0.8083
14	0.9487	0.8828	0.8635	0.8483	0.8380	0.8315	0.8279	0.8268	0.8279
15	0.9557	0.9008	0.8843	0.8698	0.8591	0.8517	0.8470	0.8449	0.8449
16	0.9614	0.9152	0.9014	0.8877	0.8770	0.8691	0.8636	0.8606	0.8598
17	0.9663	0.9268	0.9154	0.9028	0.8922	0.8840	0.8781	0.8744	0.8729
18	0.9704	0.9364	0.9269	0.9155	0.9053	0.8970	0.8906	0.8864	0.8844
19	0.9739	0.9443	0.9366	0.9262	0.9164	0.9082	0.9017	0.8971	0.8946
20	0.9768	0.9510	0.9446	0.9353	0.9261	0.9180	0.9113	0.9065	0.9036
21	0.9794	0.9566	0.9514	0.9431	0.9344	0.9265	0.9199	0.9148	0.9117
22	0.9816	0.9615	0.9572	0.9498	0.9417	0.9340	0.9274	0.9222	0.9189
23	0.9835	0.9656	0.9621	0.9555	0.9480	0.9406	0.9341	0.9288	0.9254
24	0.9852	0.9692	0.9663	0.9605	0.9534	0.9464	0.9400	0.9348	0.9312
25	0.9867	0.9723	0.9699	0.9648	0.9583	0.9516	0.9453	0.9401	0.9365
1	0.0013	0.0021	0.0033	0.0052	0.0084	0.0133	0.0207	0.0311	0.0452
2	0.0257	0.0319	0.0403	0.0512	0.0654	0.0831	0.1046	0.1299	0.1587
3	0.0913	0.1039	0.1192	0.1378	0.1596	0.1847	0.2128	0.2435	0.2761
4	0.1852	0.2008	0.2191	0.2402	0.2641	0.2904	0.3188	0.3487	0.3795
5	0.2888	0.3042	0.3219	0.3420	0.3641	0.3881	0.4135	0.4398	0.4666
6	0.3897	0.4030	0.4182	0.4352	0.4538	0.4739	0.4951	0.5170	0.5393
7	0.4816	0.4920	0.5039	0.5172	0.5318	0.5476	0.5644	0.5819	0.5998
8	0.5622	0.5696	0.5781	0.5877	0.5985	0.6102	0.6229	0.6364	0.6504
9	0.6314	0.6360	0.6414	0.6477	0.6550	0.6632	0.6724	0.6823	0.6929
10	0.6899	0.6921	0.6950	0.6985	0.7029	0.7081	0.7142	0.7212	0.7289
11	0.7390	0.7393	0.7401	0.7414	0.7434	0.7461	0.7497	0.7542	0.7595
12	0.7800	0.7789	0.7781	0.7776	0.7776	0.7784	0.7799	0.7824	0.7857
13	0.8143	0.8121	0.8100	0.8082	0.8067	0.8059	0.8058	0.8066	0.8083
14	0.8429	0.8399	0.8369	0.8340	0.8315	0.8294	0.8280	0.8275	0.8279
15	0.8667	0.8633	0.8596	0.8560	0.8526	0.8496	0.8472	0.8456	0.8449
16	0.8867	0.8829	0.8788	0.8747	0.8707	0.8670	0.8638	0.8614	0.8598
17	0.9034	0.8994	0.8951	0.8907	0.8862	0.8820	0.8782	0.8751	0.8729
18	0.9174	0.9134	0.9090	0.9043	0.8996	0.8950	0.8908	0.8872	0.8844
19	0.9292	0.9252	0.9208	0.9160	0.9111	0.9063	0.9018	0.8979	0.8946
20	0.9392	0.9352	0.9309	0.9261	0.9212	0.9162	0.9115	0.9072	0.9036
21	0.9476	0.9438	0.9395	0.9348	0.9299	0.9249	0.9200	0.9156	0.9117
22	0.9547	0.9511	0.9469	0.9424	0.9375	0.9325	0.9276	0.9230	0.9189
23	0.9608	0.9573	0.9534	0.9489	0.9441	0.9392	0.9343	0.9296	0.9254
24	0.9660	0.9627	0.9589	0.9546	0.9500	0.9451	0.9402	0.9355	0.9312
25	0.9704	0.9673	0.9637	0.9596	0.9551	0.9503	0.9455	0.9408	0.9365

Table 9. A 2-term Edgeworth Expansion Approximation $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$.

