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PHILIP RUDNICK

detection in non-gaussian background

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(10) Philip Rudnick

University of California
Marine Physical Laboratory of the
Scripps Institution of Oceanography
San Diego 52, California

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DETECTION IN NON-GAUSSIAN BACKGROUND

Philip Rudnick

ABSTRACT

The problem of detecting a small signal in a non-Gaussian background is discussed analytically for a very limited set of conditions. It is supposed that the background contains bursts of noise whose complete rejection might aid detection. For several specific detection procedures, formulas are derived by which empirically known distribution functions of the background may be used to predict a change in detection threshold resulting from burst rejection. Particular attention is given to the use of the first-order distribution function, even though it alone does not completely determine the results. A suggestion by Birdsall connecting this study with the use of the likelihood ratio is also discussed.

DETECTION IN NON-GAUSSIAN BACKGROUND

Philip Rudnick

INTRODUCTION

Current theories for the detection of a signal in noise have recently been summarized by Peterson, Birdsall and Fox.[1] These theories employ a very elaborate specification of the noise statistics, namely a probability-density function in n dimensions, where n is the number, ordinarily quite large, of parameters required to specify the receiver input in complete detail throughout the allowed integration time. The case of Gaussian statistics leads to distribution functions which can be expressed in terms of such accessible quantities as power and cross correlation coefficient. Consequently the general theory can be carried through to practical conclusions in the Gaussian case, and one has assurance that the indicated methods are the best, in some defined sense.

If the noise background has non-Gaussian statistics, the practical situation is very different. The designer of a detection apparatus is not likely to have full knowledge even of the first and second order statistics of the background, and will quite certainly have no information of n-th order complexity. He will, in addition, wonder whether these statistics, when obtained, will prove to be sufficiently stable and otherwise practically adaptable for use in apparatus design. In this situation realizable first steps in the adaption of detection processes to non-Gaussian noise are needed, even though they be neither general nor optimum.

This study deals mainly with a quite limited and specific question, that of detection in the presence of a background which has a somewhat impulsive or burst-like character, and of rejecting these impulses as a means of improving detection. Particular attention is given the first order distribution (of instantaneous values of background) as a guide to design of detection procedure. Of course, higher order statistics are also relevant and if these are not known, the mathematical foundation is necessarily incomplete.

The calculations that follow are concerned with background noise whose distribution functions are unspecified and largely arbitrary. The results are expected to be of value primarily when they are used in conjunction with empirically observed distribution functions for the background actually present in some detection problem. In this way, one may predict detection thresholds for small signals, from experimental observations on the background statistics alone, and without any actual detection experiments.

NOTATION AND FORMULATION OF PROBLEM

The diagram in Figure 1 indicates the principal class of detection processes to be discussed (though one will also be included to which Figure 1 is not appropriate).

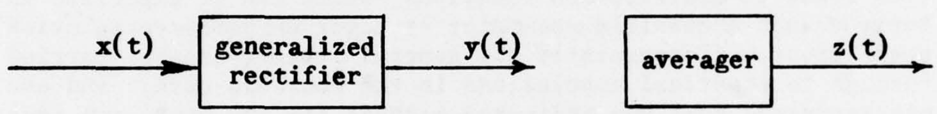

Figure 1. General block diagram of detector.

$x(t)$ represents the detector input. It will be, alternatively, either one of two stationary time series, representing background noise alone, or background plus the small added signal which is to be detected. In either case, the time average $x(t)$ is zero. No frequency-selection procedures are considered here, and it will be supposed that any appropriate filtering has already been done. $x(t)$ will be considered either a continuous function of time, or a series of discrete samples taken at equally spaced instants.

$x_e(t)$ is the envelope of $x(t)$, in the sense used in Appendix I.

$y(t)$ is the output of a generalized rectifier. The instantaneous value of $y(t)$ is either a function of the absolute value of the instantaneous input $|x(t)|$, or of $x_e(t)$, and will also be written $y[x]$. Thus the rectifier may include "integration" over one "period" of a narrow-band input, but not longer.

$z(t)$ is the result of running linear averaging of $y(t)$. The averager may be a low pass filter.

T is the effective averaging time of the averager and must be chosen consistent with some response-time stipulation in the basic detection problem. Since small signals are contemplated, T will always be long compared to the correlation time of $x(t)$, and to the sampling interval, if $x(t)$ is discrete. T is also the correlation time of the output time series $z(t)$. Loosely one may say that $z(t)$ presents one new independent value (on which a judgment of "signal present" or "signal absent" may be based) each T seconds, and that each such value is derived from the input $x(t)$ during a like interval T.

n is, if $x(t)$ or $x_e(t)$ is sampled, the number of such samples occurring in time T.

$t_0 = T/n$, is the interval between samples.

$f(u)$ is the first order probability density for instantaneous values of $|x(t)|$ for background only; $f(u)du$ is the fraction of time during which $u < |x(t)| < u + du$. The derivative of $f(u)$ is assumed to vanish as $u \to 0$.

$f(u) + g(u)$ is the corresponding probability density when $|x(t)|$ also contains the signal.

$F(u)$ and $F(u) + G(u)$ are corresponding probability densities for the envelope $x_e(t)$, without and with signal. $F(u)$ is assumed to vanish as $u \to 0$.

$f_2(u,v,\tau) = f_2(u,v,-\tau)$ is a symmetrized second order probability density for background only; $f_2(u,v,\tau)du\,dv$

is the joint probability that $u < |x(t)| < u + du$ and $v < |x(t + \tau)| < u + du$, or that $u < |x(t + \tau)| < u + du$ and $u < |x(t)| < u + du$; as an alternative f_2 has the corresponding meaning for x_e.

E(w) and V(w) are, respectively, expected value or mean, and variance of any quantity w depending on the input or its envelope, in the absence of signal. w may be a function of the instantaneous value of $|x(t)|$ or $x_e(t)$, or it may be a quantity determined by the whole set of input values extending through an integration interval T.

$E_s(w)$ and $V_s(w)$ are, respectively, mean and variance as above, except that the signal is present.

σ^2 is the variance or "power" of the signal which may be contained in x(t). This is the only property of the signal which needs enter our main discussion.

$$R = \frac{E_s(z) - E(z)}{\left[V(z)\right]^{1/2}} \tag{1}$$

is an output signal-to-noise ratio which will be used here as a criterion of detection capability. The signal is assumed small enough to make a distinction between $V_s(z)$ and V(z) unnecessary in this expression.

