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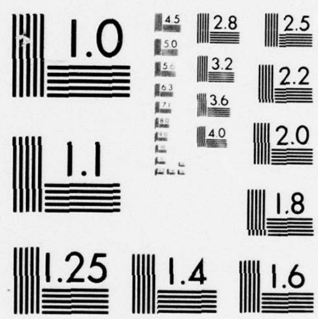

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BY

CAULTON L. IRWIN

TECHNICAL REPORT SOL 77-33

DECEMBER 1977

Systems **O**ptimization **L**aboratory

Department of Operations Research

Stanford University

Stanford California 94305

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1. INTRODUCTION

The aim of this paper is to study the technique used in the FEA-PIES energy model for approximating non-integrable vector functions. An algorithm which incorporates this technique to obtain market equilibrium in the presence of non-integrable functions is described in section 2. The main convergence, existence and uniqueness theorem is stated and a geometric interpretation of the algorithm is given for $n = 2$. Section 3 indicates some effects of a linear coordinate transformation on the algorithm and contains an illustrative example. The proofs are in section 4.

The complete report on the FEA-PIES energy model is contained in [5]. A description of the quantitative analysis as well as an interesting example problem is contained in Hogan's paper [3]. The mathematical structure, algorithms and computational experience are presented in [4]. The algorithm considered below can be viewed as a sub-algorithm of the PIES algorithm. Hopefully, the methods of proof and observations will aid in understanding the convergence of the PIES algorithm and will indicate its connection with quasi-Newton methods as surveyed in [1].

Suppose P_S is a supply price function, P_D is a demand price function and $e = P_S - P_D$. If e is integrable, i.e., $e = \nabla E$ for some function E from R^n into R^1, then, provided E is convex, calculating q^* such that $e(q^*) = (0, \ldots, 0)$ is equivalent to solving the optimization problem

$$\min_q E(q).$$

In this case, $-E$ represents a net social surplus function and so this welfare measure of the economy is maximized at q^*.

If e is not integrable, which in general is the case for $n > 1$, the problem of obtaining q^* with an optimization process can be approached by approximating e with an integrable function $\hat{e}$. A vector $\hat{q}$ such that $\hat{e}(\hat{q}) = (0, . . ., 0)$ is then obtained as a solution of

$$(1) \qquad \min_q \hat{E}(q)$$

and $\hat{q}$ may be taken as an approximation to q^*.

For the functions in PIES it is actually P_D that fails to be integrable and so the approximation technique is applied only to P_D. For the algorithm described in section 2, the same approximation technique is used, but it is applied to $P_S - P_D$. In either case, an integrable $\hat{e}$ results.

2. THE ALGORITHM AND CONVERGENCE THEOREM

Let $J_e(q)$ denote the Jacobian matrix of e at q, i.e., $J_e(q)$ is an $n \times n$ matrix whose i^{th} row is $(\nabla e_i)(q)$. The PIES technique for approximating a non-integrable vector function e with respect to to the point q^t is simply to define $\hat{e}^t$ as the vector function such that

$$J_{\hat{e}^t}(q) = \text{diag } J_e(q)$$

and

$$\hat{e}^t(q^t) = e(q^t).$$

This provides that $\hat{e}^t$ has coordinate functions given by

$$(2) \quad \hat{e}_i^t (q) = \hat{e}_i^t (q_1, . . . , q_n)$$

$$= e_i (q_1^t, . . . , q_{i-1}^t, q_i, q_{i+1}^t, . . . , q_n^t)$$

for $i = 1, . . . , n$.

$\hat{e}^t$ is integrable since it has a diagonal Jacobian matrix; in fact,

$$\hat{e}^t = \nabla \hat{E}^t$$

where

$$(3)\quad \hat{E}^t(q) = \oint_{[q^t;q]} \hat{e}^t$$

$$= \int_{q_1^t}^{q_1} e_1(\bar{q}_1, q_2^t, \ldots, q_n^t)\, d\bar{q}_1$$

$$+ \int_{q_2^t}^{q_2} e_2(q_1, \bar{q}_2, q_3^t, \ldots, q_n^t)\, d\bar{q}_2$$

$$+ \ldots$$

$$+ \int_{q_n^t}^{q_n} e_n(q_1, \ldots, q_{n-1}, \bar{q}_n)\, d\bar{q}_n.$$

The point $\hat{q}^t$ can now be obtained via (1). By letting $q^{t+1} = \hat{q}^t$ a sequence $q^1, q^2, \ldots$ is generated which may converge to a point q^* such that $e(q^*) = (0, \ldots, 0)$. Refer to this algorithm as SUB-PIES. A diagram of SUB-PIES and for comparison purposes a diagram of PIES is shown in figure 1.