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5,00,0	0.0397	0.0652	0.0561	0.0669	0.0135	0.10/8	0.1300		
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13795	0.2927	0.3141	0.3369	0.3611	0.3865	0.4128	0.4396	14.60.0	4
3454.1	0.3867	0 4078	0 4289	0 4503	0 4722	0 4944	0.5168	0.7090	
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8962.1	0.5526	0.5676	0.5816	0.5952	0.6087	0.6224	0.6362		T.
9.0.1	0.6212	0.6324	0.6425	0.6522	0.6619	0.6719	0.6821	TREASER-	8
10	0.6803	0.6878	0.6943	0.7006	0.7071	0.7138	0.7210		2
11	0.7308	0.7348	0.7383	0.7417	0.7453	0.7494	0.7541		11
12	0.7735	0.7747	0.7755	0.7765	0.7779	0.7797	0.7823	0.3373	11
13	0.8095	0.8083	0.8071	0.8061	0.8056	0.8057	0.8066	0019100	12
14	0.8398	0.8367	0.8338	0.8313	0.8293	0.8280	0.8275	0.9403	- 21
15	0.8652	0.8607	0.8565	0.8528	0.8497	0.8472	0.8456	0,9482	14
16	0.8865	0.8809	0.8758	0.8712	0.8672	0.8639	0.8614	- 12250.0	24
17	0.9044	0.3980	0.8922	0.8870	0.8823	0.8783	0.8752	0.0514	Ió
18	0.9194	0.9125	0.9063	0.9006	0.8954	0.8910	0.8873		11
19	0.9320	0.9248	0.9183	0.9123	0.9069	0.9020	0.8979		1 1 1
20	0.9426	0.9353	0.9286	0.9225	0.9168	0.9117	0.9073	1. S. A. M. C. C.	21
21	0.9515	0.9442	0.9375	0.9313	0.9255	0.9203	0.9156	-2012-0	20
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Ta	ble 10.	$\sigma_1 = \sigma_2 =$	geworth E	xpansion	Approxima	tion. Fo	r µ ₁ = µ ₂
W/p	75	50	25	0	.25	. 50	.75
1	0026	0025	0010	0.0027	0.0090	0.0184	0.0307
2	0.0347	0.0443	0.0572	0.0730	0.0914	0.1117	0.1332
3	0.1313	0.1457	0.1625	0.1814	0.2020	0.2242	0.2478
4	0.2366	0.2516	0.2681	0.2864	0.3064	0.3282	0.3517
5	0.3290	0.3443	0.3605	0.3780	0.3972	0.4182	0.4410
6	0.4075	0.4234	0.4395	0.4565	0.4748	0.4947	0.5164
7	0.4760	0.4920	0.5077	0.5239	0.5412	0.5597	0.5798
8	0.5373	0.5527	0.5674	0.5824	0.5982	0.6150	0.6332
9	0.5933	0.6070	0.6201	0.6334	0.6474	0.6623	0.6784
10	0.6444	0.6558	0.6668	0.6781	0.6900	0.7029	0.7169
11	0.6910	0.6996	0.7082	0.7173	0.7271	0.7379	0.7497
12	0.7331	0.7388	0.7450	0.7518	0.7594	0.7681	0.7780
13	0.7709	0.7738	0.7775	0.7821	0.7877	0.7944	0.8023
14	0.8045	0.8048	0.8062	0.8087	0.8124	0.8172	0.8235
15	0.8341	0.8321	0.8315	0.8322	0.8340	0.8372	0.8418
16	0.8601	0.8561	0.8538	0.8528	0.8531	0.8548	0.8579
17	0.8827	0.8772	0.8733	0.8709	0.8698	0.8702	0.8720
18	0.9023	0.8955	0.8905	0.8868	0.8845	0.8837	0.8844
19	0.9192	0.9115	0.9055	0.9008	0.8976	0.8957	0.8954
20	0.9338	0.9254	0.9186	0.9132	0.9090	0.9063	0.9051
21	0.9462	0.9374	0.9301	0.9240	0.9192	0.9157	0.9138
22	0.9568	0.9477	0.9401	0.9335	0.9282	0.9241	0.9215
23	0.9658	0.9567	0.9488	0.9419	0.9361	0.9316	0.9284
24	0.9734	0.9644	0.9564	0.9493	0.9432	0.9382	0.9345
25	0.9798	0.9710	0.9631	0.9559	0.9495	0.9442	0.9401

