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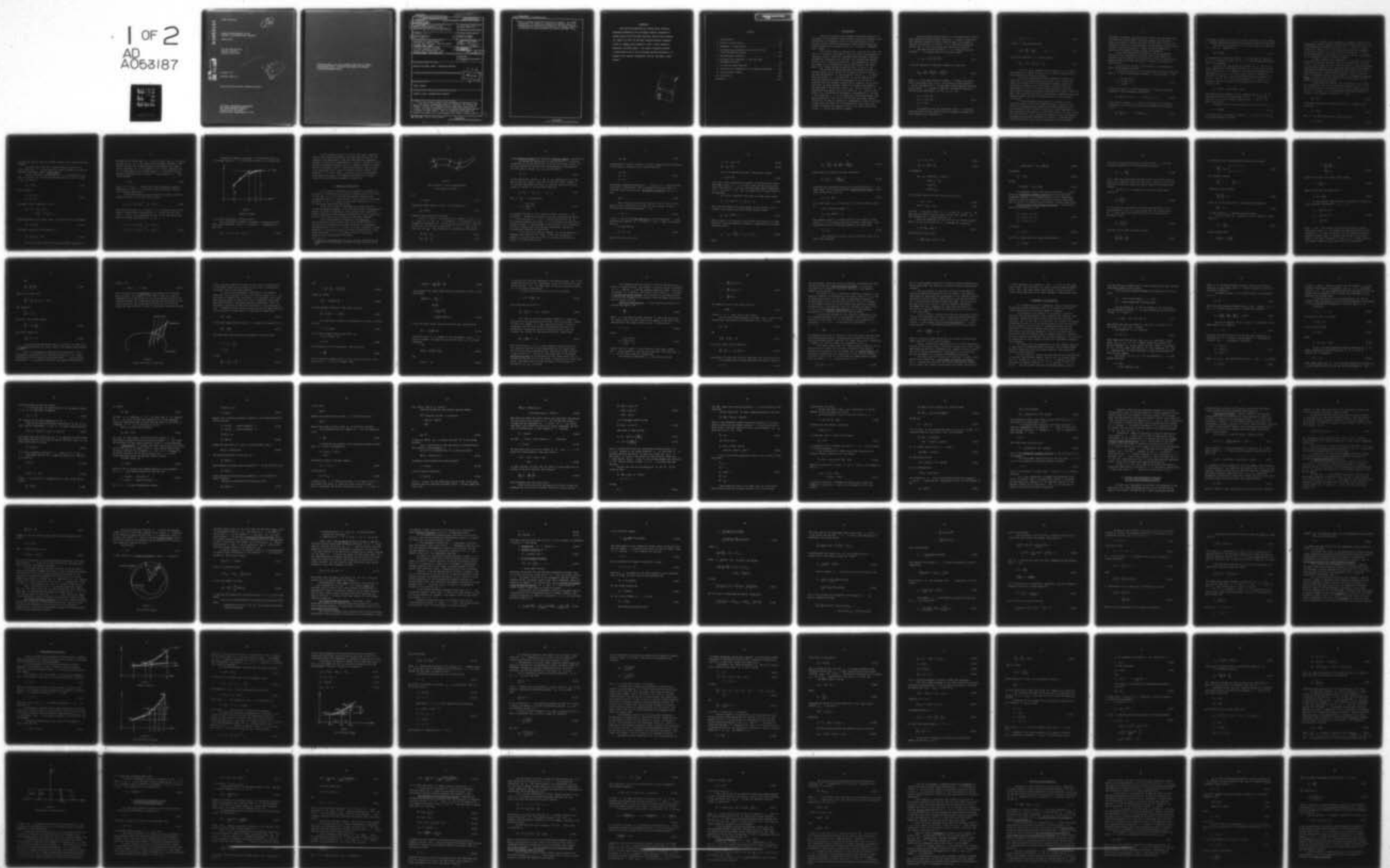

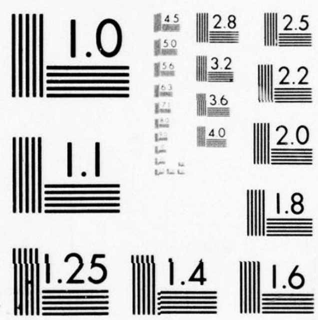
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RECENT DEVELOPMENTS IN THE GEODETIC BOUNDARY-VALUE PROBLEM

Helmut Moritz

The Ohio State University
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Columbus, Ohio 43212

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The report reviews progress in the mathematical formulation and treatment of the goedetic boundary-value problem, in particular, the existence and uniqueness theorems of L. Hörmander and the gravity space approach due to F. Sanso. The method of Hörmander uses a very advanced inverse function theorem of nonlinear functional analysis. Sanso has transformed Molodensky's free boundary-value problem into a fixed boundary-value problem in "gravity space", → next page

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thereby essentially reducing the mathematical complexity. As a linear approximation, the gravity space approach gives identical superior for treating questions of existence and uniqueness of the solution, although it is restricted to the pure gravitational case without centrifugal force.

FOREWORD

This report was prepared by Dr. Helmut Moritz, Professor, Technische Hochschule in Graz and Adjunct Professor, Department of Geodetic Science of The Ohio State University, under Air Force Contract No. F19628-77-C-0082 The Ohio State University Research Foundation Project No. 710533, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, L. G. Hanscom Field, Bedford, Massachusetts, with Mr. Bela Szabo, Project Scientist.

Contents

1. Introduction

The last five years have brought considerable advances in the theory of the boundary-value problem of physical geodesy in the formulation of Molodensky, which is the determination of the physical earth's surface from gravity. These advances have been accomplished mainly through the work of T. Krarup, L. Hörmander and F. Sansò.

The present report is devoted to a review of this work. Its aim is to introduce the reader to the basic ideas and geodetically important results, which are sometimes hidden between formidable mathematical technicalities. We shall thus attempt what mathematicians call a "heuristic exposition", for mathematical details the reader will be referred to the original papers. The treatment of the linear problem in gravity space in sec.8 is new.

The problem of Molodensky may be formulated briefly as follows: given, at all points of the physical earth's surface S, the gravity potential W and the gravity vector $\underline{g}$, to determine the surface S. The potential W can be determined by leveling combined with gravity measurements; this gives the potential up to an unknown constant which, however, can be found indirectly by other methods, especially distance measurements. The magnitude of the gravity vector $\underline{g}$, which is gravity g, is measured by gravimetry, and the direction of $\underline{g}$, which is the plumb line, is obtained by astronomical measurements of latitude Φ and longitude Λ. It is assumed that these measurements have been corrected for luni-solar tidal effects and other temporal variations, so that our problem is independent of time. We further suppose that the effect of the atmosphere has been taken into account by appropriate reduction. Hence, the space outside the surface S can be considered as empty.

We thus assume that the earth is a rigid body which rotates with constant and known angular velocity ω around a fixed axis, which passes through the earth's enter of mass. This center of mass will be taken as the origin 0 of a cartesian coordinate system, the x_3 axis coinciding with the axis of rotation.

The gravitational potential V is a harmonic function outside S. For large values of the radius vector

$$r = |\underline{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{1-1}$$

it has an expansion in spherical harmonics of the form

$$V(\underline{x}) = \frac{GM}{r} + \frac{Y_1(\theta,\lambda)}{r^2} + \frac{Y_2(\theta,\lambda)}{r^3} + \ldots , \tag{1-2}$$

where G is the gravitational constant, M denotes the total mass of the earth, and $Y_n(\theta,\lambda)$ are Laplace surface harmonics, θ (polar distance) and λ (longitude) forming together with the radius vector r a system of spherical coordinates related to the cartesian coordinates $\underline{x} = (x_1, x_2, x_3)$ by

$$\begin{aligned} x_1 &= r \sin\theta \cos\lambda , \\ x_2 &= r \sin\theta \sin\lambda , \\ x_3 &= r \cos\theta . \end{aligned} \tag{1-3}$$

The condition that the coordinate origin 0 coincides with the center of mass implies that the spherical harmonics of first degree vanish identically:

$$Y_1(\theta,\lambda) \equiv 0 \quad , \tag{1-4}$$

so that V must have the form

$$V(\underline{x}) = \frac{GM}{r} + O(\frac{1}{r^3}) \qquad \text{for} \quad r \to \infty \quad . \tag{1-5}$$

The gravity potential W is then given by

$$W(\underline{x}) = V(\underline{x}) + \frac{1}{2}\omega^2(x_1^2 + x_2^2) \quad . \tag{1-6}$$

It will also be assumed that the surface S is a one-to-one image of the sphere and that it is a smooth surface, being differentiable as often as required.

It may be questioned whether Molodensky's problem thus formulated is to-day geodetically relevant at all. On the one hand, the prerequisites for Molodensky's problem, especially continuous coverage of the whole earth's surface by gravity measurements, are still far from being realized; on the other hand, there are many more date of different kind, such as satellite data, that transcend the frame of Molodensky's problem and must be handled by data combination techniques such as least-squares collocation.

To these questions we may answer as follows. From a practical point of view, the integral formulas arising in the solution of boundary-value problems are often computationally more convenient than collocation and retain their importance if gravity data are available to a sufficient extent, at least locally (cf. Moritz,1975). From a theoretical point of view, the geodetic boundary-value problem represents an especially interesting and significant special case, whose importance for the conceptual structure of geodesy, from the time of Clairaut to the present day can hardly be overestimated (curiously enough, the theory was always far ahead of the data available at the time). In fact, the consecutive stages in the development of the boundary-value problem -- Clairaut, Stokes, Molodensky -- always served as measures of perfection for geodetic theory and set new standards.

Even today Molodensky's problem is not yet completely clarified from a mathematical point of view, with respect to existence and uniqueness of the solution, in spite of the decisive progress made in the last few years; it remains a challenge to theoreticians.

Let us now try to get a first grasp of the mathematical nature of Molodensky's problem.

The gravity vector $\underline{g}$ can be expressed in terms of measured gravity g and of astronomical latitude Φ and longitude Λ as

$$\underline{g} = \begin{bmatrix} g \cos\Phi \cos\Lambda \\ g \cos\Phi \sin\Lambda \\ g \sin\Phi \end{bmatrix} . \tag{1-7}$$

In space the vector $\underline{g}$ and the potential W may be considered functions of the rectangular coordinates:

$$\underline{g} = \underline{g}(x_1, x_2, x_3) \quad , \qquad W = W(x_1, x_2, x_3) \tag{1-8}$$

On the earth's surface S, they are functions of two surface coordinates, for which we may take the astronomical coordinates Φ and Λ :

$$\overline{\underline{g}} = \overline{\underline{g}}(\Phi,\Lambda) \quad , \qquad \overline{W} = \overline{W}(\Phi,\Lambda) \quad ; \tag{1-9}$$

the overbar denotes restriction of spatial functions to the surface S, whereas underlining characterizes vectors (and later, matrices).

Now $\bar{\underline{g}}$ may be expressed, in a certain sense, as a function of S and $\bar{W}$, symbolically

$$\bar{\underline{g}} = F(S,\bar{W}) \quad . \tag{1-10}$$

This means that, given the surface S and the gravity potential $\bar{W}$ on it, the gravity vector $\bar{\underline{g}}$ on S is then uniquely determined and can be computed.

In fact, this may be done as follows. Let S and $\bar{W}$ be given. Compute the centrifugal potential on S (which can be done since the surface S is supposed to be given and consequently the coordinates x_1, x_2, x_3 of the surface points are known) and subtract it from $\bar{W}$; this gives the gravitational potential $\bar{V}$ on S. From $\bar{V}$ on S we get the potential V outside S by solving Dirichlet's boundary value problem, which has a unique solution. Now

$$\underline{g} = \text{grad } V + \text{centrifugal force}$$

(grad denoting the gradient) can be computed outside S and, by the continuity of first derivatives, also on S, giving $\bar{\underline{g}}$. Thus $\bar{\underline{g}}$ is, in fact, uniquely determined by S and $\bar{W}$, so that (1-10) holds.

Suppose now that it were possible to solve (1-10) for S:

$$S = \Phi(\bar{W},\bar{\underline{g}}) \quad . \tag{1-11}$$

This would express the earth's surface S in terms of $\bar{W}$ and $\bar{\underline{g}}$, solving Molodensky's problem.

This is probably the conceptually simplest formulation of Molodensky's problem. However, the transition from (1-10) to (1-11) is mathematically extremely difficult. If S, $\overline{W}$ and $\overline{g}$ were simple real numbers and F were an ordinary function (supposed sufficiently smooth), then the solution of (1-10) for S would be straightforward. The existence of such a solution is guaranteed by the elementary implicit function theorem.

In fact, however, the "function" F in (1-10) is a rather complicated nonlinear operator, and the existence of a solution (1-11) is by no means obvious. There are implicit function theorems for nonlinear operators (e.g. Dieudonné,1960;Loomis and Sternberg, 1968;Schwartz,1969;Sternberg,1969), but the conditions for their application are not satisfied in the geodetic case. It was the merit of Hörmander (1975) to have found, by a mathematical tour de force, an implicit function theorem that is applicable to the geodetic boundary-value problem.

To get some first insight into the matter, let us forget all mathematical difficulties and proceed formally as if S, $\overline{W}$, and $\underline{g}$ were simply real numbers and F were a simple functions. Since $\overline{W}$ is given, it can be considered fixed once and for all, so that (1-10) becomes a function of S only:

$$\underline{g} = f(S) \quad . \tag{1-12}$$

To further simplify the notation, we write g instead of $\underline{g}$, obtaining

$$g = f(S) \quad . \tag{1-13}$$

Thus S is simply given by the inverse function

$$S = f^{-1}(g) \quad , \tag{1-14}$$

so that the implicit function problem reduces to an inverse function problem.

To practically find this inverse function, that is, to solve (1-13) for S, we may apply the usual procedure for solving nonlinear equations, namely linearization.

Let us introduce an approximation S_o to the earth's surface S and let g_o be the corresponding gravity vector, related to S_o by (1-13):

$$g_o = f(S_o) \quad . \tag{1-15}$$

Write, formally,

$$\begin{aligned} S &= S_o + \Delta S \quad , \\ g &= g_o + \Delta g \end{aligned} \tag{1-16}$$

and apply Taylor's theorem to (1-13):

$$\begin{aligned} g_o + \Delta g &= f(S_o + \Delta S) = \\ &= f(S_o) + f'(S_o)\Delta S \quad , \end{aligned}$$

omitting quadratic and higher terms. In view of (1-15) this becomes

$$\Delta g = f'(S_o)\Delta S \quad . \tag{1-17}$$

The formal solution of this equation is

$$\Delta S = \left[f'(S_o)\right]^{-1} \Delta g \quad . \tag{1-18}$$

Let us link these ideas with the conventional approach to

Molodensky's problem. Here S_o is the telluroid and g_o is normal gravity on it; Δg is the usual gravity anomaly referred to the earth's surface (it is here possible to disregard the original vector character of Δg and regard it as a scalar quantity) and ΔS is represented by the height anomaly ζ characterizing the separation between earth's surface S and telluroid S_o. Thus (1-18) becomes

$$\zeta = M\Delta g \quad , \tag{1-19}$$

where $M = \left[f'(S_o)\right]^{-1}$ denotes the linear Molodensky operator computing ζ from Δg ; practically one uses Stokes' formula with suitable corrections.

Higher approximations may be obtained by Newton's method. Combining (1-15), (1-16) and (1-18) we get

$$S_1 = S_o + \left[f'(S_o)\right]^{-1} \left[g - f(S_o)\right] \quad , \tag{1-20}$$

where we have written S_1 instead of S to indicate that by this equation we get better approximation S_1 rather than the true value S itself. By repeated application of this formula we get successive better approximations $S_2, S_3, \ldots$.

$$\begin{aligned} S_2 &= S_1 + \left[f'(S_1)\right]^{-1} \left[g - f(S_1)\right] \\ S_3 &= S_2 + \left[f'(S_2)\right]^{-1} \left[g - f(S_2)\right] \\ &\vdots \end{aligned} \tag{1-21}$$

Graphically Newton's procedure is illustrated by Fig. 1. The unknown abscissa S for the given ordinate g is approached

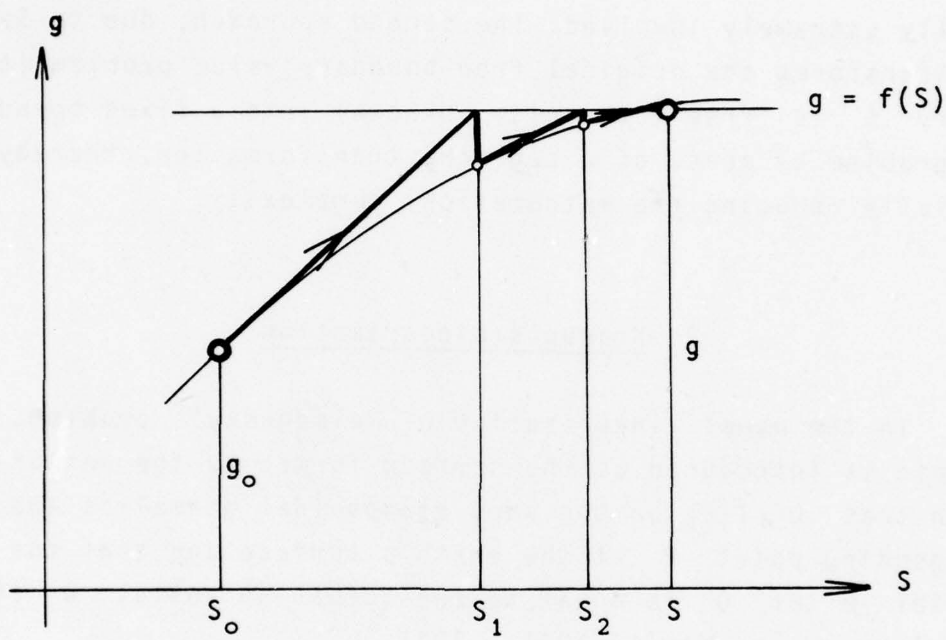

Figure 1

Newton's Method

by following the broken line with arrows.

The convergence of Newton's procedure is known to be very good, namely quadratic: there is a constant K independent of n such that

$$|S_{n+1} - S_n| \leqq K\,|S_n - S_{n-1}|^2 \quad . \tag{1-22}$$

The following sections 2 to 4 will deal with a detailed study of the linearized problem. The remaining sections will be devoted to two approaches to the nonlinear problem. Hörmander's appraoch is logically straightforward, using an iterative inverse function technique basically similar to Newton's method but mathematically extremely involved. The second approach, due to Sansò, first transforms the original free boundary value problem (the boundary S is "free", that is, unknown) into a fixed boundary-value problem by means of a Legendre transformation, thereby essentially reducing the mathematical complexity.

2. Krarup's Linearization

In the usual linearization of Molodensky's problem, the telluroid is introduced as the surface formed by the set of points Q such that Q lies on the same ellipsoidal normal as the corresponding point P at the earth's surface and that the normal potential U at Q is equal to the actual potential W at P; cf. (Heiskanen and Moritz,1967,p.292).

In his third letter on Molodensky's problem that was circulated among the members of the IAG Study Group on Mathematical Methods in Physical Geodesy but unfortunately never published, Krarup (1973) gave a more general formulation of the linearization which is also suitable for studying the nonlinear problem.[1)]

In this more general formulation, the telluroid Σ is now an arbitrary given surface close to the earth's surface S, the points Q of which are in some one-to-one correspondence with the points P of S; cf.Fig. 2. We also introduce a normal potential U which constitutes an analytic approximation to the actual potential W; U is usually taken as the gravity potential of an equipotential ellipsoid.

1) It should be mentioned that the first rigorous formulation and linearization of Molodensky's problem has been given by Meissl (1971).

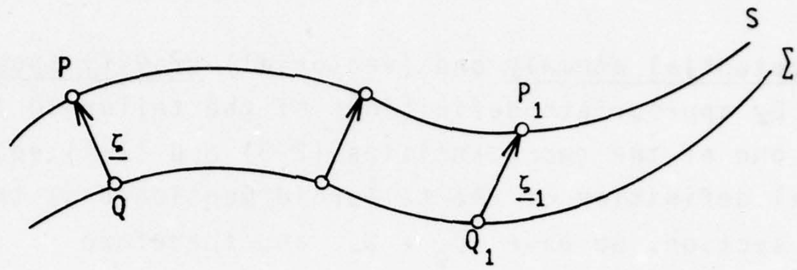

Figure 2

The telluroid Σ as an approximation to the earth's surface S

Let

$$\underline{\gamma} = \text{grad } U \tag{2-1}$$

denote the normal gravity vector, in the same way as

$$\underline{g} = \text{grad } W \tag{2-2}$$

expresses the actual gravity vector.

Since Σ and U are given, we can compute U and $\underline{\gamma}$ at Q , that is, U_Q and $\underline{\gamma}_Q$. As potential W and gravity $\underline{g}$ are supposed to be given on S (in the notation of Section 1, they are W and $\overline{\underline{g}}$), we know it at every point P on S , that is, we know W_P and $\underline{g}_P$. We, therefore, can compute the differences

$$\Delta W = W_P - U_Q \ , \tag{2-3}$$

$$\Delta \underline{g} = \underline{g}_P - \underline{\gamma}_Q \ , \tag{2-4}$$

called potential anomaly and (vectorial) gravity anomaly, respectively.

By appropriate definitions of the telluroid it is possible to make one of the two quantities (2-3) and (2-4) equal to zero. In the usual definition of the telluroid mentioned at the beginning of this section, we have $U_Q = W_P$ and therefore

$$\Delta W = 0 \quad . \tag{2-5}$$

In this definition, points P and Q are supposed to lie on the same ellipsoidal normal. Since the ellipsoidal normal through P is, strictly speaking, not known, it would be theoretically more correct to define Q by the three conditions

$$U_Q = W_P \quad , \quad \phi_Q = \Phi_P \quad , \quad \lambda_Q = \Lambda_P \quad . \tag{2-6}$$

Here ϕ and λ are defined by

$$\underline{\gamma} = \begin{bmatrix} \gamma \cos\phi \cos\lambda \\ \gamma \cos\phi \sin\lambda \\ \gamma \sin\phi \end{bmatrix} \tag{2-7}$$

in complete analogy to (1-7); thus the normal latitude ϕ and longitude λ determine the direction of the normal gravity vector $\underline{\gamma}$, in the same way as Φ and Λ define the direction of $\underline{g}$. The surface formed by the points Q in this manner has been called "normal surface" in (Moritz,1964). Krarup (1973) calls it "Marussi telluroid" because the three "Marussi coordinates" potential, latitude and longitude are identified.

