AD-A052 412 LIEGE UNIV (BELGIUM) LABORATOIRE DE TECHNIQUES AERON--ETC F/G 20/11
STRUCTURAL OPTIMIZATION BY FINITE ELEMENT.(U)
JAN 78 C FLEURY, G SANDER AFOSR-77-3118
UNCLASSIFIED L.T.A.S. NBR-SA-58 AFOSR-TR-78-0670 NL

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STRUCTURAL OPTIMIZATION BY FINITE ELEMENT

by

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Final scientific report
GRANT AFOSR-77-3118

JANUARY 1978

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR TR- 78-0670	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): STRUCTURAL OPTIMIZATION BY FINITE ELEMENT.	5. TYPE OF REPORT & PERIOD COVERED: FINAL rept. Oct 76 – Dec 77
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): C. FLEURY, G. SANDER	8. CONTRACT OR GRANT NUMBER(s): AFOSR-77-3118
9. PERFORMING ORGANIZATION NAME AND ADDRESS: UNIVERSITÉ DE LIÈGE, LABORATOIRE D'AÉRONAUTIQUE, 75, RUE DU VAL-BENOÎT/4000 LIEGE, BELGIUM	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 2307B1, 61102F
11. CONTROLLING OFFICE NAME AND ADDRESS: AIR FORCE OFFICE OF SCIENTIFIC RESEARCH/NA, BLDG 410, BOLLING AIR FORCE BASE, D C 20332	12. REPORT DATE: Jan 78
	13. NUMBER OF PAGES: 231
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): L.T.A.S. NBR-SA-58	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

OPTIMIZATION
FINITE ELEMENTS
STRUCTURAL SYNTHESIS
OPTIMALITY CRITERIA
MATHEMATICAL PROGRAMMING

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The relations existing between optimality criteria and mathematical programming methods of structural synthesis are examined. Generalized optimality criteria are proposed which are equivalent to a sequence of approximate problems in which the constraints are linearized in the space of the inverse of the design variables. From this equivalence a certain number of mixed approaches are derived which retain some of the essential properties of mathematical programming methods and of optimality criteria.

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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

18 19 6 9 10 15 16 17 11 12 14

232 p.

401 156

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CONTENTS

1. INTRODUCTION

Since the 60's an important research effort has been devoted to the development of efficient structural optimization techniques, which could amplify the power brought by modern numerical analysis tools such as the finite element method.

Structural optimization presents a large number of facets for which excellent reviews can be found in the references [G14, S6, F8, P13] . The present work concentrates on one of these aspects, that is the problem of minimum weight design of elastic structures submitted to multiple loading cases and various constraints.
It can be presented as a non linear mathematical programming problem

$$\begin{aligned} &\min \quad W(a_i) \\ &\text{with } g_j(a_i) \leq 0 \end{aligned} \qquad (1.1)$$

where W denotes the weight, a_i the design variables and g_j an explicit or implicit function of the design variables, eventually referred to a given loading case. These functions describe a number of characteristic responses of the structure such as the stresses in various components, deflections, dynamic response, stability limits, etc ...

In the last two decades essentially two main approaches have been explored to solve this problem. The first one is based on the rigorous application of a certain number of the numerical methods of non linear mathematical programming. The structural optimization programs, that have been derived, exhibit excellent convergence properties, but the cost of their application increases in a very discouraging manner with the size of the problems [K7, G13, G16, V3] .

The second approach has been developed from the concept of optimality criteria. For a given type of optimization problem, explicit expressions of the constraints in terms of the design variables are derived which are exact for isostatic structures. Their choice is guided by engineering intuition. The assumption is introduced that these explicit forms of the constraints can be used as approximations in the case of hyperstatic structures. Often the initial problem (1.1) disappears in its initial form, and is replaced by a redesign algorithm derived on the basis of the explicit approximations of the constraints.

The classical Fully Stressed Design (F.S.D) procedure is the best known example of such redesign algorithms. It is exact for an isostatic structure, but leads eventually to non optimum designs in hyperstatic structures [S7, R5, G14, K9]. However such optimality criteria reveal, in many practical situations, extremely good behavior, yielding optimum, or near optimum, designs in a number of analysis cycles which is largely independent of the number of design variables. This attractive characteristic led to the development of a second, distinct, generation of computer programs for structural synthesis [G15, D5, D6, V4]. Their application cost is reasonable, but their convergence properties are disappointing in certain cases which are difficult to identify a priori.

These two schools of thought have been traditionaly opposed in the past. The first contribution of the present work is to establish the relations that exist between the rigorous mathematical programming approaches and those based on the more intuitive optimality criteria. The discussion concentrates on the classical case where the weight is linear in the design variables and where only stress and displacement limits are imposed, plus eventually minimum and maximum sizes. The extension of the conclusions to a broader class of problems is straightforward.

An essential step was to recognize the nature of the approximations that leads to transform the original mathematical programming problem into an approximate one, from which the optimality criteria derives. The next step was to use the understanding of relations between the two approaches to improve both of them.

A generalization of the optimality criteria is proposed which consists in deriving them from the KUHN-TUCKER conditions written for an approximate problem, in which all the constraints are approximated to the first order. It is shown that the application of such a generalized optimality criterion is equivalent to solving the approximate problem exactly. An efficient way is provided by the dual algorithms of convex mathematical programming.

Hybrid optimality criteria are defined as those based on explicit approximations of the constraints that are of different orders. They are especially attractive in the presence of a large number of stress constraints where a complete linearization is costly and often not necessary.

Turning to the mathematical programming approaches, it is shown that, in the space of the inverse of the design variables, the generalized optimality criteria approach can be viewed as equivalent to a special form of the linearization methods applied to the original problem. This allows a better understanding of the reasons of the divergence sometimes met with optimality criteria.

On the other hand, a strict projection primal algorithm of mathematical programming is derived in the space of the inverse of the design variables. This space is advantageous as there the constraints are very shallow. If the first order explicit approximations of the constraints are used, the projection algorithm can take different forms. When the explicit approximations are reevaluated after each projection step, the algorithm is a rigorous application of the primal mathematical programming techniques. When the explicit approximations are kept unchanged until the exact solution of the approximate problem is reached, the method is strictly equivalent to the application of the generalized optimality criteria.

By selecting a limited number of projection steps before recomputing the explicit approximations of the constraints, which implies to analyse the structure, the projection method becomes a mixed method with properties lying between those of the optimality criteria approaches and those of pure mathematical programming. Moreover this concept of mixed method is easily extended to situations where the order of approximation of the constraints is variable, like in the hybrid optimality criteria.

The efficiency of the proposed algorithms is proved on a rather large number of applications taken from the litterature on structural optimization.

The conclusions of the present investigation are the following

- efficient algorithms for structural optimization seem to result only from the rational combination of mathematical programming techniques and engineering approximations;
- the approximations of the constraints introduced in the derivation of optimality criteria are quite often perfectly satisfactory, even those based on F.S.D., provided they are correctly used in the weight minimization algorithms;
- simple tests allow to detect a priori the situations where more elaborate approximations are necessary;
- the most efficient methods seems to be based on mixed order approximations, corresponding to what is defined in this work as hybrid optimality criteria. The solution of the corresponding approximate problems are best obtained by dual methods of mathematical programming when the number of constraints is small and when their non linearity is not too pronounced. Otherwise the primal methods are to be preferred;
- if the mixed approximations improve significantly the convergence properties of the optimality criteria, it remains that their convergence can never be taken for granted, as they are equivalent to linearization methods of mathematical programming;

- when the hybrid optimality criteria fail, the combination of the new mixed method, based on primal projection algorithms, and of various explicit approximations of the constraints, yields a very flexible and efficient tool.

2. THE TWO CLASSICAL APPROACHES

As mentionned in the introduction the present work concentrates on the solutions of the minimum weight design problem under three types of constraints, that are, limitations in the sizes of the design variables themselves, limitations in the stresses, and limitations in deflections or flexibilities. The structures are supposed to be composed only of bars and plates in a plane state of stress. Their cross sections or thicknesses are the design variables a_i and the objective function, that is the weight W, is a linear function of them. The problem can be presented mathematically in the form

problem

R

$$\min W(a_i) = \sum_1^n \rho_i \, \ell_i \, a_i \tag{2.1}$$

with the constraints

$$\underline{a}_i < a_i < \bar{a}_i \qquad i = 1, n \tag{2.2}$$

$$u_{j\ell}(a) < \bar{u}_j \qquad j = 1, n_f \quad \ell = 1, n_c \tag{2.3}$$

$$\sigma_{k\ell}(a) < \bar{\sigma}_k \qquad k = 1, n \quad \ell = 1, n_c \tag{2.4}$$

where n denotes the number of design variables, n_f the number of flexibility constraints, n_c the number of loading cases, ρ_i the mass density, ℓ_i a geometrical quantity, length of the bars or surface of the plates, such that $\ell_i \, a_i$ is the volume of the element. $u_{j\ell}$ denotes a generalized displacement under the ℓ^{th} loading case and $\sigma_{k\ell}$ a stress component, or an equivalent stress, in the k^{th} element under the ℓ^{th} loading case. The upper (lower) bar denotes an upper (lower) limit prescribed.

This problem is labelled problem R (for Reference or Real problem). It is a non linear mathematical programming problem.

Two classical approaches have been used to solve it. The first is based on more or less straight_forward applications of the numerous and various minimization techniques that have been developed in mathematical programming. The second class of approaches is more intuitive and replaces the problem R by a sequence of simpler problems obtained by introducing various approximations derived by taking advantage of the physical or intuitive knowledge of the behaviour of the structure.

In this chapter a brief review is presented of some of the techniques used in these two types of approaches. It is not intented to be exhaustive, nor to give the detailed formulations that can be found in the references. The purpose of the review is simply to recall some aspects of these methods that are useful for understanding the arguments developed in the next chapters.

2.1. Direct approaches based on mathematical programming

2.1.1. Classification of the non linear mathematical programming methods

In the following classification of the methods of non linear M.P., only the most classical methods have been retained that can be applied in the context of the present work. The statement of the problem R (2.1 to 2.4) involves a linear objective function and non linear contraints. This problem however can be transformed in different manners, so that a much broader class of methods has to be considered than those that are especially adapted to the type of functions and constraints initially considered. The classification adopted here is based on the nature of the constraints.

Unconstrained minimization

Almost all of these methods follow the same basic scheme which consists in defining a search direction and then in minimizing the function along that direction, which implies the computation of a progression step. The search for the optimal step that minimizes the objective function along that direction is called a linear search. When this step has been achieved a new search direction is defined and the process is repeated until convergence eventually occurs. An important concept in the definition of the search directions, is that of conjugate directions [H1 , B1] . Although some methods exist that do not require the computation of any derivative of the function, like the POWELL method [P2] , the only choice in our context is between methods that require an evaluation of the first or second derivatives of the function. Among those that use only the first derivatives, we find the various gradient methods like that of the steepest descent [C1, C2, G1] , the conjugate gradient methods [H1, H3, B1, F4] and the variable metric methods that are especially useful when the function has a pronounceed excentricity [F6] ; the DAVIDON-POWELL is a typical example [P1, F5, B2, P4, P5] . When the second derivatives of the objective function can be obtained, powerful methods can be constructed by using quadratic approximations of the function. Various generalizations of the NEWTON method have been proposed [C4, G2, L2, M2] .

Minimization with linear constraints

The minimization of a function under a set of linear equality or inequality constraints plays a foundamental role in M.P. Even if one does not consider linear programming, which is an important special case but not relevant to our problem, the fact is that many methods of non linear M.P. lead to a sequence of problems with linear constraints. The classical methods of unconstrained minimization are easily transposed to this type of problem from the point of view of the choice of the search directions and of the convergence properties. An essential modification in the case of inequality constraints is the addition of an algorithm which decides when a contraint becomes active or, inversely, must be abandoned .

An important class of methods is constituted by the projection methods. They can be characterized by the fact that they

progress along descent directions that remain on the boundaries of the polyhedron of the feasible domain. They present therefore the advantage of producing a sequence of feasible points that correspond to decreasing values of the objective function.
An example of such methods is the gradient projection method of ROSEN [R1] which derives from the steepest descent method.
Better convergence is obtained by using projected conjugate gradient methods [M3, F4] or projected variable metric methods [G4, G6] . These methods require only the knowledge of the first derivatives of the functions. Second order projection methods can also be applied when the second derivatives of the objective function are available. They are obtained by a generalization of the NEWTON method [G3, G5] .

Minimization with non linear constraints

In this more general case, two essentially different classes of methods have to be distinguished. The first groups the primal methods which attempt to solve directly the original problem, i.e. in our case the problem R (2.1 to 2.4). They are characterized by an iterative process that produces a sequence of feasible points corresponding to decreasing values of the objective function. This characteristic is important, as it allows to stop the process before reaching the optimum and still yield an improved design. Among the primal methods one can distinguish between two subclasses. The first one uses the concept of feasible directions introduced by ZOUTENDIJK [Z2] . The feasible directions are selected in such a way that none of the constraints are violated during the progression. The second one consists in a further generalization of the projection methods developed for linear constraints. In these methods, each minimization leads to a moderate violation of the active constraints and has therefore to be followed by a restoration phase that brings back the design point on the composite constraint surface. First order methods are available like the ROSEN's gradient projection method [R2] , or the conjugated gradient methods [A2], as well as second order methods generalizing the NEWTON method [S3, S4] .