N_A, N_B, etc. are special values of the numerator in Eq. (1) referring to specific detection processes A, B, etc., to be defined later.

$V_A(z)$, $V_A(y)$, $V_B(z)$, etc. are special values of V(z) or V(y) also referring to specific detection processes A, B, etc.

Further notation is defined in the following section.

REJECTION PROCESSES, DESCRIPTION

If a background noise contains impulses or bursts of sufficient prominence, it can be conjectured that such bursts may contribute disproportionately to the fluctuation $V_s(z)$ and their elimination may have a benefit outweighing the loss of signal which must also occur. Hence, the next two sections are devoted to calculating the reduction in signal and reduction in noise, respectively, which result when noise bursts are rejected. In this section several specific processes are described, all directed generally toward burst elimination. For this purpose, we define further notation, and assign more specific properties to the detection process.

a is a rejection limit, defined by the condition $y(t) = 0$ whenever $|x(t)| > a$, or alternatively $x_e(t) > a$.

$q = \int_0^a f(x)\,dx$ or $\int_0^a F(x)\,dx$, is the expected fraction of the input $x(t)$ or $x_e(t)$ not rejected, when no signal is present.

$p = 1-q$, is the expected fraction of input rejected.

$\bar{w} = (1/q)\int_0^a wf(x)\,dx$ or the corresponding integral with $F(x)$ replacing $f(x)$. It is the special case of $E(w)$ when averaging is over nonrejected input values only.

$\rho_A(\tau), \rho_B(\tau)$, etc. are autocorrelation coefficients, for noise only, of $y(t)$ in processes A, B, etc. They are normalized to $\rho(0) = 1$. ρ_A is given by Eq. (9). For ρ_B, see Appendix II.

τ_A, τ_B, etc. are correlation times, defined as in Eq. (8) or (11), for processes A, B, etc.

q' is an actual fraction not rejected during an interval T, for a fixed rejection limit a. It is a random variable.

a' is a variable rejection limit, fixed during any one sample (of approximate duration T) but varied from sample to sample in such a way as to reject the same fraction p from each sample.

We now specify three distinct detection processes A, B, and C which may be considered at least loosely as specializations of the scheme outlined in Figure 1, and one additional process, D, of somewhat different character. Each process is understood to include operation on $x(t)$ or $x_e(t)$, either of which may be continuous or discrete.

Process A

All values of $y(t)$, including the zero values resulting from rejection, enter the averaging process. The duration or number of such zero values contained in one integration interval T and affecting one instantaneous value $z(t)$ is of course subject to sampling fluctuations.

Process B

The zero values of $y(t)$ resulting from rejections do not enter the averaging process, and $z(t)$ is an average of non-rejected values of $y(t)$. These values are, of course, not uniformly distributed in real time and might be dealt with instrumentally by an averager capable of intermittent operation in real time. We will, however, presume the non-rejected $y(t)$ values to be stored and then re-presented uniformly on a new time scale in which the gaps arising from rejection have been closed, as illustrated in Figure 2. The labeling of the new time scale will differ from real time only by shifts of origin as one passes from one retained segment to another. On this scale the averaging will be conducted with an integration time qT, corresponding to qT/q' for the original data. The difference between this latter time and T is considered negligible. The maximum amplitude transmission factor of the integrating filter is for convenience taken to be q.

Process C

The rejection limit is the variable a', defined above, which results in a fixed fraction p of zero values of $y(t)$. These are included in a simple average, as in process A. (This same procedure might, however, equally well be considered a variant of process B.)

Process D

In this case the final output of the detection process is taken to be the rejected fraction $(1 - q')$ with q' as defined