In this way, the potential anomaly ΔW can be made zero. Somewhat surprising at first sight is that also the gravity anomaly $\Delta\underline{g}$ can be made to vanish. This requires defining the points Q of the telluroid by

$$\underline{\gamma}_Q = \underline{g}_P \quad . \qquad (2\text{-}8)$$

Expressing this vector condition in terms of magnitude and direction of the vectors involved, we get three conditions

$$\begin{aligned} \gamma_Q &= g_P \quad , \\ \phi_Q &= \Phi_P \quad , \\ \lambda_Q &= \Lambda_P \quad , \end{aligned} \qquad (2\text{-}9)$$

which again completely determine Q. Since g, Φ, Λ may be called "gravimetric coordinates", the corresponding locus of points Q has been called by Krarup the "<u>gravimetric telluroid</u>"; for it, in fact,

$$\Delta \underline{g} = 0 \quad . \qquad (2\text{-}10)$$

After these possible specializations, let us return to the general case in which both ΔW and $\Delta \underline{g}$ are nonzero. As usual, we define the disturbing potential T by

$$T = W - U \quad , \qquad (2\text{-}11)$$

W and U referring <u>to the same point</u> (this distinguishes T from the potential anomaly ΔW, in which W and U refer to different points!).

On substituting

$$W_P = U_P + T_P \qquad (2\text{-}12)$$

we get from (2-3) and (2-4)

$$T_P + U_P - U_Q = \Delta W \quad , \tag{2-13}$$

$$\underline{g}_P - \underline{\gamma}_Q = \Delta \underline{g} \quad . \tag{2-14}$$

Let us now preceed with the linearization. We put

$$\underline{\zeta} = \text{vector } QP \tag{2-15}$$

(see Fig. 2) and systematically neglect all quantities of second and higher order in ζ. It is well known and easy to see that quantities such as T and $\Delta \underline{g}$ have the same order of magnitude as $\underline{\zeta}$. So also T^2, $T\underline{\zeta}$, etc. are quantities of second order to be neglected.

By a Taylor expansion restricted to linear terms we get

$$U_P = U_Q + \text{grad } U \cdot \underline{\zeta} = U_Q + \underline{\gamma} \cdot \underline{\zeta} \quad , \tag{2-16}$$

where the dot denotes the inner product of two vectors. Let us proceed in the same way with the normal gravity vector:

$$\underline{\gamma}_P = \underline{\gamma}_Q + \text{grad } \underline{\gamma} \cdot \underline{\zeta} \quad . \tag{2-17}$$

What is grad $\underline{\gamma}$? To see this, let us write this equation in index notation, using the summation convention (summation over an index occurring twice):

$$\gamma_{P,i} = \gamma_{Q,i} + \frac{\partial \gamma_i}{\partial x_j} \zeta_j = \gamma_{Q,i} + M_{ij}\zeta_j \quad , \tag{2-18}$$

where

$$M_{ij} = \frac{\partial \gamma_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial U}{\partial x_i}\right) = \frac{\partial^2 U}{\partial x_i \partial x_j} \quad . \tag{2-19}$$

Hence grad $\underline{\gamma}$ is nothing else than the matrix

$$\underline{M} = \left[M_{ij}\right] = \left[\frac{\partial^2 U}{\partial x_i \partial x_j}\right] \tag{2-20}$$

formed by the second derivatives of the normal potential U ; in other words, it is the second-order <u>normal gradient tensor</u>. Therefore, we may write (2-17) as

$$\underline{\gamma}_Q = \underline{\gamma}_P - \underline{M\zeta} \quad . \tag{2-21}$$

It is clear that $\underline{\gamma}$ in (2-16) and $\underline{M}$ in (2-21) refer to point Q .

Let us similarly expand T_P :

$$T_P = T_Q + \text{grad } T \cdot \underline{\zeta} \quad .$$

Now, however, grad T is already small of first order, so that grad $\underline{T} \cdot \underline{\zeta}$ is of second order and, therefore, negligible. Thus, consistent with our linear approximation, we simply have

$$T_P = T_Q \; . \tag{2-22}$$

The insertion of (2-16), (2-21), and (2-22) into (2-13) and (2-14) now gives

$$T_Q + \underline{\gamma} \cdot \underline{\zeta} = \Delta W \quad , \tag{2-23}$$

$$\underline{g}_P - \underline{\gamma}_P + \underline{M\zeta} = \Delta \underline{g} \quad . \tag{2-24}$$

Furthermore,

$$\begin{aligned} \underline{g}_P - \underline{\gamma}_P &= (\text{grad } W)_P - (\text{grad } U)_P \\ &= \text{grad } (W - U)_P \\ &= (\text{grad } T)_P \\ &\doteq (\text{grad } T)_Q \quad , \end{aligned}$$

for the same reason as (2-22). We thus finally get

$$T + \underline{\gamma}^T \underline{\zeta} = \Delta W \quad , \tag{2-25}$$

$$\text{grad } T + \underline{M\zeta} = \Delta \underline{g} \quad , \tag{2-26}$$

in which T and grad T refer to Q, as well as $\underline{\gamma}$ and $\underline{M}$. We have used the matrix notation $\underline{a}^T \underline{b}$ for the inner product $\underline{a} \cdot \underline{b}$, the transpose of $\underline{a}$ being denoted by $\underline{a}^T$.

These two equations will be basic for our further developments. Let us solve (2-26) for $\underline{\zeta}$, assuming $\underline{M}$ invertible,

$$\underline{\zeta} = \underline{M}^{-1}(\Delta \underline{g} - \text{grad } T) \quad , \tag{2-27}$$

and substitute into (2-25):

$$T + \underline{\gamma}^T \underline{M}^{-1}(\Delta \underline{g} - \text{grad } T) = \Delta W$$

or

$$T - \underline{\gamma}^T \underline{M}^{-1} \operatorname{grad} T = \Delta W - \underline{\gamma}^T \underline{M}^{-1} \Delta \underline{g} \quad . \tag{2-28}$$

On putting

$$\underline{m} = - \underline{M}^{-1} \underline{\gamma} \tag{2-29}$$

we get

$$T + \underline{m}^T \operatorname{grad} T = \Delta W + \underline{m}^T \Delta \underline{g} \tag{2-30}$$

This equation, which holds on the telluroid Σ, constitutes the fundamental boundary condition for the linearized Molodensky problem. It is a generalization of the "fundamental equation of physical geodesy" (Heiskanen and Moritz,1967,p.86), just as (2-25) is a generalization of Bruns' formula (*ibid*.,p.85).

Various Forms of the Boundary Condition. - Let us introduce new coordinates q_i by

$$\begin{aligned} q_1 &= q_1(x_1, x_2, x_3) \\ q_2 &= q_2(x_1, x_2, x_3) \\ q_3 &= q_3(x_1, x_2, x_3) \end{aligned} \tag{2-31}$$

or briefly

$$q_i = q_i(x_j) \tag{2-32}$$

and let us assume that the inverse transformation

$$x_j = x_j(q_k) \tag{2-33}$$

also exists. More specifically, we shall select q_i to be the cartesian components of the normal gravity vector:

$$q_i = \gamma_i = \frac{\partial U}{\partial x_i} \quad . \tag{2-34}$$

It is clear that one-to-one relations (2-32) and (2-33) exist, at least in the spatial vicinity of the earth's surface, so that the quantities (2-34) may indeed be used as spatial curvilinear coordinates.

The matrix $\underline{M}$ introduced by (2-19) and (2-20) may be written as

$$\underline{M} = \left[\frac{\partial \gamma_i}{\partial x_j} \right] \quad ; \tag{2-35}$$

it is, therefore, nothing else than the Jacobian matrix of the transformation (2-32). It is well known that the inverse matrix $\underline{M}^{-1}$ is then simply the Jacobian matrix of the inverse transformation (2-33):

$$\underline{M}^{-1} = \left[\frac{\partial x_i}{\partial \gamma_j} \right] \quad . \tag{2-36}$$

This may also be shown directly: we have

$$\frac{\partial \gamma_i}{\partial x_j} \frac{\partial x_j}{\partial \gamma_k} = \frac{\partial \gamma_i}{\partial \gamma_k} \tag{2-37}$$

by the chain rule of differential calculus; furthermore

$$\frac{\partial \gamma_i}{\partial \gamma_k} = \delta_{ik} = \begin{cases} 1 & \text{if} \quad i = k \;, \\ 0 & \text{if} \quad i \neq k \;; \end{cases} \tag{2-38}$$

for instance, clearly

$$\frac{\partial \gamma_1}{\partial \gamma_1} = 1 \quad , \qquad \frac{\partial \gamma_1}{\partial \gamma_3} = 0 \quad .$$

Therefore, (2-37) becomes

$$\frac{\partial \gamma_i}{\partial x_j} \frac{\partial x_j}{\partial \gamma_k} = \delta_{ik} \quad , \tag{2-39}$$

which, by (2-35) and (2-36), is nothing but the equation

$$\underline{MM}^{-1} = \underline{I} \tag{2-40}$$

in index notation, $\underline{I}$ denoting the unit matrix.

Now the vector $\underline{m}$, defined by (2-29), becomes in index notation:

$$m_i = - \frac{\partial x_i}{\partial \gamma_j} \gamma_j \quad , \tag{2-41}$$

and we further have

$$\underline{m}^T \text{grad } T = m_i \frac{\partial T}{\partial x_i}$$

$$= - \frac{\partial T}{\partial x_i} \frac{\partial x_i}{\partial \gamma_j} \gamma_j$$

$$= - \frac{\partial T}{\partial \gamma_j} \gamma_j \; , \tag{2-42}$$

again be the chain rule. Hence (2-30) becomes

$$T - \gamma_i \frac{\partial T}{\partial \gamma_i} = f \tag{2-43}$$

where we have used the abbreviation

$$f = \Delta W + \underline{m}^T \Delta \underline{g} \quad . \tag{2-44}$$

An even greater simplification is achieved by introducing "quasi-spherical coordinates" ρ, ϕ, λ by

$$\gamma_1 = - \frac{1}{\rho^2} \cos\phi \, \cos\lambda \quad ,$$

$$\gamma_2 = - \frac{1}{\rho^2} \cos\phi \, \sin\lambda \quad , \tag{2-45}$$

$$\gamma_3 = - \frac{1}{\rho^2} \sin\phi \quad .$$

Here ϕ and λ are normal latitude and longitude as before, because the vector γ_i is nothing else than normal gravity. The coordinate ρ is taken as positive. If the reference ellipsoid becomes a sphere, then ρ becomes proportional to the radius vector, as we shall see below, so that ρ, ϕ, λ become spherical coordinates; hence the name, quasi-spherical coordinates.

Now

$$\frac{\partial T}{\partial \rho} = \frac{\partial T}{\partial \gamma_i} \frac{\partial \gamma_i}{\partial \rho} \quad , \tag{2-46}$$

again by the chain rule;

$$\frac{\partial \gamma_1}{\partial \rho} = \frac{2}{\rho^3} \cos\phi \cos\lambda = -\frac{2}{\rho} \gamma_1$$

and, generally,

$$\frac{\partial \gamma_i}{\partial \rho} = -\frac{2}{\rho} \gamma_i$$

by (2-45). Thus (2-46) becomes

$$\frac{\partial T}{\partial \rho} = -\frac{2}{\rho} \gamma_i \frac{\partial T}{\partial \gamma_i} \quad , \tag{2-47}$$

and (2-43) reduces to

$$\rho \frac{\partial T}{\partial \rho} + 2T = 2f \tag{2-48}$$

It should be pointed that (2-48), in spite of its simplicity, is rigorously equivalent to (2-30); there is no further approximation involved.

What is the geometrical meaning of the derivative $\partial T/\partial \rho$? According to the definition of a partial derivative, $\partial/\partial \rho$ means differentiation with respect to one coordinate ρ , the two other coordinates ϕ, λ being held constant. This means differentiation along a line

$$\phi = \text{const.}, \quad \lambda = \text{const.} \tag{2-49}$$

Such lines are called <u>isozenithals</u> (with respect to the normal gravity field). The reason for this name is that (ϕ,λ) may be considered as the coordinates of the (ellipsoidal) zenith on the celestial sphere. The isozenithals may also be looked upon as the lines along which the normal gravity vector are all parallel, having the same direction (2-49). If the plumb lines were straight

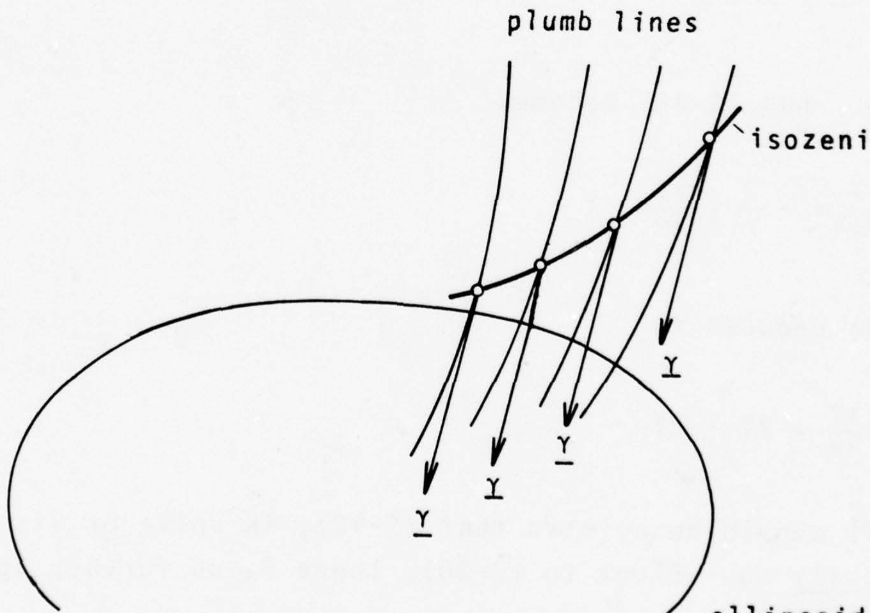

Figure 3

Plumb lines and an isozenithal

lines, then the isozenithals would coincide with the plumb lines; as the normal plumb line curvature is quite small, isozenithals and plumb lines are not very different.

In view of the fundamental importance of our boundary condition, let us approach it from still another angle. Let τ denote the arc length of the isozenithal line, measured, e.g., from the ellipsoid positive upwards (so that it represents the height above the ellipsoid , measured along the isozenithal). Then $\partial/\partial\tau$ represents a derivative along the isozenithal, in the same way as $\partial/\partial\rho$. Therefore, these two derivatives, having the same direction, can only differ in scale, that is, they must be proportional:

$$\frac{\partial}{\partial\tau} = C \frac{\partial}{\partial\rho} \quad . \tag{2-50}$$

To find the proportionality factor C we apply this equation to γ :

$$\frac{\partial\gamma}{\partial\tau} = C \frac{\partial\gamma}{\partial\rho} \tag{2-51}$$

The right-hand side can be easily evaluated, since by (2-45)

$$\gamma^2 = \gamma_i\gamma_i = \frac{1}{\rho^4} \quad ,$$

$$\gamma = \frac{1}{\rho^2} \quad , \tag{2-52}$$

so that

$$\frac{\partial\gamma}{\partial\rho} = - \frac{2}{\rho^3} = - \frac{2\gamma}{\rho} \quad , \tag{2-53}$$

and

$$C = \frac{\partial\gamma}{\partial\tau} : \frac{\partial\gamma}{\partial\rho} = -\frac{1}{2}\rho\frac{1}{\gamma}\frac{\partial\gamma}{\partial\tau} \quad . \tag{2-54}$$

Hence, by (2-50)

$$\rho\frac{\partial}{\partial\rho} = -2\left(\frac{1}{\gamma}\frac{\partial\gamma}{\partial\tau}\right)^{-1}\frac{\partial}{\partial\tau} \quad , \tag{2-55}$$

and the boundary condition (2-48) takes the form

$$\frac{\partial T}{\partial\tau} - \frac{1}{\gamma}\frac{\partial\gamma}{\partial\tau}T = -\frac{1}{\gamma}\frac{\partial\gamma}{\partial\tau}f \quad . \tag{2-56}$$

The right-hand side may be transformed as follows. By (2-44) we have

$$f = \Delta W + \underline{m}^T\Delta\underline{g} \quad . \tag{2-57}$$

Let us have a closer look at the vector $\underline{m}$.

To this effect, let

$$\underline{x} = \underline{x}(\tau) \tag{2-58}$$

be the equation of the isozenithal. Then the vector

$$\underline{e} = \frac{d\underline{x}}{d\tau} \tag{2-59}$$

will be the unit tangent vector of this curve (it will be a unit vector since τ is the arc length). Then

$$\underline{e}^T \text{grad } T = \frac{\partial T}{\partial x_i} \frac{dx_i}{d\tau} = \frac{\partial T}{\partial \tau} \tag{2-60}$$

by the chain rule. Hence there follows from (2-42), (2-47), (2-55) and (2-60):

$$\underline{m}^T \text{grad } T = - \frac{\partial T}{\partial \gamma_i} \gamma_i$$

$$= \frac{1}{2} \rho \frac{\partial T}{\partial \rho}$$

$$= - \left(\frac{1}{\gamma} \frac{\partial \gamma}{\partial \tau}\right)^{-1} \frac{\partial T}{\partial \tau}$$

$$= - \left(\frac{1}{\gamma} \frac{\partial \gamma}{\partial \tau}\right)^{-1} \underline{e}^T \text{grad } T \quad . \tag{2-61}$$

Since the vector grad T can have any direction, there must be

$$\underline{m}^T = - \left(\frac{1}{\gamma} \frac{\partial \gamma}{\partial \tau}\right)^{-1} \underline{e}^T \quad . \tag{2-62}$$

Hence the vector $\underline{m}$ is tangent to the isozenithal; since τ is positive upwards, the negative sign implies that $\underline{m}$ is directed downwards.

Thus

$$\underline{m}^T \Delta \underline{g} = -\left(\frac{1}{\gamma} \frac{\partial \gamma}{\partial \tau}\right)^{-1} \underline{e}^T \Delta \underline{g} \quad . \tag{2-63}$$

Now

$$\underline{e}^T \Delta \underline{g} = - \Delta g' \tag{2-64}$$

is nothing else than the component of the gravity vector $\Delta\underline{g}$ in the downward direction of the isozenithal. Since this direction is very nearly vertical, $\Delta g'$ is almost equal to the usual gravity anomaly Δg in the sense of Molodensky.

In view of (2-63) and (2-64), eq. (2-57) becomes

$$f = \Delta W + \left(\frac{1}{\gamma}\frac{\partial\gamma}{\partial\tau}\right)^{-1}\Delta g' \quad , \tag{2-65}$$

and (2-56) may be written as

$$\frac{\partial T}{\partial\tau} - \frac{1}{\gamma}\frac{\partial\gamma}{\partial\tau}T = -\Delta g' - \frac{1}{\gamma}\frac{\partial\gamma}{\partial\tau}\Delta W \quad . \tag{2-66}$$

This form of the basic boundary condition is rigorously equivalent to the preceding forms (2-30), (2-43) and (2-48). Though it looks less simple, it is very important because it allows a comparison with the form in which the boundary condition for Molodensky's problem was usually presented earlier. Take, for instance, eq. (8-24b) of (Heiskanen and Moritz, 1967,p.300):

$$\frac{\partial T}{\partial h} - \frac{1}{\gamma}\frac{\partial\gamma}{\partial h}T = -\Delta g \quad . \tag{2-67}$$

Here the derivative $\partial/\partial h$ is taken along the normal plumb line. This equation involves certain approximations (cf. ibid.,p.85), which are practically permissible but theoretically not rigorous. It was the merit of T.Krarup to have shown that (2-67) becomes theoretically exact if the direction of the normal plumb line is replaced by the direction of the normal isozenithal (the second term on the right-hand side of (2-66) vanishes if the telluroid is defined by $U_Q = W_P$ as usual).

The boundary condition (2-66) is valid on the telluroid Σ, which is a known surface. The problem is to solve Laplace's equation, $\Delta T = 0$, outside Σ with the boundary condition (2-66). Since the isozenithal is, in general, not normal to the surface Σ, we have an oblique derivative problem. Such problems are considerably more difficult than boundary-value problems involving normal derivatives, such as Stokes' problem.

Spherical Approximation. - If the reference ellipsoid is a nonrotating sphere, then

$$\gamma = \frac{GM}{r^2} , \qquad (2\text{-}68)$$

where G is the gravitational constant, M the total mass, and r the radius vector from the center of the sphere to the point under consideration. The normal gravity vector is then given by

$$\underline{\gamma} = - \gamma \underline{e} \qquad (2\text{-}69)$$

where

$$\underline{e} = \begin{bmatrix} \cos\phi \ \cos\lambda \\ \cos\phi \ \sin\lambda \\ \sin\phi \end{bmatrix} \qquad (2\text{-}70)$$

denotes the unit vector in the direction of the radius vector, ϕ and λ being geocentric latitude and longitude. The quantities r, ϕ, λ are the usual spherical coordinates.

The cartesian components of $\underline{\gamma}$ may thus be written

$$\gamma_1 = -\frac{GM}{r^2}\cos\phi\,\cos\lambda \quad ,$$

$$\gamma_2 = -\frac{GM}{r^2}\cos\phi\,\sin\lambda \quad , \tag{2-71}$$

$$\gamma_3 = -\frac{GM}{r^2}\sin\phi \quad .$$

The comparison with (2-45) shows that now

$$\rho = r/\sqrt{GM} \quad , \tag{2-72}$$

so that ρ is r apart from a scale factor.

