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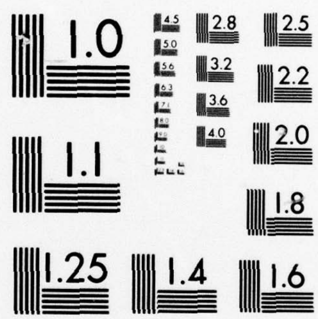

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ON THE RELATIONSHIP BETWEEN RELIABILITY AND LINEAR QUADRATIC OPTIMAL CONTROL*

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Abstract

The linear quadratic optimal control method is used today to solve many complex systems problems. As system complexity increases, and as linear quadratic optimal control is used in more demanding situations, the extension of the design methodology to cover system failures, robustness and reliability is of crucial importance. This paper documents the progress toward a theory which incorporates reliability in the performance index; a linear quadratic control problem is formulated which accounts for system effectiveness and gives an offline procedure for comparing two linear quadratic control systems on the basis of both reliability and performance.

1. Introduction

This paper deals with the formulation of and a suboptimal solution to an optimal control problem over a system with randomly varying structure. The first objective of this research is to create a procedure which can be used in offline design tradeoff studies of various system designs using different numbers of actuators with varying failure rates and which judges a given design on the basis of both performance and reliability. Secondly, we wish to establish a dividing line between reliable and unreliable systems.

Whenever one considers the optimal control of systems with varying structure, the complexity of the solution, even when it exists, is enormous. In any design technique involving these systems, there is a tradeoff between the complexity of the technique and the amount of information it produces. The approach taken in this research is to simplify the procedure as much as possible while retaining its ability to weigh system designs on the basis of both performance and reliability.

The solution presented herein serves a second purpose: It allows the researcher to study the effect of various actuator configurations and failure modes on system performance and to gain a better understanding of the relationship between reliability and control theory. This is perhaps the more important aspect of this research.

Previously, several authors have studied the optimal control of systems with randomly varying structure. Most notable among these is Wonham in [7], where he develops a solution to the linear regulator problem with randomly jumping parameters. This solution, however, assumes a priori that the controller has perfect information about the present state of the random process. Thus, the control depends on the random parameter. In reality, one could not expect to have perfect information, however, and a noisy observation of the random process leads to the dual control problem. Wonham made a major contribution, however, in deriving the form of the coupled Riccati equations to which this type of problem leads. Sworder has used this information in some specific cases [4,5], as have Ratner and Luenberger [3]. Bar-Shalom and Sivan [1] presented a variation of Wonham's work by using the optimal open loop controller at each step of a discrete-time linear system with random structure. Willner [6] developed a suboptimal control scheme, which allowed for imperfect observation of the random parameters, known as multiple-model adaptive control. In this method, the parameters could only take a discrete set of values, a cause of recent disfavor, as MMAC does not work well when the parameters vary continuously and are approximated by the mathematics. Similar work has been done by Pierce and Sworder in [2].

2. Motivation

To illustrate the dual control problem, and to motivate this work, consider the following simple example. Let the system be one dimensional with one control variable.

$$x(t+1) = ax(t) + b_k u(t) \tag{1}$$

The value of the control multiplier (b_k) is a random variable which takes on one of two discrete values at each time t.

$$b_k(t) = \begin{cases} b \text{ if } k = 0 \\ 1/b \text{ if } k = 1 \end{cases} \tag{2}$$

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The random process is governed by the Markov chain represented by

$$\underline{\pi}(t+1) = \underline{P}\,\underline{\pi}(t) \tag{3}$$

where

$$\underline{\pi}(t) \in R^2 \tag{4}$$

$$\underline{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in R^{2\times 2} \tag{5}$$

At any given time t, the following sequence of events occurs:

I) x(t) is observed exactly, $b_k(t-1)$ is computed, and k(t) is set to 0 or 1 depending on $b_k(t-1)$, where k(t) is the variable representing the Markov chain;

II) b_k may change values to $b_k(t)$;

III) u(t) is applied.