Assume for the discussion that for each $i = 1, \ldots, n$, there is an interval $I_i = [a_i, b_i]$ so that if $R = I_1 \times \ldots \times I_n$, then

(4) $e_i(q) < 0$ for $q_i = a_i$ and $q_j \epsilon I_j$ all $j \neq i$;

(5) $e_i(q) > 0$ for $q_i = b_i$ and $q_j \epsilon I_j$ all $j \neq i$;

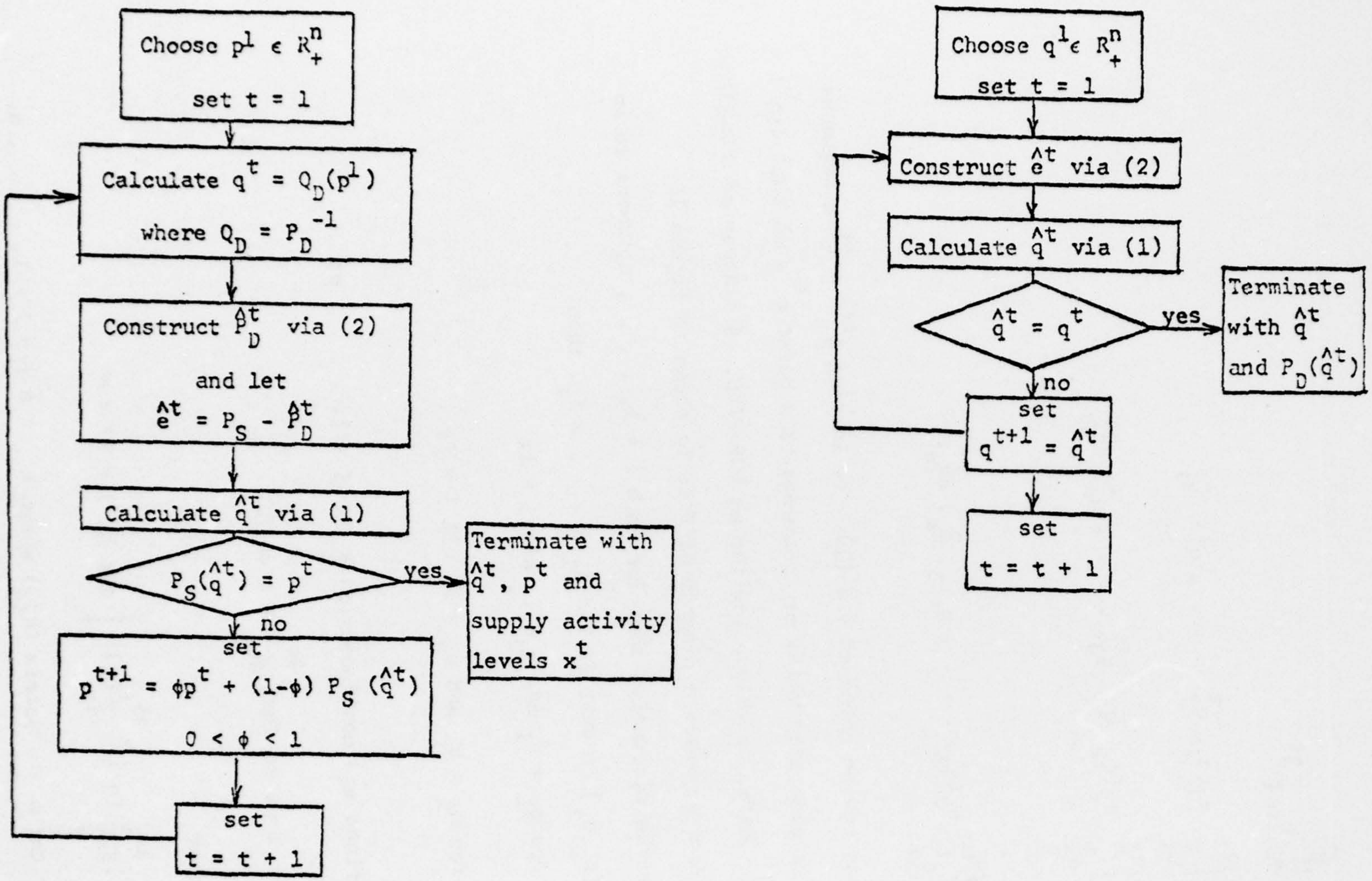
(6) $\dfrac{\partial e_i}{\partial q_j}$ is defined and continuous on R for $j = 1, \ldots, n$;

(7) there is an $\epsilon > 0$ so that $\dfrac{\partial e_i}{\partial q_i} > \epsilon$ on R.