For the non-rotating sphere, the plumb lines, as well as the isozeniths, coincide with the spherical radii. Thus, now

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial r} \quad , \tag{2-73}$$

and

$$\frac{1}{\gamma}\frac{\partial\gamma}{\partial\tau} = \frac{1}{\gamma}\frac{\partial\gamma}{\partial r} = -\frac{2}{r} \tag{2-74}$$

by (2-53). Hence (2-66) reduces to

$$\frac{\partial T}{\partial r} + \frac{2}{r}T = -\Delta g + \frac{2}{r}\Delta W \quad , \tag{2-75}$$

equivalent to (2-48) but with the right-hand side given explicitly.

The boundary-value problem expressed by Laplace's equation

$$\Delta T = 0 \tag{2-76}$$

and the boundary condition (2-75) in spherical coordinates has been called by Krarup the simple Molodensky problem; it is the one considered in virtually all practical solutions of the geodetic boundary value problem.

The reason is that, although the reference ellipsoid is not exactly a sphere, its flattening is very small, about 0.3 %, so that on tolerating an error of this order of magnitude in equations relating quantities of the anomalous gravity field, for instance, in the boundary condition, we can formally use spherical boundary condition even in the geodetic case of a reference ellipsoid. This is the so-called spherical approximation; for a more detailed explanation cf. (Heiskanen and Moritz, 1967,pp.87-88).

As Krarup has pointed out, the spherical approximation may be interpreted geometrically as the mapping of the actual point P into an auxiliary point P' by relating the quasi-spherical co-ordinates ρ, ϕ, λ of P to the spherical coordinates r', ϕ', λ' of P' by

$$r' = \rho\sqrt{GM} \quad , \quad \phi' = \phi \quad , \quad \lambda' = \lambda \quad . \tag{2-77}$$

This mapping would even be rigorous if also the Laplace equation were transformed appropriately; the approximation amounts to the use of the untransformed Laplace equation. The reader may find it interesting to compare this with the ideas to be presented in sec. 8.

As we have already said, the simple boundary condition (2-75) has been used in almost all practical solutions of the geodetic boundary value problem. This is already true for Stokes problem, the gravimetric determination of the geoid. In fact, for the reference ellipsoid itself, which is, by the spherical approximation, mapped onto the sphere $r = R$, we have with $\Delta W = 0$:

$$\frac{\partial T}{\partial r} + \frac{2}{R} T = - \Delta g \quad , \tag{2-78}$$

which is the boundary condition for Stokes' problem (Heiskanen and Moritz,1967,p.88). The solution is given by the well-known Stokes integral.

But also almost all practical solutions of Molodensky's problem presented and applied so far are based on the spherical approximation, beginning with (Molodenskii et al.,1962,pp.118-124): solutions by Arnold, Brovar, Marych, Moritz, Pellinen and others. For a review of them see (Moritz,1966 and 1969). The ellipticity has been taken into account in work by Zagrebin, Molodensky, Bjerhammar and Koch; Lelgemann has shown that the effect of ellipticity on geoidal heights and deflections of the vertical is, in fact, very small.

Let us, finally, mention that there are two senses in which "linear" and "nonlinear" are used in connection with Molodensky's problem. In theoretical work, such as the present report, "linearization" is with respect to the disturbing potential T or the height anomaly ζ ; the neglected quantities are on the order of

$$\left(\frac{\zeta}{R}\right)^2 = \left(\frac{60}{6\cdot 10^6}\right)^2 = 10^{-10} \quad ,$$

which is certainly always negligible, the present accuracy being higher than 10^{-6} only in rare instances.

Hence, for practical applications, the "linear" Molodensky problem in the present sense is practically always sufficient. Higher approximations and their convergence are, however, of basic importance for a rigorous mathematical investigation of the existence and the uniqueness of a solution to Molodensky's problem.

The other sense of "linear" and "nonlinear" is used in more practically oriented work, with respect to terrain inclination $\tan\beta$, because the practical solutions to Molodensky's problem (it is usually the "simple" problem mentioned above) are based on a series expansion with respect to $\tan \beta$. (It is in this sense that "linear" and "nonlinear" are used in (Moritz,1966 and 1969).) Since $\tan \beta$ may reach rather large values, "linear solutions" in the second sense may be no longer adequate in mountainous areas.

3. Hörmander's Linearization

As a preparation to his fundamental study of the nonlinear Molodensky problem, Hörmander (1975) first transcribes Krarup's linearization process into modern symbolic notation.

The purpose of this notation is to exploit, as much as possible, the analogy between linear operators and ordinary linear functions, between nonlinear operators and ordinary nonlinear functions, etc. In this way it is possible, for instance, to develop differential calculus in a unified way equally valid for functions of one variable, for functions in n-dimensional space and for functions in a Hilbert or Banach space (which are nothing else than linear and nonlinear operators). One can also give implicit and inverse function theorems that are equally valid for functions of one or several variables and for nonlinear operators. This explains the importance of such a modern notation for the geodetic boundary-value problem.

The reader will find presentations of this "modern analysis" in books such as (Dieudonné,1960) and (Loomis and Sternberg,1968), the first being very clear but rather abstract, the second more intuitive and accessible.

In this notation, vectors and functions are not distinguished by notation from ordinary numbers. Vectors will be written simply a or b instead of $\underline{a}$, $\underline{b}$ (or of a_i, b_i in indices notation); functions are denoted simply by f or g instead of $f(x)$ or $g(u,v)$. Thus structural similarities between numbers, vectors and functions are stressed, but it must be specified at each instance what these symbols really denote.

The following standard mathematical notations will also be used:

R .. line of real numbers x , $-\infty < x < \infty$;

R^3... three-dimensional Euclidean space.

We also denote by σ the unit sphere in R^3 (we do not use the standard notation, S^2 , to avoid confusion with the earth's surface S). There is

$$\sigma = \{ x \epsilon R^3 \ ; \ x_1^2 + x_2^2 + x_3^2 = 1 \} \ , \tag{3-1}$$

which means that the unit sphere is the set of all points of R^3 for which $x_1^2 + x_2^2 + x_3^2 = 1$ holds.

The potential W is a function

$$W : R^3 \rightarrow R \ , \tag{3-2}$$

which means that the function $W(x_1, x_2, x_3)$ associates to each point (x_1, x_2, x_3) of R^3 a number, namely the value of the gravity potential W at this point, and this number is an element of R . In this sense, the function W does indeed map R^3 into R , mapping into R being understood as associating a numerical value (a real number).

The vector (x_1, x_2, x_3) will be denoted by x , so that we may also write

$$\begin{aligned} W &= W(x) \\ &= V(x) + \tfrac{1}{2}\omega^2(x_1^2 + x_2^2) \ , \end{aligned} \tag{3-3}$$

where V is the gravitational potential and the last term the potential of centrifugal force, ω being the angular velocity of the earth's rotation.

The gravity vector g (no underlining!) is the gradient of W :

$$g = W' \ , \tag{3-4}$$

where the prime denotes differentiation; since x means (x_1, x_2, x_3) , it is quite natural that W designs the gradient:

$$W' = \left[\frac{\partial W}{\partial x_1} , \frac{\partial W}{\partial x_2} , \frac{\partial W}{\partial x_3} \right] = \text{grad } W \ . \tag{3-5}$$

How can the physical earth's surface S be defined in the modern way? It is a mapping

$$S : \sigma \rightarrow R^3 \ . \tag{3-6}$$

What does this mean? Nothing else than the usual parametric re-resentation of S in terms of the parameters Φ (astronomical latitude) and Λ (astronomical longitude):

$$\begin{aligned} x_1 &= S_1(\Phi,\Lambda) \ , \\ x_2 &= S_2(\Phi,\Lambda) \ , \\ x_3 &= S_3(\Phi,\Lambda) \ , \end{aligned} \tag{3-7}$$

where S_1, S_2, S_3 are certain functions of Φ and Λ , or briefly

$$x = S(\Phi,\Lambda) \ . \tag{3-8}$$

In fact, Φ and Λ define a point on the unit sphere σ (as we know well from spherical astronomy!), and x_1, x_2, x_3 determine a point in R^3. We are purposely using the same symbol S for the surface and its equation (3-7) or (3-8); the usefulness of this notation will be seen later.

If the functions (3-7) are differentiable (as will be assumed), then the mapping (3-6) will be differentiable: the surface S is a "differentiable embedding" of the unit sphere in R^3.

On putting

$$\Phi = u_1 \quad , \quad \Lambda = u_2 \tag{3-9}$$

we can write (3-8) in the form

$$x = S(u_1, u_2) \tag{3-10}$$

or still more briefly

$$x = S(u) \quad , \tag{3-11}$$

where

$$u = (u_1, u_2) = (\Phi, \Lambda) \quad . \tag{3-12}$$

In sec. 1 we have denoted the gravity potential of S (in modern terminology, the restriction of W to S) by $\overline{W}$. Clearly, $\overline{W}$ is a mapping

$$\overline{W} : \sigma \to R \quad , \tag{3-13}$$

which simply means that $\overline{W}$, as a function defined on a surface, is a (real-valued) function of the two surface parameters Φ and Λ

defining a point on the unit sphere σ .

In the same way, the restriction $\bar{g}$ of the gravity vector g to S is a function, or mapping,

$$\bar{g} : \sigma \rightarrow R^3 \tag{3-14}$$

(R^3 because of the three components of $\bar{g}$).

How can we relate the surface restrictions $\bar{W}$ and $\bar{g}$ to their spatial counterparts W and g ? In usual notation we have

$$\bar{W}(\Phi,\Lambda) = W(S_1(\Phi,\Lambda),\ S_2(\Phi,\Lambda),\ S_3(\Phi,\Lambda)) \quad . \tag{3-15}$$

This means that this restriction to F is obtained by substituting the surface equation (3-7) into the spatial function $W(x_1, x_2, x_3)$. More briefly this is written

$$\bar{W} = W(S(u)) \quad . \tag{3-16}$$

It is thus a composite function of u , that is, of Φ and Λ .

The modern notation for a composite function or "function of a function"

$$f(\phi(x)) \tag{3-17a}$$

is

$$f\circ\phi(x) \quad \text{or} \quad f\circ\phi \tag{3-17b}$$

(read "$\circ$" as "circle" or "composed by"), so that (3-16) may be written

$$\bar{W} = W\circ S(u) \quad , \tag{3-18}$$

or briefly

$$\overline{W} = W \circ S \quad . \tag{3-19}$$

In fact, S is a mapping $\sigma \to R^3$, by (3-6), and W is a mapping $R^3 \to R$, by (3-2), so that the composition of the two mappings (read the right-hand side of (3-19) from the right to the left!) is a mapping $\sigma \to R$, as it should be by (3-13).

Now it is clear that similarly

$$\overline{g} = g \circ S = W' \circ S \tag{3-20}$$

(by 3-4). In other terms, restriction to the surface S is equivalent to composition by the functional symbol S. This shows the usefulness of the present notation, using S in both meanings.

Linearization.- To proceed with the linearization, we consider S, W, g, $\overline{W}$ and $\overline{g}$ as smooth (that is, sufficiently often differentiable) functions of a parameter θ. Let $\theta = 0$ correspond to the normal potential and to the telluroid, and $\theta = 1$ to the actual potential and to the physical earth's surface.

In other terms, let

$$x = S(u;\theta) \tag{3-21}$$

denote a set of surfaces which depend smoothly on the parameter θ, $0 \leqq \theta \leqq 1$. The limiting surfaces of this set are:

$$\begin{aligned} x &= S(u;0) \ \ldots \text{ telluroid } \Sigma \ , \\ x &= S(u;1) \ \ldots \text{ earth's surface } S \ ; \end{aligned} \tag{3-22}$$

for $0 < \theta < 1$ we get intermediate surfaces.

Similarly, let

$$W = W(x;\theta) \tag{3-23}$$

denote a set of gravity potential functions. The limiting functions of this set are:

$$\begin{aligned} W &= W(x;0) \ \ldots \ \text{normal potential} \quad U \ ; \\ W &= W(x;1) \ \ldots \ \text{actual potential} \quad W \ . \end{aligned} \tag{3-24}$$

Finally, let

$$\overline{W} = \overline{W}(u;\theta) \tag{3-25}$$

denote the restriction of $W(x;\theta)$ to the surface $S(u;\theta)$:

$$\overline{W}(u;\theta) = W(S(u;\theta);\theta) \tag{3-26}$$

The limiting functions in this case are

$$\overline{W} = \overline{W}(u;0) \quad ,$$

which denotes the normal gravity potential U on the telluroid, and

$$\overline{W} = \overline{W}(u;1) \quad ,$$

which denotes the actual gravity potential W on the physical earth's surface.

Similarly we proceed with the gravity vector

$$\overline{g} = \overline{g}(u;\theta) \quad ; \tag{3-27}$$

in this case

$$\overline{g}(u;0)$$

denotes the normal gravity vector $\underline{\gamma}$ on the telluroid and

$$\overline{g}(u;1)$$

denotes the actual gravity vector $\underline{g}$ on the earth's surface.

The differentiation with respect to θ will be denoted by a dot:

$$\dot{F} = \frac{\partial F}{\partial \theta} \quad . \tag{3-28}$$

To establish the relation to the preceding sections we note that ΔS in sec.1 corresponds to

$$\Delta S = S(u;1) - S(u;0)$$
$$= \dot{S}\Delta\theta \quad ,$$

according to Taylor's theorem. However,

$$\Delta\theta = 1 - 0 = 1 \quad , \tag{3-29}$$

so that we get

$$\Delta S = \dot{S} = \underline{\zeta} \quad . \tag{3-30}$$

In other words, S is nothing else than ΔS as used in sec.1; it also equals the vector $\underline{\zeta}$ given by (2-15) since for fixed u , $S(u;1)$ represents, by (3-21), the position vector x of point P , and $S(u;0)$ that of Q (Fig.2).

Similarly we get for the (vector) gravity anomaly

$$\Delta \underline{g} = \underline{g}_{\text{earth's surface}} - \underline{\gamma}_{\text{telluroid}}$$

$$= \bar{g}(u;1) - \bar{g}(u;0)$$

$$\doteq \dot{\bar{g}} \Delta\theta$$

or

$$\Delta \underline{g} = \dot{\bar{g}} \quad , \tag{3-31}$$

so that our former $\Delta \underline{g}$ is nothing else than $\dot{\bar{g}}$ in the present notation.

Thus, linearization is now equivalent to differentiation with respect to the parameter θ .

Let us first differentiate $\overline{W}$ as given by (3-26):

$$\overline{W}(u;\theta) = W(S(u;\theta);\theta) \quad , \tag{3-32}$$

obtained by substituting the surface equation

$$x = S(u;\theta) \tag{3-33}$$

into the spatial expression

$$W = W(x;\theta) \quad . \tag{3-34}$$

Since θ enters into the right-hand side of (3-32) in two ways, directly and also indirectly through $S(u;\theta)$, we must apply the chain rule:

$$\dot{W}(u;\theta) = \dot{W}(S(u;\theta);\theta) +$$

$$+ \langle \text{grad } W(S(u;\theta);\theta) , \dot{S}(u;\theta) \rangle \quad . \tag{3-35}$$

What does this mean? The first term on the right-hand side denotes the derivative, with respect to θ, of the function $W(x;\theta)$, disregarding the dependence of $x = S(u;\theta)$ on θ. The second term takes into account precisely this latter dependence. We thus have to differentiate W with respect to x, getting

$$W' = \text{grad } W \tag{3-36}$$

and then $x = S(u;\theta)$ with respect to θ, obtaining

$$\dot{x} = \dot{S}(u;\theta) \quad . \tag{3-37}$$

Now both grad W and (3-37) are vectors in R^3, and $\langle \, , \, \rangle$ is to denote the inner product of two such vectors:

$$\langle a,b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= a_i b_i \tag{3-38}$$

in index notation. In fact, the last term in (3-35) might also be written in the more familiar index notation

$$\frac{\partial W}{\partial x_i} \dot{S}_i = \frac{\partial W}{\partial x_i} \frac{\partial x_i}{\partial \theta} \quad , \tag{3-39}$$

which expresses the usual chain rule.

After having clarified the meaning of (3-35), we omit all arguments but are careful to denote restriction to the surface S :

$$\dot{\overline{W}} = \dot{W} \circ S + \langle W' \circ S, \dot{S} \rangle$$

$$= \dot{W} \circ S + \langle g \circ S, \dot{S} \rangle \qquad (3\text{-}40)$$

$$= \dot{W} \circ S + \langle \overline{g}, \dot{S} \rangle \quad .$$

In an analogous manner we get

$$\dot{\overline{g}} = \dot{g} \circ S + \langle g' \circ S, \dot{S} \rangle \quad . \qquad (3\text{-}41)$$

What does g' mean? We have

$$g = W' = \text{grad } W = \left[\frac{\partial W}{\partial x_i} \right] \quad , \qquad (3\text{-}42)$$

$$g' = W'' = \left[\frac{\partial^2 W}{\partial x_i \partial x_j} \right] \quad , \qquad (3\text{-}43)$$

which is nothing else than the second-order gradient tensor. For $\theta = 0$, W reduces to the normal potential U, so that then g' is nothing else than the matrix $\underline{M}$ given by (2-19), and $g' \circ S$ is this matrix $\underline{M}$ taken at the surface of the telluroid. Note that S here denotes $S(u;\theta)$; that is, any surface between the telluroid $(\theta = 0)$ and the earth's surface $(\theta = 1)$, including these two surfaces.

We must now find out the meaning of $\dot{W}$ and $\dot{\overline{W}}$. Using (3-29) we have

$$\dot{W} = \dot{W} \Delta\theta \doteq W(x; 1) - W(x; 0)$$

$$= W - U = T \quad ,$$

so that

$$\dot{W} = T \qquad (3\text{-}44)$$

and $\dot{W}_{o}S$ means the disturbing potential T at the surface of the telluroid.

On the other hand, $\dot{\overline{W}}$ means something different. We have

$$\dot{\overline{W}} = \dot{\overline{W}}\Delta\theta \doteq \overline{W}(u;1) - \overline{W}(u;0) \quad ,$$

which is the difference between the actual potential W at the earth's surface and the normal potential U at the telluroid, in other words, using the notation of sec.2, the difference $W_P - U_Q$, which by (2-3) is ΔW . Thus

$$\dot{\overline{W}} = \Delta W \quad . \tag{3-45}$$

We finally have

$$\dot{g} = \dot{g}\Delta\theta \doteq g(x;1) - g(x,0)$$

$$= \text{grad } W - \text{grad } U = \text{grad } T \quad . \tag{3-46}$$

Let us now collect equations (3-30), (3-31), (3-44), (3-45) and (3-46):

$$\dot{S} = \underline{\zeta} \quad ,$$

$$\dot{W} = T \quad ,$$

$$\dot{g} = \text{grad } T \quad , \tag{3-47}$$

$$\dot{\overline{W}} = \Delta W \quad ,$$

$$\dot{\overline{g}} = \Delta \underline{g} \quad .$$

These equations form, so to speak, part of a dictionary that serves to translate formulas from the "old" into the new notation and vice versa.

We thus wee that (3-40) is the "translation" of (2-25), whereas (3-41) corresponds to (2-26).

We now proceed as in sec.2. We solve (3-41) for $\dot{S}$:

$$\dot{S} = (g'\circ S)^{-1}(\dot{\bar{g}} - \dot{g}\circ S) \quad , \qquad (3\text{-}48)$$

assuming that the "Marussi condition",

$$\det(g'\circ S) \neq 0$$

is satisfied, that is, that the 3x3 matrix

$$\underline{M} = g'\circ S \qquad (3\text{-}49)$$

is invertible. This is always the case for $\theta = 0$, for the normal ellipsoidal potential U .

The substitution of (3-48) into (3-40) then gives the desired boundary condition. We first obtain

$$\dot{\bar{W}} = \dot{W}\circ S + \langle g\circ S, (g'\circ S)^{-1}(\dot{\bar{g}} - \dot{g}\circ S)\rangle \quad . \qquad (3\text{-}50)$$

Generally we have for 3-vectors a and b and for a 3x3 symmetric matrix C :

$$\langle a, Cb\rangle = a^T Cb = b^T Ca = $$
$$= \langle b, Ca\rangle = \langle Ca, b\rangle \quad , \qquad (3\text{-}51)$$

using matrix notation in between and taking into account the symmetry of the matrix C and also the symmetry of the inner product $\langle \, , \, \rangle$.

By means of this formula, eq. (3-50) becomes

$$\dot{W} = \dot{W} \circ S + \langle (g' \circ S)^{-1} g \circ S, \dot{\bar{g}} - \dot{g} \circ S \rangle \quad . \tag{3-52}$$

We now put

$$m = -(g')^{-1} g \tag{3-53}$$

This 3-vector m has already been used in sec.2; cf. (2-29); it is directed along the isozenithal. Thus (3-52) becomes

$$\dot{W} = \dot{W} \circ S - \langle m \circ S, \dot{\bar{g}} - \dot{g} \circ S \rangle$$

$$= \dot{W} \circ S - \langle m \circ S, \dot{\bar{g}} \rangle + \langle m, \dot{g} \rangle \circ S \quad ; \tag{3-54}$$

as $\circ S$ means restriction to S, there is

$$\langle m \circ S, \dot{g} \circ S \rangle = \langle m, \dot{g} \rangle \circ S \quad . \tag{3-55}$$

On rearranging we get

$$(\dot{W} + \langle m, \dot{g} \rangle) \circ S = \dot{W} + \langle m \circ S, \dot{\bar{g}} \rangle \quad . \tag{3-56}$$

Let us differentiate

$$g(x;\theta) = \text{grad } W(x;\theta)$$

with respect to θ. Since the differentiations with respect to θ and to x (gradient) are independent and can be interchanged, we get

$$\dot{g} = \text{grad } \dot{W} \quad . \tag{3-57}$$

Thus (3-56) becomes

$$(\dot{W} + \langle m, \text{grad } \dot{W}\rangle)\circ S = \dot{\bar{W}} + \langle m\circ S, \dot{\bar{g}}\rangle \quad . \tag{3-58}$$

This condition differs from (2-30) only by the notation; cf., in particular, the "dictionary" (3-47).