For any given sample path, the performance index is given by

$$J = \sum_{t=0}^{T} (qx^2(t) + ru^2(t)) \tag{6}$$

where {0,1,...,T} is the time set over which the system is to be controlled. The objective of the control problem is to minimize the expected cost-to-go at t, given by

$$V(x(t),k(t),u(t),t)$$

$$= E\Big(\sum_{\tau=t}^{T} (qx^2(\tau) + ru^2(\tau))\,\Big|\,x(t)\Big) \tag{7}$$

where the expectation is taken over all possible sample paths of $k(\tau)$, $t < \tau \le T$. Using dynamic programming, we wish to minimize

$$V(x(t),k(t),u(t),t)$$

$$= E(qx^2(t) + ru^2(t) + V^*(ax(t)+b_k(t)u(t),k(t+1),t+1)\,|\,x(t)) \tag{8}$$

where $V^*(\cdot,k(t+1),t+1)$ represents the minimum cost-to-go, given k(t+1) at time t+1.

This minimization can be carried out because x(t) is known exactly at time t. Given x(t),

$$\underline{\pi}(t) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ if } \dfrac{x(t)-ax(t-1)}{u(t-1)} = b \\[2ex] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ if } \dfrac{x(t)-ax(t-1)}{u(t-1)} = 1/b \end{cases} \tag{9}$$

and k(t) = 0 if $\underline{\pi}(t)=[1\ 0]'$ or 1 if $\underline{\pi}(t)=[0\ 1]'$. The control u(t) is computed from

$$0 = \frac{\partial}{\partial u}\Big\{ qx^2(t) + ru^2(t) + \pi_0(t+1)V^*(ax(t)+bu(t),k=0,t+1) + \pi_1(t+1)V^*(ax(t)+\tfrac{1}{b}u(t),k=1,t+1) \tag{10}$$

and the assumption that

$$V^*(x(t),k=i,t) = x^2(t)S_i(t) \tag{11}$$

Thus,

$$u(t) = -\frac{[\pi_0(t+1)abS_0(t+1) + \pi_1(t+1)a/bS_1(t+1)]x(t)}{r + \pi_0(t+1)b^2S_0(t+1) + \pi_1(t+1)1/b^2S_1(t+1)} \tag{12}$$

where

$$\underline{\pi}(t+1) = \begin{pmatrix} \pi_0(t+1) \\ \pi_1(t+1) \end{pmatrix} = \underline{P}\,\underline{\pi}(t) \tag{13}$$

$S_0(t)$ and $S_1(t)$ are propagated backward in time by the following equations:

Assuming k=0 at time t, then $\underline{\pi}(t+1) = [p_{11}\ p_{21}]'$ and

$$S_0(t) = q + \frac{r[p_{11}abS_0(t+1)+p_{21}a/bS_1(t+1)]^2}{[r+p_{11}b^2S_0(t+1)+p_{21}1/b^2S_1(t+1)]^2} + p_{11}\left\{a - \frac{b[p_{11}abS_0(t+1)+p_{21}a/bS_1(t+1)]}{r+p_{11}b^2S_0(t+1)+p_{21}1/b^2S_1(t+1)}\right\}^2 S_0(t+1) + p_{21}\left\{a - \frac{p_{11}abS_0(t+1)+p_{21}a/bS_1(t+1)}{b[r+p_{11}b^2S_0(t+1)+p_{21}S_1(t+1)/b^2]}\right\}^2 S_1(t+1) \tag{14}$$

Assuming k=1 at time t, then $\underline{\pi}(t+1) = [p_{12}\ p_{22}]'$ and

$$S_1(t) = q + \frac{r[p_{12}abS_0(t+1)+p_{22}a/bS_1(t+1)]^2}{[r+p_{12}b^2S_0(t+1)+p_{22}1/b^2S_1(t+1)]^2} + p_{12}\left\{a - \frac{b[p_{12}abS_0(t+1)+p_{22}a/bS_1(t+1)]}{r+p_{12}b^2S_0(t+1)+p_{22}1/b^2S_1(t+1)}\right\}^2 S_0(t+1) + p_{22}\left\{a - \frac{p_{12}abS_0(t+1)+p_{22}a/bS_1(t+1)}{b[r+p_{12}b^2S_0(t+1)+p_{22}S_1(t+1)/b^2]}\right\}^2 S_1(t+1) \tag{15}$$

Note that u(t) switches from one form to another, depending on the value of x(t) -- thus, this solution depends on an exact knowledge of x(t). If knowledge of x(t) is corrupted by measurement noise (or, if u(t) is corrupted by control noise), then this becomes a dual control problem.