Assume further that

$$(8)\quad k_{ij} = \sup \left\{ \left|\frac{\partial e_i}{\partial q_j}(q)\right| \Big/ \left|\frac{\partial e_i}{\partial q_i}(q)\right| \;\Big|\; q \in R,\ j \neq i \right\} < \infty$$

and let k denote the $n \times n$ matrix (k_{ij}) where $k_{ii} = 0$ for $i = 1, \ldots, n$.



PIES

SUB-PIES

Figure 1

Let $\rho(k) = \sup \{|\lambda| \mid \lambda$ is an eigenvalue of $k\}$, i.e., $\rho(k)$ is the spectral radius of k.

It should be noted that in PIES, P_S is integrable, since it is obtained from a linear programming process analysis, but it is not everywhere differentiable. On the other hand, P_D is differentiable but not necessarily integrable. The result is that $e = P_S - P_D$ for PIES functions is not necessarily differentiable and not necessarily integrable. In order to concetrate on the non-integrability aspect, condition (6) will be assumed for the functions in SUB-PIES.

Our main theorem concerning SUB-PIES is :

Theorem 1. Suppose (4), (5), (6) and (7) hold. If $\rho(k) < 1$, then the sequence $q^1, q^2, \ldots$ generated by SUB - PIES converges to a point q^* such that $e(q^*) = (0, \ldots, 0)$. Furthermore q^* is unique.

Proof. See part 4.

Two situations in which $\rho(k) < 1$ can be seen by applying the Gershgorin circles Theorem, see [7]. One case is when $k_{ij} < \frac{1}{n-1}$ for $j \neq i$. This condition expresses the economic reality that the quantity demanded and produced of a good is more strongly related to its own price than to the cross prices.

Also, it has been pointed out, [2], that if we let $k_i = \max \{k_{ij} \mid j \neq i\}$ and $\bar{k}$ denote the n x n matrix $(\bar{k}_{ij})$ where $\bar{k}_{ij} = \begin{cases} k_i & \text{if } j \neq i \\ 0 & \text{if } j = i, \end{cases}$

then $\bar{k}$ is similar to the symmetric matrix $\tilde{k}$ where

$$\tilde{k}_{ij} = \sqrt{k_i\, k_j}.$$

Applying Gershgorin's Theorem again, this time on $\tilde{k}$, we find that

$$\rho(k) \leq \rho(\bar{k}) = \rho(\tilde{k}) < 1$$

provided

(9) $\sqrt{k_i}\,(\sqrt{k_1} + \ldots + \sqrt{k_{i-1}} + \sqrt{k_{i+1}} + \ldots + \sqrt{k_n}) < 1$ for $i = 1, \ldots, n$.

It is interesting to note that any one of the k_i 's can be arbitrarily large provided the other (n-1) of the k_i 's are "small enough".