It is appropriate to take the centrifugal potential in space as independent of the parameter θ . Then only the purely gravitational potential V depends on θ , and W depends on θ only through V . Thus

$$\dot{W} = \dot{V} \tag{3-59}$$

and (3-58) takes the final form

$$(\dot{V} + \langle m, \text{grad } \dot{V}\rangle)\circ S = \dot{\bar{W}} + \langle m\circ S, \dot{\bar{g}}\rangle \quad . \tag{3-60}$$

This is the <u>fundamental boundary condition</u> in the form given to it by Hörmander.

The form (3-60) clearly brings out the fact that $T = \dot{V} = \dot{W}$ is a harmonic function, satisfying Laplace's equation

$$\Delta\dot{V} = 0 \quad . \tag{3-61}$$

This is less obvious in (3-58) since V but not W is a harmonic function.

It is very interesting to compare the equivalent forms (2-30) and (3-60). There is no doubt that (2-30), its notations and the operations leading to it are much more familiar to the geodetic reader: we all know what T and Δg are and are all familiar with vector and matrix operations.

Equation (3-60) and the operations leading to it are much more abstract, apart from being unfamiliar to most geodesists at the present time (will this change in the near future?). On the other hand, the new notation possesses considerable logical consistency and rigor. We need less special symbols (no T , no ζ etc.), restriction to a surface is clearly expressed, and differentiation with respect to θ replaces Δ and other differences. It is sufficient to look at (3-47); note how clearly the new notation distinguishes between $T = W_P - U_P$ and $\Delta W = W_P - U_Q$!

Furthermore, (3-60) is formulated in terms of derivatives with respect to θ , rather than in terms of approximate finite differences. In the new notation it is obvious that we have linearization without approximation; this is less evident in the usual notation. This fact makes it possible to linearize not only at $\theta = 0$ (which would be the first stage), but also at any intermediate θ, $0 < \theta < 1$. (To repeat, S in (3-60) and similar equations denotes $S(u;\theta)$ for any θ between 0 and 1 , including $\theta = 0$ (telluroid) and $\theta = 1$ (earth's surface).) If we linearize at $\theta = 0$ and take $\Delta\theta = 1$, then (3-60) reduces, in fact, to (2-30), but it is valid for linearization at any θ and is thus much more generally useful: linearization means here simply differentiation.

In fact, it is the form (3-60) which permits the method of successive approximation, starting at $\theta = 0$ and taking consecutively θ closer and closer to 1 , which will be outlined in sec.5.

4. Existence and Uniqueness of Solution for the Linear Molodensky Problem

We shall now investigate the existence and uniqueness of the solution of the linear Molodensky problem. As an introduction we examine first Stokes' problem and the "simple Molodensky problem", following (Moritz,1971,pp.27-31),before considering the general linear problem following (Hörmander,1975,chapter I).

The Problem of Stokes.- Stokes' problem is the boundary-value problem in its simplest form: given the gravity anomaly on a sphere, to determine the anomalous potential T on and outside the sphere, assuming T to be harmonic outside this sphere. The corresponding boundary condition is (2-78); since the radial direction is normal to the bounding sphere, the oblique-derivative problem reduces in this case to a problem involving normal derivatives, which is much simpler.

The solution is given by Stokes' integral formula

$$T(\theta,\lambda) = T_o + \frac{R}{4\pi}\iint_\sigma \Delta g S(\psi) d\sigma + T_1(\theta,\lambda) \quad , \tag{4-1}$$

which expresses T on the given sphere in terms of Δg on this sphere. Here T_o is a fixed constant related to the mass of the earth, and

$$T_1(\theta,\lambda) = A_1 \sin\theta\cos\lambda + A_2 \sin\theta\sin\lambda + A_3 \cos\theta \tag{4-2}$$

is a spherical surface harmonic of the first degree. Polar distance θ and longitude λ are spherical coordinates, and A_1, A_2, A_3 are arbitrary constants which have the following physical interpretation (Heiskanen and Moritz,1967,p.99). Let ξ_1, ξ_2, ξ_3 denote the rectangular coordinates of the earth's center of gravity, the origin being the center of the ellipsoid. Then, approximately,

$$A_i = \bar{g}\xi_i \quad , \tag{4-3}$$

where $\bar{g}$ denotes a mean value of gravity over the earth. Therefore, nonzero A_i mean that the center of the reference ellipsoid does not coincide with the earth's center of mass.

A necessary and sufficient condition for Stokes' problem to be solvable for continuous boundary values is that the function Δg does not contain spherical harmonics of the first degree. In other terms, Δg must be orthogonal to any harmonic function of the first degree $Y_1(\theta,\lambda)$:

$$\iint_\sigma \Delta g(\theta,\lambda) Y_1(\theta,\lambda) d\sigma = 0 \quad ; \tag{4-4}$$

cf. (Heiskanen and Moritz,1967,p.97). Since $Y_1(\theta,\lambda)$ contains three constants, this equation comprises, in fact, three independent conditions.

The solution (4-1) contains three free constants A_1, A_2, A_3 . The solution can be made unique by putting all $A_i = 0$, which means that the first-degree harmonic (4-2) vanishes.

The fact that Δg must satisfy <u>three</u> conditions and that the solution (4-1) contains <u>three</u> free constants expresses the so-called <u>Fredholm alternative</u>; see below.

It should also be pointed out that a solution (4-1) with $A_1 \neq 0 \neq A_2$, is physically impossible, although it is mathematically valid as a solution of the boundary-value problem defined by $\Delta T = 0$ outside the sphere and by the boundary condition (2-78) on the sphere.

In fact, for T to be harmonic and zero at infinity, the centrifugal potential contained in both W and U must be equal, so as to drop out in $T = W - U$. This requires that the axis of the reference ellipsoid coincides with the earth's axis of rotation. If this common axis is taken as x_3 axis, then the centrifugal potential is

$$\frac{1}{2}\omega^2(x_1^2 + x_2^2) \quad . \tag{4-5}$$

Indeed, if the two axes were only parallel and separated by the vector

$$(\delta x_1, \delta x_2, 0) \quad ,$$

then T would contain a term

$$\omega^2(x_1 \delta x_1 + x_2 \delta x_2) \tag{4-6}$$

due to the difference of the two centrifugal potentials; this term and therefore T, would not be zero at infinity. The same would hold if the two axes were not parallel.

So the two rotation axes must coincide. Since the earth's rotation axis passes through the center of mass for physical reasons, and since the axis of the ellipsoid contains the center of the ellipsoid for reasons of symmetry, both centers must lie on the common axis, which is taken as the x_3 coordinate axis. This implies that the two centers can differ only in the x_3 coordinate, so that ξ_1 and ξ_2, and therefore A_1 and A_2 by (4-3), must be zero.

Thus, if a solution (4-1) is to be physically meaningful, only A_3 can differ from zero, so that the solution for a rotating earth has, in reality, only one degree of freedom. Since $A_1 = A_2 = 0$, it is quite natural to take also $A_3 = 0$, thus letting the center of the reference ellipsoid coincide with the earth's center of mass.

The Simple Molodensky Problem.- This is the linear Molodensky problem for a spherical reference surface (sec.2). We shall prove existence and uniqueness of the solution for this problem by establishing a one-to-one correspondence between the Stokes problem and the simple Molodensky problem.

Let us consider the telluroid Σ, on which the boundary condition (2-75) is defined, together with a sphere S' concentric to the reference sphere and such that Σ is completely inside S' (Fig.4). This sphere S' might be called Brillouin sphere, after the French scientist who proposed gravity reduction to a level surface completely outside the earth.

The function

$$F = r\Delta g \tag{4-7}$$

is well known to be a harmonic function in space, r being the

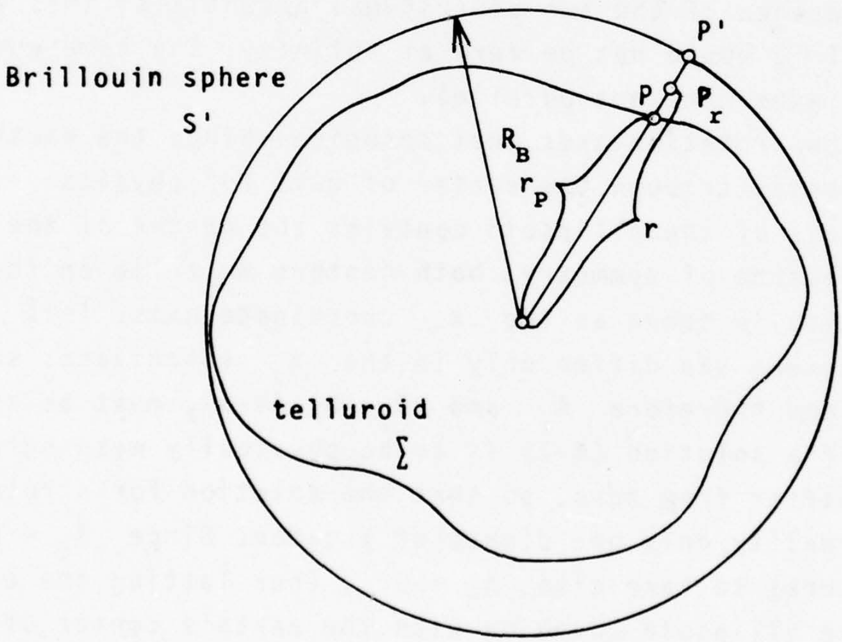

Figure 4

The Brillouin Sphere

variable radius vector of the point under consideration (<u>ibid</u>.,p.90). As the boundary values of Δg , and hence of F , are given on the surface Σ , we can compute F , and hence Δg , at every point outside Σ by solving an external Dirichlet problem, which is uniquely solvable for continuous boundary data (cf. Kellogg,1929, p.314). In particular, this gives Δg at every point P_r between the surfaces Σ and S' --to be denoted by $\Delta g(r)$ --and on the Brillouin sphere itself--to be denoted by $\Delta g'$.

From the values of Δg along a radius it is straightforward to compute radial differences of the potential T : by (Heiskanen and Moritz, 1967, p.92) we have

$$\frac{\partial}{\partial r}(r^2 T) = - r^2 \Delta g(r) \quad , \tag{4-8}$$

which on integration gives

$$(r^2 T)_P - (r^2 T)_{P'} = \int_P^{P'} r^2 \Delta g(r) dr \tag{4-9}$$

or with the symbols of Fig.4,

$$r_P^2 T - R_B^2 T' = \int_{r_P}^{R_B} r^2 \Delta g(r) dr \tag{4-10}$$

T denoting the potential on the telluroid and T' on the Brillouin sphere.

Now we can solve Molodensky's problem by the following three steps:

1. Computation of $\Delta g(r)$ and $\Delta g'$ by solving the external Dirichlet problem.

2. Determination of T' from $\Delta g'$ by solving Stokes' problem for the sphere S'.
3. Computation of T at Σ from T' at S' by (4-10).

Steps 1 and 3 are <u>one-to-one</u> because Dirichlet's problem is uniquely solvable and because different functions Δg on Σ correspond to different functions $\Delta g'$ on S' and vice versa. Thus the question of solvability of Molodensky's problem for the telluroid Σ has been reduced to the question of solvability of Stokes' problem for the sphere S', to which the answer has been given above. For the simple Molodensky problem, therefore, we have exactly the same situation concerning existence and uniqueness of solution as for Stokes' problem: Δg must satisfy three conditions, which may be expressed in the form, analogous to (4-4),

$$\iint_{\sigma} \Delta g'(\theta,\lambda) Y_1(\theta,\lambda) d\sigma = 0 \quad , \tag{4-11}$$

which means that the upward continuation of Δg to S' must not contain any first-degree spherical harmonic.

Corresponding to these <u>three</u> conditions, the solution for T', and consequently also for T, will contain <u>three</u> free constants (this is true if the linear boundary value problem is considered in itself; for physical reasons, two of these constants must be zero). Again we get a unique solution by requiring the spatial function T to have a form that contains no first-degree spherical harmonics.

<u>The Linear Molodensky Problem</u>.- The general linear Molodensky problem for an arbitrary reference surface for an arbitrary reference potential, as formulated in sections 2 and 3, is an <u>oblique-derivative problem</u>.

The classical boundary value problems--the Dirichlet problem and problems involving normal derivatives--can be formulated in terms of Fredholm integral equations of the second kind, and the well-known Fredholm alternative holds (cf. Kellogg,1929,p.298):

If the homogeneous boundary-value problem has no non-zero solution, then the corresponding nonhomogeneous problem is solvable for arbitrary continuous boundary values.

If the homogeneous problem has n independent solutions, then the boundary values must satisfy n independent conditions for the corresponding nonhomogeneous problem to be solvable, and the solution depends on n free parameters (because of the n independent solutions of the homogeneous problem).

An example is furnished by Stokes problem, in which $n = 3$.

An analogous formulation of the oblique-derivative problem leads to singular integral equations for which the Fredholm alternative is, in general, no longer valid. An example is Molodensky's integral equation which is no longer a Fredholm equation of the second kind (contrary to what is sometimes said in the literature).

However, if the oblique-derivative problem is regular, that is, if the direction of the derivative is nowhere tangential to the boundary surface, the Fredholm alternative is still valid, in spite of the singularity of the corresponding integral equation; cf. (Miranda,1970,p.86); this means that the number of conditions on the boundary data f , given by (2-57), is equal to the degree of freedom in the solution, say n .

In the simple Molodensky problem we again had $n = 3$. In the present general linear case, n must be at least three, in view of the three degrees of freedom in the spatial shift of the origin, but perhaps $n = 4$ or 5 ?

Hörmander proved that even in the general form of the linar Molodensky problem n equals 3 . First, the problem defined by (3-60) and (3-61) is reformulated as follows, putting

$$\dot{V} = T \quad , \qquad (4\text{-}12)$$

$$\dot{W} + \langle m \circ S, \dot{\bar{g}} \rangle = f \qquad (4\text{-}13)$$

(the same notations were used in sec.2). It thus becomes: to determine a function T satisfying

1. Harmonicity: $\Delta T = 0$ outside S . (4-14)

2. Boundary condition on S :

$$(T + \langle m, \text{grad } T \rangle) \circ S = f \quad . \qquad (4\text{-}15)$$

3. No first-degree harmonic:

$$T(x) = \frac{c}{r} + O\left(\frac{1}{r^3}\right) \quad , \quad r \to \infty \quad , \qquad (4\text{-}16)$$

c being some constant.

Hörmander proved that the corresponding homogeneous problem, that is, (4-14), (4-15) and (4-16) with $f \equiv 0$, has the unique solution $T \equiv 0$. The general solution of the homogeneous problem, without imposing (4-16), therefore contains the three independent spherical harmonics of degree 1. This proves that $n = 3$ also for the general linear Molodensky problem (if n were >3 , the solution of (4-14), (4-15), (4-16) would no longer be unique).

The principle of the Hörmander's proof of uniqueness may be illustrated by the corresponding proof for Stokes' problem. Take Laplace's operation in spherical coordinates r, θ, λ (cf. Heiskanen and Moritz,1967,p.19):

$$\Delta T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\lambda^2} \qquad (4\text{-}17)$$

Let us form the integral

$$J = \iiint_{r>1} 2r\frac{\partial T}{\partial r}\Delta T r^2 \sin\theta \, dr\, d\theta\, d\lambda \tag{4-18}$$

over the exterior of the reference surface, which we identify with the unit sphere σ by choosing the radius as unit of length. Since $\Delta T = 0$ outside σ, this integral will be zero:

$$J = 0 \quad . \tag{4-19}$$

Let us evaluate the integral using (4-17). We get

$$J = J_1 + J_2 + J_3 \quad , \tag{4-20}$$

where the J_i correspond to the three summands on the right-hand side of (4-18). We shall also use the abbreviations

$$dv = r^2 \sin\theta \, dr\, d\theta\, d\lambda \tag{4-21}$$

for the volume element and

$$d\sigma = \sin\theta \, d\theta\, d\lambda \tag{4-22}$$

for the surface element of σ, so that

$$dv = r^2 dr\, d\sigma \quad . \tag{4-23}$$

With these notations we have

$$J_1 = \iiint_{r>1} 2r\frac{\partial T}{\partial r}\,\frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)dv$$

$$= \iiint_{r>1} 2r\frac{\partial T}{\partial r}\,\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\sin\theta\, dr\, d\theta\, d\lambda \quad . \qquad (4\text{-}24)$$

There is

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 2rT_r + r^2T_{rr} \quad ,$$

where T_r denotes $\partial T/\partial r$ as usual, and further

$$2r\frac{\partial T}{\partial r}\,\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 4r^2T_r^2 + 2r^3T_rT_{rr}$$

$$= 4r^2T_r^2 + r^3\frac{\partial}{\partial r}(T_r^2) \quad ,$$

so that

$$\int_1^\infty 2rT_r\frac{\partial}{\partial r}(r^2T_r)dr = 4\int_1^\infty r^2T_r^2dr + \int_1^\infty r^3\frac{\partial}{\partial r}(T_r^2)dr \quad . \qquad (4\text{-}25)$$

The last term is transformed by partial integration:

$$\int_1^\infty r^3\frac{\partial}{\partial r}(T_r^2)dr = (r^3T_r^2)_{r\to\infty} - (r^3T_r^2)_{r=1} - \int_1^\infty 3r^2T_r^2dr \qquad (4\text{-}26)$$

The first term on the right-hand side is zero since $T_r = O(r^{-2})$, in the second term we have $r = 1$, so that on substituting this expression into (4-25) we get

$$\int_1^\infty 2rT_r \frac{\partial}{\partial r}(r^2 T_r)dr = \int_1^\infty r^2 T_r^2 dr - (T_r^2)_{r=1} .$$

On multiplying this equation by $d\sigma$ and integrating over σ , taking (4-23) into account, we get for (4-24):

$$J_1 = \iiint_{r>1} T_r^2 dv - \iint_\sigma T_r^2 d\sigma . \tag{4-27}$$

Now we compute J_2 . In view of (4-17) and (4-18) we have

$$J_2 = \iiint_{r>1} 2rT_r \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}(\sin\theta T_\theta)dv$$

$$= \iiint_{r>1} 2rT_r \frac{\partial}{\partial\theta}(\sin\theta T_\theta)drd\theta d\lambda \tag{4-28}$$

Let us first perform the integration with respect to θ . By partial integration we get

$$\int_o^\pi 2rT_r \frac{\partial}{\partial\theta}(\sin\theta T_\theta)d\theta = (2rT_r \sin\theta T_\theta)_{\theta=\pi} -$$

$$- (2rT_r \sin\theta T_\theta)_{\theta=o} - \int_o^\pi 2rT_{r\theta} \sin\theta T_\theta d\theta$$

$$= -\int_0^\pi 2rT_{r\theta}T_\theta \sin\theta d\theta$$

$$= -\int_0^\pi r\frac{\partial}{\partial r}(T_\theta^2)\sin\theta d\theta \quad .$$

Thus (4-28) becomes

$$J_2 = -\iiint_{r>1} r\frac{\partial}{\partial r}(T_\theta^2) dr \sin\theta d\theta d\lambda \quad . \tag{4-29}$$

The integral with respect to r is again transformed by partial integration:

$$-\int_1^\infty r\frac{\partial}{\partial r}(T_\theta^2)dr = (T_\theta^2)_{r=1} + \int_1^\infty T_\theta^2 dr \quad .$$

We multiply by $d\sigma$ and integrate over σ, obtaining in view of (4-23):

$$J_2 = \iiint_{r>1} \frac{1}{r^2} T_\theta^2 dv + \iint_\sigma T_\theta^2 d\sigma \quad . \tag{4-30}$$

The integral J_3 is transformed in exactly the same way as J_2, with the result

$$J_3 = \iiint_{r>1} \frac{1}{r^2\sin^2\theta} T_\lambda^2 dv + \iint_\sigma \frac{T_\lambda^2}{\sin^2\theta} d\sigma \quad , \tag{4-31}$$

and we are through.

We only have to collect (4-27), (4-30) and (4-31) and to take (4-19) and (4-20) into account. The result is

$$\iiint_{r>1} \left(T_r^2 + \frac{1}{r^2} T_\theta^2 + \frac{1}{r^2 \sin^2\theta} T_\lambda^2\right) dv +$$

$$+ \iint_\sigma \left(T_\theta^2 + \frac{1}{\sin^2\theta} T_\lambda^2\right) d\sigma - \iint_\sigma T_r^2 d\sigma = 0 \quad . \tag{4-32}$$

Now T_r is nothing else than the radial component of the gradient grad T , and

$$\frac{1}{r} T_\theta = \frac{\partial T}{r \partial \theta} \quad ,$$

$$\frac{1}{r \sin\theta} T_\lambda = \frac{\partial T}{r \sin\theta \partial \lambda} \tag{4-33}$$

are its horizontal, or tangential, components, the total tangential component having the squared magnitude

$$T_t^2 = \frac{1}{r^2} T_\theta^2 + \frac{1}{r^2 \sin^2\theta} T_\lambda^2 \quad . \tag{4-34}$$

Thus (4-32) may finally be written

$$\iiint_{r>1} |\text{grad } T|^2 dv + \iint_\sigma (T_t^2 - T_r^2) d\sigma = 0 \tag{4-35}$$

By means of this formula we can prove stability for Stokes' problem as follows. The boundary condition (2-78) becomes for the homogeneous problem ($\Delta g = 0$) with $R = 1$:

$$T_r + 2T = 0 \quad , \tag{4-36}$$

from which we get

$$T_r = -2T \quad \text{on} \quad \sigma \ . \tag{4-37}$$

If T on the sphere σ is expanded into a series of Laplace's surface harmonics,

$$T(\theta,\lambda) = \sum_{0}^{\infty} T_n(\theta,\lambda) \quad , \tag{4-38}$$

then

$$\iint_{\sigma} T_t^2 d\sigma = \sum_{0}^{\infty} n(n+1) \iint_{\sigma} T_n^2 d\sigma \quad ; \tag{4-39}$$

cf. (Molodenskii et al.,1962,p.87). Furthermore, by (4-36),

$$\iint_{\sigma} T_r^2 d\sigma = 4\iint_{\sigma} T^2 d\sigma =$$

$$= 4\sum_{0}^{\infty} \iint_{\sigma} T_n^2 d\sigma \quad , \tag{4-40}$$

because of the orthogonality of the spherical harmonics.