A difficulty that is common to all of these primal methods comes from the need to start the process at a feasible point. When the constraints are non linear, this is sometimes far from being evident.

In the second group of methods, the original problem is not directly solved but transformed into another problem or a sequence of auxiliary or approximate problems. Among these methods we find the dual methods, the linearization methods and the penalty methods. An advantage of these methods is that the non linearity of the constraints does not introduce any special difficulty. Their disadvantage comes eventually from the fact that the design points that are produced after each iteration are not necessarily feasible points, and therefore it is usually not possible to stop the process before reaching the optimum. It is also worth mentionning that many of these methods require certain convexity properties which are not necessary in the primal methods. The problem R, as it stands, is not necessarily convex, and therefore optimization methods requiring the convexity of the design space can only be applied to a modified problem.

The simplest method to solve a non linear M.P. problem is perhaps to transform it in a sequence of linear programming problems. Unfortunately this procedure can converge only if a local minimum exists at a vertex of the design space which is, in practice, very unusual. A certain number of more elaborate linearization methods have been developed to avoid this difficulty. Among them we shall mention the KELLEY cutting plane method [K4, W1, L1] which however applies only to convex problem. The methods of approximation programming of GRIFFITHS and STEWART [G7], of WILSON [W4] and of BEALE [B3], introduce various improvements like the definition of move limits, or the linearization of the constraints only, while a quadratic approximation of the objective function is used. An important difference with the primal projection methods, which at the current point also use a linearization of the constraints, is due to the fact that in linearization methods the design point obtained after solving an approximate linearized problem is not moved back onto the exact composite constraint surface.

The penalty methods consist in transforming the initial problem in a sequence of unconstrained problems by adding to the objective function an auxiliary function that describes the degree of non satisfaction of the constraints [F1] . The choice of the auxiliary function can be made such that the sequence of design points remain inside or outside the feasible domain [F1, C5, F11, A3, R3, L1] .

In the special case of convex programming, the lagrange multipliers,that are associate to the constraints,have the meaning of dual variables in terms of which a new formulation of the problem can be established. This dual problem consists in maximizing the lagrangian function in terms of the dual variables with the only constraint that these dual variables have to remain non negative [F1, L1, W2, F12] . These methods play an important role in the context of optimality criteria and will be discussed in more details at that occasion.

2.1.2. Gradients of the constraints

In most of the efficient minimization techniques it is necessary to compute the value of the various functions that define the problem and their gradients. As the flexibility and stress constraints are implicit functions of the design variables, their gradients can only be computed numerically.

In the context of the finite element method, a flexibility u_j can be expressed as a linear combination of the nodal D.O.F. corresponding to the displacements q

$$u_j = b_j q \tag{2.5}$$

where b_j is a line matrix of constants. The gradient of the flexibility under a loading case ℓ is therefore

$$\frac{\partial u_{j\ell}}{\partial a_i} = b_j \frac{\partial q_\ell}{\partial a_i} \tag{2.6}$$

For the stress constraints, a stress component can be written in the same manner as a linear combination of the D.O.F. of the corresponding finite element, using in practice simply a line of the stress matrix t

$$\sigma_{k\ell} = t_k \, q_\ell$$

and

$$\frac{\partial \sigma_{k\ell}}{\partial a_i} = t_k \frac{\partial q_\ell}{\partial a_i} \tag{2.7}$$

If the constraint is on an equivalent stress, the expression of its gradient can be obtained from the gradients of the 3 stress components.
The computation of the gradients of the two types of constraints reduces to the evaluation of the gradients of the generalized nodal displacements of the finite element model. The method of "pseudo-loads" [F14, G10] is rather advantageous.
Considering the equilibrium equations

$$K \, q_\ell = g_\ell \qquad \ell = 1, n_c \tag{2.8}$$

for the loads g_ℓ , one has

$$\frac{\partial K}{\partial a_i} q_\ell + K \frac{\partial q_\ell}{\partial a_i} = 0 \qquad \begin{array}{l} i = 1, n \\ \ell = 1, n_c \end{array} \tag{2.9}$$

from which

$$\frac{\partial q_\ell}{\partial a_i} = - K^{-1} \frac{\partial K}{\partial a_i} q_\ell \tag{2.10}$$

In the elements that are considered in this work, the stiffness matrices are linear in the design variables, and can be written

$$K_i = a_i \bar{K}_i \tag{2.11}$$

where $\bar{K}_i$ is the stiffness matrix for a unit value of the design variable. The pseudo loads are defined by

$$\tilde{g}_{i\ell} = - \bar{K}_i q_\ell \qquad \begin{array}{l} i = 1, n \\ \ell = 1, n_c \end{array} \tag{2.12}$$

so that the computation of the gradients (2.10) requires only the application of the $(n \times n_c)$ pseudo loads to the structure, in addition to the set of n_c actual loads.
This can be handled rather efficiently by the finite element programs, especially if the total number of loads is not too large.

2.1.3. Characteristics of some of the M.P. approaches

A large variety of solutions has been proposed to the structural synthesis problem that are based on one or the other of the M.P. methods mentionned in the classification given above. The earliest algorithms were based on primal methods, like the alternate step methods proposed by SCHMIT [S7], GELLATLY et al. [G9, G10, G11, G12] , the ROSEN's projected gradient method applied by BROWN [B4] , the ZOUTENDIJK's feasible direction method used by KARNES [K7, T2] and VANDERPLAATS [V1] etc. All of these methods solve directly the problem R in its form given by (2.1) to (2.4), that is in terms of the design variables. The linearization methods have been applied by MOSES [M6] , REINSCHMIDT [R4] , SCHMIT [S8] and others. It is worth mentionning that a certain number of these authors have noticed the advantages of using the inverses of the design variables as unknowns. The advantage comes from the fact that the stress and flexibility constraints are linear in the inverses of the design variables when the structure is isostatic, and rather shallow when hyperstatic,so that a linear approximation of the constraints corresponds to a good approximation in this design space. The penalty methods have been applied by SCHMIT [S10] , MARCAL [M9], KAVLIE [K8] and MOE [M10] .

All of these algorithms share certain common characteristics. One of them is the need for quite frequent evaluation of the derivatives of the objective function and of the constraints. These can only be obtained after a reanalysis of the structure, which is a costly operation. Moreover the number of iteration steps increases with the number of design variables,which is a well known behaviour of all the constrained or unconstrained minimization algorithms,when only the first derivatives are used. So not only the cost of the analyses increases when more complex structure are optimized,but also the number of analyses and of pseudo load cases. In practice,none of these methods revealed able to solve large problems at a cost competitive with optimality criteria based methods. However,the main advantage of the M.P. methods is their generality, their convergence behaviour which can be predicted and often guarantied due to the sound mathematical bases of the methods.

An important exception has been brought recently by the method of SCHMIT and MIURA [S18, 1976], which is characterized by the use of linear explicit approximations of the constraints in the space of the inverses of the design variables.
Each approximate problem is solved using either a feasible direction method or a penalty method. This method reveals very efficient,but,at the light of the present research,should no longer be considered strictly as a M.P. method. Due to the type of approximation introduced, it becomes in fact a mixed method and as such will be examined later on.

2.2. Indirect approaches based on optimality criteria

2.2.1. Problems with stress constraints

When only stress constraints are present in addition to the side constraints (2.2) the most popular approach consists in using the F.S.D. concept. After each analysis the structure is resized according to the stress ratio algorithm

$$
\begin{aligned}
a_i^{\nu+1} &= a_i^{\nu} \max \left(\frac{\sigma_{i\ell}^{\nu}}{\bar{\sigma}_i}\right) \\
a_i^{\nu+1} &> \underline{a}_i
\end{aligned}
\tag{2.13}
$$

where ν denotes an iteration number. It is well known [M11, M12, C7, R5, K9] that the approach leads to an optimum in the case of an isostatic structure. In such a case, the stress constraint becomes explicit and the problem R is replaced by the problem

$$
\begin{aligned}
\min \quad & W = \Sigma\, \rho_i\, \ell_i\, a_i \\
& a_i > \frac{\sigma_{i\ell}}{\bar{\sigma}_i} \\
& a_i > \underline{a}_i
\end{aligned}
\tag{2.14}
$$

Indeed the application of the F.S.D. does not require to explicit the objective function. The problem (2.14) being a linear programming problem, its solution is necessarily at a vertex of the design space which is unique and corresponds to

$$
\begin{aligned}
& a_i^{*} = \max\, (\underline{a}_i\ ,\ \tilde{a}_i) \\
\text{with} \quad & \tilde{a}_i = a_i^{o} \max \left\{ \frac{\sigma_{i\ell}^{o}}{\bar{\sigma}_i} \right\}
\end{aligned}
\tag{2.15}
$$

In the case of hyperstatic structures, the solutions obtained using the F.S.D. are no longer necessarily minimum weight designs [B7, B8, K9, R5, S12, S13] and some difficulties in convergence are observed in certain cases [B8, M13, G13, R4, G14] . In many practical situations however the F.S.D. leads very rapidly to a design that is very close to the minimum and is characterized by a number of iterations (reanalyses) that is independent of the number of the design variables.

2.2.2. Problems with flexibility constraints

A large number of publications deals with the optimality criteria approaches in the case where flexibility constraints are imposed [B9, B7, P10, P13, S14, S15, P14, B10] .

Considering a virtual loading case g_j associated to each flexibility constraint u_j, its virtual work is written

$$u_j = q_j^T g = q^T g_j = q_j^T K q \tag{2.16}$$

where q denotes the displacement vector. It is expressed as the sum of the contributions of each element

$$u_j = \sum_i q_{ij}^T K_i q_i \qquad i = 1, n \tag{2.17}$$

$$= \sum_i \frac{c_{ij}}{a_i} \tag{2.18}$$

$$\text{with} \quad c_{ij} = (q_{ij}^T K_i q_i) a_i \tag{2.19}$$

The flexibility coefficients are constant for an isostatic structure,which is the situation considered to derive the optimality criteria. Taking the case where only one flexibility constraint is imposed,which is consequently an equality constraint, it is classical to separate the n elements into a group of active members ($i = 1, n_1$) and a group of passive members ($i = n_1+1, n$) so that the flexibility can be expressed as

$$u_1 + u_2 = \bar{u} \qquad u_1 = \sum_{1}^{n_1} \frac{c_i}{a_i}$$

$$u_2 = \sum_{n_1+1}^{n} \frac{c_i}{a_i} \tag{2.20}$$

With equality constraints, a lagrangian method can be used which is based on the augmented functional

$$\mathcal{L}\,(a_i, r) = W(a_i) + r\,[\,u(a_i) - \bar{u}\,] \tag{2.21}$$

Its stationarity conditions are given by the non linear system of equations

$$a_i^2 = r\,\frac{c_i}{\rho_i \ell_i} \qquad i = 1, n_1 \tag{2.22}$$

$$\sum_{1}^{n_1} \frac{c_i}{a_i} = \bar{u} - u_2 \tag{2.23}$$

In (2.22) the coefficients c_i have to be positive, which yields a mean to distinguish between the active and passive members. Substitution of (2.22) into (2.23) yields the value of the lagrange multiplier

$$r = \left[\,\frac{1}{\bar{u}-u_2} \sum_{1}^{n_1} (\rho_k\,\ell_k\,c_k)^{1/2}\,\right]^2 \tag{2.24}$$

Reintroducing (2.24) in (2.22) yields the new values of the active design variables ($i = 1, n_1$)

$$a_i = \frac{1}{\bar{u}-u_2} \sum_1^{n_1} (\rho_k \ell_k c_k)^{1/2} \left(\frac{c_i}{\rho_i \ell_i}\right)^{1/2} \qquad (2.25)$$

This redesign formula is applied after each analysis cycle in the case of a hyperstatic structure and the choice between active and passive member is also repeated at each iteration. The relation (2.22) can be written

$$\varepsilon_i = \frac{1}{r} = \frac{e_i}{\rho_i \ell_i a_i} \qquad \text{with} \qquad e_i = \frac{c_i}{a_i} \qquad (2.26)$$

The optimality criterion states that the minimum weight design is obtained under one flexibility constraint when the virtual strain energy density e_i per unit of mass is the same for all the active elements.
The criterion is exact for isostatic structure and approximate when hyperstatic [B10, G13] . The inclusion of lower limits to the design variables is a straight_forward addition.