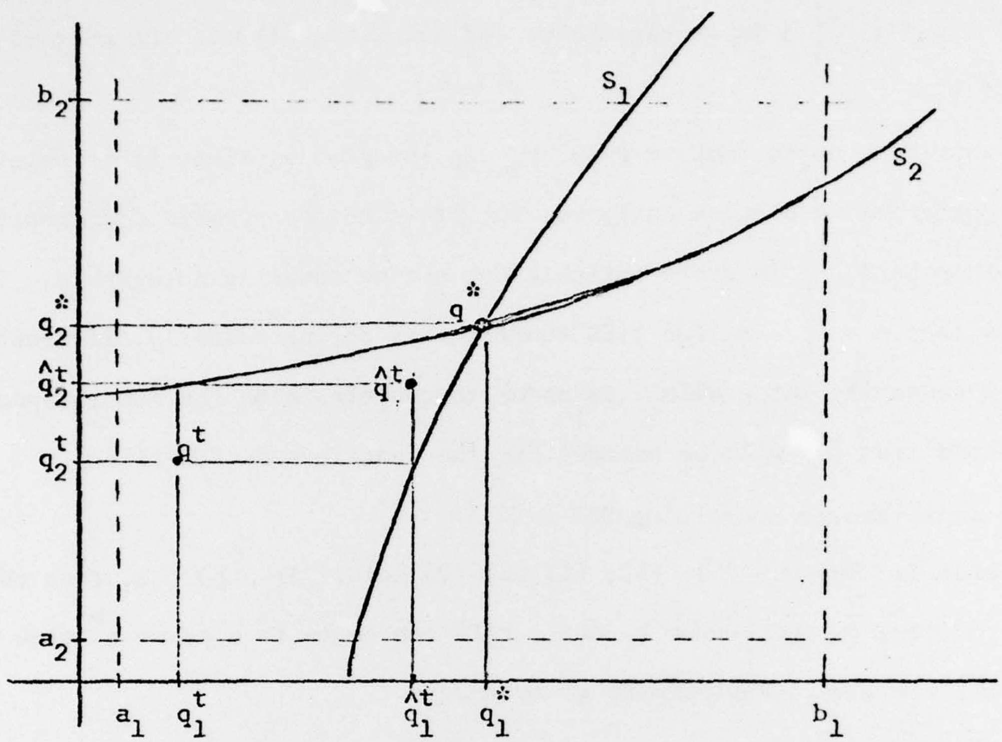


Figure 2

The geometric interpretation of the induction step in SUB-PIES can be seen clearly for $n = 2$. Let S_i denote the level set of e_i for function value 0, i.e.,

$$S_i = \{q \in R \mid e_i(q) = 0\} \qquad \text{for } i = 1,2$$

Obviously $q^* \in S_1 \cap S_2$. At q^t, as shown in figure 2, the integrable approximation $\hat{e}^t$ is defined by (2) and when the corresponding $\hat{E}^t$ is minimized a point $\hat{q}^t$ is obtained so that

$$(\nabla\hat{E}^t)(\hat{q}^t) = \hat{e}^t(\hat{q}^t) = (0,0).$$

We have

$$\begin{aligned} \hat{e}_1^t(\hat{q}^t) &= \hat{e}_1^t(\hat{q}_1^t, \hat{q}_2^t) \\ &= e_1(\hat{q}_1^t, q_2^t) \\ &= 0 \end{aligned}$$

$$\text{so } (\hat{q}_1^t, q_2^t) \in S_1 \text{ ; also,}$$

$$\hat{e}_2^t\,(\hat{q}^t) = \hat{e}_2^t\,(\hat{q}_1^t, \hat{q}_2^t)$$

$$= e_2\,(q_1^t, \hat{q}_2^t)$$

$$= 0$$

and so $(q_1^t, \hat{q}_2^t) \in S_2$.

The geometric interpretation is that $\hat{q}_1^t$ as determined by the optimization scheme in (1) is the first coordinate of the intersection of the line $q_2 = q_2^t$ with the level set S_1. Similarily, $\hat{q}_2^t$ is the second coordinate of the intersection of the line $q_1 = q_1^t$ with the level set S_2. Let $q^{t+1} = \hat{q}^t$.

In order to estimate the error $\Delta_1^t = |\, q_1^{t+1} - q_1^* \,|$ and to get an indication of the proof of theorem 1, let u_1 be defined implicitly on I_2 by $e_1(u_1\,(q_2), q_2) = 0$. Then, S_1 is the graph of u_1 and since $q_1^{t+1} = \hat{q}_1^t = u_1\,(q_2^t)$, we have

$$\Delta_1^t = |\, q_1^{t+1} - q_1^* \,| = |\, u_1\,(q_2^t) - u_1\,(q_2^*)|$$

$$= \left| \int_{q_2^*}^{q_2^t} u_1'\,(q_2)\, dq_2 \right|$$

$$\leq \left| \int_{q_2^*}^{q_2^t} \left| u_1'\,(q_2)\right| dq_2 \right|$$

$$\leq \|\, u_1' \,\| \left|\, q_2^t - q_2^* \,\right|$$

where $\|\, u_1' \,\| = \sup\, \{|u_1'(q_2)| \mid q_2 \in I_2\}$.

Recalling the notation from (8), $\|\, u_1' \,\| \leq k_{12}$ so,

$$(10) \quad \Delta_1^t \leq k_{12}\, \Delta_2^{t-1} .$$

Similarily we can obtain that

$$(11) \quad \Delta_2^t \leq k_{21} \Delta_1^{t-1} .$$

Notice that the number k_{12} represents an upper bound on the slope of S_1 with respect to q_2 as well as a bound on ratios of certain partial derivations which can be related to the demand elasticities in PIES. Similar comments hold for k_{21} as well as for k_{ij} in general.