To be specific, suppose

$$y(t) = x(t)$$
$$x(t+1) = ax(t) + b_ku(t) + \xi(t) \tag{16}$$

where b_k and k are as before. The control noise $\xi(t)$ is applied simultaneously with u(t) and is gaussian white noise with $E[\xi(t)]=0$, $E[\xi(t)\xi(\tau)] = \Xi\delta(t-\tau)$, $E[\xi(t)x(t)]=0$, and $E[\xi(t)k(t)]=0$. The optimal solution of this problem requires order of t^2 Riccati equations at any time t because exact knowledge of $\underline{\pi}(t+1)$ is not given and the control affects the optimal estimation process for x(t) and $\underline{\pi}(t)$. A suboptimal controller would likely separate the estimation function of $\underline{\pi}(t)$ and x(t) from the control gain calculation and ignore the dual control effect.

If measurement noise is present; i.e.,

$$y(t) = x(t) + \theta(t)$$
$$x(t+1) = ax(t) + b_ku(t) \tag{17}$$

a similar effect is noted.

A different approach is taken in this paper; here, it is assumed that the controller has no information about the current state of the random parameters, and that it can obtain none through observation of $\underline{x}(t)$. Therefore, the control law is independent of these parameters and $\underline{x}(t)$, and the problems of dual control do not arise. The formulation also allows a steady-state solution under certain conditions (which the optimal solution in general does not) which is structurally identical to the normal linear time invariant optimal control law. This approach is taken because the complexity of the optimal solution, when it exists, is too great to allow a numerical comparison between systems and is not constant, but depends on where various noise terms appear in the system description. The control solution presented here is still extremely complex (k coupled Riccati equations, where k is the number of possible system configurations and can be very difficult to solve in even the simplest of situations. The information presented here is in the context of reliability and, more specifically, actuator failure. The concept is applicable to a much broader range of problems.

3. Problem Formulation

A linear quadratic control problem in discrete time which accounts for system effectiveness under possible actuator failure is formulated. The steady-state solution to this problem, when it exists, yields a direct comparison of two linear systems. The comparison is based on system reliability and performance.

This paper is concerned with the discrete time linear system

$$\underline{x}(t+1) = \underline{Ax}(t) + \underline{B_0u}(t) \tag{18}$$

where

$$\underline{x}(t) \in R^n,\ \underline{u}(t) \in R^m \text{ and } \underline{A} \in R^{n \times n} \tag{19}$$

The matrix $\underline{B_0} \in R^{n \times m}$ is assumed to be of the form

$$\underline{B_0} = (\underline{b_1} \mid \underline{b_2} \mid \underline{b_3} \mid \ldots \mid \underline{b_m}) \tag{20}$$

with

$$\underline{b_i} \in R^n \tag{21}$$

corresponding to the control variable u_i of

$$\underline{u} = [u_1\ u_2\ u_3\ \ldots\ u_m]' \tag{22}$$

The system is assumed to run over the time set

$$\tau = \{-T_1, -T_1+1, -T_1+2, \ldots, T_2-2, T_2-1, T_2\}$$

Given any trajectory $(\underline{x}(t), \underline{u}(t))_{t\in\tau}$, the cost is defined as

$$J = \sum_{t=-T_1}^{T_2-1} (\underline{x}'(t)\underline{Qx}(t) + \underline{u}'(t)\underline{Ru}(t)) + \underline{x}'(T_2)\underline{Q_f x}(T_2) \tag{23}$$

4. Actuator Failures

It is assumed in this paper that each actuator may fail to zero, and may be repaired at a later time. All failures are assumed to occur after $\underline{x}(t)$ is measured and before the control input is applied. Therefore, if actuator $\underline{b_i}$ fails, then

$$\underline{B_0} \to \underline{B_\ell} \tag{24}$$

where

$$\underline{B_\ell} = [\underline{b_1} \mid \underline{b_2} \mid \ldots \mid \underline{b_{i-1}} \mid \underline{0} \mid \underline{b_{i+1}} \mid \ldots \mid \underline{b_m}] \tag{25}$$

Let $B = \{\underline{B_i}\}_{i=0}^{k}$ be the set of possible actuator configurations. Assuming the probability of failure of any set of actuators at t+1, given that they are operational at t, is constant for all t, then the vector $\underline{\pi}(t)$ is defined

$$\underline{\pi}(t) \in R^{k+1} \tag{26}$$

$$\pi_i(t) = \text{the probability that the actuator configuration is } \underline{B_i} \text{ at time } t,\ i=0,\ldots,k. \tag{27}$$