2.2.3. Problems with stress and flexibility constraints

Such problems have received different treatments. In the approaches pioneered by BERKE and GELLATLY [G13, B11] the flexibility constraints are considered as the principal ones, while the others (stress and lower limits) are treated as side constraints. The explicit problem solved after each analysis is

problem A

$$\left\{ \begin{array}{lll} \min & W = \Sigma\, \rho_i\, \ell_i\, a_i & i = 1, n \\ \text{with} & \Sigma \dfrac{c_{ij}}{a_i} < \bar{u}_j & j = 1, n_f \\ & a_i > a_i^* & \end{array} \right. \qquad (2.27)$$

where a_i^* is given by (2.15). In an isostatic structure the problem R (2.1) is identical to the problem A, while in the hyperstatic case it is expected that a sequence of approximate problems A yields to the solution of the exact problem R.
In fact, most of the classical approaches do not solve directly the problem A, but start with its simplified version

$$\begin{aligned} &\min \quad W = \Sigma\, \rho_i\, \ell_i\, a_i \\ &\text{with} \quad u_j = \bar{u}_j \end{aligned} \tag{2.28}$$

where one has dropped the side constraints and supposed that the flexibility constraints are all active at the optimum. Taking the side constraints into account and selecting the active flexibility constraints are considered as auxiliary problems. The BERKE optimality criterion (2.22), (2.23) is still applicable and can be generalized for multiple flexibility constraints by introducing a number of lagrange multipliers equal to the number of active constraints [B11]. It yields

$$a_i^2 = \frac{1}{\rho_i\, \ell_i} \sum_j r_j\, c_{ij} \qquad \begin{aligned} i &= 1, n \\ j &= 1, n_f \end{aligned} \tag{2.29}$$

$$\sum_i \frac{c_{ij}}{a_i} = \bar{u}_j \tag{2.30}$$

Unfortunately it is no longer possible to solve analytically this non linear system of equations in the case of multiple constraints. The envelope method of GELLATLY and BERKE [G13] furnishes an approximate solution to this problem. They determine for each active variable the flexibility constraint that gives it the maximum size by applying the relation (2.25) for each flexibility. The resizing relations become

$$a_i = \max_j \left[\frac{1}{\bar{u}_j - u_{2_j}} \left(\frac{c_{ij}}{\rho_i \ell_i}\right)^{1/2} \sum_k^{n_j} (\rho_k \ell_k c_{kj})^{1/2} \right] \quad (2.31)$$

$$i = 1, n_1$$

The selection between active and passive variables is different for each constraint and yields to different values of u_2 denoted u_{2_j}. The minimum size of variables is given by a_i^* defined in (2.15). In the case of multiple constraints this procedure does not yield the exact solution,even in an isostatic problem. It can be interpreted as a pseudo lagrangian method where the lagrange multipliers are approximated by

$$r_j = \left[\frac{1}{\bar{u}_j - u_{2_j}} \sum_k^{n_j} (\rho_k \ell_k c_{kj})^{1/2} \right]^2 \quad (2.32)$$

$$j = 1, n_f$$

and where the design variables are given by the envelope

$$a_i^2 = \frac{1}{\rho_i \ell_i} \max_j (r_j \; c_{ij}) \qquad i = 1, n_1 \quad (2.33)$$

instead of the exact expression,solution of (2.29).

Another approximate solution has been proposed by TAIG and KERR [T3] . They have used an exact lagrangian method but in terms of the inverse of the lagrange multipliers

$$\lambda_j = \frac{1}{r_j} \quad (2.34)$$

so that the optimality criterion (2.26) becomes

$$\sum_j \frac{\varepsilon_{ij}}{\lambda_j} = 1 \qquad i = 1, n \quad (2.35)$$

The solution of the simplified minimization problem (2.28) is given by the system of $(n + n_f)$ non linear equations

$$a_i^2 = \frac{1}{\rho_i \ell_i} \sum_j \frac{c_{ij}}{\lambda_j} \tag{2.36}$$

$$u_j = \sum_i \frac{c_{ij}}{a_i} = \bar{u}_j \tag{2.37}$$

The solution of (2.36) is obtained by an iterative NEWTON-RAPHSON procedure. A difficulty arises by the fact that initial values for the λ_j are necessary to initiate the procedure, which are sometimes difficult to find. In the case of one flexibility constraint the TAIG and KERR approach yields results identical to that of the GELLATLY-BERKE envelope method.

In both methods, most of the difficulties that appear in the applications can be related to the fact that, instead of solving the problem A, they solve the problem (2.28) with equality constraints.

Other solutions have been proposed [V4, B8, B12] which proceed from similar points of view. They are not detailed here. The two methods of BERKE [B10] and TAIG and KERR [T3] are sufficiently characteristic of the optimality criteria approach to allow to develop the comparisons that are essential for the present work.

2.3. Mixed approaches

Mixed approaches are characterized by the fact that they rely simultaneously on the minimization algorithms of mathematical programming and on the intuitive approximations of the optimality criteria.

Such approaches have been proposed by VENKAYYA [V2, V3] and others [D5, S17] . We note especially in the context of the present work, the method of "high quality explicit approximations" of the constraints proposed by SCHMIT and MIURA [S18, S20, S8] . Although in the spirit of the authors the method pertains to the line of mathematical programming methods, it will be demonstrated later that it is indeed a mixed one. The approximations of the constraints used by SCHMIT are obtained by a Taylor serie expansion limited to the first order in the inverse of the design variables. The approximate form of the problem R that they obtain is

$$\min \quad W(x_i) = \sum_i \rho_i \, \ell_i \, \frac{1}{x_i} \qquad \text{with } x_i = \frac{1}{a_i} \tag{2.38}$$

$$u_{j\ell} = u^o_{j\ell} + \sum_i \left(\frac{\partial u_{j\ell}}{\partial x_i}\right)^o (x_i - x_i^o) < \bar{u}_j \tag{2.39}$$

$$\sigma_{k\ell} = \sigma^o_{k\ell} + \sum_i \left(\frac{\partial \sigma_{k\ell}}{\partial x_i}\right)^o (x_i - x_i^o) < \bar{\sigma}_k \tag{2.40}$$

$$\frac{1}{\bar{a}_i} \leq x_i < \frac{1}{\underline{a}_i} \tag{2.41}$$

where the superscript (^{o}) denotes the values computed at a design point where the structure is analyzed.
The problem R is replaced by a sequence of such approximate problems which are solved by either a feasible directions method or a penalty method.

3. A GENERALIZATION OF THE OPTIMALITY CRITERIA

An essential ingredient of the optimality criteria is the use of explicit, and hence most often approximate, expressions of the constraints in terms of the design variables. These expressions are exact in the isostatic case only. In hyperstatic structures the degree of approximation is essentially different for stress and displacement constraints in the classical approaches. The first generalization proposed here consists in presenting a way to derive optimality criteria that are based on approximations that are consistent for all the constraints. It is to be noted that this does not imply necessarily the suggestion to use these generalized optimality criteria in that form, but they represent an essential step for the understanding of the relations between the various structural synthesis techniques and for the derivation of efficient mixed methods.

3.1. Scaling of the design variables

Before discussing the generalization of the optimality criteria, it is necessary to recall a property of the scaling operation consisting in multipliing each design variable by the same factor. Such an operation evidently does not introduce any redistribution of the internal forces if the stiffness matrices are linear in the design variables, what we have assumed in the present work. If a_i^o denotes a point in the design space, the scaling by a factor f yields a new point a_i^1

$$a_i^1 = f\ a_i^o \tag{3.1}$$

at which the displacements and stresses are

$$u_{j\ell}^1 = \frac{u_{j\ell}^o}{f} \qquad \sigma_{k\ell}^1 = \frac{\sigma_{k\ell}^o}{f} \tag{3.2}$$

$$i = 1, n \qquad j = 1, n_f \qquad \ell = 1, n_c$$

All the points obtained by (3.1) are on the line joining a_i^o to the origin in the design space. It is denoted $D(a_i^o)$. The intersections of that line with the constraint surfaces yield the scaling factors

$$f(u_{j\ell}) = \frac{u_{j\ell}^o}{\bar{u}_j} \qquad\qquad f(\sigma_{k\ell}) = \frac{\sigma_{k\ell}^o}{\bar{\sigma}_k} \tag{3.3}$$

that are to be applied to bring a design point a_i^o on those surfaces.

3.2. Order of approximation of the constraints

The stress ratio redesign algorithm used in F.S.D.

$$\tilde{a}_i = a_i^o \max_{\ell} \left\{ \frac{\sigma_{i\ell}^o}{\bar{\sigma}_i} \right\} \tag{3.4}$$

which is applied after an analysis yielding the stresses σ^o, corresponds to an explicit approximate expression of the stresses in terms of the design variables

$$\tilde{\sigma}_{i\ell} = \sigma_{i\ell}^o \frac{a_i^o}{a_i} \tag{3.5}$$

where the $\sim$ denotes an approximate expression. The exact stress constraints

$$\sigma_{i\ell} < \bar{\sigma}_i \tag{3.6}$$

are non linear implicit functions of the design variables

Their approximation (3.5) is however equal to the exact one at any point along the $D(a_i^o)$ scaling line, that is

$$\tilde{\sigma}_{i\ell} = \sigma_{i\ell} \qquad \text{on } D(a_i^o) \qquad (3.7)$$

and, in particular, at the intersection of the line with the exact constraint surface

$$\tilde{\sigma}_{i\ell} = \sigma_{i\ell} = \bar{\sigma}_i \qquad \begin{array}{l} \ell = 1,\, n_c \\ i = 1,\, n \end{array} \qquad (3.8)$$

Geometrically, the exact curved constraint surface (3.6) is approximated by a plane perpendicular to the axis a_i and passing by the intersection of the exact surface with the scaling line $D(a_i^o)$. As the approximation (3.5) preserves only the value of the function, and not of its derivatives, it is a zero-order explicit approximation of the stress constraints.

Turning now to the displacement or flexibility constraints, as introduced for instance by BERKE [B10] , their explicit approximation is given by (2.18, 2.19), that is

$$\tilde{u}_j = \sum_i \frac{c_{ij}^o}{a_i} \qquad \text{with} \qquad c_{ij}^o = (q_i^{o^T} K_i \tilde{q}_{ij}^o)\, a_i^o \qquad (3.9)$$

where q_i^o, $\tilde{q}_{ij}$ are the nodal displacements of the i^{th} element under a real or virtual loading case, at an analysis point a_i^o. In the exact constraint

$$u_j = \sum_i \frac{c_{ij}(a_i)}{a_i} < \bar{u}_j \qquad (3.10)$$

the flexibility coefficients are implicit functions of the design variables. By the same arguments as for the stress constraints, the approximate value of the flexibility (3.9) is equal to the exact value at any point along the scaling line $D(a_i^o)$. Consider now the first derivative of a given flexibility (j fixed). Recalling (2.16, 2.17) we have

$$u = q^T K \tilde{q} = q^T \tilde{g} = g^T \tilde{q} \qquad (3.11)$$

and

$$\frac{\partial u}{\partial a_i} = \frac{\partial q^T}{\partial a_i} K \tilde{q} + q^T \frac{\partial K}{\partial a_i} \tilde{q} + q^T K \frac{\partial \tilde{q}}{\partial a_i} \qquad (3.12)$$

As the loads

$$g = K q \qquad \text{and} \qquad \tilde{g} = K \tilde{q}$$

are independent of the design variables

$$\frac{\partial u}{\partial a_i} = \frac{\partial}{\partial a_i} (q^T \tilde{g}) + \frac{\partial}{\partial a_i} (g^T \tilde{q}) + q^T \frac{\partial K}{\partial a_i} \tilde{q} \qquad (3.13)$$

$$= \frac{\partial u}{\partial a_i} + \frac{\partial u}{\partial a_i} + q^T \frac{\partial K}{\partial a_i} \tilde{q}$$

and hence

$$\frac{\partial u}{\partial a_i} = - q^T \frac{\partial K}{\partial a_i} \tilde{q} \qquad (3.14)$$

Using the expansion of K in terms of the finite element stiffness matrices K_i which are supposed to be linear in the design variables

$$\frac{\partial u}{\partial a_i} = - \frac{1}{a_i} (q_i^T K_i \tilde{q}_i) \qquad (3.15)$$

At the analysis point a_i^o, using the definition of the flexibility coefficients (3.9), we have

$$\left.\frac{\partial u}{\partial a_i}\right|_{a_i^o} = -\frac{c_i^o}{(a_i^o)^2} \qquad (3.16)$$

which is identical with the value derived from the approximate explicit form (3.9)

$$\left.\frac{\partial \tilde{u}}{\partial a_i}\right|_{a_i^o} = -\frac{c_i^o}{(a_i^o)^2} \qquad (3.17)$$

We conclude therefore that, at any point along $D(a_i^o)$, the approximate values of the flexibility and its first derivative are equal to the exact ones

$$u = \tilde{u} \qquad \frac{\partial u}{\partial a_i} = \frac{\partial \tilde{u}}{\partial a_i} \qquad \text{along } D(a_i^o) \qquad (3.18)$$

Geometrically, the explicit approximation (3.9) is a surface which is tangent to the exact one at the intersection point of $D(a_i^o)$ with the exact constraint surface. It is therefore a first order approximation of the constraint.