3. EFFECTS OF A LINEAR COORDINATE CHANGE

In general, the convergence of $q^1, q^2, \ldots$ as generated by SUB-PIES depends upon how closely the level sets of e_i are approximated by the q_i = "constant" coordinate planes. This is because the q_i = "constant" coordinate planes are also the level sets of $\hat{e}_i^t$. In terms of extending the applicability of SUB-PIES and making it more comparable with PIES it is suggested to impose a new linear coordinate system $(r_1, \ldots, r_n)$ on q-space so that r_i = "constant" coordinate planes are first order approximations to the level sets of e_i at some point $w \in R$. This suggestion is illustrated in figure 3 for n = 2 in which $w = q^t$, $\tilde{q}^t$ is the approximating point determined by using the r-coordinate system and $\hat{q}^t$ is determined as before in the q-coordinate system.

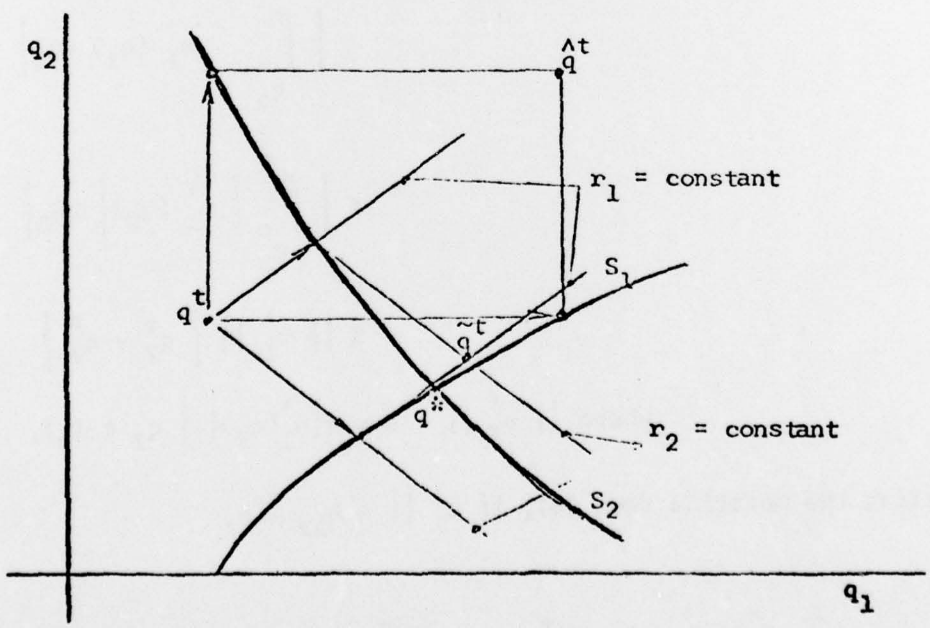


Figure 3

The coordinate transformation needed is $r = \Theta q$ where $\Theta = J_e(w)$. If Θ^{-1} exists then we can let $f(r) = e(\Theta^{-1} r)$ and attempt to apply SUB-PIES to $f(r)$. The following theorem shows that the coordinate change offers a local advantage in terms of satisfying the conditions for convergence of SUB-PIES.

Theorem 2. If Θ^{-1} exists, then there is a neighborhood N of $s = \Theta w$ so that if $R \subseteq N$, then $\rho(k) < 1$.

Proof. The theorem is true because

$$\frac{\partial f_i}{\partial r_j}(s) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

and $\frac{\partial f_i}{\partial r_j}$ is a continuous function of r. See section (4) for details.

The coordinates can be changed to "suit the problem" at each q^t where $[J_e(q^t)]^{-1}$ exists, thus suggesting a variation on the SUB-PIES algorithm. Also, a single coordinate change with respect to a point in the vicinity of the equilibrium can produce a situation in which the hypothesis of Theorem 1 is satisfied. This is illustrated by the following example.

Example 1.

Let $e = (e_1, e_2)$ be defined by

$$e_1(q_1, q_2) = \ln\, q_1 - q_2 + 10$$
$$e_2(q_1, q_2) = q_1^2 + 2q_1 - q_2$$

for $\frac{1}{2} \le q_1 \le 3$

$7 \le q_2 \le 11$.

