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VIBRATIONS OF A HELICOPTER ROTOR BLADE USING FINITE ELEMENT-UNCONSTRAINED VARIATIONAL FORMULATIONS

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In the past several years, a numerical method has been developed which is a generalized Rayleigh-Ritz - finite element discretization using the combined concept of Lagrange multipliers and adjoint variables. This approach enables one to deal with problems associated with nonconservative forces, coupling effects and all types of boundary conditions in a routine fashion; and it appears promising in solving the vibration and dynamic stability problems associated with the complicated equations of a helicopter rotor blade. This paper presents

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the first application of the general method of the vibration problem of such a rotor blade.

The basic differential equations in this paper are taken from the linear, but fully coupled set developed by Houbolt and Brooks in 1956. These equations are further reduced to a simplest possible case and yet still contain the coupling of flap and root torsion modes. An unconstrained, adjoint variational statement has been established which is both the necessary and sufficient condition for the coupled differential equations and some general, but physical meaningful boundary conditions. The finite element matrix equations are then derived from this variational statement illustrating the way that coupling terms could be handled in general.

The numerical results from some demonstrative examples show that instability of flutter can occur in the range of operational rotor speed due to the coupled motion of flapping and root torsion without any aerodynamic force, if the torsional spring (or the pitch control link) is not sufficiently stiff. This instability does not appear to have been reported previously.

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1. INTRODUCTION. An analytical investigation on vibrations and dynamic stability of helicopter rotor blades usually consists of two phases (1) the derivation of the governing differential equations to include parameters and variables considered physically important, and (2) the formulation of solutions for the equations derived and data interpretations. This paper deals with the second phase of such an investigation.

Due to the slenderness of a helicopter rotor blades, its aerodynamic cross-section and the requirements on the craft's maneuverability, there are a large number of interacting parameters and the resulting differential equations are, as a rule, nonlinear, coupled in terms of field variables. In addition, the aerodynamic forces, coriolis forces due to rotation of the blade are nonconservative in nature and the effects due to structural damping and various boundary conditions must be evaluated.

Considerable attention has been given recently to the derivation of a consistent set of nonlinear differential equations together with aerodynamic forces. This fact is amply demonstrated by the work of Friedman and Tong [1] and that of Hodges and Ormiston [2]. In both references [1] and [2], brief reviews on earlier work on helicopter blade equations can be found. As for obtaining solutions to these equations, there does not appear to exist a general and efficient method to deal with the difficulties associated with nonlinearities, nonconservative forces, coupling terms, various boundary conditions, damping effects and periodic excitations. For example, Miller and Ellis [3], Ham [4], and Friedman and Tong [1] have obtained approximate solution by including in their solution formulation only the lowest modes of vibrations, Hodges and Ormiston [2,5] employed the Galerkin technique in their numerical examples. One of the disadvantages of this approach is its inability to handle general boundary conditions.

Using the combined concept of adjoint variable and Lagrange multipliers, variational statements can be established for a wide range of linear problems with nonconservative forces and very versatile boundary conditions [6]. Thus a generalized Rayleigh-Ritz approximation scheme can be established for the obtaining of solutions of these otherwise difficult-to-solve problems. In conjunction with finite element discretization, this approach has been amply demonstrated in such applications as nonconservative stability, damping effects and very general boundary conditions [7,8,9]. In the present study, the solution formulation is limited to the blade vibration considering only the coupling of flapping and root torsion. Although the present method can conceivably be extended for solutions of nonlinear problems, it is desirable first to have a thorough understanding to solutions of linear problems. For this purpose, one can go back two decades and use the equations consistently derived by Houbolt and Brooks in 1956 [10]. The original set of equations was derived for elastic distributed torsion. It can be adapted to model root torsion if proper boundary conditions are introduced. This is shown in Sections 1 and 2. The physical justification for emphasizing root torsion over the distributed torsion was due to the pitch control link at the inboard end of the blade and was used by Miller and Ellis [3] and again by Ham [4]. It should be clear that, in the present formulation, to include distributed torsion is simply a matter of increasing the number of degrees-of-freedom of the discrete system. The basis of the solution formulation and the technique of handling the coupling

terms are given in Sections 2 and 3. Finally, the numerical results obtained indicate that "flutter instability" can occur simply due to the coupling effect considered without any aerodynamic loads.

2. STATEMENT OF THE PROBLEM - DIFFERENTIAL EQUATIONS. As a first step to demonstrate the application of the unconstrained variational - finite element formulation to helicopter rotors, the vibration of a rotor blade considering the coupling of flap and root torsion modes of motion is analyzed (Figure 1). For this purpose, the linear set of equations, derived by Houbolt and Brooks [10], including the coupling flap and distributed torsion is rewritten here:

$$(EIw'' - Te_A\phi)'' - (Tw')' - (\Omega^2 mxe\phi)' + m(\ddot{w} + e\ddot{\phi}) = L_z, \qquad (1)$$

$$- [(GJ + Tk_A^2)\phi']' - Te_A w'' + \Omega^2 mxew'$$

$$+ \Omega^2 m(k_{m2}^2 - k_{m1}^2 + ee_o)\phi + mk_m^2 \ddot{\phi} + me\ddot{w} = M \qquad (2)$$

where $w = w(x,t)$ and $\phi = \phi(x,t)$ denote the flapping deflection and distributed torsion of the rotor blade respectively. A prime (') denotes differentiation with respect to x, the coordinate along the blade elastic axis and a dot (·), differentiation with respect to the time t. Other symbols are defined in the following and are consistent with the notation in reference [10].

EI = flexural rigidity of the cross section.

GJ = torsional rigidity of the cross section.

ℓ = length of the blade.

e = chordwise distance between elastic axis (E.A.) and centre of gravity (C.G.) of a cross-section, positive if C.G. is ahead of E.A.

e_A = chordwise distance between E.A. and centre of tensile area (C.T.) of a cross section, positive if C.T. is ahead of E.A.

e_o = chordwise distance at the root between E.A. and the axis about which the blade is rotating, positive if E.A. is ahead.

k_A = polar radius of a gyration of the tensile area of a cross-section w.r.t. E.A. in a cross-section.

k_{m1} = radius of gyration of the total area of a cross-section about the major neutral axis (axis 1-1 in Figure 2).

k_{m2} = radius of gyration of the total area of a cross-section about an axis perpendicular to the major neutral axis and through E.A. (axis m2-m2 in Figure 2).

k_m = polar radius of gyration of the total area of cross-section about E.A. ($k_m^2 = k_{m1}^2 + k_{m2}^2$).

Ω = blade angular velocity.

$T = T(x) = \int_x^{\ell} m\Omega^2 x dx$, which is the tensile force at location x.

L_z = aerodynamic lift per unit length of the blade.

M = aerodynamic torque per unit length of the blade.

In reference [10], a more general set of linear equations with fully coupled flap, lag and distributed torsion modes of motion were derived. The coupling terms have been shown to be due to the noncoincidence of the elastic centre, centre of gravity and tension centre, the centrifugal force and the built-in angle of twist. For a rotor blade without the built-in twist angle, the lag mode of motion is uncoupled from the general set of equations and the remaining coupled equations are Eqs. (1) and (2) considered here.

Since only the "free vibration" of the rotor blade with a coupling between flexural and root torsion is considered in this paper, the terms due to aerodynamic forces are set to zero and the torsional displacement is only a function of time and not a function of x. Thus

$$M = L_z = 0, \quad \phi = \phi(t) \tag{3}$$

and Eqs. (1) and (2) reduce to the following:

$$EIw'''' - (Tw')' + m\ddot{w} - \Omega^2 me\phi + me\ddot{\phi} = 0 \tag{4}$$

$$\Omega^2 m(k_{m2}^2 - k_{m1}^2 + ee_o)\phi + mk_m^2 \ddot{\phi} - Te_A w'' + m\Omega^2 exw' + me\ddot{w} = 0 \tag{5}$$

In Eqs. (4) and (5), it is also assumed that the blade has constant E, I and m throughout its length. To simplfy solution formulations as much as possible, Eqs. (4) and (5) will be transformed into dimensionless forms and appropriate dimensionless parameters will be introduced. This process will also facilitate parametric studies.