Since the first term in (4-35) can never be negative, there must be

$$\iint_\sigma (T_t^2 - T_r^2)\,d\sigma \leqq 0 \quad . \tag{4-41}$$

This equation is interesting in itself, since it shows that the average square of the horizontal gradient will always be smaller than (or equal to, but only for $T \equiv 0$) the average square of the vertical gradient; this is obviously true for a sphere of any radius R .

Here we shall use (4-41) for the proof of uniqueness. We substitute (4-39) and (4-40) and obtain

$$\sum_0^\infty \left[n(n+1) - 4\right] \iint_\sigma T_n^2\,d\sigma \leqq 0 \quad . \tag{4-42}$$

The coefficients within brackets are positive for $n = 2,3,4, \ldots$ So if we can show that $T_n \equiv 0$ for $n = 0$ and $n = 1$, then all other T_n must also be zero if (4-42) is to hold.

For $n = 1$ we have, in fact, $T_1 \equiv 0$ because of (4-16). From the well-known spherical-harmonic relation (cf.Heiskanen and Moritz,1967,p.97)

$$T_n = \frac{R}{n-1}\,\Delta g_n \tag{4-43}$$

we get for $n = 0$ and $R = 1$

$$T_0 = -\,\Delta g_0 = 0$$

because Δg is identically zero, for the homogeneous Stokes problem.

Thus we have proved that

$$T \equiv 0 \tag{4-44}$$

is, indeed, the only solution of the homogeneous Stokes problem if (4-16) is prescribed.

Of course, this proof is considerably more involved than the simple reasoning concerning Stokes problem using spherical harmonics, as presented at the beginning of this section. The essential advantage of the quadratic condition (4-35) is, however, that it can be generalized to the linear Molodensky problem.

This has been done by Hörmander. His proof is, however, extremely involved and laborious and cannot be given here. Even his uniqueness theorem (Hörmander,1975,pp.22-23) is so complicated, involving many expressions and parameters, that it cannot be reproduced in the present report.

Let it be sufficient to mention that Hörmander's theorem contains a number of parameters which depend on properties of the earth's topography. Larger slopes of the terrain (say 60°) are permitted provided they do not occur too frequently. Although a detailed study of fitting Hörmander's parameters to the actual earth's topography has not yet been made, it appears that the theorem is general enough to ensure the uniqueness of solution of the linear Molodensky problem, with an ellipsoidal reference field, for the actual topography of the earth, the existence of a solution being generally guaranteed by the theory of the oblique derivative problem.

5. Nash-Hörmander Iteration

In sec.1 we have outlined Newton's method, which is widely used in advanced implicit and inverse function theorems of nonlinear functional analysis; cf. (Sternberg,1969).

In his treatment of the nonlinear Molodensky problem, Hörmander (1975,chapter II) used a different iteration method, which is a discrete scheme analogous to a continuous method used by Nash (1956).

The essence of the two methods is illustrated by Figures 5 and 6. The problem in both cases is the same: consider a function

$$y = \Phi(x) \quad , \qquad (5\text{-}1)$$

which can simply be a function of one variable, as shown in the figures, or a nonlinear operator mapping, for instance, one Banach space X into another Banach space Y, so that

$$\Phi : X \to Y \quad . \qquad (5\text{-}2)$$

Given the value $\Phi(u)$ of Φ at some unknown point $x = u$, to determine u.

We proceed as follows, writing the formulas for the simple case of an ordinary function of one variable; the Banach space case is formally quite the same. We assume an approximate value u_o for u and compute $\Phi(u_o)$. The further procedure is different for the two methods.

In Newton's method (Fig.5) we intersect the tangent at $\Phi(u_o)$ with the line

$$y = \Phi(u) = \text{const.} \qquad (5\text{-}3)$$

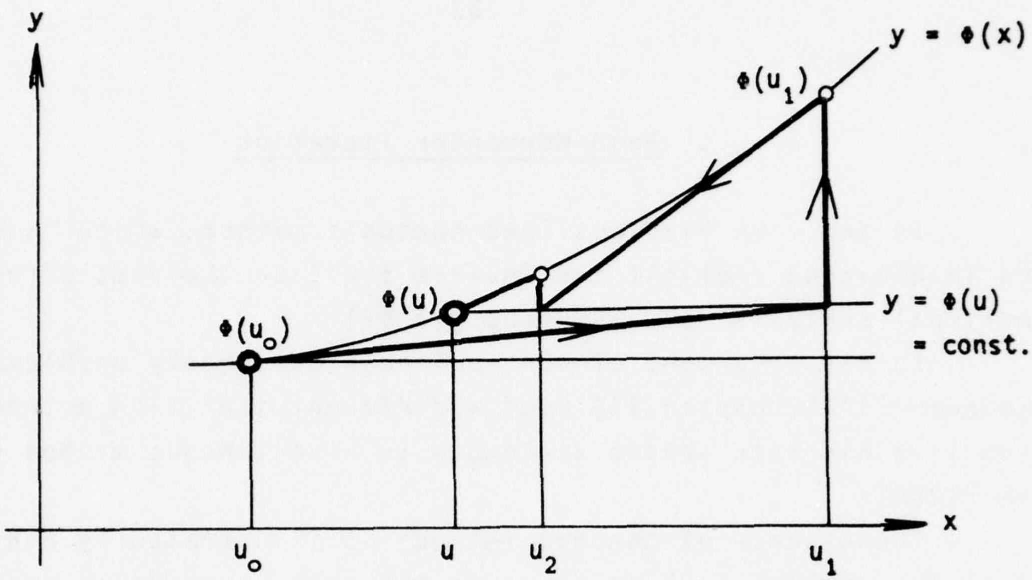

Figure 5

Newton Interation

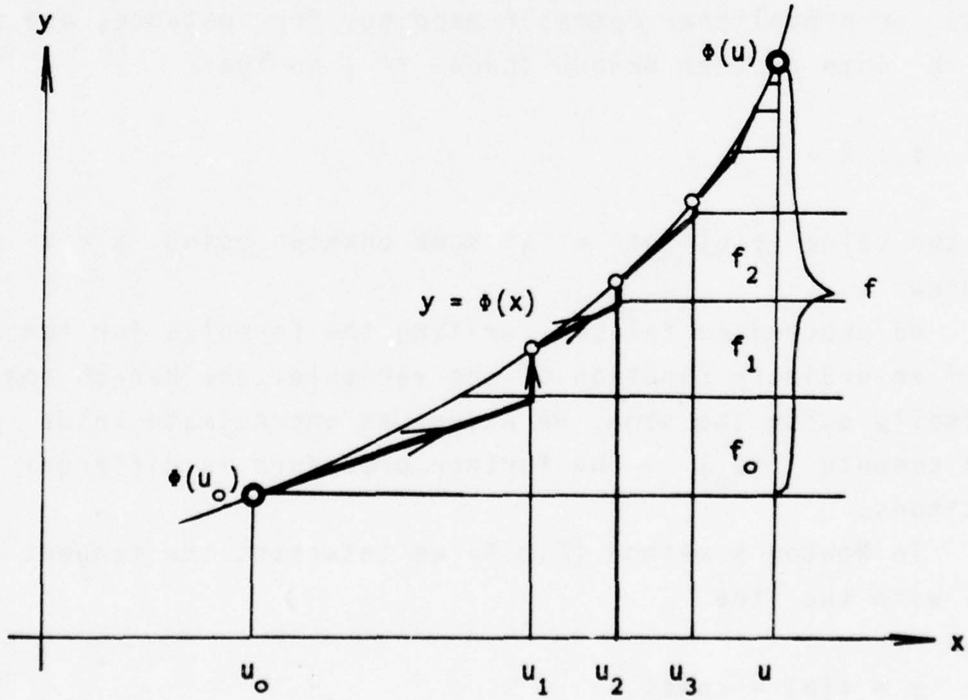

Figure 6

Nash-Hörmander Iteration

parallel to the x-axis; this line is known because $\Phi(u)$ is given. In this way we obtain u_1 and can calculate $\phi(u_1)$. The tangent at $\Phi(u_1)$ is again intersected with the line (5-3), and the procedure is repeated. The sequence $u_o, u_1, u_2, \ldots$ tends to u provided the iteration converges.

In the Nash-Hörmander scheme (Fig.6) the known difference

$$f = \Phi(u) - \Phi(u_o) \tag{5-4}$$

is split up in a suitable way into a convergent series:

$$f = f_o + f_1 + f_2 + \ldots = \sum_o^\infty f_k \quad . \tag{5-5}$$

The tangent at $\Phi(u_o)$ is now intersected with the line

$$y = \Phi(u_o) + f_o = \text{const.}, \tag{5-6}$$

which gives u_1 . The tangent at $\Phi(u_1)$ is intersected with

$$y = \Phi(u_o) + f_o + f_1 = \text{const.}, \tag{5-7}$$

which gives u_2 . The procedure is indefinitely repeated.

The respective iteration procedures are illustrated in the figures by heavy lines with arrows. The comparison of the two figures already gives an indication that the second iteration scheme may have certain advantages over Newton's method. For instance, in the Nash-Hörmander scheme, the u_1 increase more or less monotonically to u , whereas in Newton's method u_1 may be "way out" (Fig.5). In the second scheme we have, approximately ,

$$u_{k+1} - u_k = \left[\Phi'(u_k)\right]^{-1} f_k \quad , \tag{5-8}$$

so that the convergence of the approximation can be controlled almost arbitrarily by suitably selecting the terms in (5-5); in the Newton method, the convergence is quadratic according to (1-22).

The formulas for Newton's method have already been given in sec.1. In the sequel we shall be concerned exclusively with the Nash-Hörmander iteration. From Fig.7 we read off:

$$\varepsilon_{k-1} = \Phi(u_k) - \Phi(u_{k-1}) - h_{k-1} \quad , \tag{5-9}$$

$$h_k = f_k - \varepsilon_{k-1} \quad , \tag{5-10}$$

$$\delta_k = \psi(u_k)h_k \quad , \tag{5-11}$$

$$u_{k+1} = u_k + \delta_k \quad . \tag{5-12}$$

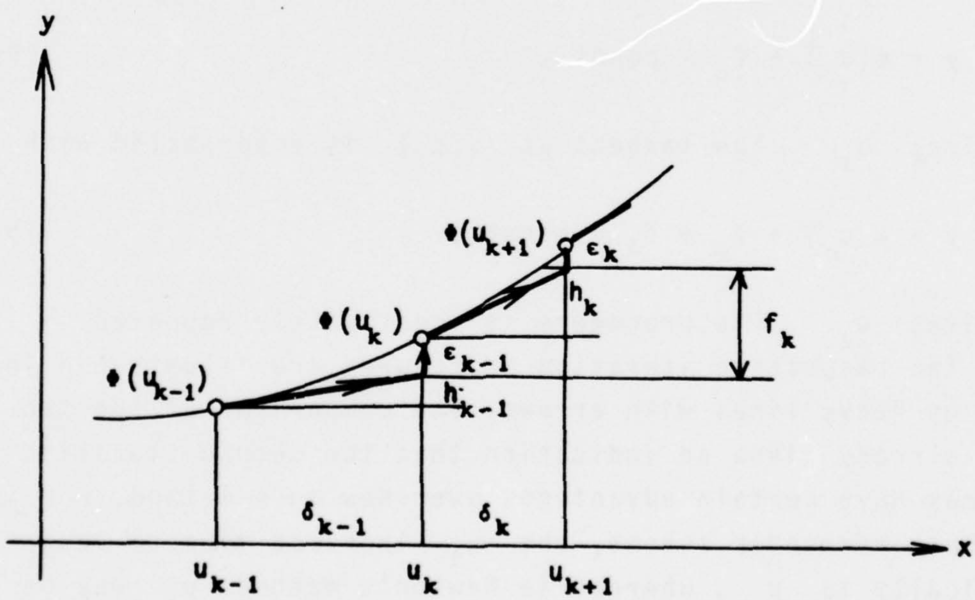

Figure 7

The iteration scheme

Here we have put

$$\psi(u_k) = \left[\Phi'(u_k)\right]^{-1} \quad , \tag{5-13}$$

where Φ' denotes the derivative with respect to u. Geometrically, ε_k is the error due to nonlinearity of the function Φ; it is the "height of the steps" in Figures 6 and 7.

According to Fig.6 we start our iteration with

$$h_o = f_o \tag{5-14}$$

by (5-10), since there is no error ε_{k-1} to start with. Then, by (5-11) and (5-12),

$$\delta_o = \psi(u_o)h_o \quad , \tag{5-15}$$

$$u_1 = u_o + \delta_o \quad , \tag{5-16}$$

Next we put $k = 1$ in (5-9) through (5-12), obtaining

$$\begin{aligned} \varepsilon_o &= \Phi(u_1) - \Phi(u_o) - h_o \\ h_1 &= f_1 - \varepsilon_o \quad , \\ \delta_1 &= \psi(u_1)h_1 \quad , \\ u_2 &= u_1 + \delta_1 \quad . \end{aligned} \tag{5-17}$$

The procedure in repeated with $k = 2,3,...$

An essential feature of the scheme just described is that the error ε_{k-1} committed at the (k-1)th step is taken into account and corrected at the next step by means of (5-10).

Unfortunately, this very simple iteration scheme cannot be proved to converge for the Molodensky problem. Therefore, Nash and Hörmander introduced a suitable smoothing process, which also provides a natural determination of the terms f_k in (5-5).

Let us introduce a sequence of smoothing operators S_o, S_1, S_2, ..., in such a way that

$$\lim_{k\to\infty} S_k = I \quad , \tag{5-18}$$

where I denotes the unit operator; in other terms, if $S_k f$ is the result of applying the smoothing operator S_k to a function f, then

$$\lim_{k\to\infty} S_k f = f \quad ; \tag{5-19}$$

so for increasing k, the smoothing becomes less and less. (There is hardly any danger to confuse the smoothing operator S_k with the boundary surface S !)

To have a simple example, let us take a harmonic function f which is developed into a series of spherical harmonics:

$$f = \sum_{n=o}^{\infty} \frac{Y_n(\theta,\lambda)}{r^{n+1}} \tag{5-20}$$

and take

$$S_k f = \sum_{n=o}^{10+k} \frac{Y_n(\theta,\lambda)}{r^{n+1}} \quad , \tag{5-21}$$

so that smoothing is obtained by truncating the spherical harmonic series or, what is the same, truncation of the spectrum. For instance,

$$S_o f = \sum_{n=o}^{10} \frac{Y_n(\theta,\lambda)}{r^{n+1}} \quad ,$$

$$S_1 f = \sum_{n=o}^{11} \frac{Y_n(\theta,\lambda)}{r^{n+1}} \quad ,$$

etc.; it is clear that (5-19) is satisfied.

The reason for introducing a smoothing in Molodensky's problem is the following. It is a well-known difficulty with many higher order solutions that the higher order terms are getting rougher and rougher. This is the case if an iteration involves differentiation: the derivative is almost always less smooth than the original function. A case in point is the well-known Molodensky series for solving the simple (linear) Molodensky problem. The calculation of higher order terms proceeds through an iteration which involves successive differentiation, with the result, that higher order terms can no longer meaningfully be computed from empirical gravity data.

The Hörmander iteration for the non-linear Molodensky problem has to struggle with a similar difficulty: assume we have an approximate solution for the potential $W(x;\theta)$ according to (3-34). The calculation of the isozenithal vector $\underline{m}$, which is required for the boundary condition (3-60), requires the calculation of the matrix (3-43), involving two differentiations. So we "lose" two derivatives at each iteration, the functions involved will get rougher and rougher, and the iteration will probably "blow up".

Therefore, we must counteract this loss of derivatives by a suitable smoothing, taking care, however, to successively reduce the amount of smoothing so that (5-19) is satisfied: otherwise we would not, in the limit, obtain the right result.

Furthermore, the smoothing operator S_k gives us a natural way of obtaining the terms f_k in (5-5): we put

$$f_o = S_o f \quad , \tag{5-22}$$

$$f_k = (S_k - S_{k-1})f = S_k f - S_{k-1} f \quad , \tag{5-23}$$

$$k = 1, 2, 3, \ldots$$

In fact,

$$\sum_o^n f_k = \left[S_o + (S_1 - S_o) + (S_2 - S_1) + \ldots + (S_n - S_{n-1})\right] f$$

$$= S_n f \quad , \tag{5-24}$$

and

$$\sum_o^\infty f_k = \lim_{n\to\infty} S_n f = f \tag{5-25}$$

by (5-19), so that (5-5) is satisfied.

Now our problem is to modify the iteration equations (5-9) through (5-12) to take smoothing into account. First of all, take equation (5-11). It is easily seen to correspond to the solution of the linearized problem in the k-th step, as ψ involves the linearized operator Φ' through (5-13). Thus it is there where loss of derivatives occurs, and smoothing is appropriate. Hence, instead of computing ψ at u_k, we compute it at

$$v_k = S_k u_k \quad , \tag{5-26}$$

and (5-11) is replaced by

$$\delta_k = \psi(v_k)h_k \quad . \tag{5-27}$$

Next, consider (5-10). The term f_k is already smoothed, being defined by (5-23), but ε_{k-1} is not, being given by (5-9) which involves $\Phi(u_k)$ and $\Phi(u_{k-1})$. The obvious thing would be to replace ε_{k-1} by $S_k\varepsilon_{k-1}$, but this will not work, as we shall see.

The right answer it to put

$$\varepsilon_{k-1} = E_k - E_{k-1} \tag{5-28}$$

where

$$E_k = \sum_{j=o}^{k-1} \varepsilon_k \tag{5-29}$$

represents the sum of all errors before the k-th step, and to replace ε_{k-1} in (5-10) by

$$S_k E_k - S_{k-1}E_{k-1} \quad ,$$

obtaining

$$h_k = f_k - S_k E_k + S_{k-1}E_{k-1} \quad . \tag{5-30}$$

Collecting these results, we replace (5-9) to (5-12) by

$$\varepsilon_{k-1} = \Phi(u_k) - \Phi(u_{k-1}) - h_{k-1} \quad , \tag{5-31}$$

$$h_k = f_k - S_k E_k + S_{k-1} E_{k-1} , \tag{5-32}$$

$$v_k = S_k u_k , \tag{5-33}$$

$$\delta_k = \psi(v_k) h_k , \tag{5-34}$$

$$u_{k+1} = u_k + \delta_k . \tag{5-35}$$

This is the Nash-Hörmander iteration scheme with smoothing.

Will this scheme converge to the right solution? A necessary condition (unfortunately it is by no means sufficient) is obtained by expressing $\Phi(u_k) - \Phi(u_{k-1})$ from (5-31):

$$\Phi(u_k) - \Phi(u_{k-1}) = h_{k-1} + \varepsilon_{k-1} , \tag{5-36}$$

replacing $k-1$ by k :

$$\Phi(u_{k+1}) - \Phi(u_k) = h_k + \varepsilon_k , \tag{5-37}$$

and summing from 0 to n :

$$\Phi(u_{n+1}) - \Phi(u_o) = \sum_o^n h_k + \sum_o^n \varepsilon_k . \tag{5-38}$$

By (5-4) this should tend to f if $n \to \infty$:

$$\sum_o^\infty h_k + \sum_o^\infty \varepsilon_k = f \tag{5-39}$$

To verify this condition for (5-32), we calculate by summing (5-32) from 0 to n :

$$\sum_{o}^{n} h_k = \sum_{o}^{n} f_k - S_n E_n \quad . \tag{5-40}$$

Now, by (5-29)

$$S_n E_n = S_n \sum_{k=o}^{n-1} \varepsilon_k$$

which because of (5-18), can be expected to tend to

$$\sum_{o}^{\infty} \varepsilon_k \quad ,$$

so that, with (5-25), condition (5-39) will hopefully be satisfied.

(It is easily seen that (5-39) would not be satisfied if we had put $h_k = f_k + S_k \varepsilon_{k-1}$ which was our first choice for modifying (5-10)).

Equations (5-31) through (5-35) are the basic formulas for our iteration. To start, we put

$$\begin{aligned} h_o &= f_o \quad , \\ v_o &= S_o u_o \quad , \\ \delta_o &= \psi(v_o) h_o \quad , \\ u_1 &= u_o + \delta_o \quad . \end{aligned} \tag{5-41}$$

Next, we put $k = 1$ in (5-31) through (5-35), then $k = 2$ and so on.

Hörmander writes these equations in a somewhat different form, introducing a parameter θ $(\theta_o \leq \theta < \infty)$ which has two tasks:

1. to represent the variable x as a function of θ :

$$x = x(\theta) \quad , \tag{5-42}$$

in such a way that

$$u_k = x(\theta_k) \tag{5-43}$$

with

$$u_o = x(\theta_o) \quad , \quad u = \lim_{\theta \to \infty} x(\theta) \quad ;$$

2. to serve as a parameter for the smoothing operator, such that

$$S_k = S_{\theta_k} \quad . \tag{5-44}$$

In both cases, $\theta_k (k=0,1,2,...)$ represents a discrete sequence of values of the parameter θ .