If the probability of failure of every set of actuators is known, then $\underline{\pi}(t)$ is propagated in time as follows, assuming independence of $\underline{\pi}$ and $\underline{x}$:

$$\underline{\pi}(t+1) = \underline{P}\,\underline{\pi}(t) \tag{28}$$

$$\underline{P} = (p_{ij}) \in R^{k+1 \times k+1} \tag{29}$$

$$p_{ij} = \{\text{prob. of actuator configuration } \underline{B_i} \text{ at time } t+1 \mid \text{actuator configuration } \underline{B_j} \text{ at time } t\} \tag{30}$$

The $\underline{P}$ matrix captures at a general level all failure rates, repair rates, and reconfiguration rates.

5. Objective

We wish our optimization problem to take into account the probabilities of transfer between various structural states [in this case, actuator configurations]. A sufficiently abstract framework is found, by using $\underline{\pi}(t)$, so that reliability can be abstracted in the context of a control problem. One system can then be ranked as more reliable than another from a control systems viewpoint. We thus understand failure by tying it to the performance index, and "more reliable" is equated with a smaller expected cost.

This framework involves open loop system design, by ranking offline various system configurations, such as redundant sensors or functional redundancy; system failures, reconfiguration and repair are represented in the objective function. In this paper are presented the first types of problems which can be handled using this procedure.

6. The Optimization Problem

We wish to find the closed-loop non-switching, non-dual feedback gain $\underline{G_0}$ such that the control $\underline{u} = \underline{G_0x}$ minimizes the expected cost-to-go, given that the actuator configuration at time $-T_1$ is $\underline{B_0}$, i.e., that $\underline{\pi}(-T_1) = [1\ 0\ 0\ .\ .\ .\ 0]'$.

We first define $C_i(t)_{u,x}$ as

$$C_i(t)_{u,x} = E\Big(\sum_{\tau=t}^{T_2-1}(\underline{x}'(\tau)\underline{Q}\underline{x}(\tau)+\underline{u}'(\tau)\underline{R}\underline{u}(\tau)) + \underline{x}'(T_2)\underline{Q}_f\underline{x}(T_2)\,\Big|\,\underline{B}_i(t),u,x,\underline{\pi}(t|x(t))=\underline{\pi}(t)\Big) \tag{31}$$

where the expectation is taken over all possible sequences of actuator configurations, given that the configuration at t is $\underline{B}_i(t)$, u and x are the sequences of control and state variables respectively, and the last condition is the assumption that $\underline{\pi}(t)$ is independent of $\underline{x}(t)$. Defining $\underline{C}(t)_{u,x}$ as the vector over all i=0,1,...,k of these expectations, then by theorem 1 (see Appendix),

$$\underline{C}(t)_{u,x} = \underline{P}^T\underline{C}(t+1) + \underline{J}(\underline{x}(t),\underline{u}(t)) \tag{32}$$

where

$$\underline{J}(\underline{x},\underline{u}) = (\underline{x}'\underline{Q}\underline{x} + \underline{u}'\underline{R}\underline{u})_{i=0}^{k} \tag{33}$$

$$\underline{C}(T_2) = (\underline{x}'(T_2)\underline{Q}_f\underline{x}(T_2))_{i=0}^{k} \tag{34}$$

The expected cost-to-go at time t, given $\underline{x}(t)$, $\underline{u}(t)$ and $\underline{\pi}(t)$, is thus

$$\langle\underline{\pi}(t),\underline{C}(t)_{u,x}\rangle \tag{35}$$

The problem is to minimize (35) w.r.t. $\underline{u}(t)$. I.e., defining $\underline{C}^*(t+1)_x$ as the vector $\underline{C}(t+1)$ resulting from the application of the minimizing control over the time interval starting at t+1,

$$\langle\underline{\pi}(t),\underline{C}^*(t)_x\rangle = \min_u \{\langle\underline{\pi}(t),\underline{P}^T\underline{C}^*(t+1)+\underline{J}(\underline{x},\underline{u})\rangle\} \tag{36}$$

where

$$\underline{C}^*(t+1) = (\underline{C}_i^*(t+1)\mid \underline{A}\underline{x} + \underline{B}_i\underline{u})_{i=0}^{k} \tag{37}$$