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Figure 4. Normal Probability Plot vs Log W or the Exact and the Log-Normal Distributions, for $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$ and $\rho = .75$.

-24-

5. The Moments of Z = log W in the Correlated Case.

Hamdan [2] developed formulae for the expectation and variance of $Z = \log W$ in the case of correlated random variables, with possibly different variances. His formula for $E \{Z\}$ in the standard bivariate case $(\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 1)$ can be written in the form

(5.1)
$$\mu_1^{\rho}(Z) = \sqrt{\frac{1-\rho}{\pi}} + 2 \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j} e^{j^2(1-\rho)} \phi(-j \sqrt{2(1-\rho)}).$$

It is easy to prove that this series is absolutely convergent, since for large values of $j \Phi(-j \sqrt{2(1-\rho)}) \approx \frac{1}{j \sqrt{2\pi}} e^{-j^2(1-\rho)}$ (see Feller [1; pp. 166]).

One should be careful in the computation of $\mu_1^{\rho}(Z)$ according to (5.1) since $e^{j^2(1-\rho)}$ grows very fast with j and $\Phi(-j\sqrt{2(1-\rho)})$ decreases very fast. We have found that the polynomial approximation for $\Phi(z)$ given by Zelen and Severo [7] to be very effective. This approximation is given by

(5.2)
$$\phi(z) = 1 - \phi(z) \sum_{j=1}^{5} b_{j} (1 + pz)^{-j}, \quad z > 0$$

where p = .2316419; $b_1 = .319382$; $b_2 = -.356564$; $b_3 = 1.781478$; $b_4 = -1.821256$; and $b_5 = 1.330274$. By substituting (5.2) in (5.1) and since $\Phi(-z) = 1 - \Phi(z)$, we obtain the formula

(5.3) $\mu_1^{\rho}(Z) = \sqrt{\frac{1-\rho}{\pi}} + \sqrt{\frac{2}{\pi}} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j} \sum_{j=1}^{5} \frac{b_i}{(1+p(j\sqrt{2(1-\rho)})^i)}.$

The convergence is of $0(\frac{1}{2})$. Our experience has shown that between 10 and 20 terms are sufficient in most cases to obtain stable results. The error in (5.2) is smaller in magnitude than 1.5×10^{-8} for all z. We therefore consider the values obtained from (5.3) as close to the exact ones. This approximation is better than the one given by Hamdan in [2].

We could not obtain similar formulae for the higher moments of Z. Although Hamdan provides in [2] a formula for $E_{\rho}\{Z^2\}$ we have not been able to apply it (the series expression given by Hamdan does not converge absolutely!). We therefore provide the following numerical approximation formula for the determination of the moments of z:

(5.4)
$$\mu_{\mathbf{r}}^{\rho}(z) \approx \sum_{i=1}^{m} \sum_{j=1}^{m} [\log(e^{n'i} + e^{n'j})]^{\mathbf{r}} \cdot [\Phi(n_{i}, n_{j}; \rho) - \Phi(n_{i-1}, n_{j}; \rho) - \Phi(n_{i}, n_{j-1}, \rho) + \Phi(n_{i-1}, n_{j-1}; \rho)],$$

where $\Phi(z_1, z_2; \rho)$ is the standard bivariate normal integral; m is the number of subintervals for each variable. We compute the moments over a grid of m x m squares. The range in each dimension is from -4.5 to +4.5 and the length of each subinterval is $\Delta = g/m$.

In Table 11 we compare the values of the first moment of Z obtained by (5.3) and by (5.4) with m = 7.