A choice for θ is

$$\theta_k = (\theta_o^\mu + k)^{1/\mu} \quad , \tag{5-45}$$

θ_o and μ denoting some large constants. Then the difference

$$\Delta_k = \theta_{k+1} - \theta_k \tag{5-46}$$

$$= (\theta_o^\mu + k + 1)^{1/\mu} - (\theta_o^\mu + k)^{1/\mu}$$

$$= \theta_k \left[(1 + \frac{1}{\theta_o^\mu + k})^{1/\mu} - 1 \right]$$

$$= \theta_k (1 + \frac{1}{\mu} \theta_k^{-\mu} + \ldots - 1)$$

is

$$\Delta_k = \mu^{-1}\theta_k^{1-\mu}(1 + O(\theta_k^{-\mu})) \tag{5-47}$$

and can thus be made as small as any desired power of θ_k^{-1}.

The difference quotient

$$\dot{u}_k = \frac{u_{k+1} - u_k}{\Delta_k} = \frac{\delta_k}{\Delta_k} \tag{5-48}$$

can, therefore, be made as close as desired to a derivative, so that the discretization error becomes very small as $k \to \infty$.

Hörmander writes the basic iteration scheme in a different form, which explicitly involves the parameter θ. Using (5-23) and (5-48) and putting

$$\varepsilon_k = \Delta_k e_k \ ,$$

$$h_k = \Delta_k g_k \ ,$$

we transform (5-31) through (5-35) into

$$g_k = \Delta_k^{-1}\left[(S_k - S_{k-1})(f - E_{k-1}) - \Delta_{k-1} S_k e_{k-1}\right] \ ,$$

$$v_k = S_k u_k \ ,$$

$$\dot{u}_k = \psi(v_k) g_k \ ,$$

$$u_{k+1} = u_k + \Delta_k \dot{u}_k \ ,$$

$$e_k = e'_k + e''_k \quad ,$$

$$e'_k = \left[\Phi'(u_k) - \Phi'(v_k)\right] \dot{u}_k \quad , \tag{5-49}$$

$$e''_k = \Delta_k^{-1}\left[\Phi(u_{k+1}) - \Phi(u_k) - \Phi'(u_k)\Delta_k \dot{u}_k\right] \quad .$$

Here e'_k denotes the error due to smoothing, and e''_k expresses the error due to nonlinearity. The quantity

$$E_k = \sum_{j=o}^{k-1} \Delta_k e_k \tag{5-50}$$

represents again the sum of all errors before the k-th step.

The smoothing operator S_θ used by Hörmander is rather similar to the simple example (5-21). It also amounts to a truncation of the spectrum. In (5-21) we had a discrete spectrum, n taking the values $0,1,2,\ldots,$ and the operator S_k was also defined only for integers k . Now, however, we have a parameter θ that runs continuously from θ_o to ∞ , and S_θ has to be defined for continuous θ . Therefore, one considers functions defined on a plane, or generally in R^n , which have a continuous spectrum (2 or n-dimensional Fourier transform). Now the smoothing simply consists in the application of a low-pass filter, which suppresses higher frequencies, leaving lower frequences unchanged. In other words, the spectrum of the function to be smoothed is multiplied by a function

$$H_\theta(\omega) = H(\frac{\omega}{\theta}) \quad , \tag{5-51}$$

where $H(\omega)$ is a symmetric function of the frequency ω , which is 1 for $|\omega| \leqq \omega_1$ and 0 for $|\omega| \geqq \omega_2$; between ω_1 and ω_2 , $H(\omega)$ is interpolated by an infinitely differentiable function,

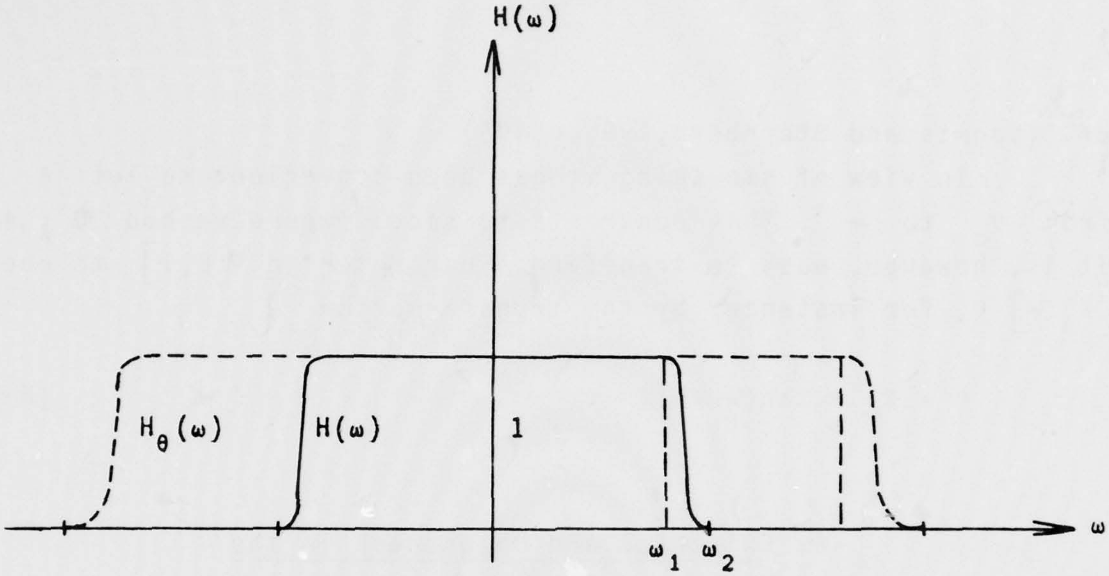

Figure 8

Smoothing by a low-pass filter

so that $H(\omega)$ is everywhere infinitely often differentiable. This is shown, in one dimension, by Fig.8; the generalization to higher dimensions is straightforward.

Equivalent to a multiplication of the spectrum by (5-51) is, of course, a convolution of the function to be smoothed with a function whose Fourier transform is (5-51); cf. (Papoulis,1968,p.74).

The smoothing just described can be applied, with slight modifications, to functions with support in a compact set K, that is, to functions which are zero outside K. This finally gives the possibility to extend the smoothing operator S_θ to functions defined on a compact manifold, for instance, on the earth's surface or the telluroid. This is done by covering the manifold by several coordinate patches and splitting up the given function into a sum of functions with compact support by a "partition of the unity";

cf. (Loomis and Sternberg,1968,p.405).

In view of smoothing it has been convenient to let θ run from θ_o to ∞. This contrasts to sec.3, where we had $0 \leqq \theta \leqq 1$. It is, however, easy to transform a parameter in $[0,1]$ to one in $[\theta_o,\infty]$, for instance, by the transformation

$$\theta = \theta_o + \tan(\tfrac{1}{2}\pi\theta') \quad . \tag{5-52}$$

6. Existence and Uniqueness in the Nonlinear Molodensky Problem

The iteration procedure outlined in the preceding section is now applied to Molodensky's problem; now

$$u = S \tag{6-1}$$

represents the earth's surface to be determined, and

$$\Phi(u) = \overline{g} \tag{6-2}$$

is the given gravity vector on S ; cf. sec.1.

The crucial problem is, of course, to prove convergence of the iteration scheme for conditions applicable to Molodensky's problem. This is so difficult and complicated that it cannot be done here; the interested reader is referred to (Hörmander,1975).

We shall restrict ourselves to stating the main results. First we must say a few words about the norms which have been used in this problem. A norm is needed to characterize, in some sense, the "size" of the function under consideration; it can be considered as a generalization of the norm of a vector, e.g.

$$\|a\| = (a_1^2 + a_2^2 + a_3^2)^{1/2} \tag{6-3}$$

for a vector $a = (a_1, a_2, a_3)$.

A well-known norm of a continuous function $f(x)$ defined in a convex compact set B in R^n is

$$\|f\|_o = \sup_{x\in B} |f(x)| \quad , \tag{6-4}$$

where "sup" denotes the supremum (which, for continuous functions, equals the maximum value). The functions f for which the norm (6-4) is finite form a Banach space; cf. (Loomis and Sternberg, 1968,p.218). This space will be denoted here by H^o .

An appropriate norm for functions that are differentiable as well as continuous is

$$\|f\|_1 = \sup_{x\in B} |f(x)| + \sup_{x\in B} \left|\frac{\partial f}{\partial x_i}\right| \quad , \tag{6-5}$$

where $\partial f/\partial x_i$ denotes any partial derivative. The functions with fintie norm (6-5) form a Banach space H^1 .

Now it is of basic importance for Molodensky's problem, as for many problems in potential theory, to define norms $\|f\|_\alpha$ for $0<\alpha<1$, that is, spaces H^α intermediate between H^o and H^1 . For this purpose we consider continuous functions that satisfy a Hölder condition with exponent α ; they are functions for which

$$\sup_{x,y\in B} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \tag{6-6}$$

is finite. These functions form a Banach space H^α ; the norm is given by

$$\|f\|_\alpha = \sup_{x\in B}|f(x)| + \sup_{x,y\in B} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \quad , \tag{6-7}$$

It can be shown that

$$H^0 \supset H^{\alpha 1} \supset H^{\alpha 2} \supset H^1 \tag{6-8}$$

for

$$0 < \alpha_1 < \alpha_2 < 1 \quad ; \tag{6-9}$$

that is, there are more functions in H^0 than in $H^{\alpha 1}$, more functions in $H^{\alpha 1}$ than in $H^{\alpha 2}$, and more functions in $H^{\alpha 2}$ than in H^1. So satisfying a Hölder condition with exponent α is a stronger condition than mere continuity and weaker than differentiability.

We may also consider a Hölder condition (6-6) with exponent $\alpha = 1$; it is seen that this is almost (although not completely) the same as differentiability. In fact, we shall use H^1 for the space of functions satisfying a Hölder condition with $\alpha = 1$ rather than for functions with finite norm (6-5).

So far, we have defined spaces H^a for $0 \leq a \leq 1$. For $a > 1$ we proceed as follows. Let k be a positive integer such that $k < a \leq k+1$ (for instance, for $a = 5.75$ there is $k = 5$). Denote by $D^k f$ any derivative of k-th order (for instance,

$$\frac{\partial^5 f}{\partial^2 x_1 \partial^3 x_2}$$

for $k = 5$). Then the norm $\|f\|_a$ is defined as

$$\|f\|_a = \sup_{x\in B}|f(x)| + \sup_{x,y\in B} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^{a-k}} . \tag{6-10}$$

It is clear that (6-7) is a special case of (6-10) with $k = 0$.

In other words, the space H^a consists of continuous functions which are k times differentiable and whose k-th derivatives satisfy a Hölder condition with exponent $a-k \leq 1$.

Reformulation of the Molodensky Problem.- Hörmander has reformulated Molodensky's problem in the following way: to determine a closed surface S in R^3, which is a one-to-one image of the unit sphere, from given values $\overline{g}$ and $\overline{W}$, such that the following conditions are satisfied:

$$\overline{W} = W \circ S + \sum_1^3 a_j A_j , \tag{6-11}$$

$$\overline{g} = g \circ S = W' \circ S , \tag{6-12}$$

$$W(x) = V(x) + \frac{1}{2}\omega^2(x_1^2 + x_2^2) , \tag{6-13}$$

$$\Delta V = 0 \quad \text{outside } S , \tag{6-14}$$

$$V(x) = \frac{\text{const.}}{|x|} + O\left(\frac{1}{|x|^3}\right) . \tag{6-15}$$

Uniqueness of the solution is achieved by postulating that the harmonic function $V(x)$, which represents the external gravitational potential, contains no first-degree spherical harmonics; this is expressed by (6-15),

$$|x| = r \tag{6-16}$$

denoting the radius vector. The second term on the right-hand side of (6-13) denotes, of course, the centrifugal force potential, ω being the angular velocity of the earth's rotation.

The new feature is (6-11) instead of simply taking $\overline{W} = W \circ S$, which would be the restriction of W to S . In the modified expression, the a_j are three constants to be determined, and the A_j are three suitably assumed functions. The purpose of adding the linear combination $\sum_1^3 a_j A_j$ is to ensure the solvability of Molodensky's problem for arbitrary boundary data.

This is to be understood as follows. Assume the earth's surface S to be known, and consider the given function $\overline{W} = W \circ S$. We can now solve the exterior Dirichlet problem

$$\Delta V = 0 \quad , \quad V \to \infty \quad \text{for} \quad r \to \infty \quad ,$$

$$V \circ S = \overline{W} - \frac{1}{2}\,\omega^2(S_1^2 + S_2^2) \quad , \qquad (6\text{-}17)$$

which gives, for every data function $\overline{W}$, a unique solution $V(x)$. For arbitrary data $\overline{W}$, the spatial function $V(x)$ will, in general, contain spherical harmonics of first degree, contrary to the condition (6-15).

Let now (6-11) be used instead of $\overline{W} = W \circ S$. Then (6-17) is replaced by

$$V \circ S = \overline{W} - \frac{1}{2}\,\omega^2(S_1^2 + S_2^2) - \sum_1^3 a_j A_j \quad . \qquad (6\text{-}18)$$

Again, the solution of the exterior problem with boundary data $V \circ S$ gives a unique solution $V(x)$ which will contain three linearly independent spherical harmonics of the first degree. Now, however, the three constants a_j can be chosen in such a way that these three first-degree harmonics vanish.

This is readily seen to lead to three linear equations for the three unknowns a_1, a_2, a_3 . These equations will have a unique solution, provided the boundary-value problem

$$\Delta v = 0 \quad , \quad v \circ S = \sum_{1}^{3} a_j A_j \qquad (6\text{-}19)$$

has a solution $v(x)$ which contains three linearly independent forst-degree harmonics

$$(c_1 \sin\theta \cos\lambda + c_2 \sin\theta \sin\lambda + c_3 \cos\theta)/r^2 \quad . \qquad (6\text{-}20)$$

In fact, c_1 can then be chosen to be equal to the c_1 of the boundary-value problem (6-17), and similar for c_2 and c_3. Since the problem (6-18) is the difference of problems (6-17) and (6-19), all first-degree harmonics will cancel in the solution of (6-18).

To achieve this, we may select the functions A_i as follows:

$$A_1 = \left(\frac{\sin\theta \cos\lambda}{r^2}\right) \circ \Sigma \quad , \quad A_2 = \left(\frac{\sin\theta \sin\lambda}{r^2}\right) \circ \Sigma \quad , \quad A_3 = \left(\frac{\cos\theta}{r^2}\right) \circ \Sigma$$

(6-21)

If the earth's surface S coincided with the telluroid Σ, then the spatial functions corresponding to the A_i by a solution of the exterior Dirichlet problem would be the first-degree harmonics

$$\frac{\sin\theta \cos\lambda}{r^2} \quad , \text{ etc.} \qquad (6\text{-}22)$$

themselves, for which the condition (6-20) is certainly satisfied (with $c_i = a_i$). If S does not deviate too much from Σ, then (6-20) is still satisfied because of continuity, even if the functions $A_i(\Phi,\Lambda)$ are now considered as functions on S instead of Σ (now, of course, $c_i \neq a_i$ in general). More generally, if the $A_i(\Phi,\Lambda)$ as given by (6-21) independently of the parameter θ, are considered as functions of θ with $S(\theta_o) = \Sigma$ (telluroid) and $S(\infty) = S$

(earth's surface), then

$$a_j = a_j(\theta) \tag{6-23}$$

will be functions of θ .

The linearization of the modified Molodensky problem defined by (6-11) through (6-15) is now done as in sec.3. The only difference is the additional term $\sum_1^3 a_j A_j$, so that the boundary condition (3-60) is replaced by

$$(\dot{V} + \langle m, \text{grad } \dot{V}\rangle) \circ \Sigma = \dot{W} + \langle m \circ \Sigma, \dot{\bar{g}}\rangle - \sum_1^3 \dot{a}_j A_j \ , \tag{6-24}$$

where $\dot{a}_j$ is the derivative of (6-23) with respect to θ . Now for arbitrary boundary data $\dot{W}$ and $\dot{\bar{g}}$ there exist unique constants $\dot{a}_1, \dot{a}_2, \dot{a}_3$, so that the corresponding boundary-value problem with $\Delta\dot{V} = 0$ outside $\Sigma(\theta)$ admits a unique solution $\dot{V}(x)$ that does not contain spherical harmonics of first degree.

To the Molodensky problem reformulated in this way, the Nash-Hörmander method is applied. Convergence of this method can be proved using various Hölder norms and estimates for the linear problem and also for the nonlinearity. All this is extremely difficult and laborious. Finally one obtains the

<u>Theorem of Hörmander</u>. Let $\varepsilon > 0$, then:

(1) For all $\bar{W}$ *and* $\bar{g}$ *in a* $H^{2+\varepsilon}$ *neighborhood of* $\bar{W}_0$ *and* $\bar{g}_0$ *, then the modified Molodensky problem defined by equations (6-11) to (6-15) has a solution* S *close to* $S_0 = \Sigma$ *in* $H^{2+\varepsilon}$ *and* (a_1, a_2, a_3) *close to* 0 *in* R^3 .

(2) If $\bar{W}$ *and* $\bar{g}$ *are in* H^a *for some* $a > 2+\varepsilon$ *which is not an integer, then* $S \in H^a$.

(3) One can find a $H^{3+\varepsilon}$ *neighborhood of* S_0 *which cannot contain two solutions of the problem.*

Let us look at this theorem more closely and explain its meaning. A $H^{2+\varepsilon}$ neighborhood of $\bar{W}_o = \bar{W}(u;\theta_o)$ consists of all functions $\bar{W}$ for which

$$\|\bar{W} - \bar{W}_o\|_{2+\varepsilon} < \delta \quad , \qquad (6\text{-}25)$$

where δ is sufficiently small and the norm is defined by (6-10) with $a = 2 + \varepsilon$. Smallness of this norm implies that not only the maximum deviation of $\bar{W}$ from $\bar{W}_o$,

$$\max|\bar{W} - \bar{W}_o|$$

is small, but also that

$$\max|\bar{W}' - \bar{W}_o'|$$

and

$$\max|\bar{W}'' - \bar{W}_o''|$$

is small, so that not only $\bar{W}$ must be close to $\bar{W}_o$, but also the first and the second derivatives of $\bar{W}$ must be close to those of $\bar{W}_o$. In addition to this, something more is required. If $\varepsilon = 1$, then also closeness of the third derivatives must hold; if $0<\varepsilon<1$, the intermediate Hölder condition is stronger than mere closeness of the second and weaker than closeness ot the third derivatives: the difference of the second derivatives must satisfy a Hölder condition.

Closeness of the telluroid Σ and the earth's surface S in $H^{2+\varepsilon}$ means that the maximum deviation of the surface is small and that, in addition, slopes (first derivatives) and curvatures (second derivatives) are also similar for S and Σ ; in addition, there is a Hölder condition for the difference of the second derivatives.

Part (1) of Hörmander's theorem asserts the existence of a solution provided we have a good approximation Σ for the earth's surface S not only with respect to the maximum deviation between S and Σ, but also with respect to first and second derivatives (plus a Hölder condition), and also a good approximation to potential and gravity.

This condition is obviously very strong. If one uses an ellipsoidal reference field and the telluroid according to the usual definition, then the actual gravity field and the earth's surface probably fall short of this condition. It is, however, not required that the initial approximations for S and W satisfy this condition; it would be sufficient if any intermediate approximation would meet it (because then this intermediate approximation could be considered as the starting point). Still it is not clear even then whether Hörmander's theorem could be applied to the actual earth.

Part (2) of the theorem assures that the resulting surface S will be as smooth as the data: if the data are n times differentiable and if the n-th derivatives satisfy a Hölder condition, then the same will hold true for S.

Part (3) ensures uniqueness but under an even stronger condition, ($H^{3+\varepsilon}$ neighborhood) than for the existence theorem of Part (1) ($H^{2+\varepsilon}$ neighborhood). However, Hörmander thinks it highly probable that $H^{3+\varepsilon}$ could be replaced by $H^{2+\varepsilon}$, so that the condition for uniqueness would be the same as for existence.

In Part 2, integer values of ε are excluded; this reflects the well-known fact that Hölder conditions with $\varepsilon \neq 0$ are essential in potential-theoretical considerations. In Parts 1 and 3, also integer ε are admitted.

In conclusion we may say that Hörmander's theorem, although not directly applicable to the real earth, gives the first mathematically exact results on existence and uniqueness for Molodensky's problem and is thus of fundamental importance.

7. The Gravity Space Approach

Recently, F. Sansò (1977 a,b,c,d) has given a completely different approach to the nonlinear Molodensky problem. The idea is to use the three cartesian components of actual gravity, g_1, g_2, g_3, as new curvilinear coordinates, instead of the cartesian coordinates x_1, x_2, x_3 themselves. Thus the potential W becomes a function of the g_i,

$$W = W(\underline{g}) = W(g_1, g_2, g_3) \quad . \qquad (7\text{-}1)$$

On the physical earth's surface S, the three components g_i of the vector $\underline{g}$ are given, as well as the potential W; therefore the three curvilinear coordinates g_1, g_2, g_3 of each point of the surface S are known, or <u>S is a known surface if expressed in terms of coordinates g_i</u>.

There are two ways of interpreting g_i: either they may be considered as curvilinear coordinates in ordinary space, or as cartesian coordinates in an auxiliary space, called <u>gravity space</u>. Using the second interpretation, we may say that S becomes a known surface in gravity space or <u>the free boundary-value problem is transformed into a fixed boundary-value problem</u>. The simplification which is achieved in this way for the nonlinear Molodensky problem is so decisive that inconveniences and difficulties arising with this indirect approach are more than compensated as far as theoretical investigations on existence and uniqueness of the solution are concerned.

The main inconvenience is that a one-to-one correspondence between cartesian coordinates x_j and g_j outside and on S, which is the region in which we work, is possible only if the earth is nonrotating. To see this, consider ellipsoidal gravity $\underline{\gamma}$ along a radius vector in the equatorial plane. As the height increases,

γ first decreases but then it increases again because the centrifugal force becomes dominant. So at a certain elevation, γ will be the same as on the ground, which violates a one-to-one correspondence between gravity vector and position. For the actual gravity field the situation is similar as in the ellipsoidal case.

If the earth is considered as nonrotating, then the correspondence between gravity and position is seen to be one-to-one (provided the Marussi condition holds, see below). In other terms, the correspondence is unique if we work with the gravitational potential V and the gravitational vector grad V instead of the gravity potential W and the gravity vector g = grad W .