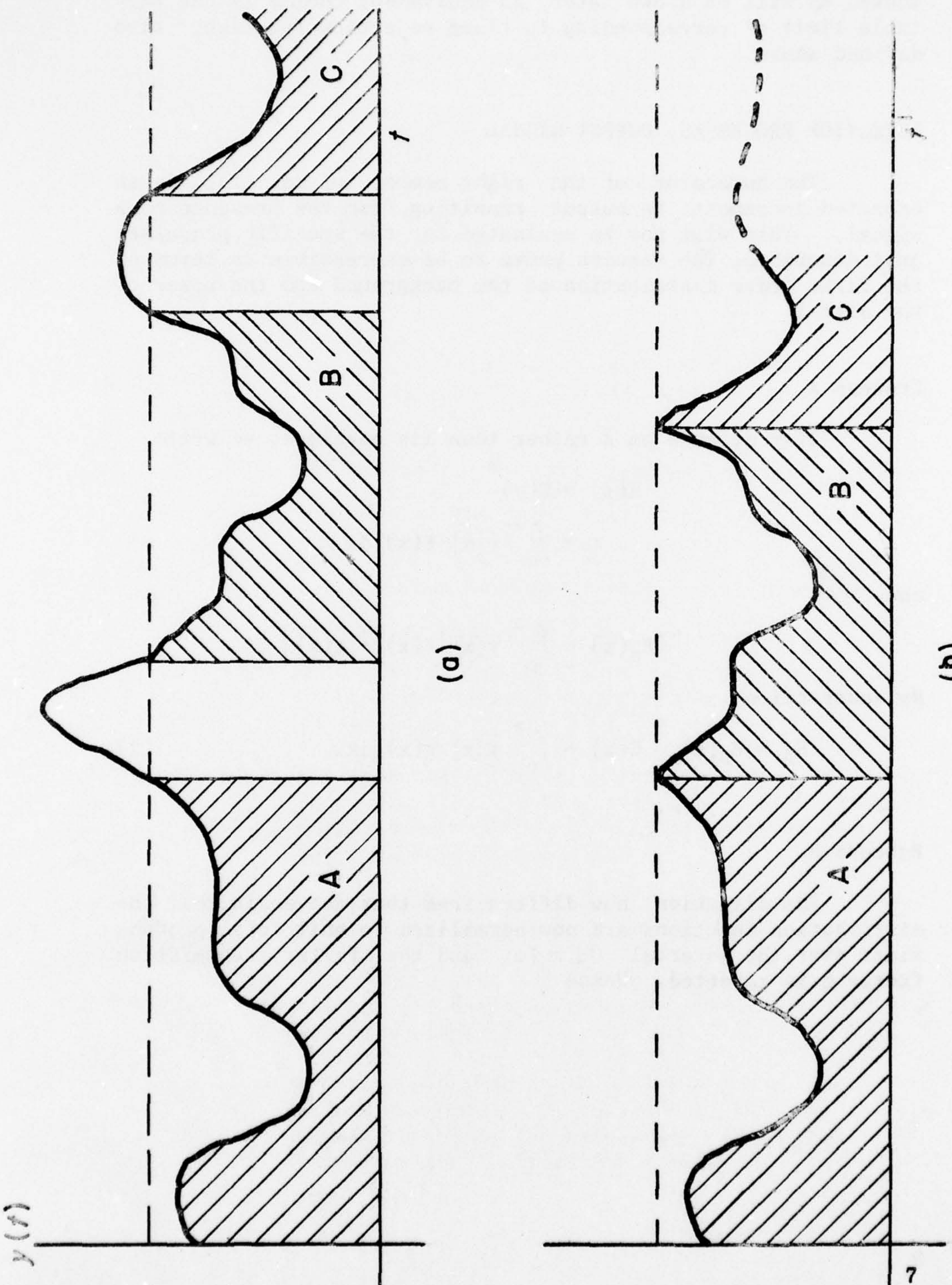

Figure 2. Process B. Illustration of rejection (a) and closure of resulting gaps (b).

above. As will be shown later, an equivalent choice is the variable limit a' corresponding to fixed rejection fraction, also defined above.

REJECTION PROCESSES, OUTPUT SIGNAL

The numerator of the right member of Eq. (1), is an expected increment in output resulting from the presence of a signal. This will now be evaluated for the specific processes just described. The results prove to be expressible in terms of the first order distribution of the background and the power of the signal.

Process A

If y depends on x rather than its envelope, we write

$$E(z) = E(y)$$
$$= \int_0^a y[x]\ f(x)\ dx$$

and

$$E_s(z) = \int_0^a y[x] \left[f(x) + g(x)\right] dx.$$

By subtraction

$$N_A = E_s(z) - E(z) = \int_0^a y[x]\ g(x)\ dx. \qquad (2)$$

Process B

The situation now differs from that above in that the distribution functions are now normalized to unit total probability over the interval $0 \leq x \leq a$, and the filter transmission factor q is inserted. Hence

$$E(z) = q \frac{\int_0^a y\, f(x)\, dx}{\int_0^a f(x)\, dx}$$

$$E_S(z) = q \frac{\int_0^a y[f(x) + g(x)]\, dx}{\int_0^a [f(x) + g(x)]\, dx} .$$

Treating g(x) as a small quantity, we obtain

$$N_B = E_S(z) - E(z) = \int_0^a [y[x] - \bar{y}] g(x)\, dx. \quad (3)$$

Process C

Here one may recognize for each value of a' a mean value of z which depends on a'. The final E(z) is, strictly, a mean of these means, but to a satisfactory approximation when the relative fluctuations in a' are small, we will calculate E(z) as a single mean associated with the limit a, just as in A.

$$E(z) = \int_0^a y(x)\, f(x)\, dx.$$

A similar approximation will be made in the presence of signal, but with a higher limit.

$$E_S(z) = \int_0^{a + \delta} y(x) [f(x) + g(x)]\, dx$$

with δ determined by

$$q = \int_0^a f(x)\, dx = \int_0^{a + \delta} [f(x) + g(x)]\, dx$$

or

$$f(a)\,\delta \cong -\int_0^a g(x)\,dx. \tag{4}$$

Hence

$$N_C = E_S(z) - E(z)$$

$$= \int_0^a \left[y(x) - y(a)\right] g(x)\,dx. \tag{5}$$

For later convenience, Eq. (2), (3), and (5) are integrated by parts to give:

$$N_C = \int_0^a \left[-\int_0^a g(x)\,dx\right] \frac{dy[x]}{dx}\,dx \tag{5'}$$

$$N_A = N_C - y[a] \int_0^a \left[-g(x)\right] dx \tag{2'}$$

$$N_B = N_C - \left[y[a] - \bar{y}\right] \int_0^a \left[-g(x)\right] dx. \tag{3'}$$

These results apply for either continuous or discrete input. When y depends on x_e, $g(x)$ is replaced by $G(x)$.

Process D

If $(1 - q')$ is the chosen detector output, we have

$$E(1 - q') = 1 - q$$

and

$$E_S(1 - q') = 1 - \int_0^a \left[f(x) + g(x)\right] dx.$$

Hence

$$N_D = \int_0^a \left[-g(x)\right] \; dx. \tag{6}$$

If a' is the chosen output, we have approximately

$$E_s(a') - E(a') = (a + \delta) - a = \delta$$

$$= \left[1/f(a)\right] \int_0^a \left[-g(x)\right] \; dx$$

where δ is the quantity in Eq. (4). This, of course, differs from Eq. (6) only by a constant factor.

Next, we draw on results developed in Appendix I, which express $g(x)$ and $G(x)$ in terms of $f(x)$ or $F(x)$, and the signal power σ^2. We may note the typical behavior for the increment $g(x)$ indicated in Figure 3. The net area under the curve of $g(x)$ against x is zero, with the negative portion lying to the left of the positive. Hence the integral

$$\int_0^a \left[-g(x)\right] \; dx$$

which appears in Eqs. (2'), (3'), (5') and (6) is normally a positive quantity. This leads to the inequalities

$$N_A < N_B < N_C \tag{7}$$

These relations are readily understood intuitively. Eq. (6) shows that the presence of signal causes a systematic increase in the rejected fraction of input. N_B measures an increase in the mean non-rejected y, due to signal. N_A is less because it is also affected by an increased proportion of zeros caused by the presence of signal. N_C is greater because it is also affected by the increased rejection limit.