Let

$$\bar{w} = \frac{w}{\ell} \ , \quad \bar{\phi} = \phi$$

$$\bar{x} = \frac{x}{\ell} \ , \quad \bar{t} = \frac{t}{c} \tag{6}$$

where the constant c has a real time dimension and will be defined later in Eqs. (12). Eqs. (4) and (5) become

$$\frac{EI\ell}{\ell^4}\frac{\partial^2\bar{w}}{\partial\bar{x}^4} - \frac{\ell}{\ell^4}\frac{\partial}{\partial\bar{x}}\left(T\frac{\partial\bar{w}}{\partial\bar{x}}\right) + \frac{m\ell}{c^2}\frac{\partial^2\bar{w}}{\partial\bar{t}^2} - m\Omega^2 e\bar{\phi} + \frac{me}{c^2}\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2} = 0 \qquad (7)$$

and

$$m\Omega^2 k^2\bar{\phi} + \frac{mk_m^2}{c^2}\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2} - \frac{Te_A\ell}{\ell^2}\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + \frac{m\Omega^2 e\ell^2}{\ell}\bar{x}\frac{\partial\bar{w}}{\partial\bar{x}} + \frac{me\ell}{c^2}\frac{\partial^2\bar{w}}{\partial\bar{t}^2} = 0 \qquad (8)$$

where

$$k^2 = k_{m2}^2 - k_{m1}^2 + ee_o \qquad (9)$$

Multiplying Eq. (7) by $\frac{\ell^3}{EI}$, one has

$$\frac{\partial^4\bar{w}}{\partial\bar{x}^4} - \frac{\partial}{\partial\bar{x}}\left(\frac{T\ell^2}{EI}\frac{\partial\bar{w}}{\partial\bar{x}}\right) + \frac{m\ell^4}{EI}\frac{1}{c^2}\frac{\partial^2\bar{w}}{\partial\bar{t}^2} - \frac{m\ell^4}{EI}\frac{e}{\ell}\frac{1}{c^2}c^2\Omega^2\bar{\phi}$$

$$+ \frac{m\ell^4}{EI}\frac{1}{c^2}\frac{e}{\ell}\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2} = 0 \qquad (10)$$

Multiplying Eq. (8) by $\frac{c^2}{m\ell^2}$, one has

$$c^2\Omega^2\frac{k^2}{\ell^2}\bar{\phi} + \frac{k_m^2}{\ell^2}\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2} - \frac{c^2T}{m\ell^2}\frac{e_A}{\ell}\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + c^2\Omega^2\frac{e}{\ell}\bar{x}\frac{\partial\bar{w}}{\partial\bar{x}} + \frac{e}{\ell}\frac{\partial^2\bar{w}}{\partial\bar{t}^2} = 0 \qquad (11)$$

Now let

$$\left.\begin{aligned} c^2 &= \frac{m\ell^2}{EI} \\ \bar{e} = \frac{e}{\ell}\ ,\ \bar{e}_A &= \frac{e_A}{\ell}\ ,\ \bar{e}_o = \frac{e_o}{\ell} \\ \bar{k} = \frac{k}{\ell}\ ,\ \bar{k}_m &= \frac{k_m}{\ell} \\ \bar{\Omega} &= c\Omega \\ \bar{T}(\bar{x}) = \frac{T(x)\ell^2}{EI} &= \frac{1}{2}\bar{\Omega}^2(1-\bar{x}^2) \end{aligned}\right\} \qquad (12)$$

Eqs. (10) and (11) then become

$$\frac{\partial^4\bar{w}}{\partial\bar{x}^4} - \frac{\partial}{\partial\bar{x}}\left(\bar{T}(\bar{x})\frac{\partial\bar{w}}{\partial\bar{x}}\right) + \frac{\partial^2\bar{w}}{\partial\bar{t}^2} - \bar{\Omega}^2\bar{e}\bar{\phi} + \bar{e}\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2} = 0 \qquad (13)$$

and

$$\bar{\Omega}^2\bar{k}^2\bar{\phi} + k_m^2 \frac{\partial^2\bar{\phi}}{\partial\bar{t}^2} - \bar{T}(\bar{x})\,\bar{e}_A \frac{\partial^2\bar{w}}{\partial\bar{x}^2} + \bar{\Omega}^2\bar{e}\bar{x} \frac{\partial\bar{w}}{\partial\bar{x}} + \bar{e} \frac{\partial^2\bar{w}}{\partial^2\bar{t}^2} = 0 \tag{14}$$

With all quantities in Eqs. (13) and (14) in dimensionless forms, one can omit the bars altogether and write:

$$w'''' - (Tw')' + \ddot{w} - \Omega^2 e\phi + e\ddot{\phi} = 0 \tag{15}$$

and

$$\Omega^2 k^2\phi + k_m^2 \ddot{\phi} - Te_A w'' + \Omega^2 exw' + e\ddot{w} = 0 \tag{16}$$

Furthermore, it is assumed that

$$w(x,t) = w(x)e^{\lambda t} \tag{17}$$

$$\phi(t) = \phi e^{\lambda t}$$

Thus the final set of equations upon which the present solution formulations are based, is the following:

$$w'''' - (Tw')' + \lambda^2 w - \Omega^2 e\phi + \lambda^2 e\phi = 0 \tag{18}$$

and

$$\Omega^2 k^2\phi + \lambda^2 k_m^2 \phi - Te_A w'' + \Omega^2 exw' + \lambda^2 ew = 0 \tag{19}$$

3. AN UNCONSTRAINED VARIATIONAL STATEMENT AND BOUNDARY CONDITIONS. Some of the unique features of the unconventional variational formulation are that all the boundary conditions are natural boundary conditions and that the set of the differential equations, together with all the boundary conditions, is the direct consequence of a variational statement and vice versa. The construction of such a variational statement is simply a process of integration-by-parts from a bilinear functional of the original differential equations multiplied by the variation of the adjoint field variable, into some other bilinear functional with lowest possible derivations of both the original field variable and the variations of the adjoint variable. With a proper choice of the generalized Lagrange multipliers, any physical meaningful boundary conditions consistent with the physical meaning of the given differential equations themselves can be resulted from the variational statement as natural boundary conditions. This general process was treated elsewhere [7] and will not be reported here. Presently, an unconstrained variational statement will be given which leads to the original differential equations and a set of a very general boundary conditions.

Let us consider the variational statement

$$\delta I = 0 \tag{20a}$$

with

$$I = I_1 + I_2 \tag{20b}$$

$$I_1 = \int_0^1 [w''w^{*''} + Tw'w^{*'} + \lambda^2 ww^* + e(\lambda^2 - \Omega^2)\phi w^*]dx$$

$$+ k_1 w(0)w^*(0) + k_2 w'(0)w^{*'}(0) \tag{20c}$$

and

$$I_2 = \int_0^1 [\Omega^2 k^2 + \lambda^2 k_m^2)\phi\phi^* + e_A T'w'\phi^* + e(\Omega^2 xw'\phi^* + \lambda^2 w\phi^*)]dx + k_3\phi\phi^* \tag{20d}$$

where a star(*) denotes the adjoint variables and the variations are totally unconstrained. The Lagrange multipliers k_1, k_2 and k_3 are the spring constants, in dimensionless form, for deflection, bending and torsion at the hub ($x = 0$) respectively. To show that the unconstrained variational statement of Eqs. (20) leads to the original differential equations and the necessary boundary conditions, one considers

$$(\delta I)_{w,\phi} = 0 \tag{21a}$$

where $(\delta I)_{w,\phi}$ means taking the variation of I with ω,ϕ not varied. Thus

$$(\delta I)_{w,\phi} = (\delta I_1)_{w,\phi} + (\delta I_2)_{w,\phi} = 0 \tag{21b}$$

with

$$(\delta I_1)_{w,\phi} = \int_0^1 [w''\delta w^{*''} + Tw'\delta w^{*'} + \lambda^2 w\delta w^* + e(\lambda^2 - \Omega^2)\phi\delta w^*]dx$$

$$+ k_1 w(0)\delta w^*(0) + k_2 w'(0)\delta w^{*'}(0) \tag{21c}$$

and

$$(\delta I_2)_{w,\phi} = \int_0^1 [(\Omega^2 k^2 + \lambda^2 k_m^2)\phi\delta\phi^* + e_A T'w'\delta\phi^*$$

$$+ e(\Omega^2 xw' + \lambda^2 w)\delta\phi^*]dx + k_3\phi\delta\phi^* \tag{21d}$$

Performing integration-by-parts, one arrives at

$$(\delta I_1)_{w,\phi} = \int_0^1 [w'''' - (Tw')' + \lambda^2 w + e(\lambda^2 - \Omega^2)\phi]\delta w^* dx$$

$$+ w''(1)\delta w^{*\prime}(1) - [w''(1) - T(1)w'(1)]\delta w^*(1)$$

$$- [w''(0) - k_2 w'(0)]\delta w^{*\prime}(0)$$

$$+ [w'''(0) - T(0)w'(0) + k_1 w(0)]\delta w^*(0) \qquad (22a)$$

and

$$(\delta I_2)_{w,\phi} = \int_0^1 [(\Omega^2 k^2 + \lambda^2 k_m^2)\phi - e_A T w'' + e(\Omega^2 x w' + \lambda^2 w)]\delta\phi^* dx$$

$$+ [e_A T(1)w'(1) - e_A T(0)w'(0) + k_3\phi]\delta\phi^* \qquad (22b)$$

Thus Eqs. (21) is the necessary and sufficient condition for the following differential equations and boundary conditions:

D.E.:

$$w'''' - (Tw')' + \lambda^2 w + e(\lambda^2 - \Omega^2)\phi = 0 \qquad (23a)$$

$$(\Omega^2 k^2 + \lambda^2 k_m^2)\phi - e_A T w'' + e(\Omega^2 x w' + \lambda^2 w) = 0 \qquad (23b)$$

B.C.:

$$w''(1) = 0, \quad w'''(1) = 0 \qquad (24a,24b)$$

$$w''(0) - k_2 w'(0) = 0 \qquad (24c)$$

$$w'''(0) - \frac{\Omega^2}{2} w'(0) + k_1 w(0) = 0 \qquad (24d)$$

and

$$-\frac{e_A}{2}\Omega^2 w'(0) + k_3\phi = 0 \qquad (24e)$$

Note that in Eqs. (24) the fact that

$$T(1) = 0$$

and

$$T(0) = \frac{1}{2}\Omega^2 \qquad (25)$$

has been used.