It is, of course, clear that only W and grad W (including the centrifugal force) are directly measurable. However, the effect of centrifugal force can be calculated with sufficient accuracy on the basis of our current knowledge of the earth's surface (the error in the centrifugal force is less than $\pm$ 0.005 mgal for a position error of $\pm$ 10 meters), and subtracted from W and grad W to give their gravitational counterparts V and grad V . Therefore, Sansò's boundary-value problem, which uses gravitation instead of gravity, is practically as meaningful as the original Molodensky problem.

In the sequel we shall thus work with the gravitational potential V , which is a harmonic function, and take $\underline{g}$ as

$$\underline{g} = \text{grad } V \quad , \tag{7-2}$$

so that $\underline{g}$ is the vector of gravitation rather than gravity. We shall, however, continue to call g , even if defined by (7-2), the gravity vector, to be in agreement with Sansò's terminology and with the term "gravity space" (this is consistent with current terminology if we consider the earth as nonrotating).

We can thus reformulate Molodensky's problem in terms of V as follows: to find a function $V(\underline{x})$ which is harmonic outside a unknown closed surface S,

$$\Delta V = 0 \tag{7-3}$$

and which, together with its gradient, assumes on S the given boundary values

$$V \circ S = \overline{V}(u) \tag{7-4}$$

$$(\text{grad } V) \circ S = \overline{\underline{g}}(u) \tag{7-5}$$

where

$$u = (\Phi, \Lambda) \tag{7-6}$$

as in sec.3.

We now introduce the components g_i of $\underline{g}$ as new coordinates, which are functions of the rectangular coordinates (x_1, x_2, x_3) :

$$g_i = g_i(x_j) \quad . \tag{7-7}$$

If this transformation is to have an inverse,

$$x_j = x_j(g_k) \quad , \tag{7-8}$$

then the Jacobian determinant

$$\det \left[\frac{\partial g_i}{\partial x_j} \right]$$

must be nonzero everywhere on and outside S . Since

$$g_i = \frac{\partial V}{\partial x_i} \quad , \tag{7-9}$$

this condition is

$$\det \left[\frac{\partial^2 V}{\partial x_i \partial x_j} \right] \neq 0 \quad , \tag{7-10}$$

which is nothing else than the Marussi condition; this condition (restricted to S) has already been used before; cf.eq. (3-48). It will be assumed that the Marussi condition is satisfied everywhere outside and on S .

Now the potential V becomes a function of the vector $\underline{g}$:

$$V = V(\underline{g}) = V(g_1, g_2, g_3) \tag{7-11}$$

As we have mentioned, this would reduce Molodensky's problem to a fixed boundary-value problem (actually a Dirichlet problem) in gravity space. Since V as a function of $\underline{x}$ satisfies a linear partial differential equation of second order, which is Laplace's equation $\Delta V = 0$, it does the same as a function of $\underline{g}$ since the transformation (7-7) or (7-8) transforms Laplace equation into another linear second-order partial differential equation. However, since the transformation (7-8) is actually unknown, the coefficients of this differential equations are not known, and therefore this approach appears hopeless.

Sansô has found an ingenious way out of this difficulty by transforming not only the coordinates but also the potential, introducing an <u>adjoint potential</u>

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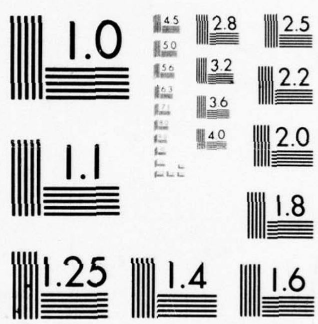

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$$\Psi = \underline{x} \cdot \underline{g} - V = x_k g_k - V \quad ; \tag{7-12}$$

this is a Legendre transformation familiar from other fields (ordinary differential equations, analytical mechanics, thermodynamics, etc.).

Differentiating (7-12) with respect to g_i we get

$$\frac{\partial \Psi}{\partial g_i} = \frac{\partial x_k}{\partial g_i} g_k + x_i - \frac{\partial V}{\partial x_k} \frac{\partial x_k}{\partial g_i}$$

$$= x_i$$

in view of (7-9). Thus

$$x_i = \frac{\partial \Psi}{\partial g_i} \tag{7-13}$$

or

$$\underline{x} = \text{grad}_g \Psi \quad , \tag{7-14}$$

which shows a striking symmetry between x_i and V on the one hand and g_i and ψ on the other hand.

Also (7-12) is completely symmetric

$$V + \Psi = x_k g_k \quad , \tag{7-15}$$

and permits to express one potential in terms of the other:

$$\Psi = x_k \frac{\partial V}{\partial x_k} - V \quad , \tag{7-16}$$

$$V = g_k \frac{\partial \Psi}{\partial g_k} - \Psi \quad . \tag{7-17}$$

The matrix of second gradients of Ψ ,

$$\underline{M}_\Psi = \left[\frac{\partial^2 \Psi}{\partial g_i \partial g_j} \right] = \left[\frac{\partial x_i}{\partial g_j} \right] \quad , \tag{7-18}$$

(by (7-13)) is inverse to the matrix of second gradients of V :

$$\underline{M}_V = \left[\frac{\partial^2 V}{\partial x_i \partial x_j} \right] = \left[\frac{\partial g_i}{\partial x_j} \right] \quad , \tag{7-19}$$

cf. (2-35) and (2-36); that is,

$$\underline{M}_V = \underline{M}_\Psi^{-1} \quad . \tag{7-20}$$

Now Laplace's operator

$$\Delta V = \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} \tag{7-21}$$

is nothing else than the trace Tr of the matrix $\underline{M}_V$, and Laplace's equation may be written

$$\mathrm{Tr}\, \underline{M}_V = 0 \tag{7-22}$$

This gives us a possibility to find the corresponding partial differential equation for $\Psi(g_1, g_2, g_3)$: by combining (7-20) and (7-22) we get

$$\mathrm{Tr}(\underline{M}_\Psi^{-1}) = 0 \tag{7-23}$$

On introducing

$$\Psi_{ij} = \frac{\partial^2 \Psi}{\partial g_i \partial g_j} , \tag{7-24}$$

the matrix $\underline{M}_\Psi$ becomes

$$\underline{M}_\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} \end{bmatrix} \tag{7-25}$$

Inverting this matrix and taking the trace gives

$$\Psi_{11}\Psi_{22} - \Psi_{12}^2 + \Psi_{22}\Psi_{33} - \Psi_{23}^2 + \Psi_{11}\Psi_{33} - \Psi_{13}^2 = 0 , \tag{7-26}$$

which is a partial differential equation for $\Psi(g_1, g_2, g_3)$ with known coefficients (all $\pm$ 1), but unfortunately a nonlinear one. (In two dimensions, we would again get Laplace's equation for $\Psi(g_1, g_2)$.)

The basic differential equation (7-26) may also be written in the form

$$(\mathrm{Tr}\ \underline{M}_\Psi)^2 - \mathrm{Tr}(\underline{M}_\Psi^2) = 0 , \tag{7-27}$$

which is verified by direct calculation.

In gravity space, the vector $\underline{g}$ is the position vector, the components g_i serve as rectangular coordinates and gravity g serves as radius vector. In fact, we have from (1-7)

$$g_1 = g \cos\Phi \cos\Lambda \;,$$
$$g_2 = g \cos\Phi \sin\Lambda \;, \qquad (7\text{-}28)$$
$$g_3 = g \sin\Phi \;,$$

where Φ and Λ are the astronomical coordinates (for a non-rotating earth or after removal of centrifugal effects). This shows that g, Φ, Λ are nothing else than spherical polar coordinates in gravity space. The derivative $\partial/\partial g$ is thus a radial derivative in gravity space; we have

$$\frac{\partial\Psi}{\partial g} = \frac{\partial\Psi}{\partial g_k}\frac{\partial g_k}{\partial g} = \frac{\partial\Psi}{\partial g_k}\frac{g_k}{g} \;,$$

using (7-28). Thus

$$g\frac{\partial\Psi}{\partial g} = g_k\frac{\partial\Psi}{\partial g_k} \qquad (7\text{-}29)$$

and (7-17) may be written as

$$V = g\frac{\partial\Psi}{\partial g} - \Psi \;. \qquad (7\text{-}30)$$

The boundary condition in gravity space thus becomes

$$\left(g\frac{\partial\Psi}{\partial g} - \Psi\right)\circ S_g = \overline{V}(u) \quad , \tag{7-31}$$

where the known function $\overline{V}(u)$ is given as a function of the parameter (7-6) which in gravity space denotes the two angular spherical coordinates; S_g is the image of the earth's surface in gravity space.

For large values of the spatial radius vector

$$r = \sqrt{x_k x_k} = |\underline{x}| \tag{7-32}$$

we have

$$V = \frac{\mu}{r} + O\left(\frac{1}{r^3}\right) \quad , \tag{7-33}$$

$$g = \frac{\mu}{r^2} + O\left(\frac{1}{r^4}\right) \quad , \tag{7-34}$$

where

$$\mu = GM \tag{7-35}$$

denotes the product of the gravitational constant G and the earth's mass M ; we have taken the coordinate origin at the earth's center of mass.

For $r \to \infty$ we have $g \to 0$, so that the spatial infinity corresponds to the origin in gravity space. Solving (7-34) for $1/r$,

$$\frac{1}{r} = \mu^{-\frac{1}{2}} g^{\frac{1}{2}} + O(g^2) \quad , \tag{7-36}$$

and substituting this into (7-33) we get

$$V = \mu^{\frac{1}{2}} g^{\frac{1}{2}} + O(g^{\frac{3}{2}}) \quad , \tag{7-37}$$

which expresses the behavior of V as $g \to 0$. Finally,

$$\Psi = - 2\mu^{\frac{1}{2}} g^{\frac{1}{2}} + O(g^{\frac{3}{2}}) \quad , \tag{7-38}$$

which is verified by substitution into (7-30), taking (7-37) into account.

We thus arrive at the following formulation of the geodetic boundary-value problem in gravity space: to find the solution of the partial differential equation (7-26) outside S_g with the boundary condition (7-31) on S_g ; the earth's surface S will then be given by (7-14):

$$\underline{x} \circ S = (\text{grad}_g \Psi) \circ S_g \quad , \tag{7-39}$$

where $\underline{x} \circ S$ denotes the position vector $\underline{x}$ restricted to the surface S , that is, the position vector of any surface point, $\underline{x}(\Phi, \Lambda)$.

Since the direction $\partial/\partial g$ is the direction of the radius vector in gravity space, in general different from the normal to S_g , we have an oblique-derivative problem with a known surface S_g and a linear boundary condition (7-31), but for a nonlinear partial differential equation (7-26).

8. Linearization; Comparison to the Standard Approach

The linearized equation (2-43) shows a striking formal analogy with (7-17). To take a closer look at this analogy, we shall also linearize (7-17) and other relations in gravity space.

We shall use the concept of the gravimetric telluroid explained in sec.2: there is a one-to-one correspondence between the points P of the earth's surface S and Q of the gravimetric telluroid S by postulating

$$\gamma_i(Q) = g_i(P) \quad , \tag{8-1}$$

that is, the normal gravity vector at Q is to be equal to the actual gravity vector at P ; cf. Fig.2.

As always in the gravity space approach, we assume the earth as nonrotating or, which is the same, the potential is the gravitational potential V . The normal gravitational potential will be denoted by $\tilde{V}$. Then the disturbing potential T is

$$T = V - \tilde{V} \quad ; \tag{8-2}$$

it is the same as in the usual definition T = W - U since the centrifugal potential cancels in the difference.

The adjoint potentials corresponding to V and $\tilde{V}$ are given by (7-12):

$$\psi(g_i) = g_k x_k(g_i) - V\left[x_j(g_i)\right] \quad , \tag{8-3}$$

$$\tilde{\psi}(g_i) = g_k \xi_k(g_i) - \tilde{V}\left[\xi_j(g_i)\right] \quad . \tag{8-4}$$

Here we have been careful in specifying the arguments. The gravity space for normal gravity is identified with the gravity space of actual gravity: equal numerical values of g_i and γ_i correspond to the same point in gravity space. It is, therefore, possible to denote the independent variable in gravity space simply by g_i, also when the normal potential is under consideration, for instance, in (8-4).

Equations (7-7) and (7-8) give the transformation between ordinary space and gravity space for actual gravity. The correspon-ding transformations, between ordinary space and gravity space, for normal gravity are given by

$$g_i = \gamma_i(x_j) \quad , \tag{8-5}$$

$$x_i = \xi_i(g_j) \quad . \tag{8-6}$$

In (8-5), g_i denote the <u>coordinates in gravity space</u>, and $\gamma_i(x_j)$ are the <u>functions</u> which express normal gravity in terms of the coordinates x_j; the $\xi_i(g_j)$ in (8-6) are the <u>inverse functions</u> of $\gamma_i(x_j)$. This will explain the notation used in (8-3) and (8-4).

It is clear now that

$$x_i(g_j) = x_i(P) \tag{8-7}$$

are the coordinates of the point P and

$$\xi_i(g_j) = x_i(Q) \tag{8-8}$$

are the coordinates of the point Q, in view of (8-1); for the same reason, P and Q are mapped into the same point in gravity space:

$$Q_g = P_g \quad , \tag{8-9}$$

to which both $\Psi(g_i)$ and $\tilde{\Psi}(g_i)$ in (8-3) and (8-4) refer.

Let us now calculate the difference

$$\tau = \Psi - \tilde{\Psi} \quad , \tag{8-10}$$

which is the gravity space equivalent of the disturbing potential T as given by (8-2). Subtracting (8-3) and (8-4) we get

$$\tau(g_i) = g_k \left[x_k(g_i) - \xi_k(g_i) \right] - - V\left[x_j(g_i)\right] + \tilde{V}\left[\xi_j(g_i)\right] \quad . \tag{8-11}$$

In agreement with (2-15) we put

$$x_j = \xi_j + \zeta_j \tag{8-12}$$

(we omit the argument g_i : x_j denotes the coordinates of P and ξ_j those of Q). Now to $V\left[x_j(g_i)\right]$ we apply Taylor's theorem:

$$\begin{aligned} V(x_j) &= V(\xi_j + \zeta_j) \\ &= V(\xi_j) + \frac{\partial V}{\partial x_k}\zeta_k \\ &= V(\xi_j) + g_k\zeta_k \quad . \end{aligned} \tag{8-13}$$

The substitution of (8-12) and (8-13) into (8-11) gives

$$\tau(g_i) = g_k \zeta_k - V(\xi_j) - g_k \zeta_k + \tilde{V}(\xi_j)$$

$$= - V(\xi_j) + \tilde{V}(\xi_j)$$

or

$$\tau(g_i) = - T \left[\xi_j(g_i) \right] \quad . \tag{8-14}$$

In geometrical terms, τ at $P_g = Q_g$ equals the negative of T at Q .

We have thus obtained the result that the adjoint potential of T is simply the negative of T . This is certainly surprising at first sight, and it indicates a deep relation between gravity space and ordinary space: gravity space is not just an artifice introduced <u>ad hoc</u> , but a natural expression of the mathematical structure of the geodetic boundary-value problem.

This will even become more evident if we consider the boundary condition. In view of (8-1) we have

$$S_g = \Sigma_g \, , \tag{8-15}$$

so that the earth's surface S and the telluroid Σ are mapped into the same surface S_g in gravity space.

By (7-29), the boundary condition (7-31) becomes

$$(g_k \frac{\partial \Psi}{\partial g_k} - \Psi) \circ S_g = \overline{V}(u) \quad . \tag{8-16}$$

The corresponding condition for the normal potential $\tilde{V}$ at the telluroid Σ is

$$(g_k \frac{\partial \tilde{\Psi}}{\partial g_k} - \tilde{\Psi}) \circ S_g = \bar{\bar{V}}(u) \tag{8-17}$$

The subtraction of these two equations, which are linear in Ψ and $\tilde{\Psi}$, gives by (8-10):

$$(g_k \frac{\partial \tau}{\partial g_k} - \tau) \circ S_g = \bar{V}(u) - \bar{\bar{V}}(u) \quad . \tag{8-18}$$

Now,

$$\bar{V}(u) - \bar{\bar{V}}(u) = \bar{W}(u) - \bar{U}(u)$$

$$= W_P - U_Q$$

because $\bar{W}$ refers to S and $\bar{U}$ to $\sum$ and because the difference of the centrifugal potentials at P and at Q is negligibly small. By (2-3) this is

$$\bar{V}(u) - \bar{\bar{V}}(u) = \Delta W \quad . \tag{8-19}$$

Furthermore τ on S_g equals $-T$ on $\sum$. Thus (8-18) becomes

$$- g_k \frac{\partial T}{\partial g_k} + T = \Delta W \quad , \tag{8-20}$$

which is now a boundary condition on the telluroid $\sum$. The replacement of g_k by γ_k changes (8-20) only be second-order quantities, which are to be neglected. Thus the boundary condition on $\sum$ finally takes the form

$$T - \gamma_k \frac{\partial T}{\partial \gamma_k} = \Delta W \quad . \tag{8-21}$$

This is nothing else than (2-43) with $\Delta \underline{g} = 0$ for the gravimetric telluroid, and with (2-44). We thus have recovered the fundamental boundary condition of sec.2 via gravity space.

What about the differential equation which τ must satisfy? We could derive it from (7-26), but there is a much simpler way, using (8-14). In this equation we substitute (8-5) and (8-6), obtaining

$$\tau\left[\gamma_i(x_j)\right] = -T(x_i) \tag{8-22}$$

Since T satisfies Laplace's equation

$$\Delta T = 0 \quad , \tag{8-23}$$

$\tau = -T$ will also satisfy it:

$$\Delta\tau = 0 \quad ; \tag{8-24}$$

if τ is considered as a function of γ_i, then the Laplacian is to be expressed in terms of γ_i, which here are to be regarded as curvilinear coordinates in ordinary space related to the x_i by (8-5). It is not difficult to transform the Laplacean to curvilinear coordinates; cf. (Hotine,1969,p.19); the important thing to note is that it does <u>not</u> have the "cartesian" form:

$$\frac{\partial^2}{\partial\gamma_1^2} + \frac{\partial^2}{\partial\gamma_2^2} + \frac{\partial^2}{\partial\gamma_3^2} \; . \tag{8-25}$$

Thus, as far as the <u>linear</u> problem goes, the gravity space approach simply amounts to the use of curvilinear coordinates in ordinary space. It is, therefore, not essentially different from the usual approach outlined in sec.2; it is even less general as it supposes a nonrotating earth. The situation is quite different for the nonlinear problem where the gravity space approach introduces essentially new features and a considerable simplification.

Different as the ordinary approach and the use of gravity space are, the linearized problem is the same in both methods. This is practically important because the linearized Molodensky problem is probably sufficient for all present applications, as we have pointed out at the end of sec.2.

Even for the linear problem, however, the gravity space approach provides a deeper insight into the problem; in particular, the structure of the operator that acts on T in (2-43),

$$T - \gamma_i \frac{\partial T}{\partial \gamma_i} , \tag{8-26}$$

is interpreted by the relation between potential and adjoint potential as expressed by (7-17).

Spherical Approximation.- Let us finally introduce a spherically symmetric normal potential; this corresponds to the "spherical approximation" outlined in sec.2.

For a spherically symmetric mass configuration we have

$$\tilde{V} = \frac{\mu}{r} ;$$

cf. (7-33). By a simple change of scale of length and without loss of generality we can make $\mu = 1$, obtaining

$$\tilde{V} = \frac{1}{r} . \tag{8-27}$$

Differentiation with respect to x_i gives

$$\gamma_i = -\frac{x_i}{r^3} \quad ,$$

so that

$$\gamma = \frac{1}{r^2} \tag{8-28}$$

with

$$\gamma^2 = \gamma_k \gamma_k \quad , \quad r^2 = x_k x_k \quad . \tag{8-29}$$

It is, therefore, possible to express the x_i in terms of γ_i by

$$x_i = -\gamma^{-\frac{3}{2}} \gamma_i \quad ; \tag{8-30}$$

in the case of a spherically symmetric mass configuration, cartesian coordinates x_i and gravimetric coordinates γ_i are thus related in a simple way.

Another possibility to convert gravimetric coordinates into cartesian coordinates, denoted by y_i , is by putting

$$y_i = \gamma^{-\frac{1}{2}} \gamma_i \quad . \tag{8-31}$$

These coordinates y_i can be interpreted in the following way. Let us consider an <u>inversion in the unit sphere</u> $r = 1$; cf. (Kellogg, 1929,p.231). This inversion transforms a point with coordinates x_i into a point with coordinates x_i' given by

$$x_i' = \frac{1}{r^2} x_i \tag{8-32}$$

the inverse transformation being

$$x_i = \frac{1}{r'^2} x_i' \quad \text{with} \quad r'^2 = x_k' x_k' \ . \tag{8-33}$$

On substituting (8-30) and comparing the result with (8-31) we see that

$$y_i = - x_i' \ , \tag{8-34}$$

so that, apart from the sign, y_i are the cartesian coordinates of the image of the point x_i under an inversion in the unit sphere.

The corresponding transformation of harmonic functions is called a Kelvin transformation (ibid.,p.232). The basic principle is that if $U(x_i)$ is a harmonic function of x_i in a domain T, then

$$V(x_i') = \frac{1}{r'} U\left(\frac{x_i'}{r'^2}\right) \tag{8-35}$$

is a harmonic function of x_i' in the domain T' into which T is carried by the inversion.

So far, we have interpreted this transformation as a point transformation, which transforms a point $P(x_i)$ into a point $P'(x_i')$, the coordinates x_i and x_i' referring to the same cartesian coordinate system. We may, however, interpret it also as a coordinate transformation, by which the same point in space is referred to different coordinate systems x_i and x_i' . Then the Kelvin transformation implies that if $U(x_i)$ satisfies Laplace's equation in "cartesian" form using x_i :

$$\Delta_x U = \frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} = 0 \quad , \tag{8-36}$$

then the function (8-35) satisfies Laplace's equations in cartesian form using x'_i :

$$\Delta_{x'} V = \frac{\partial^2 V}{\partial x_1'^2} + \frac{\partial^2 V}{\partial x_2'^2} + \frac{\partial^2 V}{\partial x_3'^2} = 0 \quad . \tag{8-37}$$

In view of (8-34), Laplace's operator will then have cartesian form also in coordinates y_i :

$$\Delta_y V = 0 \quad . \tag{8-38}$$

The symbol Δ_x, Δ_y, etc. will be reserved for Laplace's operator in cartesian form.