Equations (2'), (3'), (5'), and (6) have been combined with (37) or (38), and y has been specialized to either x^2 or

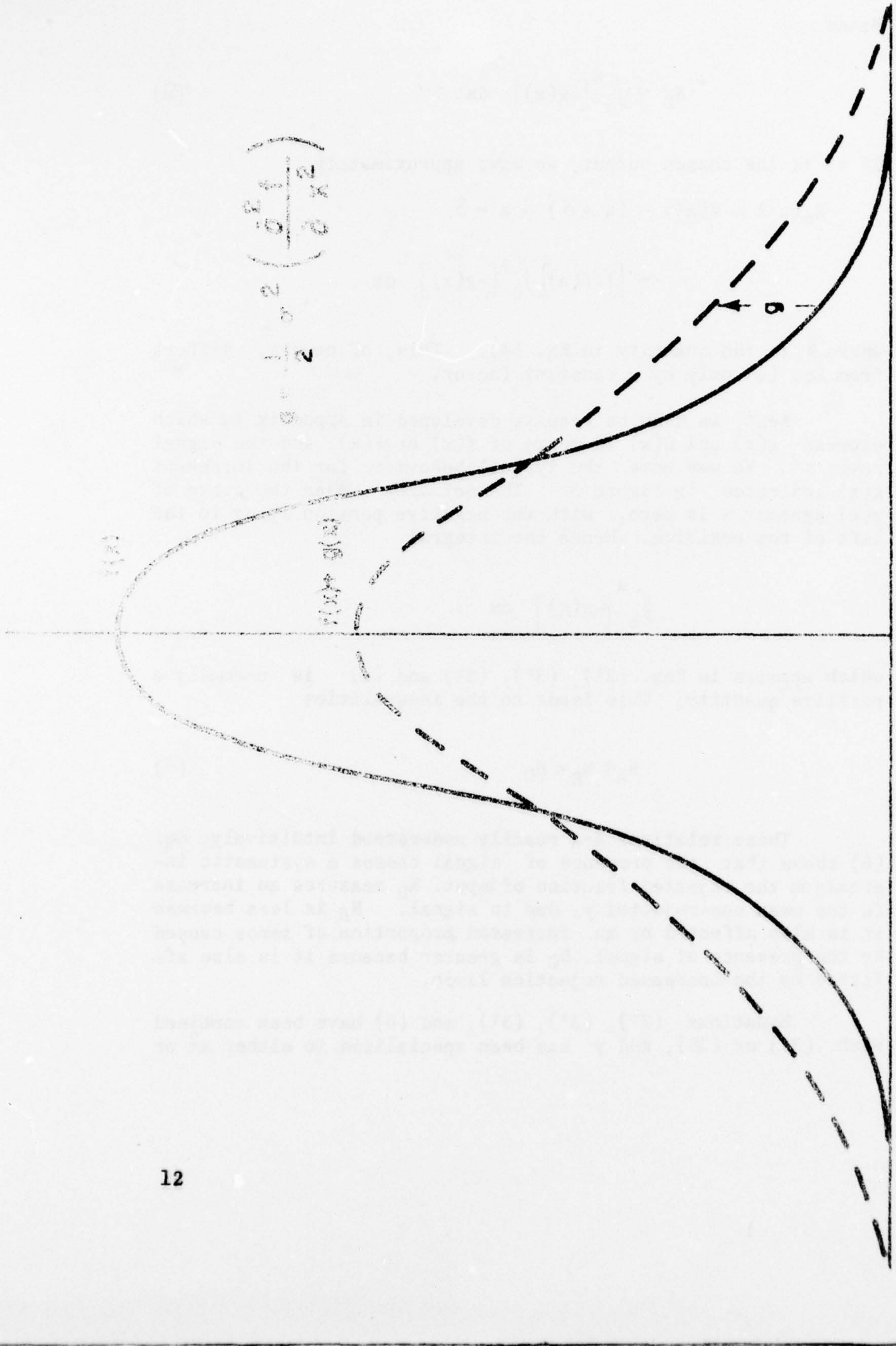

Figure 3. Distribution functions for noise only, $f(x)$, and for signal-plus-noise $f(x)+g(x)$.

$|x|$, appropriate to a square law or linear detector. The resulting specific expressions for the numerator in Eq. (1) are summarized in Table 1.

REJECTION PROCESSES, OUTPUT NOISE

The denominator in Eq. (1) represents the root-mean-square fluctuation in the output z in the absence of signal. Unlike the output signal evaluated in the preceding section, this output noise does not in general depend solely on the first order distribution of y, but on higher statistics as well. In the special case where the input consists of discrete and independent values of the variable x, the first order distribution does determine all the statistics. For this reason, this special case will be included in the following, even though in most real detection problems, spacing of input values would be so chosen as to yield at least moderate serial correlations.

Process A

For a continuous input, the mean squared output fluctuation V(z) may be calculated as the product of two factors, one the effective band width of the averaging filter, (1/T), and the other the power spectrum density of the filter input y(t) at zero frequency. The latter is obtained through the Fourier transform of the autocorrelation of y. Thus

$$V_A(z) = \frac{1}{T} V_A(y) \int_{-\infty}^{\infty} \rho_A(\tau) \, d\tau = \frac{\tau_A}{T} V_A(y) \tag{8}$$

with

$$V_A(y) \rho_A(\tau) = \int_0^a \int_0^a y[u] \; y[v] \; f_2(u,v,\tau) \, du \, dv - q^2 \, \bar{y}^2 \tag{9}$$

and

$$V_A(y) = q \, \overline{y^2} - (q \, \bar{y})^2 = q\left[(\overline{y^2} - \bar{y}^2) + p \, \bar{y}^2\right]. \tag{10}$$

When the input is discrete, the integration in Eq. (8) is replaced by a summation; we then have

Table I.

VALUES OF OUTPUT DUE TO SIGNAL
FOR PROCESSES A, B, C, AND D

$y = x^2$	$N_A = q\sigma^2 \left\{1 - \frac{af(a)}{q} - \frac{1}{2}\frac{a^2}{q}\left[-\frac{df(a)}{dx}\right]\right\}$ $N_B = q\sigma^2 \left\{1 - \frac{af(a)}{q} - \frac{1}{2}\frac{a^2 - \overline{x^2}}{q}\left[-\frac{df(a)}{dx}\right]\right\}$ $N_C = q\sigma^2 \left\{1 - \frac{af(a)}{q}\right\}$
$y = \|x\|$	$N_A = \frac{1}{2}\sigma^2 \left\{f(o) - f(a) - a\left[-\frac{df(a)}{dx}\right]\right\}$ $N_B = \frac{1}{2}\sigma^2 \left\{f(o) - f(a) - (a - \overline{\|x\|})\left[-\frac{df(a)}{dx}\right]\right\}$ $N_C = \frac{1}{2}\sigma^2 \left\{f(o) - f(a)\right\}$
form of y not relevant	$N_D = \frac{1}{2}\sigma^2 \left[-\frac{df(a)}{dx}\right]$

Table I. (Continued)