It is observed that the boundary conditions of Eqs. (24c) and (24d) represent very general support conditions at the hub. The special case $k_1 = \infty$ and $k_2 = 0$ corresponds to a hinged blade, while the case $k_1 = \infty$, $k_2 = \infty$ corresponds to a hingeless blade. These two special cases are the only ones considered in the literature. The boundary condition (24e) indicates a coupling between the flexural motion ω and root torsion at the hub due to the centrifugal forces and the noncoincidence of the elastic axis and the tension axis.

4. MATRIX EQUATIONS FROM FINITE ELEMENT DISCRETIZATION. In this section, we shall briefly describe the formulation of the matrix equation of the approximate solution from the variational statement given in the previous section. From Eqs. (21), one can write

$$(\delta I)_{w,\phi} = 0 ;$$

or,

$$(\delta I)_{w,\phi} = \int_0^1 (w'\delta w^{*\prime\prime} + Tw'\delta w^{*\prime} - e\Omega^2\phi\delta w^*)dx$$

$$+ k_1 w(0)\delta w^*(0) + k_2 w'(0)\delta w^{*\prime}(0) + \lambda^2 \int_0^1 (w\delta w^* + e\phi\delta w^*)dx$$

$$+ \int_0^1 [\Omega^2 k^2 \phi\delta\phi^* + e_A T'w'\delta\phi^* + e\Omega^2 xw'\delta\phi^*]dx + k_3\phi\delta\phi^*$$

$$+ \lambda^2 \int_0^1 (k_m^2\phi\delta\phi^* + ew\delta\phi^*)dx \qquad (26)$$

It should be noted that Eq. (26) is a quite general equation. Various types of approximate solutions can be obtained depending on the choice of the coordinate functions. The motivation of using the finite element discretizations, which corresponds to the choice of a set of piecewise analytic functions, is twofold: for the ease of extending this formulation to problems of irregular geometry and for the adaptations to general finite element computer systems.

In the present formulation, however, only blades of uniform cross-sections will be considered and the elements are assumed to be of the same length. Thus, one introduces a local (element) coordinate ξ which relates to the global (entire blade) coordinate x such that

$$\xi = \xi^{(i)} = L(x - \frac{i-1}{L}) \qquad (27)$$

where L is the number of elements and i denotes the i-th element. Using the notation

$$w^{(i)}(\xi) = w[x(\xi^{(i)})], \text{ etc.} \qquad (28)$$

for simplicity and noting that

$$d\xi = Ldx$$

$$w'(x) = Lw^{(i)\prime}(\xi), \text{ etc.} \qquad (29)$$

one then has, from Eq. (26):

$$(\delta I)_{w,\phi} = 0$$

$$= \sum_{i=1}^{L} \{\int_0^1 L^3 w^{(i)''}(\xi)\delta w^{*(i)''}(\xi)d\xi$$

$$+ \int_0^1 L\, T^{(i)}(\xi)w^{(i)'}(\xi)\delta w^{*,(i)}(\xi)d\xi - \frac{e\Omega^2\phi}{L}\int_0^1 \delta w^{*(i)}d\xi\}$$

$$+ k_1 w^{(1)}(0)\delta w^{*(1)}(0) + k_2 L^2 w^{(i)'}(0)\delta w^{*(i)'}(0)$$

$$+ \lambda^2 \sum_{i=1}^{L} \{\frac{1}{L}\int_0^1 w^{(i)}(\xi)\delta w^{*(i)}(\xi)d\xi + \frac{e\phi}{L}\int_0^1 \delta w^{*(i)}(\xi)d\xi\}$$

$$-\sum_{i=1}^{L} \{\Omega^2 k^2 \phi\delta\phi^* \frac{1}{L}\int_0^1 d\xi^{(i)} + e_A L \int_0^1 T^{(i)'}(\xi)w^{(i)'}(\xi)d\xi\delta\phi^*$$

$$+ \frac{e\Omega^2}{L}\int_0^1 [\xi + (i-1)]w^{(i)'}(\xi)d\xi\,\delta\phi^*\} + k_3\phi\delta\phi^*$$

$$+ \lambda^2 \{\frac{k_m^2}{L}\phi\delta\phi^* \int_0^1 d\xi^{(i)} + \sum_{i=1}^{L}\frac{e}{L}\int_0^1 w^{(i)}(\xi)d\xi\}\delta\phi^* \qquad (30)$$

where

$$T^{(i)}(\xi) = -b[\xi^2 + 2(i-1)\xi + (i-1)^2 - L^2] \qquad (31)$$

and

$$b = \frac{\Omega^2}{2L^2}\,, \qquad (32)$$

At this point, it is appropriate to introduce the shape function and generalized coordinates. Let

$$w^{(i)}(\xi) = \underset{\sim}{a}^T(\xi)\underset{\sim}{W}^{(i)}$$

$$w^{*(i)}(\xi) = \underset{\sim}{a}^T(\xi)\underset{\sim}{W}^{*(i)} \qquad (33)$$

where the superscript T denotes the transpose of a matrix,

$$\underset{\sim}{W}^{(i)T} = \{ W_1^{(i)} \quad W_2^{(i)} \quad W_3^{(i)} \quad W_4^{(i)} \}$$

$$\underset{\sim}{W}^{*(i)T} = \{ W_1^{*(i)} \quad W_2^{*(i)} \quad W_3^{*(i)} \quad W_4^{*(i)} \} \tag{34}$$

and

$$\underset{\sim}{a}^T(\xi) = \{1 - 3\xi^2 + 2\xi^3, \quad \xi - 2\xi^2 + \xi^3, \quad 3\xi^2 - 2\xi^3, \quad -\xi^2 + \xi^3\} \tag{35}$$

Using Eqs. (31), (32) and in terms of $W^{(i)}$, $W^{*(i)}$, Eq. (30) becomes

$$(\delta I)_{w,\phi} = 0$$

$$= \sum_{i-1}^{L} \delta W^{*(i)T} \{L^3 \int_0^1 a'' a''^T d\xi - bL[\int_0^1 \xi^2 a' a'^T d\xi + 2(i-1)\int_0^1 \xi a' a'^T d\xi$$

$$+ \{L^2 - (i-1)^2\} \int_0^1 a' a'^T d\xi] \underset{\sim}{W}^{(i)}\}$$

$$+ \sum_{i=1}^{L} \delta \underset{\sim}{W}^{*(1)T} (-2ebL) \int_0^1 \underset{\sim}{a} d\xi \ \phi$$

$$+ \delta \underset{\sim}{W}^{*(1)T} k_1 \underset{\sim}{a}(0) \underset{\sim}{a}^T(0) \underset{\sim}{W}^{(1)} + \delta \underset{\sim}{W}^{*(1)} k_2 L^2 \underset{\sim}{a}'(0) \underset{\sim}{a}'^T(0) \underset{\sim}{W}^{(1)}$$

$$+ \lambda^2 \{ \sum_{i=1}^{L} \delta \underset{\sim}{W}^{*(i)T} \frac{1}{L} \int_0^1 \underset{\sim\sim}{a a}^T d\xi \ W^{(i)} + \frac{e}{L} \int_0^1 \underset{\sim}{a} d\xi \ \phi\}$$

$$+ \delta\phi^* \Omega^2 k^2 \phi + \delta\phi^* k_3 \phi$$

$$+ \delta\phi^* \sum_{i=1}^{L} 2(e - e_A) bL \int_0^1 \xi a'^T(\xi) d\xi + (i-1) \int_0^1 \underset{\sim}{a}'^T d\xi \ \underset{\sim}{W}^{(i)}$$

$$+ \lambda^2 \delta\phi^* [k_m^2 \phi + \sum_{i=1}^{L} \frac{e}{L} \int_0^1 \underset{\sim}{a}^T d\xi \ \underset{\sim}{W}^{(i)}] \tag{36}$$