3.3. Interpretation as approximations of the problem R

A graphical interpretation is presented in figure 3.1 for a two dimensional problem. The exact stress and flexibility constraints are indicated which define the exact problem (problem R) and lead to the optimum design in R. From a design point A^o at which the structure is analysed, the optimality criteria approach, using the F.S.D. concept, consists in using the explicit approximations of the constraints examined above.

They are represented on the figure by a straight line perpendicular to the axis for the stress constraint, and a curved line tangent to the exact flexibility constraint. The problem R is clearly replaced by a problem A as defined in (2.27) and yields a solution which is indicated by the point P.

The example is easily generalized and leads to the conclusion that the classical methods using optimality criteria replace the problem R by a sequence of approximate problems which are characterized by the use of explicit expressions of the constraints. In the case where the stress constraints are treated by the F.S.D. concept, the approximation corresponds to the problem A. It is expected that a sequence of problem A leads to the solution of the problem R. This is not necessarily the case and it is clear that it does depend of the quality of the approximations introduced for the constraints.
It is important to note that most of the optimality criteria methods proposed in the past, do not always solve exactly the problem A, especially when multiple inequality flexibility constraints are imposed.

3.4. First order approximation of the stress constraints

The zero order approximations of the stress constraints introduced by the F.S.D. can easily be replaced by first order approximations. A given stress component in an element k can be written as a linear combination of the nodal degrees of freedom of the element

$$\sigma_{k\ell} = t_k \, q_\ell \qquad \ell = 1, n_c \quad k = 1, n \qquad (3.19)$$

where t_k is a line of the stress matrix of the element.
n_c is the number of load cases. This expression, already introduced in (2.7), is formally identical to that of the flexibility (2.5) and therefore the same treatment can be applied to compute

first order explicit approximations $\tilde{\sigma}_{k\ell}$

$$\tilde{\sigma}_{k\ell} = \sum_i \frac{d^o_{ik\ell}}{a_i} \tag{3.20}$$

with the stress constraint coefficients

$$d^o_{ik\ell} = (q^T_{ik}\ K_i\ q_{i\ell})^o\ a^o_i \tag{3.21}$$

The first derivative of the explicit approximation of the stresses are

$$\left.\frac{\partial\tilde{\sigma}_{k\ell}}{\partial a_i}\right|_{a^o_i} = -\frac{d^o_{ik\ell}}{(a^o_i)^2} \tag{3.22}$$

Such approximation exhibits obviously the same property as mentionned for the flexibility constraints

$$\left.\begin{array}{l} \sigma_{k\ell} = \tilde{\sigma}_{k\ell} \\ \\ \dfrac{\partial\sigma_{k\ell}}{\partial a_i} = \dfrac{\partial\tilde{\sigma}_{k\ell}}{\partial a_i} \end{array}\right\} \quad \text{along } D(a^o_i) \tag{3.23}$$

which characterizes first order approximations of the constraints.

If the concept is easily introduced, a certain number of difficulties arise in practice. The most important one is that the computation of the stress constraint coefficients $d^o_{ik\ell}$ requires to take into account a number of virtual loading cases which is equal to the number of first order approximated stress constraints, and can be quite large. The solution to that difficulty will come from a selection of those stress constraints that require a first order approximation and will be examined later in the context of hybrid methods.

A second difficulty comes when the need exists to use equivalent stresses to express the constraints. They can not be expressed as linear combinations of the nodal displacements as in (3.19). Two methods are proposed in that case.

First order approximation of a quadratic form of the displacements

Consider the quadratic form of the displacements

$$\lambda^2 = q^T H q \tag{3.24}$$

where H is a (n × n) positive definite matrix of constants. A virtual loading case is defined by

$$g^v = \frac{1}{\lambda} H q \tag{3.25}$$

and its virtual work is given by

$$\lambda = q^T g^v = q^{v^T} g = q^T K q^v = (q^T H q)^{1/2} \tag{3.26}$$

At a given analysis point a_i^o in the design space, it yields the exact value of the form (3.24)

$$\lambda_o^2 = q_o^T H q_o \tag{3.27}$$

Just as for the linear form in the displacements, one can write a first order explicit approximation

$$\tilde{\lambda} = \sum_i \frac{d_i^o}{a_i} \tag{3.28}$$

with

$$d_i^o = (q_i^T K_i q_i^v)^o a_i^o \tag{3.29}$$

where q_i^v are the displacements of the i^{th} element for the virtual loading case (3.25) evaluated in a_i^o.

The exact first derivative of λ are given by

$$\frac{\partial \lambda}{\partial a_i} = \frac{\partial q^T}{\partial a_i} K q^v + q^T \frac{\partial K}{\partial a_i} q^v + q^T K \frac{\partial q^v}{\partial a_i} \qquad (3.30)$$

but

$$\frac{\partial \lambda}{\partial a_i} = g^T \frac{\partial q^v}{\partial a_i} \qquad \text{and} \qquad \frac{\partial \lambda}{\partial a_i} = \frac{\partial q^T}{\partial a_i} g^v$$

so that

$$\frac{\partial \lambda}{\partial a_i} = - q^T \frac{\partial K}{\partial a_i} q^v \qquad (3.31)$$

and, at an analysis point a_i^o

$$\left.\frac{\partial \lambda}{\partial a_i}\right|_{a_i^o} = - \frac{(q_i^T \; K_i \; q_i^v)^o}{a_i^o} \qquad (3.32)$$

which shows that the exact first derivatives are equal to the ones derived from the approximate expression (3.28).
We conclude that

$$\lambda = \tilde{\lambda} \qquad \text{and} \qquad \frac{\partial \lambda}{\partial a_i} = \frac{\partial \tilde{\lambda}}{\partial a_i} \qquad \text{along } D(a_i^o) \qquad (3.33)$$

that means that (3.28) is a first order approximation as announced.

This procedure can be applied when a constraint exists on the module of a displacement or in the case of an equivalent stress. Taking the example of the von MISES equivalent stress in two dimensions

$$\sigma_c^2 = \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3 \tau_{xy}^2 \qquad (3.34)$$

we introduce the matrix V

$$V = \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{3.35}$$

so that

$$\sigma_c^2 = \sigma^T V \sigma \tag{3.36}$$

if

$$\sigma^T = (\sigma_x , \sigma_y , \tau_{xy})$$

The stresses in an element are given by the stress matrix

$$\sigma = T q \tag{3.37}$$

and the square of the equivalent stress is

$$\sigma_c^2 = q^T T^T V T q = q^T H q \tag{3.38}$$

which allows to compute the stress constraint coefficients d_i^o. This implies to take into account one virtual loading case for each equivalent stress constraint and each loading case.

Direct computation of the gradients of the equivalent stresses

It is also possible to compute the first order approximations for each stress component

$$\frac{\partial \sigma_{x_{k\ell}}}{\partial a_i} = - \frac{e_{ik\ell}^o}{a_i^2} \qquad \frac{\partial \sigma_{y_{k\ell}}}{\partial a_i} = - \frac{b_{ik\ell}^o}{a_i^2}$$

$$\frac{\partial \tau_{xy_{k\ell}}}{\partial a_i} = - \frac{t_{ik\ell}^o}{a_i^2} \tag{3.29}$$

where $e^o_{ik\ell}$, $b^o_{ik\ell}$, $t^o_{ik\ell}$ are defined by analogy with (3.29). In the case of the von MISES stress (3.24), it turns out that

$$\frac{\partial \sigma_{c_k}}{\partial a_i} = - \frac{d^o_{ik\ell}}{a_i^2} \tag{3.30}$$

with

$$d^o_{ik\ell} = \frac{1}{\sigma^o_{c_{k\ell}}} \left[e^o_{ik\ell}(\sigma^o_{x_{k\ell}} - \frac{1}{2}\sigma^o_{y_{k\ell}}) + b^o_{ik\ell}(\sigma^o_{y_{k\ell}} - \frac{1}{2}\sigma^o_{x_{k\ell}}) \right.$$
$$\left. + 3\, t^o_{ik\ell}\, \sigma^o_{xy_{k\ell}} \right] \tag{3.31}$$

This procedure requires three virtual loading cases for each equivalent stress constraint.

3.5. A rigorous first order optimality criterion

When all the constraints defined in the original problem R (2.1 to 2.4) are replaced by their first order explicit approximations, a first order approximate problem is obtained which is denoted problem A1 and is written

$$\text{problem A1} \quad \begin{cases} \min \quad W = \sum_i \rho_i\, \ell_i\, a_i \\ \text{with} \quad \sum_i \frac{c_{ij\ell}}{a_i} < \bar{u}_j \\ \qquad \sum \frac{d_{ik\ell}}{a_i} < \bar{\sigma}_k \\ \qquad \underline{a}_i < a_i < \bar{a}_i \end{cases} \tag{3.32}$$

$$i = 1, n \qquad j = 1, n_f \qquad k = 1, n \qquad \ell = 1, n_c$$

A graphical representation of the problem A1 is given in figure 3.2, which is to be compared with the figure 3.1 illustrating the problem A.
The first order optimality conditions for the problem R are given by the classical KUHN-TUCKER conditions

$$\frac{\partial W}{\partial a_i} + \sum_j \sum_\ell r_{j\ell} \frac{\partial u_{j\ell}}{\partial a_i} + \sum_k \sum_\ell s_{k\ell} \frac{\partial \sigma_{k\ell}}{\partial a_i} - t_i + w_i = 0 \tag{3.33}$$

with

$$\begin{aligned} & r_{j\ell} \, (u_{j\ell} - \bar{u}_{j\ell}) = 0 \\ & s_{k\ell} \, (\sigma_{k\ell} - \bar{\sigma}_k \,) = 0 \qquad \begin{array}{l} i = 1, n \\ j = 1, n_f \\ \ell = 1, n_c \\ k = 1, n \end{array} \\ & t_i \quad (\underline{a}_i - a_i \,) = 0 \\ & w_i \quad (a_i - \bar{a}_i \,) = 0 \end{aligned} \tag{3.34}$$

and

$$r_{j\ell} \geq 0 \qquad s_{k\ell} \geq 0 \qquad t_i \geq 0 \qquad w_i \geq 0 \tag{3.35}$$

The variables a_i are the primal variables, while r, s, t, w are the dual variables which have the meaning of lagrange multipliers. They are positive for an active constraint and zero otherwise. They are not allowed to be negative like in the problems with equality constraints. Geometrically, these conditions means that the gradient of the objective function is orthogonal to the plane tangent to the active constraints at the optimum. The KUHN-TUCKER conditions are only necessary. They become also sufficient in a convex problem which is not the case in general for the problem R.

The KUHN-TUCKER conditions can be written for the first order approximate problem A1 (3.32) by using the explicit expressions of the constraint gradients (3.17 and 3.22).

It yields a rigorous first order optimality criterion. Its expression is clarified by defining three classes of variables

$$\begin{array}{lll} C_1 & \text{for} & (i : a_i = \underline{a}_i) \\ C_2 & \text{for} & (i : \underline{a}_i < a_i < \overline{a}_i) \\ C_3 & \text{for} & (i : a_i = \overline{a}_i) \end{array} \tag{3.36}$$

corresponding respectively to the variables that have reached their minimum values, those that are between the upper and lower limits, and those that have reached their upper limits. This classification is analogous to the classical separation in active and passive members. With these notations, we obtain

$$\frac{1}{\rho_i \ell_i a_i^2} \left(\sum_j \sum_\ell r_{j\ell}\, c_{ij\ell} + \sum_k \sum_\ell s_{k\ell}\, d_{ik\ell} \right) \begin{array}{lll} \leq 1 & \text{for} & i \text{ in } C_1 \\ = 1 & \text{for} & i \text{ in } C_2 \\ \geq 1 & \text{for} & i \text{ in } C_3 \end{array}$$

with

$$\begin{array}{llll} r_{j\ell} & = 0 & \text{if} & u_{j\ell} < \overline{u}_j \\ & \geq 0 & \text{if} & u_{j\ell} = \overline{u}_j \\ s_{k\ell} & = 0 & \text{if} & \sigma_{k\ell} < \overline{\sigma}_k \\ & \geq 0 & \text{if} & \sigma_{k\ell} = \overline{\sigma}_k \end{array} \tag{3.37}$$

The essential differences between this expression of the optimality criteria and the classical ones are the following.

First, it is established for the first order approximate problem A1 formulated in terms of inequality constraints. The selection between active and inactive constraints is therefore built in the formulation and can be treated rigorously, which is not always the case in classical formulations.

Second, it uses a consistent first order approximation for all the constraints while the classical F.S.D. concept leads to consider the stress constraints as side constraints. This step is essential for convergence considerations. At the optimum the necessary conditions (3.33) are in general not satisfied using the F.S.D. which is known to lead sometimes to non optimum solutions. At the opposite, the expression (3.37) is deduced from these necessary conditions and the problem is reduced to the approximation of the problem R by a sequence of problems A1, which does not necessarily yield a converging process.

It is to be mentionned that the expression (3.37) corresponds exactly to previously derived optimality criteria [B10, G13, B11, T3] when only one flexibility constraint is imposed and no stress constraints. In the case of multiple constraints the difficulties of solving exactly the system of non linear inequalities (3.37) joined to the computational handicap of using first order approximations for the stress constraints, led many authors to using various additional approximations. These are responsible for the poor convergence properties, or even divergence, encountered in certain cases.