Then

$$J_e(q) = \begin{pmatrix} \frac{\partial e_1}{\partial q_1}(q) & \frac{\partial e_1}{\partial q_2}(q) \\ \frac{\partial e_2}{\partial q_1}(q) & \frac{\partial e_2}{\partial q_2}(q) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{q_1} & -1 \\ 2q_1+2 & -1 \end{pmatrix}$$

and

$$k_{12} = \sup_{q} \frac{|-1|}{\left|\frac{1}{q_1}\right|} = \sup_{q_1} |q_1| = 3$$

$$k_{21} = \sup_{q} \frac{|2q_1+2|}{|-1|} = \sup_{q_1} |2q_1+2| = 8.$$

Clearly the conditions for convergence are not satisfied in the q- coordinates. Let $w = (2,10)$, then $\Theta = J_e(w) = \begin{pmatrix} \frac{1}{2} & -1 \\ 6 & -1 \end{pmatrix}$ and $\Theta^{-1} = \frac{1}{11}\begin{pmatrix} -2 & 2 \\ -12 & 1 \end{pmatrix}$.

In r-coordinates determined by $r = \Theta q$ we have $f(r) = e(\Theta^{-1} r)$ and so

$$\begin{aligned}(\nabla f)(r) &= (\nabla e)(\Theta^{-1} r)\,\Theta^{-1} \\ &= \frac{1}{11}\begin{pmatrix} -\frac{2}{q_1} + 12 & \frac{2}{q_1} - 1 \\ -4q_1 + 8 & 4q_1 + 3 \end{pmatrix}\end{aligned}$$

where $q_1 = q_1(r)$.

In the r-coordinates,

$$k_{12} = \sup \frac{\left|\frac{2}{q_1} - 1\right|}{\left|-\frac{2}{q_1} + 12\right|} < 1 \quad \text{for } q_1 \geq \frac{1}{2}$$

$$k_{21} = \sup \frac{|-4q_1 + 8|}{|4q_1 + 3|} < 1 \quad \text{for } q_1 \geq \frac{1}{2}$$

which means that $\rho(k) < 1$ for any R such that $R \subseteq \{q \mid q_1 \geq \frac{1}{2}\}$.

Also

$$\frac{\partial f_1}{\partial r_1}(r) = \frac{1}{11}\left(-\frac{2}{q_1} + 11\right) > \frac{3}{11}$$

and

$$\frac{\partial f_2}{\partial r_2}(r) = \frac{1}{11}(4q_1 + 3) > \frac{3}{11}$$

for any $R \subseteq \{q \mid q_1 \geq \frac{1}{2}\}$, which means $\varepsilon = \frac{3}{11}$ fulfills condition (7).

It can be show that

$$f_1(-11,r_2) < 0 \qquad r_2 \in [-4,5]$$

$$f_1(-7, r_2) > 0 \qquad r_2 \in [-4,5]$$

$$f_2(r_1, -4) < 0 \qquad r_1 \in [-11, -7]$$

$$f_2(r_1, 5) > 0 \qquad r_1 \in [-11, -7]$$

i.e., conditions (4) and (5) are satisfied for $I_1 = [-11, -7]$ and $I_2 = [-4, 5]$. $R = I_1 \times I_2$ in r-coordinates is contained in $\{q| \ q_1 \geq \frac{1}{2}\}$ so all the conditions for convergence of SUB-PIES are satisfied. The equilibrium point q^* can now be obtained.

The following theorem is a "local" converse of Theorem 1.

Theorem 3. Suppose q^* is an equilibrium point, i.e. $e(q^*) = (0, \ldots, 0)$, and $\Theta^{-1} = [J_e(q^*)]^{-1}$ exists, then, locally, SUB-PIES converges to q^* in the r-coordinate system given by $r = \Theta q$.

Proof. Theorem 2 and an implicit function Theorem provide that the conditions for convergence of SUB-PIES are satisfied. See section (4) for details.