Or,

$$(\delta I)_{w,\phi} = 0$$

$$= \sum_{i=1}^{L} \delta \underset{\sim}{W}^{*(i)T} \{L^3 \underset{\sim}{C} - bL[\underset{\sim}{E} + 2(i-1)\underset{\sim}{D} - (L^2 - (i-1)^2)\underset{\sim}{B}]\} \underset{\sim}{W}^{(i)}$$

$$+ \sum_{i=1}^{L} \delta \underset{\sim}{W}^{*(i)T} (-2\, ebL)\underset{\sim}{p}\,\phi + \delta \underset{\sim}{W}^{*(i)T} [k_1 \underset{\sim}{F} + k_2 L\, \underset{\sim}{G}] \underset{\sim}{W}^{(1)}$$

$$+ \lambda^2 [\sum_{i=1}^{L} \delta \underset{\sim}{W}^{*(i)T} [\frac{1}{L} \underset{\sim}{A}\, \underset{\sim}{W}^{(i)} + \frac{e}{L} \underset{\sim}{p}\, \phi]$$

$$+ \delta\phi^* \sum_{i=1}^{L} 2(e - e_A)bL\, (\underset{\sim}{s}^T + (i-1)\underset{\sim}{r}^T)\, \underset{\sim}{W}^{(i)}$$

$$+ \delta\phi^* (\Omega^2 k^2 + k_3)\phi + \lambda^2\, \delta\phi^* (\sum_{i=1}^{L} \frac{e}{L} p^T \underset{\sim}{W}^{(i)} + k_m^2 \phi) \qquad (37)$$

where the element matrices are defined as follows,

$$\underset{\sim}{A} = \int_0^1 \underset{\sim}{a}\underset{\sim}{a}^T d\xi, \qquad \underset{\sim}{B} = \int_0^1 \underset{\sim}{a}'\underset{\sim}{a}'^T d\xi, \qquad \underset{\sim}{C} = \int_0^1 \underset{\sim}{a}''\underset{\sim}{a}''^T d\xi$$

$$\underset{\sim}{D} = \int_0^1 \xi \underset{\sim}{a}'\underset{\sim}{a}'^T d\xi, \qquad \underset{\sim}{E} = \int_0^1 \xi^2 \underset{\sim}{a}'\underset{\sim}{a}'^T d\xi$$

$$\underset{\sim}{F} = \underset{\sim}{a}(0)\, \underset{\sim}{a}^T(0), \qquad \underset{\sim}{G} = \underset{\sim}{a}'(0)\underset{\sim}{a}'^T(0)$$

$$\underset{\sim}{p} = \int_0^1 \underset{\sim}{a}\, d\xi, \qquad \underset{\sim}{r} = \int_0^1 \underset{\sim}{a}'\, d\xi, \qquad \underset{\sim}{s} = \int_0^1 \xi\, \underset{\sim}{a}'\, d\xi \qquad (38)$$

The numerical values of these matrices are given in the Appendix.

In terms of the global coordinates defined in the following

$$\underset{\sim}{W} = \begin{Bmatrix} \underset{\sim}{W} \\ \phi \end{Bmatrix}, \qquad \underset{\sim}{\bar{W}}^* = \begin{Bmatrix} \underset{\sim}{W}^* \\ \phi \end{Bmatrix} \qquad (39)$$

and

$$\underset{\sim}{W}^T = \{W_1^{(1)}\ W_2^{(1)}\ W_3^{(1)}\ W_4^{(1)}\ W_3^{(2)}\ W_4^{(2)} \ldots W_3^{(L)}\ W_4^{(L)}\}$$

$$\underset{\sim}{W}^{*T} = \{W_1^{*(1)}\ W_2^{*(1)}\ W_3^{*(1)}\ W_4^{*(1)}\ W_3^{*(2)}\ W_4^{*(2)} \ldots W_3^{*(L)}\ W_4^{*(L)}\} \qquad (40)$$

One has from Eq. (37):

$$(\delta I)_{w,\phi} = 0$$

$$= \delta\bar{\underset{\sim}{W}}^{*T}(\bar{\underset{\sim}{K}} + \lambda^2\bar{\underset{\sim}{M}})\bar{\underset{\sim}{W}}$$

$$= \{\delta\underset{\sim}{W}^{*T}\delta\phi^*\} \begin{bmatrix} \bar{K}_{11} + \lambda^2\bar{M}_{11} & \bar{K}_{12} + \lambda^2\bar{M}_{12} \\ \bar{K}_{21} + \lambda^2\bar{M}_{21} & \bar{K}_{22} + \lambda^2\bar{M}_{22} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{W} \\ \phi \end{Bmatrix} \tag{41}$$

The global matrices $\bar{K}_{ij}$, $\bar{M}_{ij}$, i,j = 1,2 are assembled from the element matrices defined in Eqs. (38) in the following manner.

$$\delta\underset{\sim}{W}^{*T}\bar{K}_{11}\underset{\sim}{W} = \sum_{i=1}^{L} \delta\underset{\sim}{W}^{*(i)T}\{L^3(\underset{\sim}{C} + b\underset{\sim}{B}) + bL(2\underset{\sim}{D} - B - E)$$

$$+ 2ibL(\underset{\sim}{B} - \underset{\sim}{D}) - i^2bL\underset{\sim}{B}\}\underset{\sim}{W}^{(i)} + \delta W^{*(1)T}\{k_1\underset{\sim}{F} + k_2L^2\underset{\sim}{G}\}\underset{\sim}{W}^{(1)} \tag{42a}$$

$$\delta W^{*T}\bar{K}_{12}\phi = \sum_{i=1}^{L} \delta\underset{\sim}{W}^{*(i)T}(-2ebL\ \underset{\sim}{p})\phi \tag{42b}$$

$$\delta\phi^*\bar{K}_{21}\underset{\sim}{W} = \delta\phi^* \sum_{i=1}^{L} 2(e - e_A)bL[\underset{\sim}{s}^T - \underset{\sim}{r}^T + i\underset{\sim}{r}^T]\underset{\sim}{W}^{(i)} \tag{42c}$$

$$\delta\phi^*\bar{K}_{22}\phi = \delta\phi^*(\Omega^2k^2 + k_3)\phi \tag{42d}$$

$$\delta\underset{\sim}{W}^*\bar{M}_{11}\underset{\sim}{W} = \sum_{i=1}^{L} \delta\underset{\sim}{W}^{*(i)T}\frac{1}{L}\underset{\sim}{A}\ \underset{\sim}{W}^{(i)} \tag{42e}$$

$$\delta\underset{\sim}{W}^*\bar{M}_{12}\phi = \sum_{i=1}^{L} \delta\underset{\sim}{W}^{*(i)T}\frac{e}{L}\underset{\sim}{p}\ \phi \tag{42f}$$

$$\delta\phi^*\bar{M}_{21}\underset{\sim}{W} = \delta\phi^* \sum_{i=1}^{L} \frac{e}{L}\underset{\sim}{p}^T\ \underset{\sim}{W}^{(i)} \tag{42g}$$

$$\delta\phi^*\bar{M}_{22}\phi = \delta\phi^*\ k_m^2\ \phi \tag{42h}$$

Since $\delta \underset{\sim}{W}^{*T} = \{\delta \underset{\sim}{W}^{*T}\ \delta\phi^{*}\}$ in Eq. (41) is unconstrained, Eq. (41) leads directly to

$$(\bar{\underset{\sim}{K}} + \lambda^2 \bar{\underset{\sim}{M}})\bar{\underset{\sim}{W}} = 0 \tag{43}$$

which is the final matrix eigenvalue equation to be solved.

5. RESULTS AND DISCUSSION. Prior to the presentation of the demonstrative numerical results, it will be worthwhile to make some observations on the nondimensionalized differential equations (15) and (16) and the boundary conditions (24).

a. For $e = 0$, the flapping motion $w(x,t)$ and the root torsion $\phi(t)$ are essentially uncoupled. The eigenvalue solutions of w reduce to that of a rotating beam and agree well with the available data of Boyce, DiPrima and Handelman [11]. The torsional vibration frequency has only one eigenvalue solution and, if $e_A = 0$ also, it varies linearly with the rotor speed Ω as expected.

b. For $e \neq 0$ and $e_A = 0$, the motions are generally coupled. It is observed that if e and ϕ both change sign, the equation remains unaltered. Thus, as far as eigenvalues are concerned, they depend only on the absolute values of e. The solution of ϕ for a negative e, however, is 180° out of phase compared with the one for a positive e of the same magnitude.

c. For $e \neq 0$ and $e_A \neq 0$ the motions are generally coupled. It is observed that if e, e_A and ϕ all three change signs, the governing equations remain unaltered.

Some demonstrative calculations will now be given*. The eigenvalue λ is generally a complex number.

$$\lambda = \lambda_R + i\lambda_I \tag{44}$$

From equations (17) and (44), it is clear that the system is unstable of divergence when λ_R is nonzero positive and $\lambda_I = 0$. When $\lambda_R = 0$, on the other hand, λ_I (LAMBDA)** represents the nondimensional frequency of vibration and it can then be plotted against the nondimensional blade rotating speed Ω (OMEGA), as shown in Figures 3 through 7. When λ is complex, one of the square roots of λ^2 must have nonzero positive real part. The system is then unstable since λ appears in the equations only as λ^2. The value of λ generally become complex as the two branches of the frequency curve coalesce. The "critical" speeds can be located in these figures by noting these points of coalescence.

*An extensive parametric study of a rotor blade instability in vacuum due to the coupled flap-(root)torsion motion will be presented in a forthcoming paper.