Let us now apply these considerations to the present problem. We have seen that $T(x_i)$ satisfies Laplace's equation $\Delta_x T = 0$; cf. (8-23). $\tau(\gamma_i)$ also satisfies Laplace's equation (8-24), but not in cartesian form (8-25). If in $\tau(\gamma_i)$ we introduce new coordinates y_i , defined by (8-31), putting

$$\gamma_i = y_i y \quad \text{with} \quad y^2 = y_k y_k \quad , \tag{8-39}$$

then the new function

$$\phi(y_i) = \frac{\tau}{y} = - \frac{T}{y} \tag{8-40}$$

will satisfy

$$\Delta_y \phi = 0 \tag{8-41}$$

because of (8-35) with $U = -T$, $V = \phi$ and $y_i = -x_i'$, since T satisfies $\Delta_x T = 0$.

Also the function v defined by

$$v(y_i) = \frac{1}{2}\left(y_i \frac{\partial \phi}{\partial y_i} - \phi\right) \tag{8-42}$$

is harmonic:

$$\Delta_y v = 0 \quad . \tag{8-43}$$

This can be easily verified by direct calculation: there is

$$2\Delta_y v = \Delta_y \phi + y_i \frac{\partial}{\partial y_i} \Delta_y \phi \quad . \tag{8-44}$$

The interpretation of v is as follows. Consider ΔW as given by (8-21). (Of course, Δ has here nothing to do with Laplace's operator!) It may also be expressed in terms of τ by

$$\Delta W = \gamma_k \frac{\partial \tau}{\partial \gamma_k} - \tau \quad . \tag{8-45}$$

In (8-21) , ΔW has been considered as defined on the telluroid Σ . It may, however, also be regarded as a spatial function, defined outside and on Σ , since τ is a function of the γ_k which can be interpreted as curvilinear coordinates in space. If now ΔW , regarded as a spatial function, is expressed in terms of y_i, we can also transform (8-45) to these coordinates. This is best done by transforming it first to the form

$$\Delta W = \gamma \frac{\partial \tau}{\partial \gamma} - \tau \quad , \tag{8-46}$$

using (7-29) with γ_i instead of g_i. Now

$$\gamma = \sqrt{\gamma_k \gamma_k} \tag{8-47}$$

is related to

$$y = \sqrt{y_k y_k} \tag{8-48}$$

by

$$\gamma = y^2 \quad , \quad y = \sqrt{\gamma} \quad , \tag{8-49}$$

by (8-39). Therefore,

$$\gamma \frac{\partial \tau}{\partial \gamma} = y^2 \frac{\partial \tau}{\partial y} \frac{dy}{d\gamma} = \frac{1}{2} y \frac{\partial \tau}{\partial y} \quad , \tag{8-50}$$

and (8-46) takes the form

$$\Delta W = \frac{1}{2} y \frac{\partial \tau}{\partial y} - \tau \quad . \tag{8-51}$$

Substituting

$$\tau = y\phi \tag{8-52}$$

according to (8-40), we get

$$\Delta W = \frac{1}{2}\, y\frac{\partial(y\phi)}{\partial y} - y\phi$$

$$= \frac{1}{2}\, y\left(y\frac{\partial\phi}{\partial y} - \phi\right) \quad . \qquad (8\text{-}53)$$

Using again (7-29) with y_k instead of g_k we obtain

$$\Delta W = \frac{1}{2}\, y\left(y_i\frac{\partial\phi}{\partial y_i} - \phi\right) \quad , \qquad (8\text{-}54)$$

and the comparison with (8-42) shows that

$$v = \frac{\Delta W}{y} \quad . \qquad (8\text{-}55)$$

This furnishes the desired physical interpretation of v .

These two auxiliary functions

$$\phi(y_i) = \frac{\tau}{y} = -\frac{T}{y} \quad , \qquad (8\text{-}56)$$

$$v(y_i) = \frac{1}{2}\left(y_i\frac{\partial\phi}{\partial y_i} - \phi\right) = \frac{\Delta W}{y} \quad , \qquad (8\text{-}57)$$

which, in the case of spherical symmetry, satisfy Laplace's equation:

$$\Delta_y\phi = 0 \quad , \qquad (8\text{-}58)$$

$$\Delta_y v = 0 \quad , \qquad (8\text{-}59)$$

will play a basic role in the next section.

9. The Nonlinear Problem

Reformulation of the Problem.- If ψ is a solution of the boundary-value problem defined by the differential equation (7-26) and the boundary condition (7-31), which, in view of (7-29), may be written in the form

$$g_k \frac{\partial \Psi}{\partial g_k} - \psi = \nabla(u) \quad \text{on} \quad S_g \quad , \tag{9-1}$$

then the function

$$\hat{\Psi} = \Psi + c_i g_i \quad , \tag{9-2}$$

with an arbitrary constant vector c_i, is also a solution of the problem. In fact,

$$\hat{\Psi}_{ij} = \frac{\partial^2 \hat{\Psi}}{\partial g_i \partial g_j} = \frac{\partial^2 \Psi}{\partial g_i \partial g_j} = \Psi_{ij} \quad , \tag{9-3}$$

so that $\hat{\Psi}$ satisfies (7-26) if Ψ does, and

$$g_k \frac{\partial \hat{\Psi}}{\partial g_k} - \hat{\Psi} = g_k \frac{\partial \Psi}{\partial g_k} - \Psi_k \quad , \tag{9-4}$$

so that the boundary condition is also satisfied.

It is easily seen that the addition of the term $c_i g_i$ to Ψ represents a translation by the vector c_i in ordinary space: by (7-13) we get

$$\hat{x}_i = \frac{\partial \hat{\Psi}}{\partial g_i} = \frac{\partial \Psi}{\partial g_i} + c_i = x_i + c_i \quad . \tag{9-5}$$

We obtain a unique solution by requesting Ψ to have the form (7-38), which places the x-coordinate system at the earth's center of mass. This is in complete correspondence with the usual treatment of Molodensky's problem.

However, the solution will not exist for arbitrary boundary values $\overline{V}$ but only for those functions $\overline{V}(u)$ which satisfy n conditions; from the discussion of the linearized problem we expect $n = 3$. It is true that if we had idealized conditions, especially absence of measuring errors, then the data function $\overline{V}(u)$ would satisfy these conditions because the solution exists for physical reasons. In practice, however, especially because of measuring and interpolation errors, we cannot expect that the actual $\overline{V}(u)$ will exactly satisfy these conditions.

This suggests a reformulation of the boundary-value problem in gravity space along the lines of Hörmander's formulation; cf.sec.6, especially eq. (6-11): we replace the boundary condition (9-1) by

$$g_k \frac{\partial \Psi}{\partial g_k} - \Psi = \overline{V}(u) + a_i g_i \quad \text{on} \quad S \quad . \tag{9-6}$$

The new boundary-value problem can now be expected to have a solution for arbitrary data functions $\overline{V}(u)$. The three constants a_1, a_2, a_3 are determined as unknowns and, so to speak, take care of the three conditions.

<u>Transformation of the Differential Equation</u>.- The main difficulty in the gravity space approach lies in the differential equation for the adjoint potential Ψ . This equation, given by (7-26) or (7-27), is unfortunately considerably more complicated than Laplace's equation for the original potential V .

The consideration of the spherical approximation in the preceding section suggests, however, that it may be possible to reduce, at least approximately, this differential equation to Laplace's equation.

First, in agreement with (7-38), we split off the main part in Ψ by putting

$$\Psi(g_i) = -2\mu^{\frac{1}{2}} g^{\frac{1}{2}} + \tau(g_i) \quad . \tag{9-7}$$

where

$$g^2 = g_k g_k \quad . \tag{9-8}$$

This may be interpreted by (8-10) as using a spherically symmetric reference potential

$$\tilde{\Psi} = -2\mu^{\frac{1}{2}} g^{\frac{1}{2}} \tag{9-9}$$

in gravity space (the reader will find it best to consider all transformations to follow as <u>transformations in gravity space</u> and to forget, for the time being, about ordinary space). In contrast to the linear treatment in the preceding section we shall not introduce any approximations, so that the transformed differential equations will be as rigorous as the original one, eq. (7-26).

The reference potential (9-9), which is spherically symmetric in gravity space, is the adjoint potential of a potential $\tilde{V}$ that is spherically symmetric in ordinary space. In fact, by (7-13) and (7-15),

$$x_k = \frac{\partial \tilde{\Psi}}{\partial g_k} = -\mu^{\frac{1}{2}} g^{-\frac{3}{2}} g_k \quad , \tag{9-10}$$

$$r = \sqrt{x_k x_k} = \mu^{\frac{1}{2}} g^{-\frac{1}{2}} , \qquad (9\text{-}11)$$

$$\tilde{V} = g_k x_k - \tilde{\Psi} = - \mu^{\frac{1}{2}} g^{\frac{1}{2}} + 2\mu^{\frac{1}{2}} g^{\frac{1}{2}}$$

$$= \mu^{\frac{1}{2}} g^{\frac{1}{2}} = \frac{\mu}{r} . \qquad (9\text{-}12)$$

Eq. (8-31) suggests the substitution

$$y_i = g^{-\frac{1}{2}} g_i . \qquad (9\text{-}13)$$

(Now, however, the y_i are to be considered as curvilinear coordinates in gravity space, having no direct relation with cartesian coordinates in ordinary space.) This transforms the reference potential into

$$\tilde{\Psi} = - 2\mu^{\frac{1}{2}} y , \qquad (9\text{-}14)$$

eliminating the singularity $g^{\frac{1}{2}}$ at the origin $g = 0$.

We now introduce the new function

$$\phi = \frac{\tau}{y} , \qquad (9\text{-}15)$$

so that

$$\tau = y\phi . \qquad (9\text{-}16)$$

If we neglected all squares and higher powers of τ , we should have the linear spherical approximation discussed in the preceding section since, apart from a scale factor, (9-12) is identical to (8-27). This shows that ϕ , as a function of y , must satisfy a differential equation of form

$$\Delta_y \phi = O(\phi^2) \quad ; \tag{9-17}$$

there can be no term $O(\phi)$ on the right-hand side since $\Delta_y \phi = 0$ as a linear approximation, by (8-58).

In fact, Sansò has calculated the exact differential equation which ϕ must satisfy. This is done by substituting

$$\Psi = - 2\mu^{\frac{1}{2}} y + y\phi \tag{9-18}$$

into (7-27) and performing some lengthy but straightforward transformations. The result is (Sansò,1977a,p.69):

$$\Delta_y \phi = \mu^{-\frac{1}{2}} B_1(\phi,\phi) \quad , \tag{9-19}$$

where $B_1(\phi,\phi)$ is a quadratic operator given by

$$B_1(\phi,\phi) = \frac{1}{2}(\phi - y\phi')\Delta_y \phi + y^2 \left[(\mathrm{Tr}\underline{L})^2 - \mathrm{Tr}(\underline{L}^2) \right] \tag{9-20}$$

(it must be quadratic since the original equation (7-26) is). The matrix $\underline{L}$ has elements

$$L_{ij} = (\delta_{ik} - \frac{3}{4} \frac{y_i y_k}{y^2}) \phi_{kj} \tag{9-21}$$

where δ_{ij} denotes the elements of the unit matrix and

$$\phi_{ij} = \frac{\partial^2 \phi}{\partial y_i \partial y_j} \quad ; \tag{9-22}$$

ϕ' is defined by

$$\phi' = \frac{\partial \phi}{\partial y} \quad ,$$

and $\Delta_y \phi$ expresses the Laplace operator in the "cartesian" form

$$\Delta_y \phi = \frac{\partial^2 \phi}{\partial y_1^2} + \frac{\partial^2 \phi}{\partial y_2^2} + \frac{\partial^2 \phi}{\partial y_3^2} \quad ; \tag{9-23}$$

needless to say, y_i are not rigorously to be interpreted as cartesian coordinates in ordinary space.

The boundary operator (9-1)

$$g_k \frac{\partial \Psi}{\partial g_k} - \Psi = g \frac{\partial \Psi}{\partial g} - \Psi \tag{9-24}$$

is transformed as follows. Using (9-9) we find

$$g \frac{\partial \tilde{\Psi}}{\partial g} - \tilde{\Psi} = \mu^{\frac{1}{2}} g^{\frac{1}{2}} \quad , \tag{9-25}$$

so that

$$g \frac{\partial \Psi}{\partial g} - \Psi = \mu^{\frac{1}{2}} g^{\frac{1}{2}} + g \frac{\partial \tau}{\partial g} - \tau \quad . \tag{9-26}$$

Since

$$y = g^{\frac{1}{2}} \tag{9-27}$$

by (9-13), we have

$$\frac{\partial \tau}{\partial g} = \frac{\partial \tau}{\partial y}\frac{dy}{dg} = \frac{1}{2} g^{-\frac{1}{2}} \frac{\partial \tau}{\partial y} \quad . \tag{9-28}$$

In view of these relations we get

$$g_k \frac{\partial \Psi}{\partial g_k} - \Psi = \mu^{\frac{1}{2}} y + \frac{1}{2} y \frac{\partial \tau}{\partial y} - \tau \tag{9-29}$$

On substituting (9-16) and taking (9-6) into account we find as boundary condition for ϕ :

$$y \frac{\partial \phi}{\partial y} - \phi = 2 \left[\bar{v}(u) + a_i \bar{y}_i \right] \quad \text{on } S_g \tag{9-30}$$

where

$$\bar{v}(u) = \frac{\bar{V}(u)}{\bar{y}(u)} - \mu^{-\frac{1}{2}} \tag{9-31}$$

is a function of the data, and

$$\bar{y}(u) = y \circ S_g \quad , \quad \bar{y}_i(u) = y_i \circ S_g \tag{9-32}$$

denote the values of y and y_i calculated for that point of the surface S_g which has the parameter u .

Since the direction of $\partial/\partial y$, as well as the direction of $\partial/\partial g$, is the direction of the radius vector in gravity space, we still have an oblique derivative problem as in the original formulation given at the end of sec.7; the problem is, however, simplified because we now have a "quasilinear" differential equation (9-19), which has a form suitable for an iterative solution.

It is, nevertheless, appropriate to transform the problem still further by introducing a new potential v by

$$v = \frac{1}{2}\left(y_k \frac{\partial \phi}{\partial y_k} - \phi\right) = \frac{1}{2}\left(y\frac{\partial \phi}{\partial y} - \phi\right) \quad . \tag{9-33}$$

This substitution has been motivated in the preceding section; cf. (8-57). As a linear approximation, $v(y_i)$ is harmonic and is, furthermore, related to the potential anomaly ΔW .

In fact, we have even rigorously

$$V(g_i) = \mu^{\frac{1}{2}} g^{\frac{1}{2}} + yv \quad , \tag{9-34}$$

so that yv represents the perturbation in the potential V , if expressed in gravimetric coordinates g_i , in the same way as we had

$$\Psi(g_i) = - 2\mu^{\frac{1}{2}} g^{\frac{1}{2}} + y\phi \quad , \tag{9-35}$$

$\tau = y\phi$ representing the perturbation in the adjoint potential Ψ . It is easy to verify (9-34) by substituting (9-35) into (7-30).

By means of (9-33), eq. (9-19) is finally transformed into a differential equation for v :

$$\Delta_y v = \mu^{-\frac{1}{2}} B_2(v,v) \tag{9-36}$$

where the quadratic operator B_2 is given by

$$B_2(v,v) = -v\Delta_y v - v'\int_o^y \Delta_y v dy + $$

$$+ 2\left[(Tr\underline{N})^2 - Tr(\underline{N}^2)\right]$$

$$+ 4y\left[Tr\underline{M}\cdot Tr\underline{N} - Tr(\underline{MN})\right] \quad . \qquad (9\text{-}37)$$

$\underline{M}$ and $\underline{N}$ are 3x3 matrices with elements

$$M_{ij} = (\delta_{ik} - \frac{3}{4}\frac{y_i y_k}{y^2})v_{kj} \quad , \qquad (9\text{-}38)$$

$$N_{ij} = (\delta_{ik} - \frac{3}{4}\frac{y_i y_k}{y^2})\int_o^y v_{kj} dy \quad , \qquad (9\text{-}39)$$

where δ_{ij} denotes the elements of the unit matrix and

$$v_{ij} = \frac{\partial^2 v}{\partial y_i \partial y_j} \quad ; \qquad (9\text{-}40)$$

v' is defined as

$$v' = \frac{\partial v}{\partial y} \qquad (9\text{-}41)$$

and Δ_y again denotes the "cartesian form" of the Laplace operator. The solution of (9-36) has to satisfy the boundary condition

$$v \circ S_g = \overline{v} + a_i \overline{y}_i \qquad (9\text{-}42)$$

$\bar{v} = \bar{v}(u)$ and $\bar{y} = \bar{y}(u)$ being given by (9-31) and (9-32). This is simply the boundary condition for a Dirichlet problem.

By means of the substitution (9-33) it has thus been possible to transform Sansò's problem into a Dirichlet problem for the nonlinear equation (9-36). The price to be paid is that this equation is a nonlinear integro-differential equation, as (9-37) shows. However, since the principal part of (9-36) is simply Laplace's equation, the quadratic right-hand side being relatively small, our equation is still relatively manageable (it is hardly necessary to remind the reader that (9-36) is as rigorous as the original equation (7-26), no neglections are involved).

This reduction to a Dirichlet problem is similar to methods used in the linear Molodensky problem, cf. (Brovar,1964), (Krarup, 1973: the "Prague method") and the reduction the Brillouin sphere in the present sec.4. The enormous advantage of the gravity space approach is that the boundary condition (9-1) is linear even for the nonlinear problem, so that methods can be used that are applicable to the Molodensky problem only in its linearized form.

A necessary condition for the existence of the solution is

$$\left.\frac{\partial v}{\partial y_i}\right)_{y=0} = 0 \quad . \tag{9-43}$$

In fact, the differentiation of (9-33) gives

$$2\frac{\partial v}{\partial y_i} = - y_k \frac{\partial^2 \phi}{\partial y_i \partial y_k} \quad . \tag{9-44}$$

If the solution ϕ is to be regular with finite second derivatives at the origin $y = \sqrt{y_j y_j} = 0$, then (9-44) must tend to zero as $y_k \to 0$. The condition (9-43) is to be provided for by suitably disposing of the free constants a_1, a_2, a_3 in the boundary condition (9-30).

If a solution v satisfying (9-43) has been found, then ϕ is obtained by

$$\phi = -2v(0) + 2y \int_0^y \left[v(y) - v(0)\right] y^{-2} dy \quad ; \tag{9-45}$$

it is easy to verify by direct substitution that this solution satisfies (9-33). Then the adjoint potential Ψ is given by (9-35), and finally the earth's surface is obtained by (7-39). A check is provided by (9-34).

Compared with the Nash-Hörmander approach to the nonlinear Molodensky problem, it is relatively simple and straightforward to solve (9-36) by a Newton iteration scheme, considering $\mu^{-1/2}$ as a small parameter. In this way, Sansò has obtained first results on existence and uniqueness of the solution. He has proved that a unique solution exists provided

$$\|\bar{v}(u)\|_{2+\varepsilon} < \delta \quad , \tag{9-46}$$

where the constant δ is sufficiently small. The norm is a Hölder norm very similar to the norm used in sec.6.

The condition (9-46) is directly comparable to (6-25); the constants δ will be different. It means that $\bar{v}$ as given by (9-31) should be small, as well as the first and second derivatives, including a Hölder condition on the second derivatives. Since v denotes the deviations from a spherical symmetrical solution, this condition is again very restrictive.

10. Conclusions

During the recent years, the problem of existence and uniqueness of the solution for Molodensky's problem has for the first time been treated with adequate mathematical rigor. Certainly, existence and uniqueness have been proved only under very restrictive conditions on smoothness and smallness of the deviations from a "normal" solution, conditions which are hardly met in the actual geodetic situation. However, these results have been obtained rigorously.

The treatment by Hörmander uses a very advanced inverse function theorem and is mathematically extremely complicated; it applies to a rotating earth. The mathematical complexity is mainly due to the fact that Molodensky's problem is a free boundary-value problem, the boundary surface being unknown.

The gravity space approach due to Sansò transforms the free boundary problem into a fixed one, although for a nonlinear partial differential equation. It nevertheless reduces essentially the mathematical complexity. The limitation of the gravity space approach is the restriction to a nonrotating earth; practically this amounts to the use of gravitation instead of gravity by reducing for the effect of centrifugal force.

From this point of view, an extension of Sansò's approach to a rotating earth by an iterative procedure (using the fact that ω is small) appears less urgent; at any rate it seems to be not quite easy to prove convergence of such an iteration.

The results obtained so far by Hörmander for $\omega \neq 0$ and by Sansò for $\omega = 0$ are comparable; the conditions are similarly restrictive. It would, of course, be desirable to obtain stronger results, for norms $\| \ \|_{1+\varepsilon}$ or even better $\| \ \|_{\varepsilon}$ with $0<\varepsilon<1$.

For this purpose the gravity space approach appears to be more promising since it is so much easier.

The impact of the gravity space approach to the theory of Molodensky's problem appears to be enormous; it may well be comparable to the impact of Hamiltonian methods to Newtonian classical mechanics (both apply a Legendre transformation!).

From a practical point of view it is important to note that the linear approximation (linear in the anomalous potential T), which is probably sufficient for almost all present purposes, is the same in the usual approach and in gravity space. Therefore, the number of the usual methods for practically solving Molodensky's problem, such as Molodensky's series and related solutions, is not augmented by the new developments.

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