In Table 12 we present the values of the first four moments of Z computed according to (5.4) with m = 10, and also the values of the standard deviation, $\sqrt{\gamma_1}$ and γ_2 of Z.

-26-

Table :	11. The	Expectation	of 2.	1948年1月1日日月月	一般之影温养。10	achine The	alder
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Net of 2 - log W.	(5.3)	(.54)	alors of r_{11} and r_{2} for $e =75$ i
soliditistb VI	1.0295	1.0254	hes) is close to -1, can be appare
ot time.50	.9893	.9860	see Johnson and Fors [31 pp 121]
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Table 12. Moments, Standard Deviation and the Coefficients of Skewness and Kurtosis of ^Z in the Standard Case.

ρ	75	50	25	0	.25	.50	.75
μ	1.0396	0.9998	0.9583	0.9148	0.8690	0.8203	0.7679
μ2	1.4047	1.4165	1.4297	1.4447	1.4620	1.4824	1.5074
μ3	2.2543	2.3395	2.4126	2.4739	2.5234	2.5601	2.5818
μ4	4.1885	4.4522	4.7300	5.0318	5.3648	5.7353	6.1522
ρ	75	50	25	0	.25	.50	.75
S.D.	0.5692	0.6457	0.7151	0.7796	0.8407	0.8997	0.9580
$\sqrt{Y_1}$	0.6533	0.3330	0.1708	0.0848	0.0411	0.0220	0.0169
Y2	3.9928	3.4180	3.1729	3.0648	3.0161	2.9936	2.9826

Table 12 shows again the observation previously discussed that the distribution of Z, for $\rho \ge 0$, is approximately normal. Considering the values of $\sqrt{\gamma_1}$ and γ_2 for $\rho = -.75$ it seems that the distribution of Z = log W, when ρ is close to -1, can be approximated by the Pearson type IV distribution (see Johnson and Kotz [3; pp. 12]). However, it is quite difficult to compute the c.d.f. of the Pearson type IV, while the c.d.f. of Z can be computed numerically very well according to (2.5) or (2.10).

Kortoels of Z in the Standard Case.

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6. References

- Feller, W. (1957) <u>An Introduction to Probability Theory and Its Applications</u>, Vol. 1. <u>New York</u>: John Wiley & Sons.
- [2] Hamdan, M.A. (1971)
 "The logarithm of the sum of two correlated log-normal variates", Journal of the American Statistical Association, 66: 105-106.
- Johnson, N.L. and Kotz, S. (1970)
 <u>Distributions In Statistics: Continuous Univariate Distributions 1,</u> Boston: Houghton Mifflin Co.
- [4] Naus, J.I. (1969)
 "The distribution of the logarithm of the sum of two log-normal variates", Journal of the American Statistical Association, 64: 655-659.
- [5] Tsokos, C.P. and Lowrimore, G.R. The Probability Distribution of the Logarithm of the Sum of Two Log-Normal Variates. In preparation.
- [6] Tsokos, C.P. The Probability Distribution of the Logarithm of the Sum of Two Log-Normally Distributed Random Variables. In preparation.
- [7] Zelen, M. and Severo, N.C. (1968)
 Probability Functions, Chapter 27 of: <u>Handbook of Mathematical Functions</u> with Formulas, Graphs, and Mathematical Tables; M. Abramowitz and I.A.
 Segun (eds.); New York: Dover Publications, Inc.
- [8] Abramowitz, M. and Segun, I.A. (1968) <u>Handbook of Mathematical Functions with Formulas, Graphs and Mathematical</u> Tables, New York: Dover Publications, Inc.

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methods are compared by numerical results in standard and non-standard cases. The moments of the distribution of the sum are given explicitly and also the coefficients of skewness and kurtosis. It is shown that for for positive correlations the distribution of the sum is approximately log-normal. For negative values of the correlation the log-normal becomes ineffective. Another approximation is given for these cases, based on the first few terms of an Edgeworth expansion. Finally, methods for computing the moments of the logarithm of the sum are developed.

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C. P. ISONOS, UNIVERSITY OF SOUTH TEOFICIA

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