**The symbols enclosed in parentheses are those used in Figures 3 through 7.

For the following sample calculations, the parameters used are typical for a medium size rotor blade of 20 ft. in length, for example. It is further assumed that (the nondimensionalized quantities)

$$e(E) = 0.003, \qquad e_o(E_o) = 0$$

$$k_{m1}(KM1) = 0.0025, \qquad k_{m2}(KM2) = 0.0100$$

The boundary conditions at the hub are those of a hingeless blade. The values of k_1(K-1) and k_2(K-2) are set to 10^8 as approximations to infinity. The torsional spring constant k_3(K-3) has been set to zero.

In Figure 3, $e_A(E_A)$ is taken to be 0.003. Thus

$$\gamma \text{ (GAMMA)} = \frac{e_A}{e} = \frac{-e_A}{-e} = 1.00 \ .$$

Here the lowest branch of λ_I is essentially for the torsional motion. The second and third lowest branches are essentially the first and the second for flapping motion. For the range of rotor speed shown*, $0 \leq \Omega \leq 25$, the coupling is not sufficient to have instability. In the subsequent figures as e_A = 0.0015, 0, -0.0015 and -0.003 (and γ = 0.5, 0, -0.5 and -1.0), the effect of coupling becomes more and more severe. In Figure 4, the two lowest branches of eigenvalues appear to draw closer compared with those in Figure 3. They actually coalesce in Figure 5 at a critical speed about Ω = 16.5. As e_A continue to decrease (increase) in algebraic sense while holding e a positive (negative) constant the critical speed of flutter instability tends to decrease and thus the structure becomes more critical. This is observed in Figures 6 and 7.

To obtain some idea on the effect due to the torsional spring k_3, the calculations are repeated for the same set of data as in Figure 5 except that the value of k_3 is changed from zero to 0.01. The new frequency curves are shown in Figure 8. It appears in this figure that the essentially torsional branch has moved upward while the other branches remain largely unaltered. The new critical speed prior to flutter instability, however, has changed from 16.46 (Figure 5) to 17.08.

A rather quick review of the existing literature suggests that the flutter instability due to the coupling flapping and root torsion motions may have not been reported previously. The practical significance of such instability within the range of real design parameters is obvious. An extensive parametric study on this instability is being carried out. The results will be reported shortly.

ACKNOWLEDGEMENT. The authors wish to express their appreciation to Mr. Richard F. Haggerty for his assistance in obtaining the numerical results. They also wish to thank Miss Ellen Fogarty for her patience and skill in typing the manuscript.

*At Ω (OMEGA) = 25, which is nondimensional, the real rotor speed is roughly 1000 rpm for a typical value of $c = \sqrt{(\rho A \ell^4/EI)} = 0.0040$min.

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APPENDIX

The numerical value of some of the matrices used in Section 4 are given here.

$$\underset{\sim}{A} = \int_0^1 \underset{\sim\sim}{a}\underset{\sim\sim}{a}^T d\xi = \begin{bmatrix} \frac{13}{25} & \frac{11}{210} & \frac{9}{70} & -\frac{13}{420} \\ \frac{11}{210} & \frac{1}{105} & \frac{13}{420} & -\frac{1}{140} \\ \frac{9}{70} & \frac{13}{420} & \frac{13}{35} & -\frac{11}{210} \\ -\frac{13}{420} & -\frac{1}{140} & -\frac{11}{210} & \frac{1}{105} \end{bmatrix} \qquad \text{(A-1)}$$

$$\underset{\sim}{B} = \int_0^1 \underset{\sim}{a}'\underset{\sim}{a}'^T d\xi = \begin{bmatrix} \frac{6}{5} & \frac{1}{10} & -\frac{6}{5} & \frac{1}{10} \\ \frac{1}{10} & \frac{2}{15} & -\frac{1}{10} & -\frac{1}{30} \\ -\frac{6}{5} & -\frac{1}{10} & \frac{6}{5} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{1}{30} & -\frac{1}{10} & \frac{2}{15} \end{bmatrix} \qquad \text{(A-2)}$$

$$\underset{\sim}{C} = \int_0^1 \underset{\sim}{a}''\underset{\sim}{a}''^T d\xi = \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \qquad \text{(A-3)}$$

$$\underset{\sim}{D} = \int_0^1 \xi \underset{\sim}{a}' \underset{\sim}{a}'^T d\xi = \begin{bmatrix} \frac{3}{5} & \frac{1}{10} & -\frac{3}{5} & 0 \\ \frac{1}{10} & \frac{1}{30} & \frac{1}{10} & \frac{1}{60} \\ -\frac{3}{5} & -\frac{1}{10} & \frac{3}{5} & 0 \\ 0 & -\frac{1}{60} & 0 & \frac{1}{10} \end{bmatrix} \tag{A-4}$$

$$\underset{\sim}{E} = \int_0^1 \xi^2 \underset{\sim}{a}' \underset{\sim}{a}'^T d\xi = \begin{bmatrix} \frac{12}{35} & \frac{1}{14} & -\frac{12}{35} & -\frac{1}{35} \\ \frac{1}{14} & \frac{1}{105} & -\frac{1}{14} & -\frac{1}{70} \\ -\frac{12}{35} & -\frac{1}{14} & \frac{12}{35} & \frac{1}{35} \\ -\frac{1}{35} & -\frac{1}{70} & \frac{1}{35} & \frac{3}{35} \end{bmatrix} \tag{A-5}$$

$$\underset{\sim}{F} = \underset{\sim}{a}(0)\underset{\sim}{a}^T(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A-6}$$

$$\underset{\sim}{G} = \underset{\sim}{a}'(0)\underset{\sim}{a}'^T(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A-7}$$

$$\underset{\sim}{p} = \int_0^1 \underset{\sim}{a} d\xi = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{12} \\ \frac{1}{2} \\ -\frac{1}{12} \end{bmatrix} \tag{A-8}$$

$$\underset{\sim}{r} = \int_0^1 \underset{\sim}{a}' d\xi = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{A-9}$$

$$\underset{\sim}{s} = \int_0^1 \xi \, \underset{\sim}{a}' d\xi = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{12} \\ \frac{1}{2} \\ \frac{1}{12} \end{bmatrix} \tag{A-10}$$