If we assume

$$\underline{c}_i^*(t) = \underline{x}'(t)\underline{S}_i(t)\underline{x}(t) \tag{38}$$

then we obtain

$$\underline{u}^*(t) = -[\underline{R} + \sum_{i=0}^{k}\pi_i(t)[\sum_{j=0}^{k}p_{ji}\underline{B}_j'\underline{S}_j(t+1)\underline{B}_j]]^{-1} \cdot \sum_{\ell=0}^{k}\pi_\ell(t)[\sum_{m=0}^{k}p_{m\ell}\underline{B}_m'\underline{S}_m(t+1)]\underline{A}\,\underline{x}(t) \tag{39}$$

$$\begin{aligned}\underline{S}_i(t) = \underline{A}'\Big(\sum_{j=0}^{k}p_{ji}\Big[\underline{S}_j(t+1) &- \Big(\sum_{\ell=0}^{k}\pi_\ell(t)\Big[\sum_{m=0}^{k}p_{m\ell}\,\underline{S}_m(t+1)\,\underline{B}_m\Big]\Big)\\ &\cdot\Big[\sum_{\ell=0}^{k}\pi_\ell(t)\Big[\sum_{m=0}^{k}p_{m\ell}\underline{B}_m^T\underline{S}_m(t+1)\underline{B}_m\Big] + \underline{R}\Big]^{-1}\\ &\cdot \underline{B}_j^T\underline{S}_j(t+1) - \underline{S}_j(t+1)\underline{B}_j\Big[\underline{R} + \sum_{\ell=0}^{k}\pi_\ell\Big(\sum_{m=0}^{k}p_{m\ell}\underline{B}_m^T\underline{S}_m(t+1)\underline{B}_m\Big)\Big]^{-1}\\ &\cdot \sum_{\ell=0}^{k}\pi_\ell(t)\Big[\sum_{m=0}^{k}p_m\,\underline{B}_m^T\underline{S}_m(t+1)\Big]\\ &+\Big[\sum_{\ell=0}^{k}\pi_\ell(t)\Big[\sum_{m=0}^{k}p_{m\ell}\underline{S}_m(t+1)\underline{B}_m\Big]\Big]\\ &\cdot\Big[\underline{R} + \sum_{\ell=0}^{k}\pi_\ell(t)\Big[\sum_{m=0}^{k}p_{m\ell}\underline{B}_m^T\underline{S}_m(t+1)\underline{B}_m\Big]\Big]^{-1}\\ &\cdot\Big(\underline{R} + \underline{B}_j^T\underline{S}_j(t+1)\underline{B}_j\Big)\\ &\cdot\Big[\underline{R} + \sum_{\ell=0}^{k}\pi_\ell(t)\Big[\sum_{m=0}^{k}p_{m\ell}\underline{B}_m^T\underline{S}_m(t+1)\underline{B}_m\Big]\Big]^{-1}\\ &\cdot\sum_{\ell=0}^{k}\pi_\ell(t)\Big[\sum_{m=0}^{k}p_{m\ell}\underline{B}_m^T\underline{S}_m(t+1)\Big]\Big]\Big)\,\underline{A}^T + \underline{Q}\end{aligned} \tag{40}$$

$$\underline{S}_i(T_2) = \underline{Q}_f \tag{41}$$

Note that $\underline{u}^*(t)$ is structurally like the usual linear quadratic optimal control law; however, the control law takes account of the different possible actuator configurations using a weighted sum of forms containing the solutions to a set of coupled Riccati equations (40). The control law does not account for the information in $\underline{x}(t)$ about $\underline{\pi}(t)$, and therefore does not have switching gains which are dependent on a function of $\underline{x}(t)$.

7. The Steady-State Solution

We conjecture in this section sufficient conditions for the existence of a steady-state solution. The existence of a solution depends on I) the existence of a steady-state value $\underline{\pi}$, i.e., such that $\underline{\pi} = \underline{P}\,\underline{\pi}$, and II) the stabilizability of the system $[\underline{A},\underline{B}_i]$, i=0,...,k. It is interesting to note that condition I is necessary, whereas condition II is not.

Conjecture

[A] A steady-state solution to equations (39)-(41) exists if

I) there exists a $\underline{\pi}$ such that $\underline{\pi} = \underline{P}\,\underline{\pi}$ and $\underline{\pi}(t) \to \underline{\pi}$ as $t \to \infty$.