$y = x_e^2$	$N_A = 2q\sigma^2 \left\{ 1 - \frac{3}{4}\frac{aF(a)}{q} - \frac{1}{4}\frac{a^2}{q}\left[-\frac{dF(a)}{dx}\right]\right\}$ $N_B = 2q\sigma^2 \left\{ 1 - \frac{3}{4}\frac{aF(a)}{q}\left[1 - \frac{1}{3}\frac{\overline{x_e^2}}{a^2}\right] - \frac{1}{4}\frac{a^2 - \overline{x_e^2}}{q}\left[-\frac{dF(a)}{dx}\right]\right\}$ $N_C = 2q\sigma^2 \left\{ 1 - \frac{1}{2}\frac{aF(a)}{q}\right\}$
$y = x_e$	$N_A = \frac{1}{2}\sigma^2 \left\{ \int_o^a \frac{1}{x} F(x)\, dx - 2F(a) - a\left[-\frac{dF(a)}{dx}\right]\right\}$ $N_B = \frac{1}{2}\sigma^2 \left\{ \int_o^a \frac{1}{x} F(x)\, dx - 2F(a)\left[1 - \frac{1}{2}\frac{\overline{x_e}}{a}\right] - (a - \overline{x_e})\left[-\frac{dF(a)}{dx}\right]\right\}$ $N_C = \frac{1}{2}\sigma^2 \left\{ \int_o^a \frac{1}{x} F(x)\, dx - F(a)\right\}$
form of y not relevant	$N_D = \frac{1}{2}\sigma^2 \left\{ \frac{1}{a} F(a) + \left[-\frac{dF(a)}{dx}\right]\right\}$

$$\tau_A = t_o \left[1 + 2 \sum_{i=1}^{\infty} \rho_A(it_o)\right] . \tag{11}$$

Combining Eqs. (8) and (10) gives

$$V_A(z) = \frac{\tau_A}{T} q\left[(\overline{y^2} - \bar{y}^2) + p\,\bar{y}^2\right] . \tag{12}$$

In the special case of discrete, independent input values, $\tau_A = t_o$ and we have

$$V_A(z) = \frac{q}{n}\left[(\overline{y^2} - \bar{y}^2) + p\,\bar{y}^2\right] . \tag{13}$$

In order to calculate $V_A(z)$ as a function of the rejection limit, knowledge of the first order distribution f(x) or F(x) serves to evaluate all factors but τ_A in Eq. (12). In the event that the first order distribution, but not the second, is known for some particular background, it would appear to be of some use to evaluate Eq. (12) as a function of the rejection limit a in so far as possible, but to regard the correlation time τ_A as constant, independent of a. In so doing one is neglecting a factor which may easily be substantial, but does not appear at all likely to be overriding. Any reduction in fluctuation which might be indicated in this way would be an overestimate only if an effect of rejection were to increase autocorrelation coefficients and hence τ_A.

Process B

$V_B(z)$ is calculated in the same general way as above, with some specific changes. The averaging time of the filter is qT rather than T; its maximum power transmission factor is q^2 rather than 1. The autocorrelation of y has been altered (see Figure 2), so that a different correlation time τ_B is introduced.

We now have

$$V_B(y) = \overline{y^2} - \bar{y}^2,$$

$$V_B(z) = \frac{q^2}{qT} \tau_B V_B(y) = \frac{\tau_B}{T} q (\overline{y^2} - \bar{y}^2). \tag{14}$$

Again, for discrete, independent inputs,

$$V_B(z) = \frac{q}{n} (\overline{y^2} - \bar{y}^2). \tag{15}$$

In this extreme case, Eq. (13) is greater than Eq. (15) by the last term which can be interpreted as resulting from the fluctuation in the number of zeros which enter the process A average. At the other extreme, for continuous input it is shown in Appendix II that the autocorrelations of y (and hence τ_A and τ_B) differ in processes A and B in such a way that Eqs. (12) and (14) give identical fluctuation values, and $V_A(z) = V_B(z)$.

Process C

$V_C(z)$ is calculated in Appendix III for the discrete-independent input case, with the result:

$$V_C(z) = \frac{q}{n}\left[\overline{y^2} - \bar{y}^2 + p(y[a] - \bar{y})^2\right]. \tag{16}$$

No formal analysis has been found for the case of continuous input, but one may suppose that the result will usually resemble the above in containing terms involving respectively $\langle V(y,a')\rangle$ (see Appendix III for notation) and $V(a')$, and further that the two terms will have, rather than a common coefficient $1/n$, two distinct coefficients involving different effective correlation times, but both of the order of τ_A/T.

Process D

Depending on whether q' or a' is chosen as the final output, the relevant variance is $V(q')$ or $V(a')$. These are, to a sufficient approximation, related by a common factor.

$$\frac{V(q')}{V(a')} = f^2(a) \text{ or } F^2(a) \tag{17}$$

In the discrete-independent input case, $V(q')$ is essentially the variance of a binomial distribution,[2]

$$V(q') = \frac{pq}{n} . \tag{18}$$

Again, for continuous input, formal analysis is lacking. One may, however, consider that n will be replaced by an effective number of "independent opportunities" in time T for the alternative events whose probabilities are q and p. This number may be of the order of T/τ_A, as suggested in connection with process C.

COMPARISON OF DETECTION PROCESSES

In the foregoing, $V_A(z) \gtrless V_B(z)$ has been demonstrated for the cases of discrete-independent, and continuous, input. Taken in conjunction with Eq. (7), this indicates that process B generally has a lower threshold than process A. No general comparison of B and C has been found possible. The greater signal from C is offset, surely in the discrete-independent case, and probably also in the continuous, to some degree by increased output noise arising from the fluctuations in the rejection limit a'. In view of the analytical difficulties in treating process C, actual comparison with B in specific cases will probably need to be done empirically. Similarly with process D, though one may suppose here that the greater information loss involved will usually operate against the effectiveness of the process.

GENERALIZATION OF PROCESS A

One may generalize the rejection process by replacing the rejected values of y with a fixed number b, rather than with zero. This then includes "clipping" as the special case $b = y[a]$.

Formally, we define process A' as like process A except that y = b when $|x(t)| > a$. One finds, for output signal:

$$N_{A'} = \int_0^a (y[x] - b)\, g(x)\, dx$$

$$= N_C - (y[a] - b) \int_0^a \left[-g(x)\right] dx . \tag{19}$$

This result includes Eqs. (2'), (3'), and (5') as special cases.