4. PROOFS

In Theorem 1, assume there are intervals I_i so that (4), (5), (6) and (7) hold and let q^t be a point of R. Let the functions $\hat{e}^t$ and $\hat{E}^t$ be defined by equations (2) and (3) respectively. Condition (7) guarantees that the Hessian matrix, $\nabla^2\hat{E}^t$, is positive definite; therefore, $\hat{e}^t$ is strictly convex on R and so has a unique minimum at $\hat{q}^t$ where

$$\hat{e}^t(\hat{q}^t) = (\nabla\hat{E}^t)(\hat{q}^t) = (0, \ldots, 0). \tag{9}$$

Letting $S_i = \{q \mid e_i(q) = 0\}$, it follows from (9) that

$$(10) \qquad \hat{e}_i^t(\hat{q}^t) = \hat{e}_i^t\,(\hat{q}_1^t, \ldots, \hat{q}_n^t)$$

$$= e_i\,(q_1^t, \ldots, q_{i-1}^t, \hat{q}_i^t, q_{i+1}^t, \ldots q_n^t)$$

$$= 0$$

and so $(q_1^t, \ldots, q_{i-1}^t, \hat{q}_i^t, q_{i+1}^t, \ldots, q_n^t) \in S_i$. This point can be interpreted as a partial equilibrium with respect to the i^{th} quantity. Equation (10) also guarantees that $\hat{q}_i^t \in I_i$ and so $\hat{q}^t \in R$. By an implicit function theorem there is a function u_i defined on $R_i = I_1 \times \ldots \times I_{i-1} \times I_{i+1} \times \ldots \times I_n$ so that

$$e_i(q_1, \ldots, q_{i-1}, u_i(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n), q_{i+1}, \ldots, q_n) = 0$$

for all

$$\alpha_i = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) \in R_i.$$

Note that S_i is the graph of u_i.

Letting $q^{t+1} = \hat{q}^t$ we obtain

$$(11) \qquad \Delta_i^t = |q_i^{t+1} - q_i^t|$$

$$= |u_i(\alpha_i^t) - u_i(\alpha_i^{t-1})|$$

$$= \left| \oint_{[\alpha_i^{t-1};\, \alpha_i^t]} \nabla u_i \right|$$

$$= \left| \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_{q_j^{t-1}}^{q_j^t} \frac{\partial u_i}{\partial q_j}\, dq_j \right|$$

$$\leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \int_{q_j^{t-1}}^{q_j^t} \left| \frac{\partial u_i}{\partial q_j} \right| dq_j \right|$$

$$= \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \int_{q_j^{t-1}}^{q_j^t} \frac{\left|\frac{\partial e_i}{\partial q_j}\right|}{\left|\frac{\partial e_i}{\partial q_i}\right|} dq_j \right|$$

$$\le \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} \left| q_j^t - q_j^{t-1} \right|$$

$$= \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} \Delta_j^{t-1}$$

where the k_{ij} 's are defined by (8).

Inequality (11) can be formulated as

(12) $\Delta^t \le k \, \Delta^{t-1}$ where $\Delta^t = (\Delta_1^t, \ldots, \Delta_n^t)^T$.

We can now obtain from (12) that Δ^t, the vector of successive differences satisfies

(13) $\Delta^t \le k^t \Delta^\circ$.

Using (13) and standard results in matrix analysis, we obtain that the sequence $q^1, q^2, \ldots$ generated by SUB-PIES is a Cauchy sequence in R provided $\rho(k) < 1$. Let q^* be the point to which $q^1, q^2, \ldots$ converges. To see that $e_i(q^*) = 0$, let $x_i^t = (q_1^t, \ldots, q_{i-1}^t, q_i^{t+1}, q_{i+1}^t, \ldots, q_n^t)$ and recall from (10) that $e_i(x_i^t) = 0$. It can be shown that for each i, $x_i^1, x_i^2, \ldots$ converges to q^* and so by the continuity assumption on e_i, $e_i(q^*) = 0$; hence $e(q^*) = (0, \ldots, 0)$.

It is interesting that uniqueness of the equilibrium point also follows from the condition $\rho(k) < 1$. Suppose $\bar{q}$ is another point in R so that $e(\bar{q}) = (0, \ldots, 0)$. In a manner similar to how (13) was obtained we get

$$\Delta_i = | \bar{q}_i - q_i^* |$$

$$= | u_i(\bar{\alpha}_i) - u_i(\alpha_i^*) |$$

$$= \left| \oint_{[\bar{\alpha}_i;\ \alpha_i^*]} \nabla u \right|$$

$$\leq \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} \quad \Delta_j \qquad .$$

And so $\Delta \leq k\Delta$, which implies that

(14) $$\Delta \leq k^n \Delta \qquad \text{for all } n$$

where $\Delta = (\Delta_1, \ . \ . \ . \ , \Delta_n)^T$.