The difficulty of solving (3.37) is in fact essentially related to the fact that the problem A1 is presented in its primal form. Efficient, rigorous and systematic solutions can however be derived by using the dual formulation of the same problem as shown in the next chapter. Therefore the presentation in the form (3.37) is only intended to serve as a definition of the optimality criterion associated to the problem A1.

3.6. Hybrid optimality criteria

The computational handicap of using first order approximations for the stress constraints is due to need to introduce an eventually large number of virtual loading cases in the analysis, which is avoided when the zero order F.S.D. approximation is used.

In practical structures it is observed that most of the stress constraints are approximated with sufficient accuracy by the F.S.D. while some other, in limited number most of time, reveal critical and require a first order approximation. It is not possible in general to decide a priori which constraint is going to be critical and the number of such critical constraints can vary at each iteration, that is after each reanalysis.

An automatic selection of these critical constraints can be based on the preceding interpretations and presented as follows.
If

$$\frac{\sigma_{k\ell}}{\bar{\sigma}_k} \simeq 1 \qquad \text{AND} \qquad \frac{d_{kk\ell}}{a_k} << \sigma_{k\ell} \tag{3.38}$$

the constraint is potentially critical. The first condition is rather evident and means that the stress in the element is equal or close to the prescribed limit which implies that the constraint is active or potentially active. The second condition arises from the fact that a first order approximation reduces to a zero order approximation when

$$d_{ik\ell} = 0 \qquad\qquad i \neq k \tag{3.39}$$

Hence if, in the sum

$$\sigma_{k\ell} = \sum_i \frac{d_{ik\ell}}{a_i} \tag{3.40}$$

the term $d_{kk\ell}$ is dominant, the situation is not far from that existing in the isostatic case where $d_{kk\ell} = \sigma_{k\ell}\, a_k$.
In other words, if in the sum (3.40) the contribution of the other elements ($i \neq k$) is negligible, the F.S.D. zero order

approximation is a good approximation.

After each analysis step, the conditions (3.38) can be used to define which stress constraints are to be first order approximated while the others are treated by F.S.D., that means are treated as side constraints by using the definition (2.15) for the minimum size. This procedure leads to a sequence of problems A1 which involve each a different number of constraints. The corresponding optimality conditions can be derived just as in (3.37).

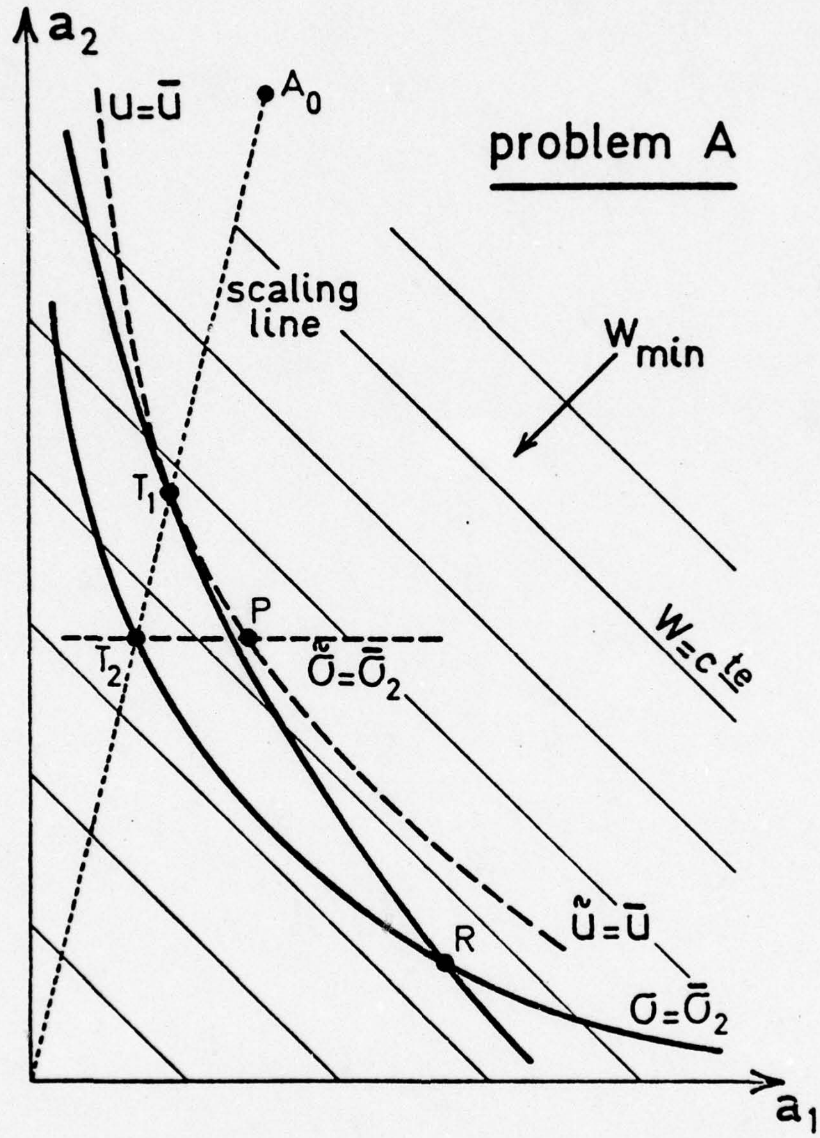
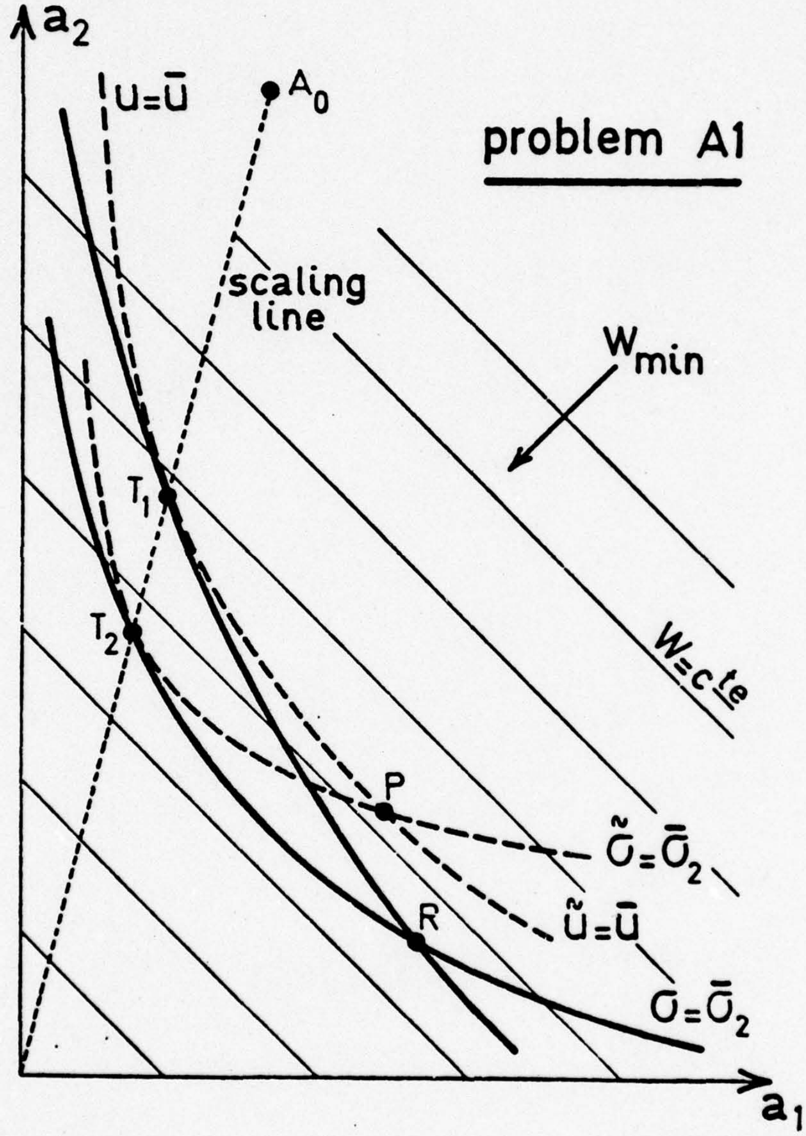


FIGURE 3.1



a_2
$u=\bar{u}$
A_0
problem A1
scaling line
W_{min}
T_1
T_2
$W=c^{te}$
P
$\tilde{\sigma}=\bar{\sigma}_2$
$\tilde{u}=\bar{u}$
R
$\sigma=\bar{\sigma}_2$
a_1

FIGURE 3.2

4. DUAL FORMULATION AND OPTIMALITY CRITERIA

4.1. The primal problem

The classical optimality criteria have been derived by using a lagrangian formulation to solve a simplified version of the problem A (2.27) where a selection between active and inactive constraints allows to define the problem with equality constraints. The generalized versions presented in the preceding chapter are based on the problem A1 (3.32) and treat the case with inequality constraints. Eventually a hybrid situation is considered where only a limited number of stress constraints are first order approximated. To take these various situations into account we write the basic problem in the form

$$\text{problem } A' \quad \left\{ \begin{array}{llll} \min & W = \sum_i \rho_i \ell_i a_i & i = 1, n & (4.1) \\ \text{with} & u_j = \sum_i \dfrac{c_{ij}}{a_i} \leq \bar{u}_j & j = 1, m & (4.2) \\ & a_i \leq \bar{a}_i & i = 1, n & (4.3) \\ & a_i \geq \underline{a}_i & i = 1, n & (4.4) \end{array} \right.$$

where we consider that the m constraints (4.2) group all the first order approximated constraints, that is, the true flexibility constraints and the first order approximated stress constraints (3.20). The minimum variable sizes $\underline{a}_i$ are defined as the a_i^* (2.15), that is, result from the choice of the maximum between the true minimum sizes or the values computed by the stress ratio algorithm of F.S.D.

In such a way, all the above mentionned situations are covered. The problem A' is called the primal problem because it is expressed in the primal variables a_i. It is not necessarily convex as the c_{ij} coefficients in (4.2) are not always all positive. However if a change of variables

$$x_i = \frac{1}{a_i} \tag{4.5}$$

is introduced the problem becomes strictly convex and separable, which allows to derive easily a dual formulation.

4.2. The dual problem

The fact that the classical optimality criteria are based on a lagrangian method suggests to use a dual method to solve the problem A'. The change of variables (4.5), which is necessary to have a strictly convex problem, is not introduced explicitely as they disappear, and the solution will be expressed in terms of the dual variables only. These are the lagrange multipliers associated to the various constraints.

4.2.1. The complete dual formulation

In the sense of WOLFE [W2] the dual problem associate to the problem A' is

$$\max \mathcal{L} (a, r, w, t)$$

$$= \sum_i \rho_i \ell_i a_i - \sum_j r_j (\bar{u} - u_j) - \sum_i w_i (\bar{a}_i - a_i)$$

$$- \sum_i t_i (a_i - \underline{a}_i) \tag{4.6}$$

with the constraints

$$\frac{\partial \mathcal{L}}{\partial a_i} = 0 \tag{4.7}$$

$$r_j \geq 0 \qquad w_i \geq 0 \qquad t_i \geq 0 \tag{4.8}$$

The equality constraints (4.7) insuring the stationarity of the lagrangian function $\mathcal{L}$ yields, as the problem is separable, the explicit relations

$$a_i^2 = \frac{\sum_j r_j \, c_{ij}}{\rho_i \ell_i + w_i - t_i} \qquad (4.9)$$

allowing to compute the primal variables a_i when the dual variables r_j, w_i, t_i are known. If the a_i given by (4.9) are substituted into the functional (4.6), the problem reduces to a maximization problem in the $(m + 2n)$ dual variables r_j, w_i, t_i subjected to the $(m + 2n)$ non negativity constraints.

Formulated in this way the dual approach is not advantageous as it involves more variables than in the primal problem. However the simple form of the primal constraints (4.3) and (4.4) allows to use a formulation reduced in terms of the only variables r_j.

4.2.2. The reduced dual formulation

The foundation of the method used in this section is due to FALK [F12] . Consider the primal problem

$$\min \quad W = \sum_i \rho_i \, \ell_i \, a_i \qquad i = 1, n \qquad (4.10)$$

$$\text{with } \bar{u}_j = \sum_i \frac{c_{ij}}{a_i} > 0 \qquad j = 1, m \qquad (4.11)$$

$$a_i \in C$$

where C is the convex and compact set of the variables defined between the minimum and maximum values by

$$C = (a : \underline{a}_i < a_i < \bar{a}_i) \qquad (4.12)$$

The definition is necessary to exclude the possibility of infinite solutions in the minimization process.