II) The systems $[\underline{A},\underline{B}_i]$ are stabilizable for all i=0,...,k.

[B] $\underline{\pi}$ of part I exists iff one of the following three conditions is satisfied for each element α_i of the Jordon normal form $\underline{\Lambda}$ of $\underline{P}$, where

$$\underline{P} = \underline{T}\,\underline{\Lambda}\,\underline{T}^{-1} \tag{42}$$

$$\underline{\Lambda} = \begin{pmatrix}\alpha_0 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_k\end{pmatrix} \tag{43}$$

For each i,

i) $|\alpha_i| < 1$

ii) $\alpha_i = 1$

iii) $|\alpha_i| = 1,\ \alpha_i \neq 1,\ (\underline{T}^{-1}\underline{\pi}_0)_i = 0$

If the steady-state solution exists, then $\underline{S}_i$, $i=0,1,\ldots,k$ satisfy the following set of equations.

$$\underline{S}_i = \underline{A}^T \Bigg(\sum_{j=0}^{k} p_{ji} \Bigg(\underline{S}_j - \Bigg(\sum_{\ell=0}^{k} \pi_\ell \underline{S}_\ell \underline{B}_\ell \Bigg) \cdot \Bigg(\sum_{\ell=0}^{k} \pi_\ell \underline{B}_\ell^T \underline{S}_\ell \underline{B}_\ell + \underline{R} \Bigg)^{-1} \underline{B}_j^T \underline{S}_j$$

$$- \underline{S}_j \underline{B}_j \Bigg(\underline{R} + \sum_{\ell=0}^{k} \pi_\ell \underline{B}_\ell^T \underline{S}_\ell \underline{B}_\ell \Bigg)^{-1} \Bigg(\sum_{\ell=0}^{k} \pi_\ell \underline{B}_\ell^T \underline{S}_\ell \Bigg)$$

$$+ \Bigg(\sum_{\ell=0}^{k} \pi_\ell \underline{S}_\ell \underline{B}_\ell \Bigg) \Bigg(\underline{R} + \sum_{\ell=0}^{k} \pi_\ell \underline{B}_\ell^T \underline{S}_\ell \underline{B}_\ell \Bigg)^{-1} \cdot \Bigg(\underline{B}_j^T \underline{S}_j \underline{B}_j + \underline{R} \Bigg) \Bigg(\underline{R} + \sum_{\ell=0}^{k} \pi_\ell \underline{B}_\ell^T \underline{S}_\ell \underline{B}_\ell \Bigg)^{-1}$$

$$\cdot \Bigg(\sum_{\ell=0}^{k} \pi_\ell \underline{B}_\ell^T \underline{S}_\ell \Bigg) \Bigg) \Bigg) \underline{A} + \underline{Q} \qquad (44)$$

The corresponding steady-state control is

$$\underline{u}^* = -\Big[\underline{R} + \sum_{i=0}^{k} \pi_i \underline{B}_i^T \underline{S}_i \underline{B}_i\Big]^{-1} \Big(\sum_{i=0}^{k} \pi_i \underline{B}_i^T \underline{S}_i \Big) \underline{A}\,\underline{x} \qquad (45)$$

The steady-state cost-to-go at time t, given $\underline{x}(t)$, is

$$\langle \underline{\pi}, \underline{C}(\underline{x}(t)) \rangle \qquad (46)$$

where

$$\underline{C}(\underline{x}(t)) = (\underline{x}^T(t)\underline{S}_i\,\underline{x}(t))_{i=0}^{k} \qquad (47)$$

8. A Method for Comparing Two Linear Feedback Control Laws

In the last section, the equations were given for the steady-state optimal cost-to-go as a function of $\underline{x}$. In this section, these equations are extended for the case of the control law

$$\underline{u} = -\underline{G}\,\underline{x} \qquad (48)$$

The solution of these equations yields a valid comparison between two linear feedback control laws when it exists. The cost of control $u = -Gx$ is defined as a function of x

$$V(\underline{G}) = \langle \underline{\pi}, \underline{C}(\underline{G}, \underline{x}) \rangle \qquad (49)$$

where $\underline{C}$ now depends on $\underline{G}$, as well as $\underline{x}$. Assuming $\underline{C}$ is of the form

$$\underline{C}(\underline{G},\underline{x}) = [\underline{x}'\underline{S}_0(\underline{G})\underline{x} \,|\, \ldots \,|\, \underline{x}'\underline{S}_k(\underline{G})\underline{x}]^T \qquad (50)$$

where $\underline{S}_i$ depends on $\underline{G}$, then if a steady state solution exists, $\underline{S}_i$ for $i=0,1,\ldots,k$ satisfy the following set of equations.