It is clear from (14) that $\rho(k) < 1$ implies $\Delta = (0, \ . \ . \ . \ , 0)$, i.e., $\bar{q} = q^*$.

In Theorem 2, we have $\Theta = J_e(w)$, $s = \Theta w$ and $f_i(r) = e_i(\Theta^{-1} r)$, so

(15) $$\frac{\partial f_i}{\partial q_j}(r) = (\nabla e_i)(\Theta^{-1} r) \cdot C_j(\Theta^{-1})$$

where we are using the notation that $C_\ell(M)$ = the ℓ^{th} column of the $m \times n$ matrix M.

From (15),

(16) $$\frac{\left|\frac{\partial f_i}{\partial r_j}(s)\right|}{\left|\frac{\partial f_i}{\partial r_i}(s)\right|} = \frac{\left|(\nabla e_i)(w) \cdot C_j(\Theta^{-1})\right|}{\left|(\nabla e_i)(w) \cdot C_i(\Theta^{-1})\right|}$$

$$= \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases} \quad .$$

With the continuity of $\frac{\partial f_i}{\partial r_j}$ at s, we can guarantee a neighborhood N of s so that for any $R = I_1 \times \ldots \times I_n \subseteq N$, $\rho(k) < 1$. Again reference [7] is appropriate here.

In Theorem 3, let $r^* = Oq^*$; Theorem 2 provides a neighborhood N of r^* so that $\rho(k) < 1$ for any $R \subseteq N$. Since $f_i(r^*) = 0$ and

$$\frac{\partial f_i}{\partial r_j}(r^*) = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases}$$

for i=1, . . . , n there will be a set R in the form $I_1 \times \ldots \times I_n \subset N$ so that (4), (5), (6) and (7) hold in the r - coordinate system. Since $R \subset N$, $\rho(k) < 1$ and so a sequence $r^1, r^2, \ldots$ generated by SUB-PIES must converge to r^*.

5. MISCELLANEOUS

In terms of future work it would be nice to be able to deal with functions in SUB-PIES which are not everywhere differentiable and to find explicit relations between the k_{ij}'s and the demand elasticities. There are also other possibilities for approximating non-integrable functions, eg. define $\hat{e}^t$ such that

$$J_{\hat{e}^t}(q) = \tfrac{1}{2}[J_e(q) + J_e(q)^T]$$

$$\text{and} \quad \hat{e}^t(q^t) = e(q^t).$$

It is interesting to note that one step in John Neuberger's iterative method of solving non-linear partial differential equations is to approximate a non-conservative vector field by its "nearest" conservative vector field, see [6]. This method would perhaps offer another means of obtaining an $\hat{e}^t$. It has been suggested that using SUB-PIES in the price space would facilitate relating the demand elasticities to the conditions for convergence.

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REFERENCES

1. Dennis, J.E., "A Brief Survey of Convergence Results for Quasi-Newton Methods," Non-linear Programming, Edited by Richard W. Cottle and C.E. Lemke, Volume IX SIAM-AMS Proceedings, 1976.

2. Derr, J.B., Department of Mathematics, West Virginia University, private communication.

3. Hogan, W.W., "Energy Policy Models for Project Independence," Computers and Operations Research, Vol. 2, Pergammon Press, 1975.

4. Hogan, W.W., "Project Independence Evaluation System: Structure and Algorithms," Presented at: American Mathematical Society Short Course on Mathematical Aspects of Production and Distribution of Energy, 1976 annual meeting.

5. National Energy Outlook, Federal Energy Administration, GPO Stock No. 041-018-00097-6.

6. Neuberger, J.W., "Iteration for Systems of Non-linear Partial Differential Equations," preprint, North Texas State University.

7. Ortega, J.M., Numerical Analysis; A Second Course, Academic Press, New York, New York, 1972.

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SOL 77-33 "Convergence Properties of a Pies-Type Algorithm for Non-Integrable Functions"

An algorithm for determining the market equilibrium in the presence of non-integrable but differentiable excess demand functions is developed. This can be reviewed as a variant of the Project Independence Evaluation System Algorithm. A sequence of approximate market equilibria are obtained by constructing integrable excess demand functions. Conditions for the existence and uniqueness of the solutions are demonstrated. It is shown further that the sequence converges to the true market equilibrium if a matrix related to the demand elasticities has a spectral radius less than one. There is a close analogy to known methods for iterative solution of nonlinear equations. Geometric interpretations and some effects of coordinate transformation are discussed.