Turning to the output fluctuations, we will make only the following remarks. Equation (10) is replaced by

$$V_{A'}(y) = q\left[(\overline{y^2} - \bar{y}^2) + p(b - \bar{y})^2\right]$$

and Eq. (13) by

$$V_{A'}(z) = (1/n)\, V_{A'}(y). \tag{20}$$

The final signal-to-noise ratio R for the discrete-independent input case, obtained from Eqs. (1), (19), and (20) has a maximum with respect to b, at fixed a, which is readily calculated and occurs for $b > \bar{y}$.

The modification of Eq. (9) for this process is straightforward, but will not be given here.

LIKELIHOOD SHAPING

The whole of this section rests on a suggestion made by T. G. Birdsall, of the University of Michigan, in conversation with the writer. We observe that the computations of Appendix I determine the likelihood ratio for an instantaneous value of x(t), say L(x), and introduce the following form for y:

$$y[x] = \log L(x) = \log\left[1 + \frac{g'(x)}{f'(x)}\right]. \tag{21}$$

The resulting value of z is an exact likelihood-detector output if the input x-values are discrete and independent, and is proposed for consideration in the case of autocorrelated input also. Further analysis or empirical investigation is needed to determine the extent to which this result differs from the true likelihood output for an autocorrelated input.

Simplification of Eq. (21) is permissible. From Eq. (28), for example, we have

$$L(x) = 1 + \sigma^2 \frac{d^2 f'(x)/dx^2}{2f'(x)}. \tag{22}$$

The appropriate value for σ^2 in this expression is the power of a signal which the process is designed to detect, that is to say, a threshold signal. Setting N_A from the first line of Table I, for example, equal to $V_A^{1/2}(z)$ from Eq. (8) shows the threshold σ^2 to be of the order

$$\left[\frac{\tau_A}{T} V_A(x^2)\right]^{1/2}.$$

Inserting this in Eq. (22) gives

$$L(x) = 1 + \left(\frac{\tau_A}{T}\right)^{1/2} w(x) \tag{23}$$

where

$$w(x) \cong V_A^{1/2}(x^2) \frac{d^2 f'(x)}{dx^2} \Big/ 2f'(x)$$

is of order of magnitude unity or less. Since we suppose $T >> \tau_A$, $L(x)$ differs little from unity and the logarithm in Eq. (21) can be expanded in a power series, of which only the first degree term need be retained.

$$y[x] \cong \left(\frac{\tau_A}{T}\right)^{1/2} w(x)$$

To this approximation, constant factors have no effect and may be omitted. Thus we may take, finally

$$y[x] = \frac{d^2 f'(x)}{dx^2} \Big/ f'(x) \tag{24}$$

or, if the input envelope is employed,

$$y[x] = \left[\frac{d^2F(x)}{dx^2} - (1/x)\frac{dF(x)}{dx}\right]\frac{1}{F(x)} + \frac{1}{x^2}\ . \tag{25}$$

These functions may be called likelihood shapes for the detector characteristic. When $f'(x)$ is the Gaussian distribution, Eqs. (24) and (25) become, essentially, $y[x] = x^2$.

From Eq.(24) one may readily find the final output signal-to-noise ratio.

$$R = \left(\frac{T}{\tau_L}\right)^{1/2} \frac{\sigma^2}{2} \left\{\int_{-\infty}^{\infty}\left[\frac{d^2f'(x)}{dx^2}\right]^2 \frac{dx}{f'(x)}\right\}^{1/2} . \tag{26}$$

τ_L is the correlation time for the time series y determined by Eq. (24). Here again we have a factor depending only on the first-order distribution, and the factor τ_L involving also the second order distribution unless the input values are discrete and independent.

SUMMARY REMARKS

The foregoing results may be applied to study small-signal detection problems in which the background statistics are not necessarily Gaussian. Empirically determined distribution functions for the background noise may be used to calculate or estimate detection thresholds. Effects of the following characteristics of the detection process may be studied:

a. peak rejection or clipping with various limits (values of a).

b. various detector characteristics (forms of the function $y[x]$).

c. the special two-valued detector characteristic ($y = 0$ or 1) implied in Process D.

d. the special detector characteristic, Eq. (24) or (25) determined by likelihood shaping.

As previously pointed out, detection thresholds depend only on a first-order distribution if the input values are discrete and independent. In the more usual case of autocorrelated input, most of the processes discussed are fully determined by the second-order distribution, but even in these cases it is to be hoped that the first-order distribution alone may be roughly indicative. Processes C and D involve statistics beyond the second order.

ACKNOWLEDGMENT

This investigation of peak rejection was originally conceived as largely of practical significance and without apparent connection with the likelihood theory of detection. The suggestion by Birdsall which is elaborated in the foregoing provides such a connection and thus adds much perspective to the discussion. This major contribution is gratefully acknowledged.

APPENDIX I. EVALUATION OF g(x) AND G(x).

Let $f'(x)$, or $f'(x) + g'(x)$ be the probability density for x itself, defined over $-\infty < x < \infty$. The even parts of these functions will be $f(x)$ and $f(x) + g(x)$. The density $f'(x) + g'(x)$ will be the convolution of $f'(x)$ and the probability density for signal alone, $s(x)$.[3]

$$f'(x) + g'(x) = \int_{-\infty}^{\infty} f'(x-u)\, s(u)\, du. \tag{27}$$

$f'(x-u)$ is now expanded as three-term power series about $f'(x)$ and it is assumed that almost all values of u (instantaneous signal) are small enough, and the function $f'(x)$ smooth enough to justify the use of the expansion in Eq. (27). Thus we have

$$f'(x) + g'(x) = \int_{-\infty}^{\infty} \left[f'(x) - \frac{df'(x)}{dx} u + 1/2 \frac{d^2 f'(x)}{dx^2} u^2 \right] s(u)\, du$$

or

$$g'(x) = 1/2 \frac{d^2 f'(x)}{dx^2} \sigma^2 \tag{28}$$

and

$$g(x) = 1/2 \frac{d^2 f(x)}{dx^2} \sigma^2 \tag{29}$$

recalling that the mean signal is supposed zero.