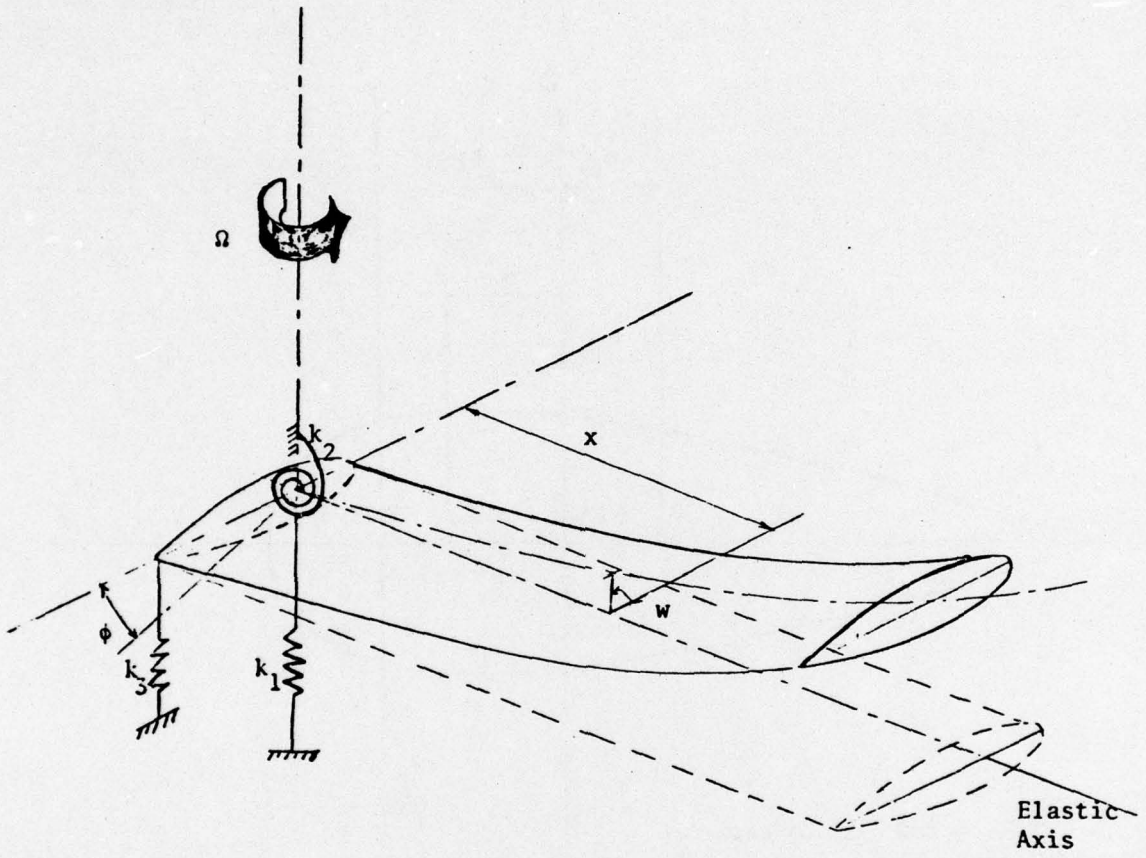
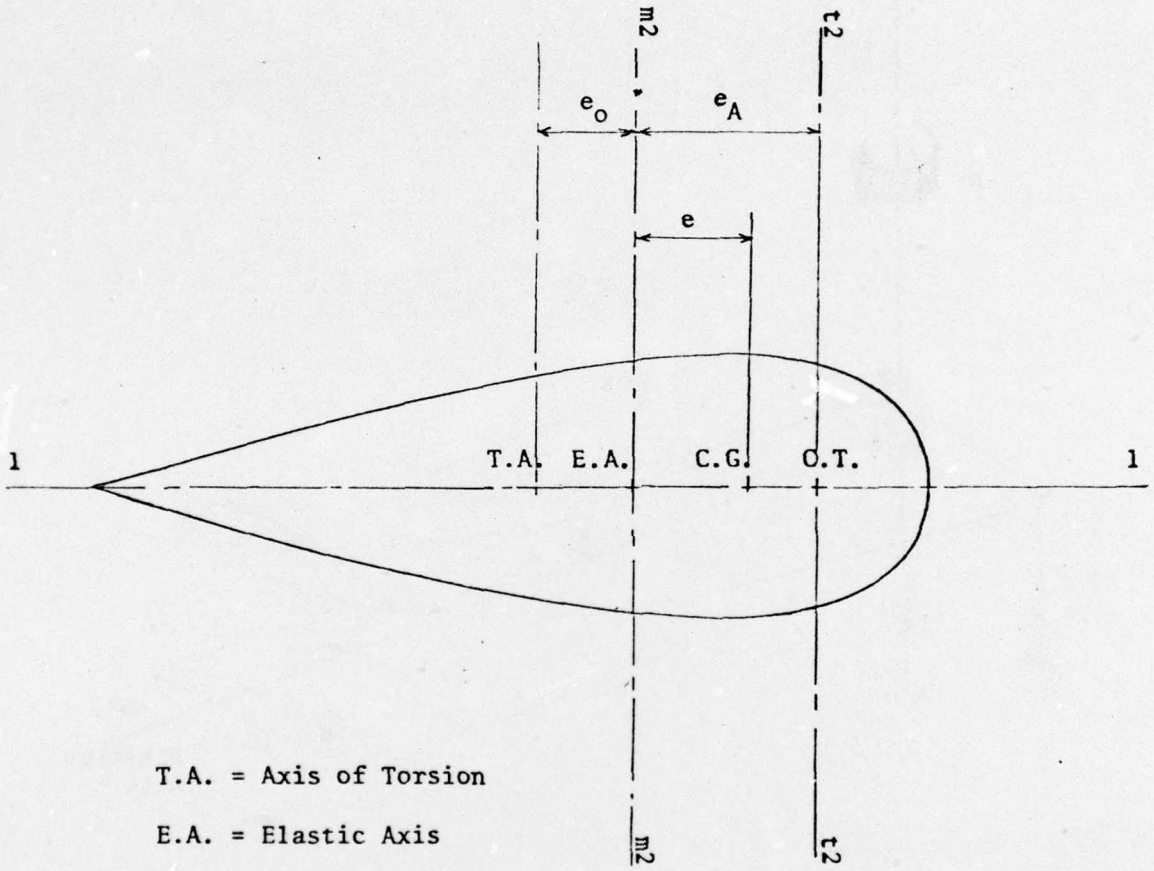


Figure 1. Problem configuration. (Elastic axis is shown to coincide with the axis of torsion in the figure.)



T.A. = Axis of Torsion

E.A. = Elastic Axis

C.G. = Centre of Gravity

C.T. = Centroid of Tensile Area of a Cross Section

Figure 2. Parameters related to off-sets of various axes.

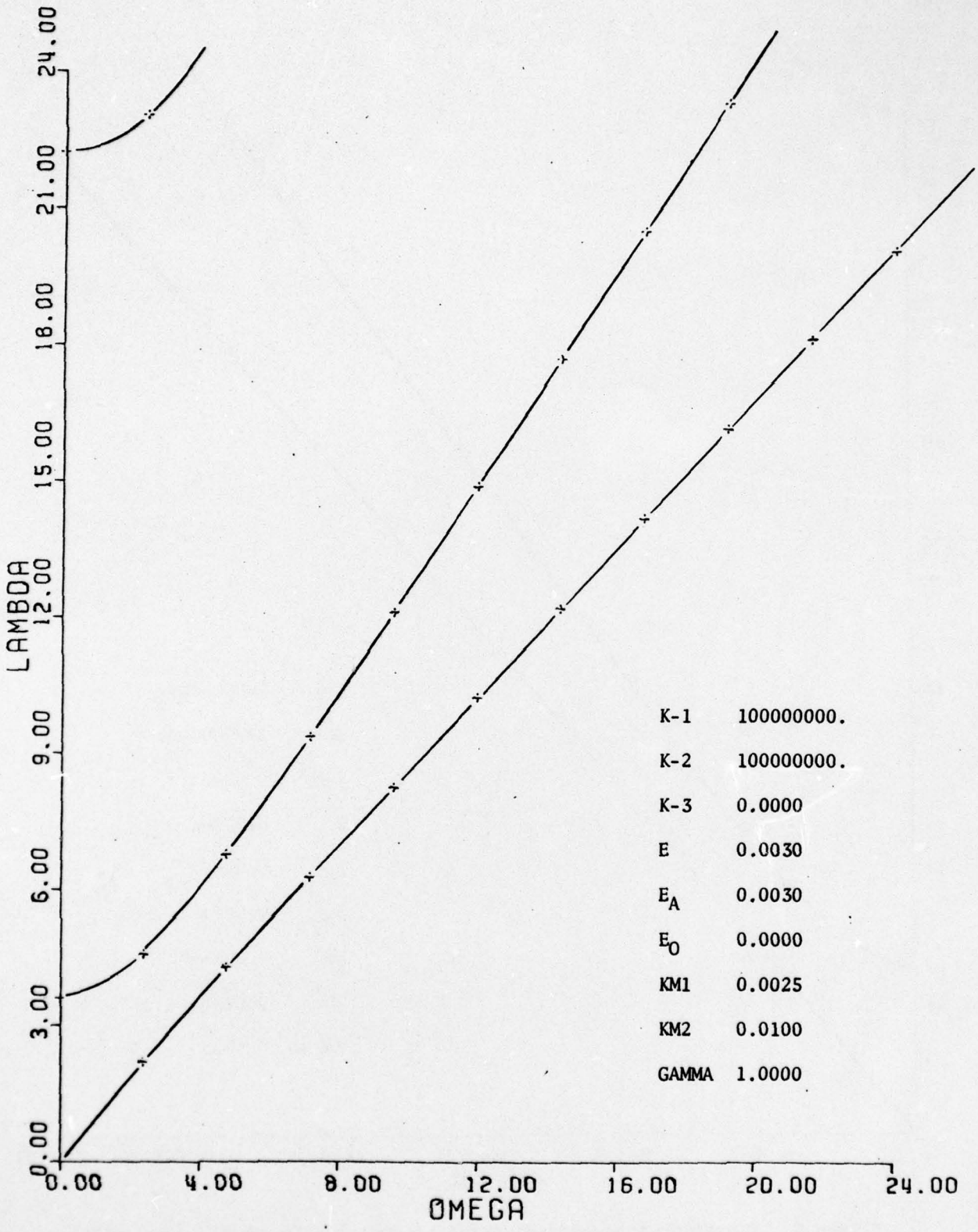


Figure 3. Vibration frequency vs speed of rotation (e_A=0.0030, e=0.0030). No instability in the range of speed shown.