$$\underline{S}_i = \sum_{j=0}^{k} p_{ji} [\underline{A}'\underline{S}_j\underline{A} + \underline{G}'\underline{B}'_j\underline{S}_j\underline{B}_j\underline{G} - \underline{G}'\underline{B}'_j\underline{S}_j\underline{A} - \underline{A}'\underline{S}_j\underline{B}_j\underline{G}] + \underline{Q} + \underline{G}'\underline{R}\underline{G} \qquad (51)$$

9. Conclusion

This paper reports on a class of linear quadratic control problems in discrete-time, where actuator failure may occur. The major contribution of this work is that a steady-state, structure-independent control law can be defined, along with its associated cost. This allows a numerical comparison of different control methodologies to be performed.

The case for a suboptimal solution to be used for comparisons is made by the decrease in complexity over any optimal solution which might exist. Although this solution is still quite complex, it is solvable and easier conceptually than the dual control problem, whose solution has little application due to the increased complexity and non-existence of a steady-state solution.

The concept behind this paper is much broader in scope than is shown here. The special structure of the B-matrix allows one to tie together the concepts of stabilizability of individual structures and of existence of a solution. This cannot be done in general. There are very simple examples that clearly demonstrate, for instance, the uncertainty threshold principle.

10. Appendix

__Theorem 1__:

For the system given in section (3), if $c_i(t)$ is the cost-to-go, given the structure i (B-matrix = $\underline{B}_i$) as a function of $\{(\underline{x}(\ell),\underline{u}(\ell))_{\ell=t}^{T_2-1}, \underline{x}(T_2)\}$, then $c_i(t)$ can be computed recursively as

$$c_i(t) = \sum_{j=0}^{k} \text{probability}(i \rightarrow j \text{ at } t)\,c_j(t+1) + j(t) \qquad (A.1)$$

where

$$j(t)(\underline{x},\underline{u}) = \begin{cases} \underline{x}'(t)\underline{Q}\,\underline{x}(t) + \underline{u}'(t)\underline{R}\,\underline{u}(t); & t < T_2 \\ \underline{x}'(T_2)\underline{Q}_f\underline{x}(T_2) & ;\ t = T_2 \end{cases} \qquad (A.2)$$

and, if $\underline{C}(t) = (c_i(t))_{i=0}^{k}$, $\underline{J}(t) = (j(t))_{i=0}^{k}$, then

$$\underline{C}(t) = \underline{P}^T\,\underline{C}(t+1) + \underline{J}(t) \qquad (A.3)$$

__Proof__:

$c_i(t)$ is the expectation of $J(\{(\underline{x}(\ell),\underline{u}(\ell))_{\ell=t}^{T_2-1}, \underline{x}(T_2)\})$

where $J(\cdot) = \sum_{t \in T} j(t)(\cdot)$ as in equation (8), given that the structure is i at time t. Since this expectation is a sum of sums, and since the space of sequences $\{i_\ell \in \{0,1,\ldots,k\}\}_{\ell=t}^{T_2}$ is countable, then Fubini's theorem applies and

$$c_i(t) = [\text{cost over } [t,t+1)] + \sum_{j=0}^{k} \text{prob.}(\text{structure } j \text{ at time } t+1) \cdot [\text{cost-to-go at time } t+1, \text{ given structure } j] \qquad \text{(A.4)}$$

which is exactly (A.1). Q.E.D.

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The linear quadratic optimal control method is used today to solved many complex systems problems. As system complexity increases, and as linear quadratic optimal control is used in more demanding situations, the extension of the design methodology to cover system failures, robustness and reliability is of crucial importance. This paper documents the progress toward a theory which incorporates reliability in the performance index; a linear quadratic control problem is formulated which accounts for system effectiveness and gives

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20. Abstract

an offline procedure for comparing two linear quadratic control systems on thebasis of both reliability and performance.