When envelopes of noise and signal are considered, the rule of combination becomes a vector addition. Let the noise only and signal only be represented by $N(t)\,[\cos \omega t + \phi(t)]$ and $S(t)\,[\cos \omega t + \psi(t)]$, respectively. The envelopes $N(t)$ and $S(t)$, and the phase difference $\theta = \phi(t) - \psi(t)$ are assumed to be three independent random variables, with Θ uniformly distributed over the circle. They serve to determine the envelope of noise-plus-signal, say E, by the relation:

$$E^2 = N^2 + 2NS\cos\Theta + S^2 \quad \text{with } N, S, E \text{ all } \geq 0. \tag{30}$$

Let the joint probability density in the (N, S, Θ) space be $(1/2\pi)\, F(N)\, s_e(S)$, which then serves to determine the distribution of E by

$$\left[F(E_1) + G(E_1)\right] dE = \int_R (1/2\pi)\, F(N)\, s_e(S)\, dN\, dS\, d\Theta. \tag{31}$$

The region of integration R is the volume within which $E_1 < E < E_1 + dE$.

The integral in Eq. (31) is now transformed to new independent variables (E, u, v) given by $u = S\cos\Theta$, $v = S\sin\Theta$, and Eq. (30). The Jacobian is

$$\frac{\partial(N, S, \Theta)}{\partial(E, u, v)} = \left(1 - \frac{v^2}{E^2}\right)^{-1/2} \frac{1}{S} \tag{32}$$

and we write $F(N) = F_1(E,u,v)$. Then

$$\left[F(E_1) + G(E_1)\right] dE = \int_R \frac{1}{2\pi} F_1(E,u,v)\, s_e(S) \left(1 - \frac{v^2}{E^2}\right)^{-1/2} \frac{1}{S}\, dE\, du\, dv$$

or with slight simplification, replacing E_1 by x

$$F(x) + G(x) = \int_{E=x} F_1(x,u,v)\left(1 - \frac{v^2}{x^2}\right)^{-1/2} \left[\frac{s_e(S)}{2\pi S}\right] du\, dv. \tag{33}$$

The further argument is restricted to x satisfying

$$x^2 >> v^2 \tag{34}$$

that is to x with magnitude greater than that of almost all signal values. F_1 is expanded as a power series in u and v, including quadratic terms, about (x, 0, 0). After some reduction, one obtains:

$$F(x) + G(x)$$
$$= \int_{E=x} \left[F(x) - \frac{dF(x)}{dx} u + 1/2 \frac{d^2F(x)}{dx^2} u^2 - \frac{1}{2x} \frac{dF(x)}{dx} v^2\right]$$
$$\cdot \left[1 + 1/2 \frac{v^2}{x^2}\right] \left[\frac{s_e(S)}{2\pi S}\right] du\, dv. \tag{35}$$

It is now to be noted that the third square bracket in the integrand is the probability density in the (u, v) plane. Because of the restriction (34), the integration is over almost all values of u and v. Hence the effect of the integration is merely to replace u, u^2, and v^2 by their respective mean values 0, σ^2, and σ^2 (taking the mean of S^2 to be $2\sigma^2$). We have, finally,

$$G(x) = 1/2 \left[\frac{d^2F(x)}{dx^2} - \frac{1}{x} \frac{dF(x)}{dx} + \frac{1}{x^2} F(x)\right] \sigma^2. \tag{36}$$

In view of the limitation (34), Eq. (36) is not correct, for a fixed σ^2, at x = 0. However, without further analysis, the relative error involved in using Eq. (36) over the range $0 \leq x \leq \infty$ will be assumed small, for small σ^2. The limiting behavior of Eq. (36) as $x \rightarrow 0$ seems satisfactory.

For use of these results we require the integrals:

$$\int_0^x \left[-g(x)\right] dx = -1/2 \frac{df(x)}{dx} \sigma^2 \tag{37}$$

since

$$\frac{df(0)}{dx} = 0$$

and

$$\int_0^x \left[-G(x)\right] dx = 1/2\left[-\frac{dF(x)}{dx} + \frac{1}{x}F(x)\right]\sigma^2 \qquad (38)$$

if

$$F(0) = 0 \text{ and } \frac{dF(0)}{dx} \text{ and } \frac{d^2F(0)}{dx^2} \text{ are finite.}$$

APPENDIX II. EFFECT OF CLOSING GAPS

Equation (12) gives the output variance V(z) for a continuous input containing gaps, as in the upper part of Figure 2. To show the effect of closing the gaps, we separate the probability density f_2 into components as follows:

$$f_2(u,v,\tau) = f_0(u,v,\tau) + \int_0^\tau f_1\,(u,v,\tau,\,w)\,dw. \tag{39}$$

$f_0\,(u,v,\tau)$ is the probability density for ordinate pairs (u,v) separated by an interval τ containing no rejection gaps; $f_1\,(u,v,\tau,\,w)\,dw$ is the probability density for such pairs when the interval τ contains rejection gaps of total duration lying between w and w + dw. Equations (8) and (9) may then be written

$$V_A(z) = \frac{1}{T}\int_{-\infty}^{\infty} d\tau \int_0^a \int_0^a y[u]\;y[v]\;\Big[f_2(u,v,\tau) - f(u)\;f(v)\Big]\,du\,dv$$

$$= \frac{1}{T}\int_0^a \int_0^a du\,dv\;y[u]\;y[v]\int_0^L d\tau\Big[-f(u)\;f(v) + f_0(u,v,\tau)$$

$$+ \int_0^\tau f_1(u,v,\tau,\,w)\Big]\,dw\,. \tag{40}$$

For large τ, x(t) and $x(t+\tau)$ are presumed to vary independently. The square bracket in the integrand of Eq. (40) then vanishes for large τ; the integration with respect to τ is assumed to converge and the upper limit may be set at a sufficiently large positive number L rather than ∞.

The double integral in Eq. (40) with respect to w and τ will now be transformed to new independent variables w and $\tau' = \tau - w$. (An ordinate pair originally characterized by $(\tau,\,w)$ will have the separation τ' after gaps are closed.) The Jacobian is unity. The original region of integration is shown in Figure 4, together with contours without which the density f_1 effectively vanishes. These represent the assumed behavior, that as $\tau \to \infty$, the rejected fraction within the interval τ becomes increasingly