The dual formulation of FALK is simply

$$\max \ \gamma(r)$$
$$\text{with } r_j > 0 \qquad\qquad j = 1, m \tag{4.13}$$

where $\gamma(r)$ is an auxiliary function given by

$$\gamma(r) = \min_{a_i} \left[\sum_i \rho_i \ell_i a_i - \sum_j r_j \left(\bar{u}_j - \sum_i \frac{c_{ij}}{a_i}\right) \right]$$
$$\text{for} \quad \underline{a}_i < a_i < \bar{a}_i \tag{4.14}$$

or, more simply, as the problem is separable,

$$\gamma(r) = \sum_i \rho_i \ell_i a_i(r) - \sum_j r_j \left(\bar{u}_j - u_j [\ a_i(r)\ ]\ \right) \tag{4.15}$$

The solution of (4.14) leads to introduce the quantities

$$\beta_i(r) = \frac{1}{\rho_i \ell_i} \sum_j c_{ij} r_j \qquad j = 1, m \quad i = 1, n \tag{4.16}$$

The primal variables $a_i(r)$ are then expressed in terms of the dual variables r_j, using the quantities β_i

$$\begin{aligned} a_i &= \sqrt{\beta_i} && \text{if} \quad \underline{a}_i^2 < \beta_i < \bar{a}_i^2 \\ a_i &= \underline{a}_i && \text{if} \quad \beta_i < \underline{a}_i^2 \\ a_i &= \bar{a}_i && \text{if} \quad \bar{a}_i^2 < \beta_i \end{aligned} \tag{4.17}$$

The explicit expression of the dual problem still requires the definition of the flexibilities

$$u_j \, [\, a_i(r) \,] = \sum_i \frac{c_{ij}}{a_i(r)} \qquad j = 1, m \qquad (4.18)$$

The gradient of lagrangian function can be obtained as follows

$$\frac{\partial\gamma}{\partial r_j} = \sum_i^n \rho_i \, \ell_i \, \frac{\partial a_i}{\partial r_j} + \sum_k^m \frac{\partial u_k}{\partial r_j} + u_j - \bar{u}_j \qquad (4.19)$$

but, using (4.16) and (4.17)

$$\frac{\partial a_i}{\partial r_j} = \frac{1}{2} \, \frac{c_{ij}}{\rho_i \ell_i a_i} \qquad i = 1, \tilde{n} \qquad (4.20)$$

$$\frac{\partial a_i}{\partial r_j} = 0 \qquad i = \tilde{n}+1, n \qquad (4.21)$$

if $\tilde{n}$ denotes the number of free variables, that is belonging to the interval $] \, \underline{a}_i \, , \, \bar{a}_i \, [$. From (4.20) and (4.21) it follows that

$$\sum_i^n \rho_i \, \ell_i \, \frac{\partial a_i}{\partial r_j} = \frac{1}{2} \sum_i^{\tilde{n}} \frac{c_{ij}}{a_i} \qquad (4.22)$$

and from (4.18)

$$\frac{\partial u_k}{\partial r_j} = - \sum_i^n \frac{c_{ik}}{a_i^2} \, \frac{\partial a_i}{\partial r_j} \qquad (4.23)$$

and

$$\sum_k^m r_k \, \frac{\partial u_k}{\partial r_j} = - \frac{1}{2} \sum_i^{\tilde{n}} \frac{c_{ij}}{a_i} \qquad (4.24)$$

Finally the gradient (4.19) reduces, using (4.22) and (4.24) to

$$\frac{\partial\gamma}{\partial r_j} = u_j - \bar{u}_j \qquad (4.25)$$

All the ingredients necessary to the solution of the dual problem (4.10) (4.11) have been evaluated.
The reduced dual problem involves the maximization of a lagrangian function (4.10) under the constraints (4.11) in a space of dimension m equal to the number of first order approximated constraints. A certain number of methods can be applied to this maximization problem and some of them are described in the next section.

It must be emphasized that when the number of dual variables is much smaller than the number of design variables, which is often the case, a significant advantage exists for the dual formulations. An important property is their ability to solve exactly a problem A' associated to a given rigorous or hybrid optimality criterion,without introducing any approximation in the selection of active and passive constraints,nor in the treatment of the side constraints.

Domains of definition and planes of discontinuity

The relations (4.17) indicate the existence,in the dual space, of planes of discontinuity for the second derivatives of the dual objective function. They are given by

$$\sum_{j}^{m} c_{ij}\, r_j = \rho_i\, \ell_i\, \underline{a}_i^2 \tag{4.26}$$

and

$$\sum_{j}^{m} c_{ij}\, r_j = \rho_i\, \ell_i\, \bar{a}_i^2 \tag{4.27}$$

When crossing these planes,the definition of the lagrangian function (4.15) is modified according to (4.17). In each subregions of the dual space, called domains definition, the dual objective function is modified due to different distributions of the primal variables in free and fixed variables.

4.3. The dual algorithms

Various maximisation methods can be applied to solve the reduced dual problem (4.10) (4.11) corresponding to the problem A' (4.1 to 4.4). Two types of algorithms have been developped and used in the numerical applications. The first one is a first order algorithm which requires only the first derivatives of the objective function, the second one is a second order algorithm requiring the second derivatives of the dual objective function.

4.3.1. A first order dual algorithm

The simple algorithm proposed here is based on the projected gradient method described by ROSEN in [R1] . It takes a very simple form due to the fact that the constraints reduce to the non negativity of the dual variables.

Initialization

(i) define an admissible starting point such that

$$r_j^o > 0 \qquad j = 1, m$$

Iterative procedure

(ii) Let r^λ the point observed at the iteration λ

Compute - the primal variables by (4.16) and (4.17)

- the flexibilities u_j^λ by (4.18)

The search direction z^λ is given by (4.25) that is

$$z_j^\lambda = 0 \qquad \text{if} \qquad r_j^\lambda = 0 \quad \text{and} \quad u_j^\lambda < \bar{u}_j$$

$$z_j^\lambda = u_j^\lambda - \bar{u}_j \quad \text{if} \qquad r_j > 0 \quad \text{or} \quad u_j^\lambda > \bar{u}_j$$

(iii) if $\|z^{\lambda}\| > \varepsilon$ (given tolerance) go to (iv)

otherwise r^{λ} is the global minimum (of the problem A')

(iv) determine the optimal step length τ^{λ} along z^{λ} by the linear search

$$\gamma(r^{\lambda} + \tau^{\lambda} z^{\lambda}) = \max_{\tau \in S^{\lambda}} [\gamma(r^{\lambda} + \tau z^{\lambda})]$$

where $S^{\lambda} = [\tau > 0 : (r^{\lambda} + \tau z^{\lambda}) \geqslant 0]$

During this linear search, a dual variable may eventually reach the assigned limit $r_j^{\lambda} = 0$

(v) Progress to the step $\lambda+1$ by computing

$$r^{\lambda+1} = r^{\lambda} + \tau^{\lambda} z^{\lambda}$$

go to (ii)

This procedure is very simple. It can be improved significantly by using an algorithm of conjugate direction,which is not presented here for simplicity,as it is rather classical [F4] . In the application described later in this work, such a conjugate direction algorithm is used.

4.3.2. A second order dual algorithm

The algorithm developed here is based on the generalized NEWTON method particularized for the case of linear constraints. A preliminary problem in such a method is to determine the direction of the search. Consider the case where the $\tilde{m}$ first dual variables are positive (active primal constraints) and the $\bar{m}=m-\tilde{m}$ next ones equal to zero (inactive primal constraints)

$$\tilde{r}_j > 0 \qquad j = 1, \tilde{m} \qquad \bar{r}_j = 0 \qquad j = \tilde{m}+1, m$$

$$(4.28)$$

The gradient g and the hessian matrix F are partitionned accordingly

$$g = \nabla\gamma = \begin{bmatrix} \tilde{g} \\ \bar{g} \end{bmatrix} \qquad F = \nabla^2\gamma = \begin{bmatrix} \tilde{F} & F^* \\ F^{*T} & \bar{F} \end{bmatrix} \qquad (4.29)$$

If N denotes the matrix defining the base of the active constraints so that (4.28) is written

$$N^T r = 0 \qquad (4.30)$$

the matrix N has the simple structure

$$N = \begin{bmatrix} 0 \\ I \end{bmatrix} \qquad (4.31)$$

where I is a $(\bar{m} \times \bar{m})$ unit matrix. The NEWTON search direction is given by [G3, G5, F9]

$$z = \hat{P} F^{-1} g$$

with (4.32)

$$\hat{P} = I - F^{-1} N (N^T F^{-1} N)^{-1} N^T$$

Introducing the notation

$$v = (N^T F^{-1} N)^{-1} N^T F^{-1} g \qquad (4.33)$$

it can also be written

$$z = F^{-1} (g - N v) \tag{4.34}$$

The inverse of the Hessian matrix terms out to be

$$F^{-1} = \begin{bmatrix} (\tilde{F}^{-1} + \tilde{F}^{-1} F^{*} G^{-1} F^{*^T} \tilde{F}^{-1}) & (- \tilde{F}^{-1} F^{*^T} G^{-1}) \\ (- G^{-1} F^{*^T} \tilde{F}^{-1}) & (G^{-1}) \end{bmatrix} \tag{4.35}$$

where $\quad G = \bar{F} - F^{*^T} \tilde{F}^{-1} F^{*}$

from which we obtain successively

$$N^T F^{-1} = \begin{bmatrix} - G^{-1} F^{*^T} \tilde{F}^{-1} \\ G^{-1} \end{bmatrix}$$

$$(N^T F^{-1} N)^{-1} = G$$

$$v = \bar{g} - F^{*^T} \tilde{F}^{-1} \tilde{g}$$

so that

$$g - N v = \begin{bmatrix} \tilde{g} \\ F^{*^T} \tilde{F}^{-1} \tilde{g} \end{bmatrix} \tag{4.36}$$

and the search direction (4.34) reduces to the very simple form

$$z = \begin{bmatrix} \tilde{z} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \tilde{F}^{-1} \tilde{g} \\ 0 \end{bmatrix} \tag{4.37}$$

while (4.33) is simply written

$$v = \bar{g} - F^{*^T} \tilde{z} = \bar{g} - F^{*^T} \tilde{F}^{-1} \tilde{g} \qquad (4.38)$$

As expected in a second order method, the determination of the search direction z requires the inversion of the hessian matrix of the lagrangian considered as a function of the $\tilde{m}$ free dual variables only.

To determine the active constraints, a very simple selection rule can be established if the sign of the multipliers (4.38) is examined only at a stationary point of the objective function where, by definition, the gradient $\tilde{g}$ is zero, which implies

$$v = \bar{g} \qquad (4.39)$$

which does not require the evaluation of the matrix F^* present in (4.38). The inversion of $\tilde{F}$ is therefore only achieved when the basis of active constraints is known, which avoids the numerical difficulties that could appear in case of linearly dependent flexibility constraints.

Algorithm

Initialization

(i) select an admissible starting point

$$\tilde{r}_j^o > 0 \qquad j = 1, \tilde{m}$$

$$\tilde{r}_j^o = 0 \qquad j = \tilde{m}+1, m$$

such that the hessian matrix $\tilde{F}^o$ is non singular

Iterative procedure

(ii) Let r^λ be the point obtained at iteration λ

. Compute the primal variables $a_i^\lambda(r^\lambda)$ by (4.16) and (4.17) and the flexibilities u_j^λ by (4.18)

. Determine

$$\tilde{g}^\lambda = u_j^\lambda - \bar{u}_j \qquad j = 1, \tilde{m}$$

. If $|| \tilde{g}^\lambda || < \varepsilon$ (given tolerance) go to (v)

otherwise compute

$$\tilde{z}^\lambda = (\tilde{F}^\lambda)^{-1} \tilde{g}^\lambda$$

(iii) Compute the optimal step length along $\tilde{z}^\lambda$ by the linear search in the space of the $\tilde{m}$ dual variables

$$\gamma(\tilde{r}^\lambda + \tau^\lambda \tilde{z}^\lambda) = \max_{\tau \in S^\lambda} \left[\gamma(\tilde{r}^\lambda + \tau \tilde{z}^\lambda) \right]$$

where

$$S^\lambda = \{ \tau > 0 : (\tilde{r}^\lambda + \tau z^\lambda) \geq 0 \}$$

(iv) Compute

$$\tilde{r}^{\lambda+1} = \tilde{r}^\lambda + \tau^\lambda z^\lambda$$

go to (ii) with eventually $\hat{m} = \tilde{m}-1$

(v) Compute

$$\bar{g}_j^\lambda = u_j^\lambda - \bar{u}_j \qquad j = \tilde{m}+1,\ m$$

If $\max \{ g_j^\lambda \} < 0$ then $\tilde{r}^\lambda$ is the global maximum (of the problem A')

Otherwise go to (ii) with $\tilde{m} = \tilde{m}+1$ by incorporating in $\tilde{r}^{\lambda+1}$ the dual variable which has the largest positive component $\bar{g}_j^\lambda$.

To fully determine the procedure, it is still necessary to give the explicit form of the Hessian matrix of the objective function. From (4.19) to (4.25) we obtain

$$(\tilde{F})_{jk} = \frac{\partial^2 \gamma}{\partial r_j \partial r_k} = - \frac{1}{2} \sum_i \frac{c_{ij} c_{ik}}{\rho_i \ell_i a_i^3} \qquad (4.40)$$

where $i = 1,\ \tilde{n}$ $\quad j = 1,\ \tilde{m}$ $\quad k = 1,\ \tilde{m}$

and $\tilde{n}$ is the number of free primal variables as defined by (4.17).