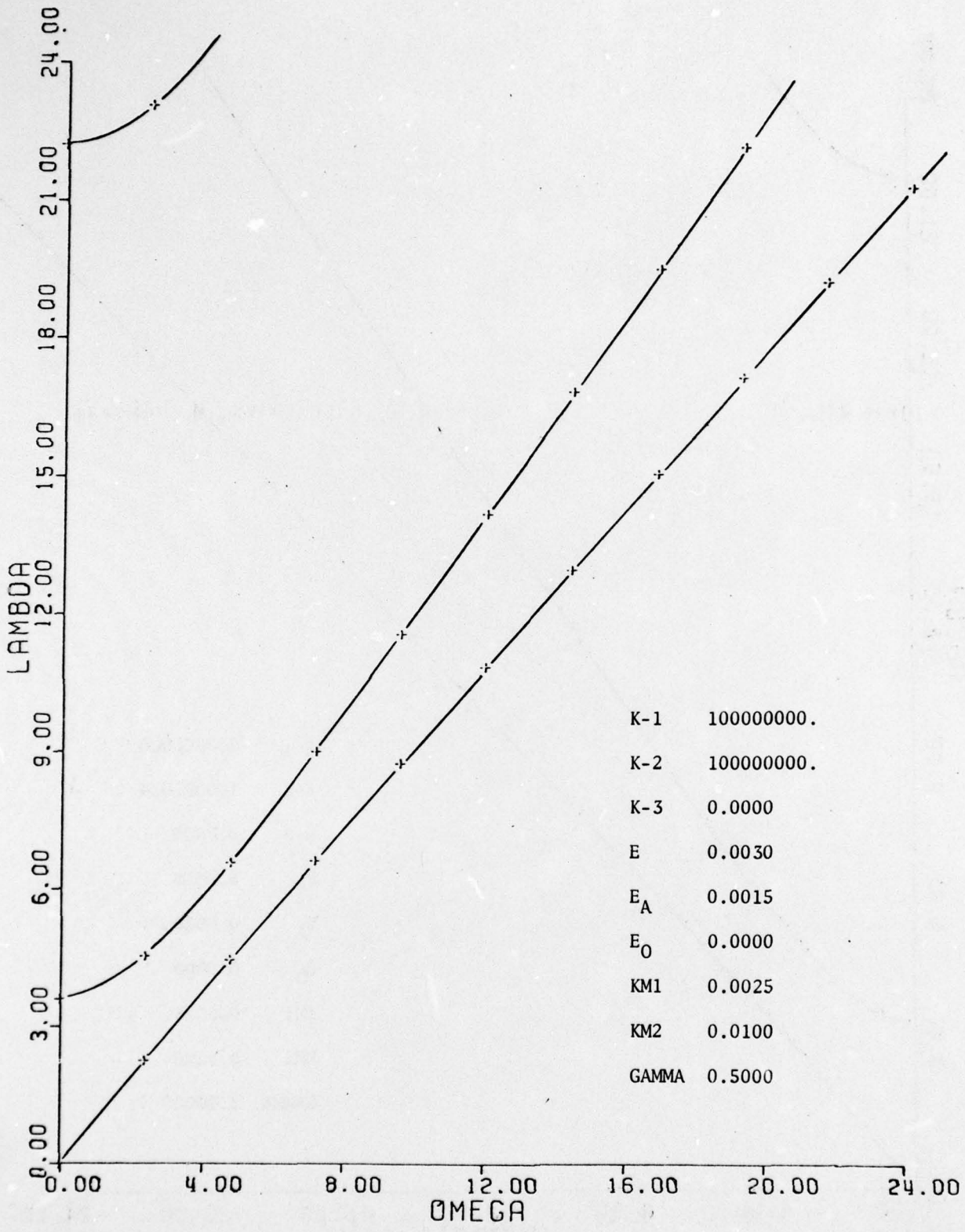


Figure 4. Vibration frequency vs speed of rotation (e_A=0.0015, e=0.0030). No instability in the range of speed shown.

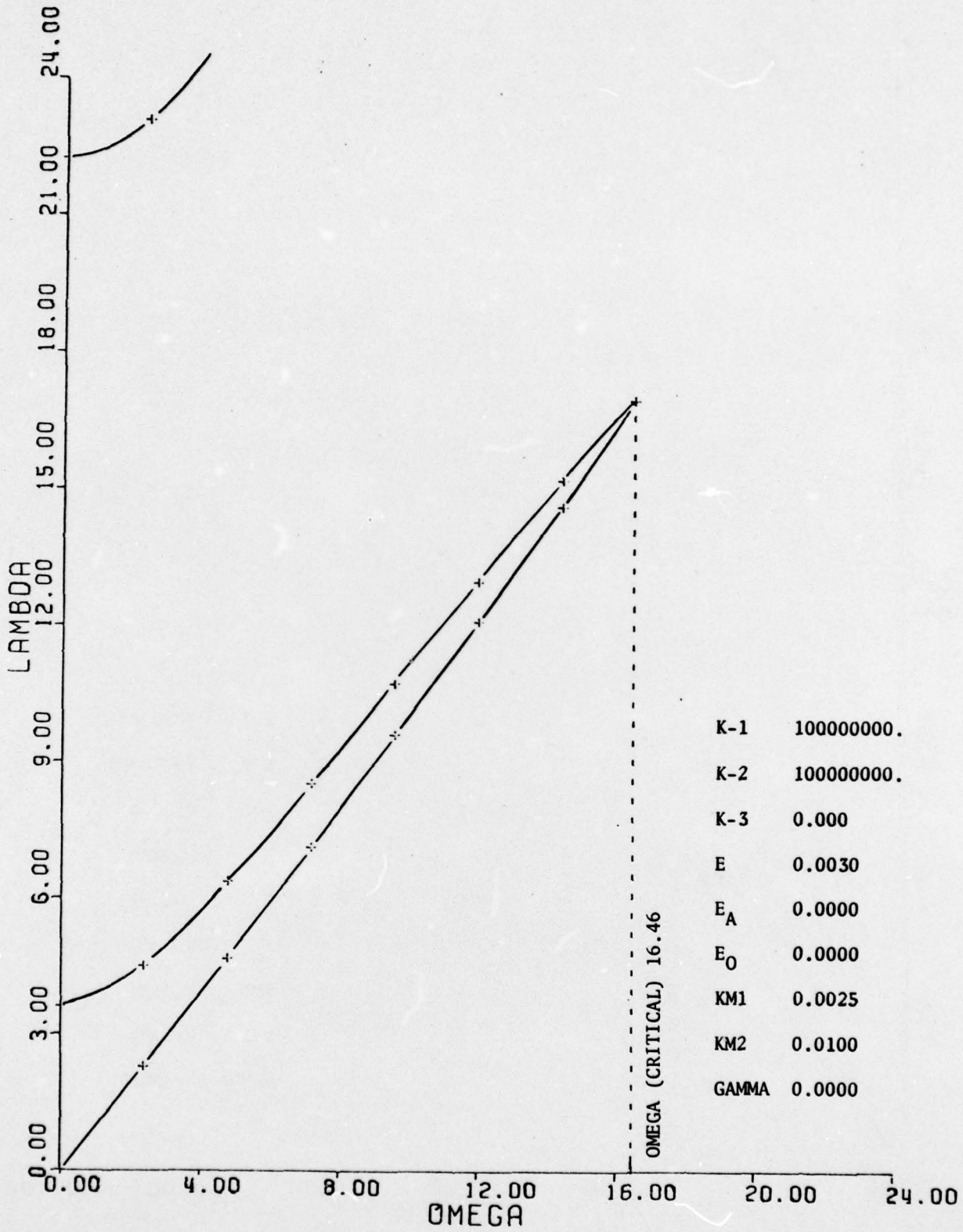


Figure 5. Vibration frequency vs speed of rotation (e_A=0, e=0.0030). Flutter instability at Ω=16.46.

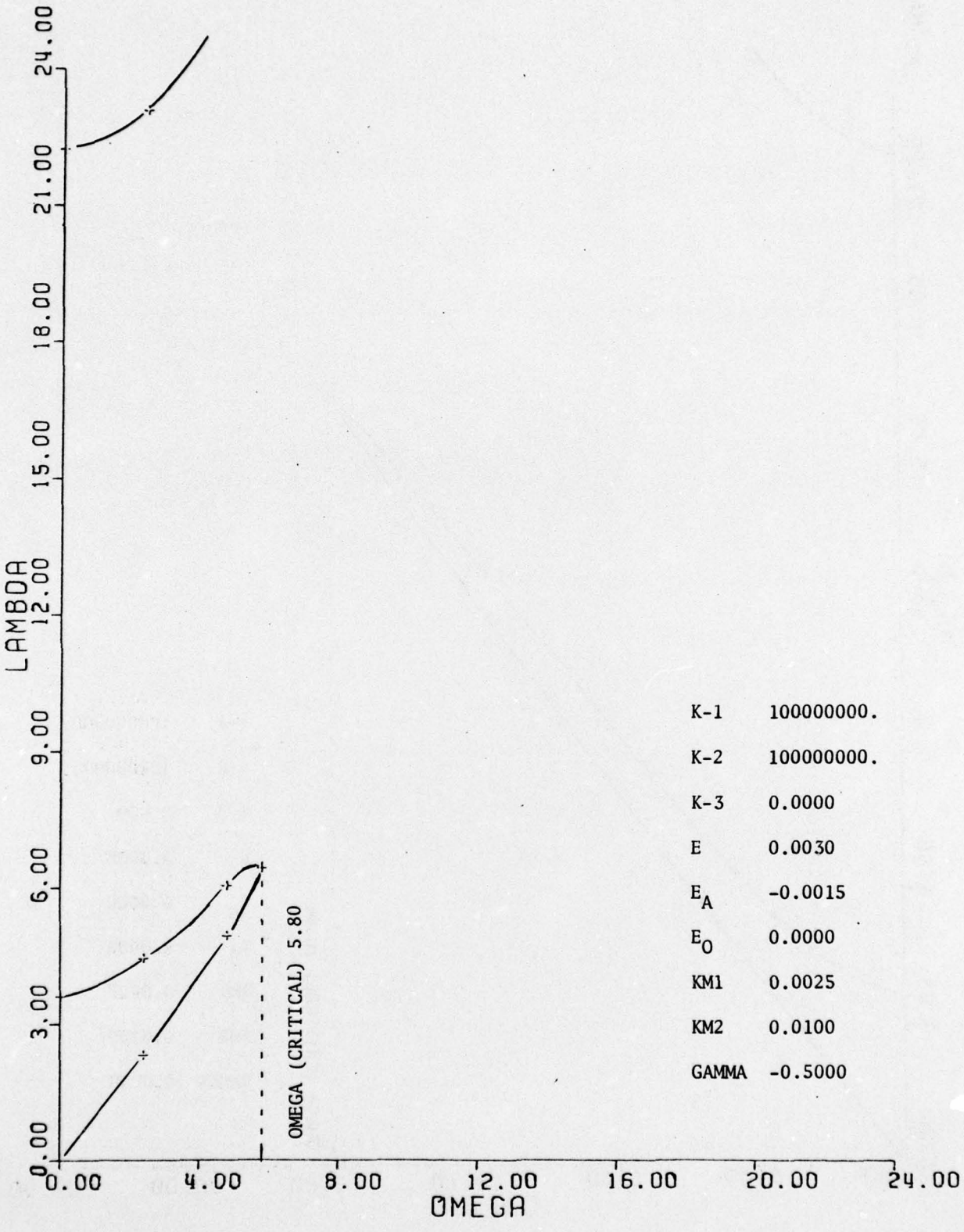


Figure 6. Vibration frequency vs speed of rotation (e_A=-0.0015, e=0.0030). Flutter instability at Ω=5.80.