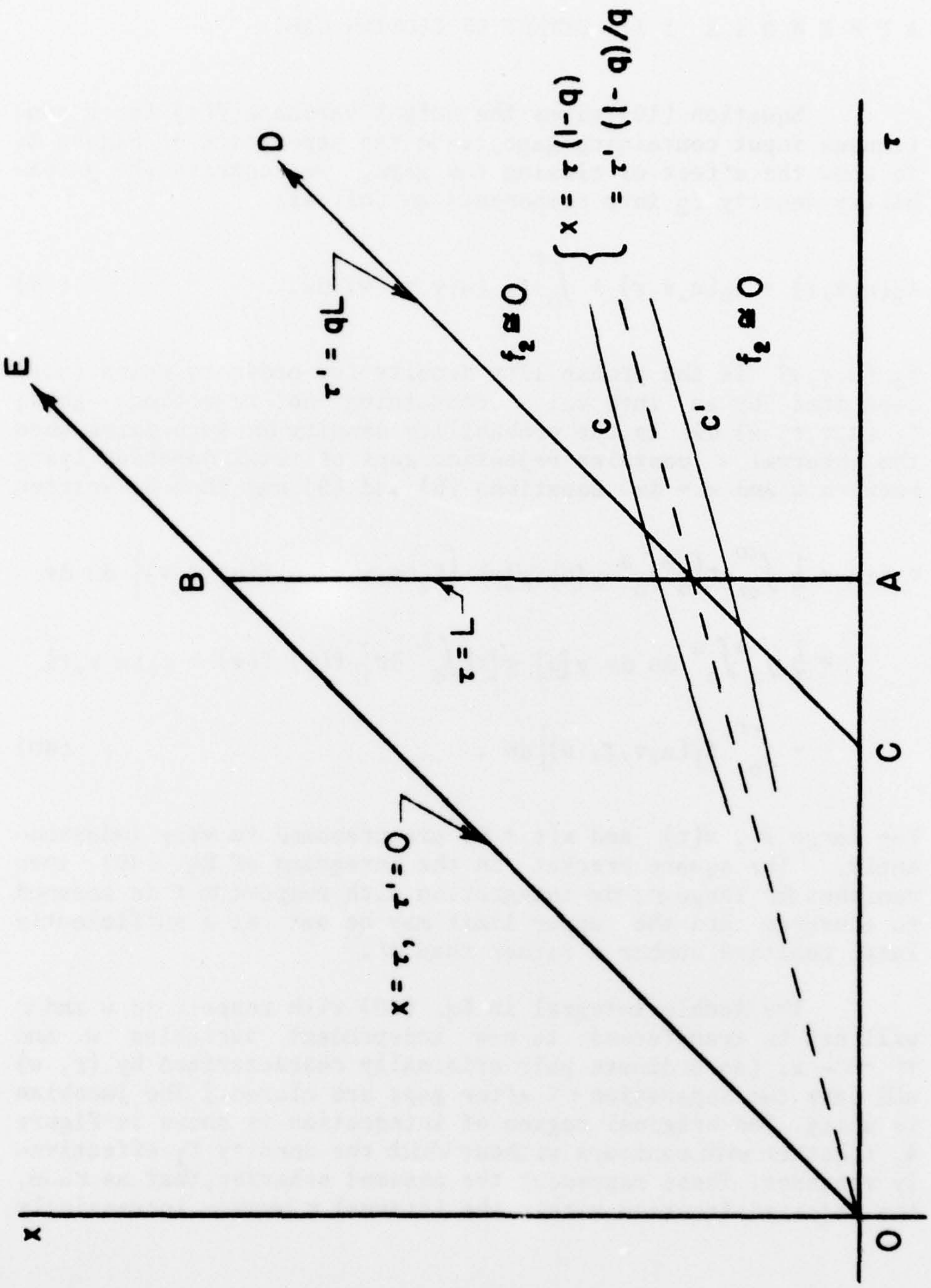

Figure 4. Regions of integration. Eq. (40): AOB; Eq. (41): DCOE. The probability density f_2 vanishes approximately above the contour c and below c'.

predictable and $w/\tau \rightarrow (1-q)$ is the only region of nonvanishing f_1. Hence the region of integration may be altered to $0 < x < \infty$, $0 < \tau' < qL$ as also shown in Figure 4. For the term involving the constant $f(u)\ f(v)$ we write $d\tau = d\tau'/q$, but because $f_o(u,v,L)$ vanishes approximately, we assume

$$\int_o^L f_o(u,v,\tau)\, d\tau = \int_o^{qL} f_o(u,v,\tau')\, d\tau'.$$

Hence the final transformed expression can be written (using τ for τ'):

$$V_A(z) = \frac{q^2}{qT} \int_o^a \int_o^a du\, dv\, y[u]\, y[v] \int_o^\infty d\tau$$

$$\cdot \left\{ -\frac{f(u)\ f(v)}{q^2} + \left[\frac{1}{q} \left\{ f_o(u,v,\tau) + \int_o^\infty f_1(u,v,\tau+w,w)\, dw \right\} \right] \right\}$$

$$= V_B(z) \tag{41}$$

The relation to V_B follows when one observes that the term in the integrand which is in square brackets is the probability density analogous to Eq. (39) into which f_2 must be changed by the gap-closing operation.

APPENDIX III. CALCULATION OF $V_C(z)$.

The case of discrete, independent input values is considered. Of the n values of x occurring during T, the lowest nq are retained and will be referred to as an x-sample. These samples are classified into subgroups according to the value of a'. Within one subgroup the x-values are distributed independently over the interval $0 < x < a'$, yielding the following mean and variance for y:

$$E(y,a') = \frac{\int_0^{a'} y[x]\, f(x)\, dx}{\int_0^{a'} f(x)\, dx}$$

$$V(y,a') = \frac{\int_0^{a'} y^2[x]\, f(x)\, dx}{\int_0^{a'} f(x)\, dx} - E^2(y,a').$$

We use $< \; >$ to denote a mean with respect to a', and note that in this case z can be regarded as a mean of (nq) values of (qy), in which y means a retained value. Hence

$$E_C(z) = q < E(y,a')>$$

$$V_C(z) = <V(z,a')> + <E^2(z,a')> - <E(z,a')>^2$$

$$= q^2 \left[\frac{1}{nq} <V(y,a')> + <E^2(y,a')> - <E(y,a')>^2\right].$$

We take, approximately:

$$<V(y,a')> = V(y,a) = \overline{y^2} - \bar{y}^2$$

$$\langle E^2(y, a')\rangle - \langle E(y, a')\rangle^2 = \left(\frac{dE(y,a)}{da'}\right)^2 V(a')$$

$$\frac{dE(y,a)}{da'} = \frac{f(a)}{q}\,(y[a] - \bar{y})$$

and

$$V(a') = \frac{pq}{nf^2(a)}$$

using Eqs. (17) and (18). Combination of the last five equations leads to Eq. (16).

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