4.3.3. Linear search algorithm

The difficulty of the linear search along a given direction arises from the fact that the dual space is subdivided into various definition domains where the objective function takes different forms. Across a plane of discontinuity the number of free primal variables $\tilde{n}$ is modified and the second derivatives (4.40) jump to other values.
The linear search can be represented indifferently by the maximization of the lagrangian

$$\max_\tau \gamma(\tau) = \sum_i \rho_i \ \ell_i \ a_i(\tau) + \sum_j^m r_j(\tau) \ [\ u_j(\tau) - \bar{u}_j \ ]$$

or by the problem of finding a vanishing first derivative

$$\gamma'(\tau) = z^T \nabla \gamma(\tau) = \sum_j^m z_j [u_j(\tau) - \bar{u}_j] = 0 \quad (4.41)$$

This last formulation presents some advantages and can be used as follows. Introduce the constants

$$X = \sum_j^m z_j \bar{u}_j \qquad\qquad Y_i = \frac{1}{\rho_i \ell_i} \sum_j^m c_{ij} \hat{r}_j$$

$$Z_i = \frac{1}{\rho_i \ell_i} \sum_j^m c_{ij} z_j \qquad (4.42)$$

where $\hat{r}_j$ denotes the current value of the dual variables. The problem consists in finding the zero of the function

$$\gamma'(\tau) = \sum_i^n \frac{Z_i \rho_i \ell_i}{a_i(\tau)} - X \qquad (4.43)$$

where the design variables $a_i(\tau)$ are given by

$$a_i^2(\tau) = Y_i + \tau Z_i \qquad \text{if} \qquad \underline{a}_i^2 \leq Y_i + \tau Z_i < \bar{a}_i^2$$

$$\text{(that is if } i = 1, \tilde{n}) \quad (4.44)$$

$$a_i = \underline{a}_i \qquad \text{or} \qquad a_i = \bar{a}_i \qquad \text{otherwise .}$$

This problem can be solved by the NEWTON-RAPHSON or by the TCHEBYCHEV method [B13] . The higher order derivatives required are given by

$$\gamma''(\tau) = - \frac{1}{2} \sum_i \frac{Z_i^2 \rho_i \ell_i}{a_i^3}$$

$$i = 1, \tilde{n} \qquad (4.45)$$

$$\gamma'''(\tau) = \frac{3}{4} \sum_i \frac{Z_i^3 \rho_i \ell_i}{a_i^5}$$

The function $\gamma'(\tau)$ is continuous across the planes of discontinuity,but the higher derivatives are only piecewise continuous. They exhibit a discontinuity each time a change in the number of free variables occurs. This is the case for the following values of the step :

$$\underline{\tau}_i = \frac{\underline{a}_i^2}{Z_i} - \frac{Y_i}{Z_i} \qquad \bar{\tau}_i = \frac{\bar{a}_i^2}{Z_i} - \frac{Y_i}{Z_i} \tag{4.46}$$

$$i = 1, n$$

It is therefore required to take some measure to avoid a divergence of the NEWTON-RAPHSON procedure. A simple technique consists in finding by (4.46) the values of the step where a discontinuity occurs,and select the interval in which the maximum is obtained.

4.4. Relation with the classical optimality criteria

The optimality criterion for a single flexibility constraint as proposed by BERKE [B10] appears to be equivalent to a dual formulation of a problem A1, in the sense of FALK [F12] .
In this simple case the maximization problem (4.13) is solved analytically as the auxiliary function $\gamma(r)$ depends only of one variable.

Turning to the optimality criteria of TAIG and KERR [T3],a change of variables in the dual problem (4.13)

$$\lambda_j = \frac{1}{r_j} \tag{4.47}$$

and the application of the second order NEWTON method to the new expression of the problem yields the iterative scheme

$$\lambda^{\nu+1} = \lambda^{\nu} + \tau^{\tau}(X^{\nu})^{-1}\ (\bar{u} - u^{\nu}) \tag{4.48}$$

where ν is the iteration number and X the Hessian matrix of the new dual objective function, deduced from (4.40)

$$X_{jk} = \frac{\partial^2\gamma}{\partial\lambda_j\,\partial\lambda_k} = \frac{1}{2\lambda_k^2}\ \sum_i \frac{c_{ij}c_{ik}}{\rho_i \ell_i a_i^3} \tag{4.49}$$

$$i = 1, \tilde{n} \qquad j = 1, \tilde{m} \qquad k = 1, \tilde{m}$$

In the process (4.48), the step length has to be computed to maximize the lagrangian along the direction $(X^{\nu})^{-1}\ (\bar{u} - u^{\nu})$ while remaining in the admissible dual domain $\lambda_j \geq 0$.
With the present notation, the original algorithm of TAIG and KERR replaces (4.48) and (4.49) by

$$\lambda^{\nu+1} = \lambda^{\nu} + (X^{\nu})^{-1}\ (\bar{u} - u^{\nu})$$

and (4.50)

$$X_{jk} = \frac{1}{2\lambda_k^2}\ \sum_i\ \frac{c_{ij}c_{ik}}{\rho_i \ell_i a_i^3}$$

$$i = 1, n \qquad j = 1, m \qquad k = 1, m$$

This algorithm consists in maximizing the lagrangian using the simple NEWTON method without computing the optimal step length which is equivalent to $\tau^{\nu} = 1$ in (4.48). In the TAIG and KERR method the reference strain energy densities are interpreted as the inverses of the dual variables associated to the flexibility constraints. The reasons of the instability of this more intuitive algorithm becomes clear. First the method can not avoid

the λ_j to become negative, that is, ignores the non negativity constraints of the dual variables.
Second, the fact that the optimal step length is not computed can lead to a divergence when the starting point is chosen too far from the optimum. It is only close to the optimum that the step length becomes close to unity in the generalized NEWTON method.
Third, they ignore the fact that the dual space is subdivided into various definition domain. The consequence is that the choice of free and fixed design variables can not always be correct.

The dual methods proposed in this chapter appears as a sound mathematical way to solve the problems corresponding to the rigorous (or hybrid) optimality criteria.
Their computational performances are excellent and will be demonstrated in the chapter dealing with applications.
They present the fundamental advantages of allowing any combination of zero or first order approximated constraints, and of a correct selection between active and passive constraints.
The determination of the values of the lagrange multipliers remains possible even when the constraints are linearly dependent.
The convergence properties are dependent of the validity of the approximation of the problem R by a sequence of problems A'.
This implies that if the convergence to the solution of the problem A' is always guaranteed, the convergence of the sequence of problems A' to the solution of the problem R can not.
This remains an essential difficulty using optimality criteria which will be given additional attention after having introduced the mixed methods in the next chapter.

4.5. Two simple illustrations

A two dimensional problem is used to illustrate graphically the concepts introduced in the dual methods.
The two bar truss schematized on figure 4.1 can be analytically formulated as follows :

$$\min \quad W(a_i) = \rho \, \ell \, \sqrt{2} \, (a_1 + a_2) \tag{4.51}$$

$$\text{with} \qquad u = \frac{P\ell\sqrt{2}}{2E} \left(\frac{1}{a_1} + \frac{1}{a_2}\right) < \frac{3}{2} \, \frac{P\ell\sqrt{2}}{2E} \tag{4.52}$$

$$v = \frac{P\ell\sqrt{2}}{2E} \left(\frac{1}{a_1} - \frac{1}{a_2}\right) < \frac{1}{2} \, \frac{P\ell\sqrt{2}}{2E} \tag{4.53}$$

$$1 < a_1 < 2 \tag{4.54}$$

$$1 < a_2 < 2 \tag{4.55}$$

The design space and the constraints are also illustrated in the figure 4.1. The optimum corresponds to the joint $a_1 = \frac{4}{3}$, $a_2 = \frac{4}{3}$ at which only one constraint is active (horizontal, u, displacement). The formulation of the dual problem involves two lagrange multipliers r_1 and r_2 associate to the constraints (4.52) and (4.53). The planes of discontinuity defined in (4.26) (4.27) are illustrated in figure 4.2 and correspond to

$$\begin{array}{ll} r_1 + r_2 = 1 & \qquad r_1 + r_2 = 4 \\ r_1 - r_2 = 1 & \qquad r_1 - r_2 = 4 \end{array} \tag{4.56}$$

It yields 6 subdomains in the dual space. In each of them, the expression of the objective function in terms of the dual variable is different and given in explicit form in table 4.3.
The optimum in the dual space occurs in the subdomain II for $r_1 = \frac{16}{9}$ and $r_2 = 0$ and a value of the dual lagrangian objective function $\gamma(r) = \frac{8}{3}$. The linear search along a direction $r_2 = r_o =$ constant is illustrated on figure 4.4 as well as the continuity properties of the lagrangian function $\gamma(\tau)$ and its derivatives defined in (4.41).

Another simple example is provided by the 10 bar truss defined in figure 4.5. In this problem no flexibility constraints are imposed but the stress constraints are $\sigma_i \leq 25000$ psi in all the bars but in bar 8 when the limit $\bar{\sigma}_8$ is given different values.

Using the classical F.S.D. approach, as reported by BERKE in a similar problem [B8] , the solution converges rapidly to the optimum for $\bar{\sigma}_8 < 36800$ psi while for $\bar{\sigma}_8 > 45000$ psi a fast convergence is observed but to a non optimum F.S.D. design. For $36800 < \bar{\sigma}_8 < 45000$ the convergence is very slow and does not lead to the optimum solution after 30 reanalyses nor even to a design satisfying the F.S.D. concept. An exception exists for $\bar{\sigma}_8 = 37500$ psi where the convergence to a non optimal F.S.D. design is obtained in 11 reanalyses. This behavior is illustrated in figure 4.6. The same problem has been solved using the generalized optimality criterion based on first order approximation of the stress constraints and using the dual algorithm described above. The results are given in figure 4.7. Besides the fact that the optimum is obtained for $\bar{\sigma}_8 > 36800$ psi, the number of reanalyses remains approximately the same for all the values of $\bar{\sigma}_8$ and corresponds to a very fast convergence. The designs corresponding to $\bar{\sigma}_8 = 25000$, $\bar{\sigma}_8 = 30000$, $\bar{\sigma}_8 \geq 37500$ are presented in table 4.8.