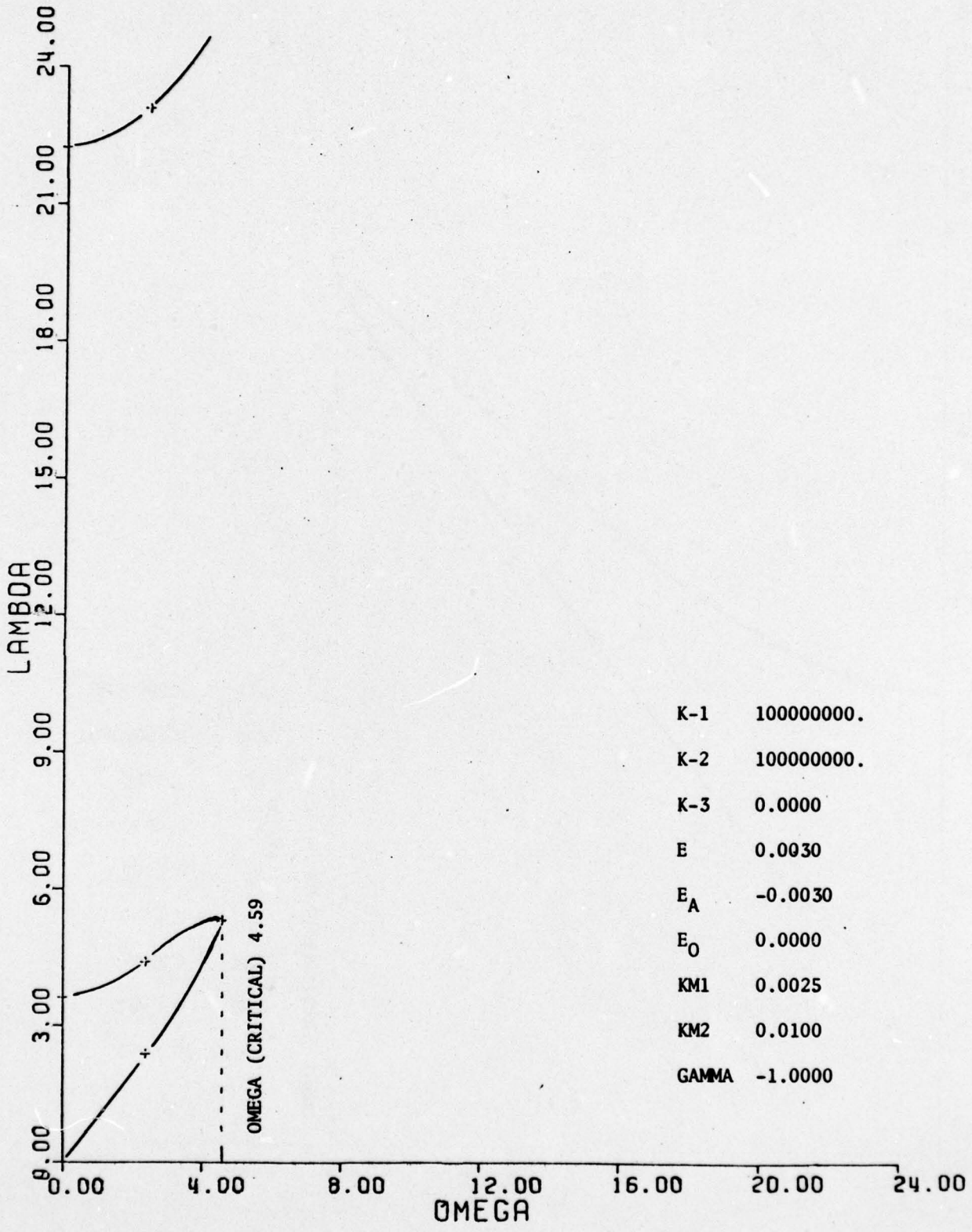


Figure 7. Vibration frequency vs speed of rotation (e_A=-0.0030, e=0.0030). Flutter instability at Ω=4.59.

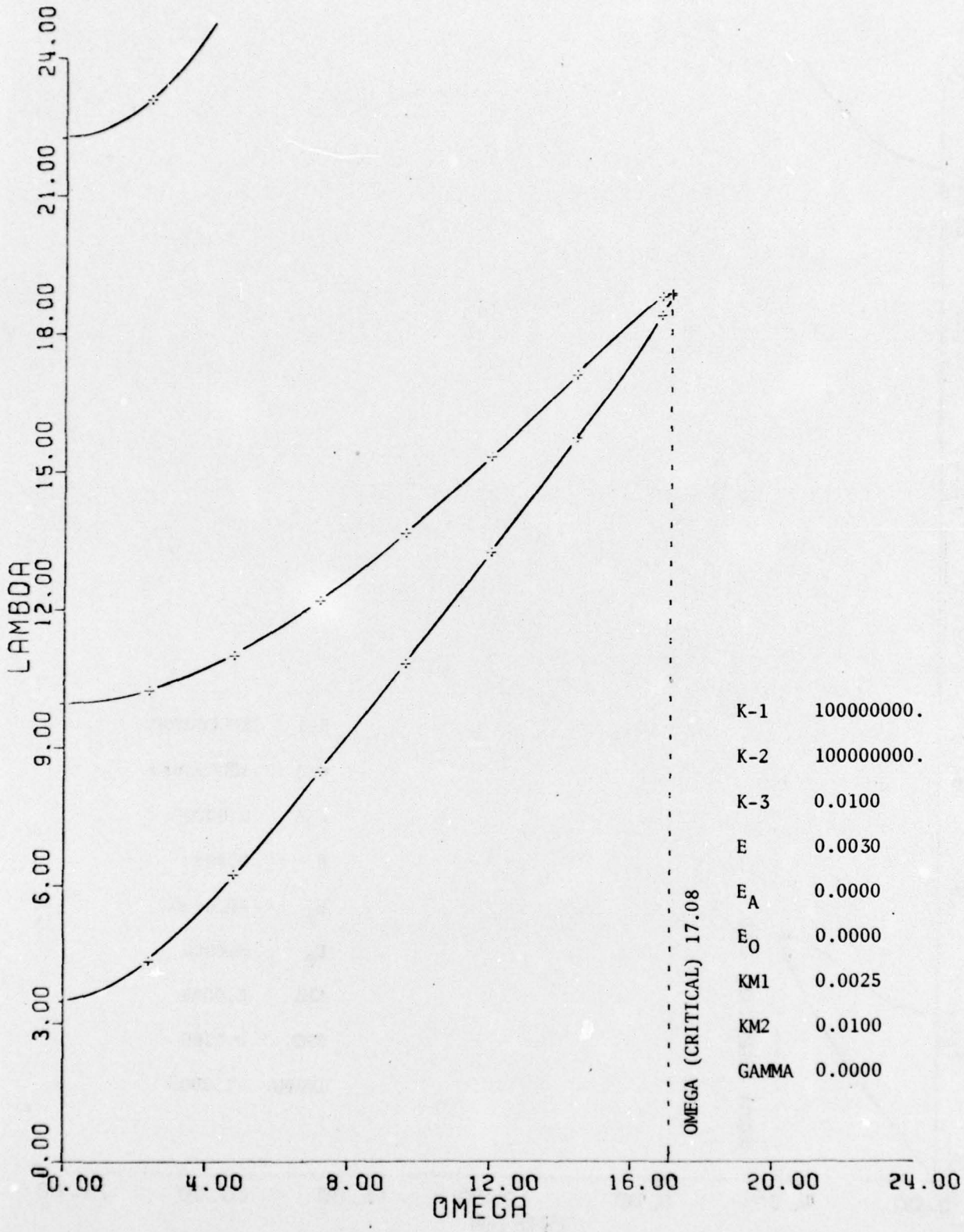


Figure 8. Vibration frequency vs speed of rotation. (Same data as in Figure 5, except k_3 has been changed from zero to 0.01). Flutter instability at Ω=17.08.

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