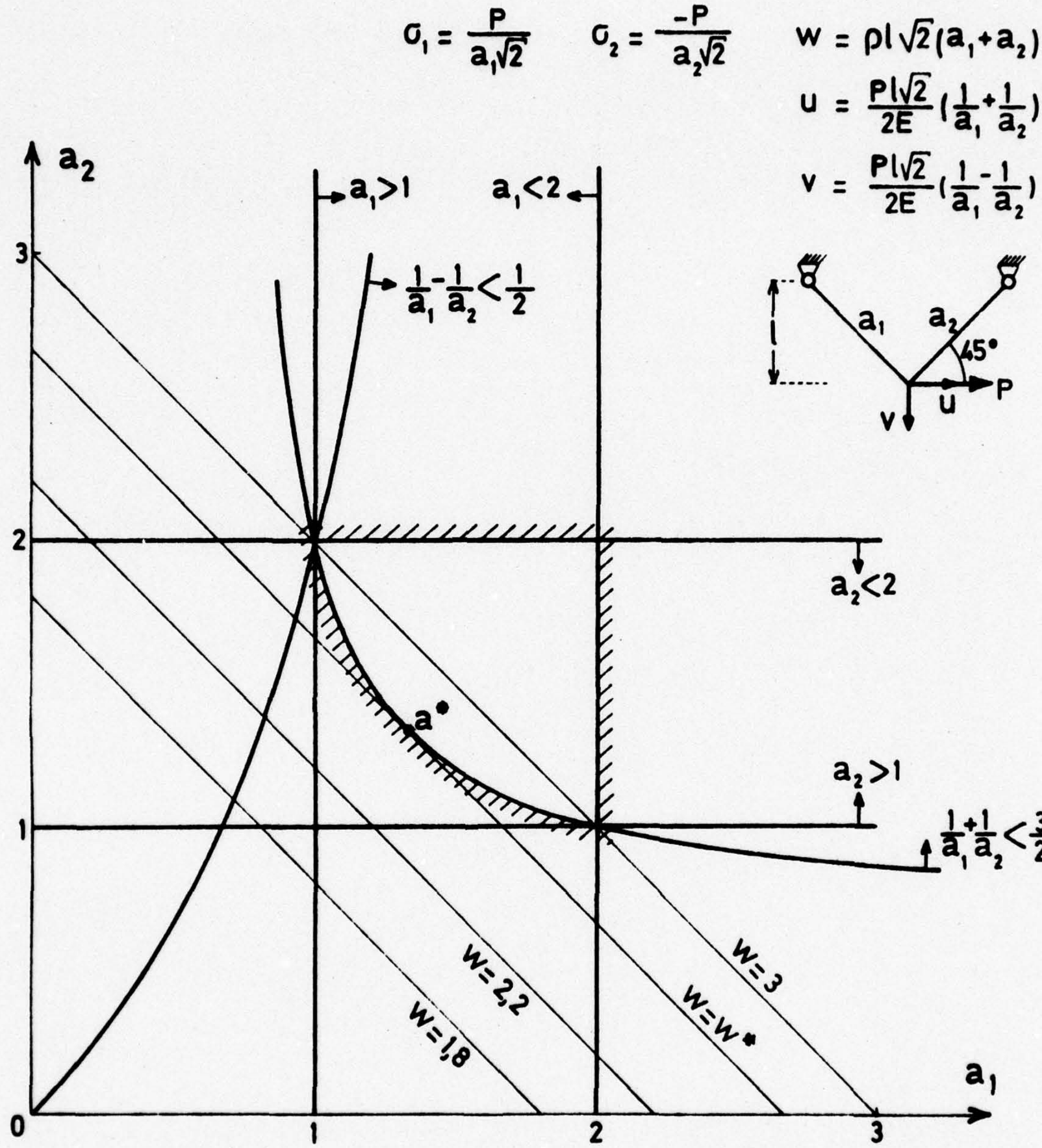
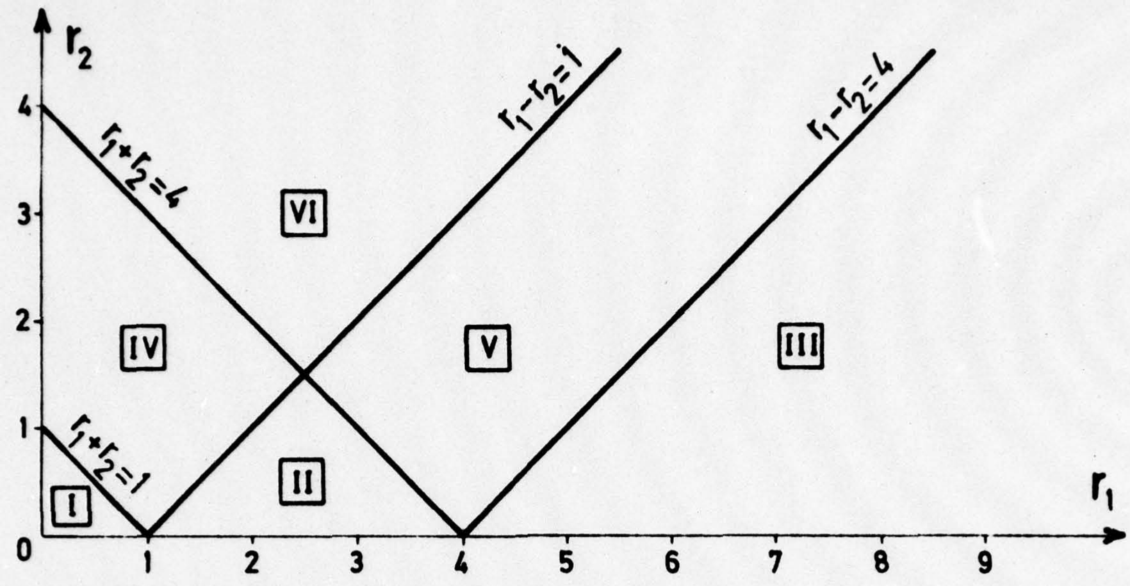


FIG. 4.1 : 2 BAR TRUSS - PRIMAL PROBLEM



(a) DOMAINS OF DEFINITION AND PLANES OF DISCONTINUITY

(b) CONTOURS OF THE DUAL OBJECTIVE FUNCTION

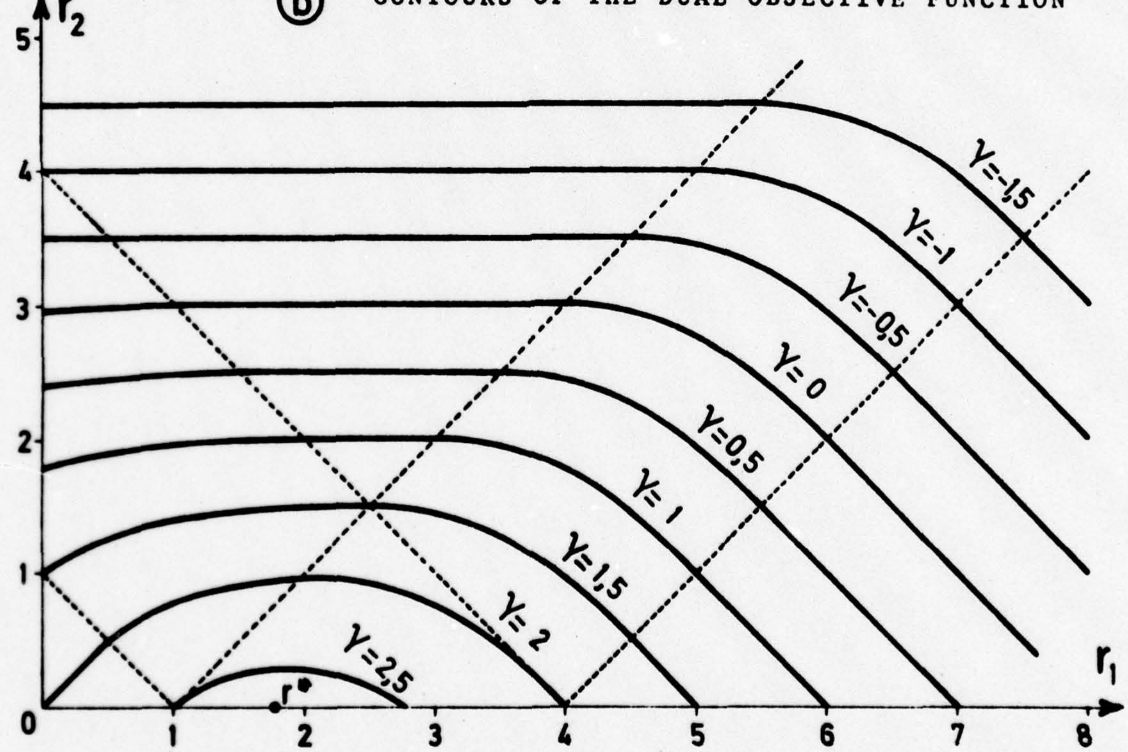


FIG. 4.2 : 2 BAR TRUSS - DUAL SPACE

domains of definition	primal variables		flexibilities		dual objective function
	a_1	a_2	u_1	u_2	γ
I	1	1	2	0	$2+\frac{r_1-r_2}{2}$
II	$\sqrt{r_1+r_2}$	$\sqrt{r_1-r_2}$	$\frac{1}{\sqrt{r_1+r_2}}+\frac{1}{\sqrt{r_1-r_2}}$	$\frac{1}{\sqrt{r_1+r_2}}-\frac{1}{\sqrt{r_1-r_2}}$	$2(\sqrt{r_1+r_2}+\sqrt{r_1-r_2})-\frac{3r_1+r_2}{2}$
III	2	2	1	0	$4-\frac{r_1+r_2}{2}$
IV	$\sqrt{r_1+r_2}$	1	$\frac{1}{\sqrt{r_1+r_2}}+1$	$\frac{1}{\sqrt{r_1+r_2}}-1$	$1+2\sqrt{r_1+r_2}-\frac{r_1+3r_2}{2}$
V	2	$\sqrt{r_1-r_2}$	$\frac{1}{2}+\frac{1}{\sqrt{r_1-r_2}}$	$\frac{1}{2}-\frac{1}{\sqrt{r_1-r_2}}$	$2+2\sqrt{r_1-r_2}-r_1$
VI	2	1	$\frac{3}{2}$	$-\frac{1}{2}$	$3-r_2$

TABLE 4.3

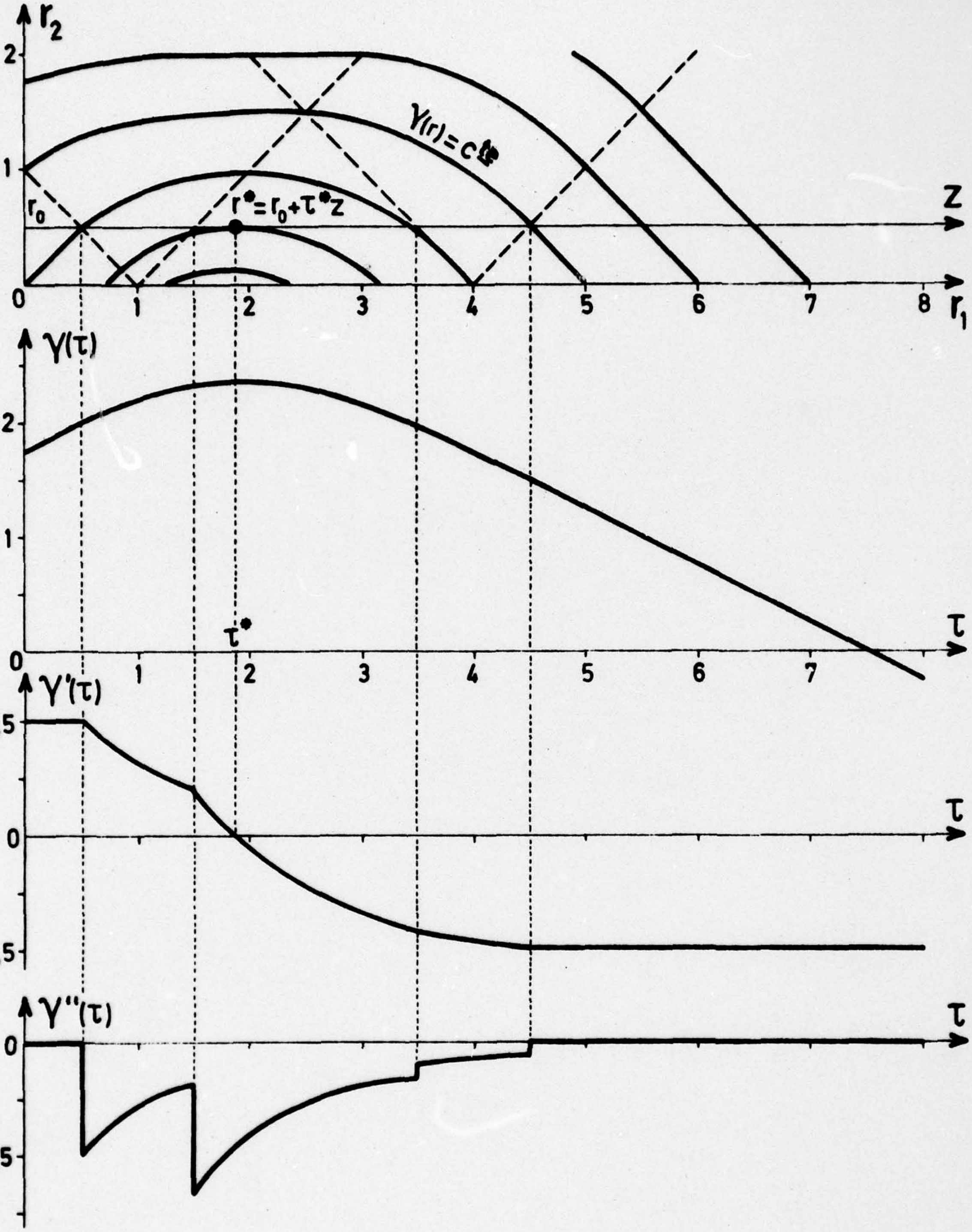
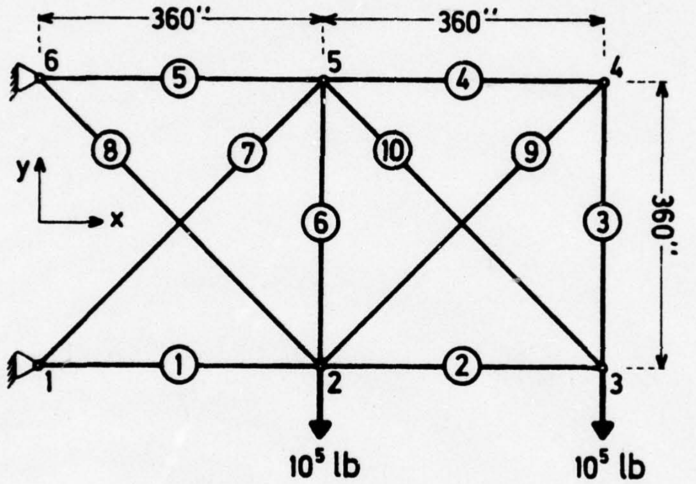


FIG. 4.4 : 2 BAR TRUSS - DUAL LINEAR SEARCH



$\bar{\sigma}$ = 25000 psi
E = 10^7 psi
ρ = 0.1 lb/in^3
$\underline{a}$ = 0.1 in^2
$\bar{u}$ = 2.0 in

FIGURE 4.5 10 BAR TRUSS

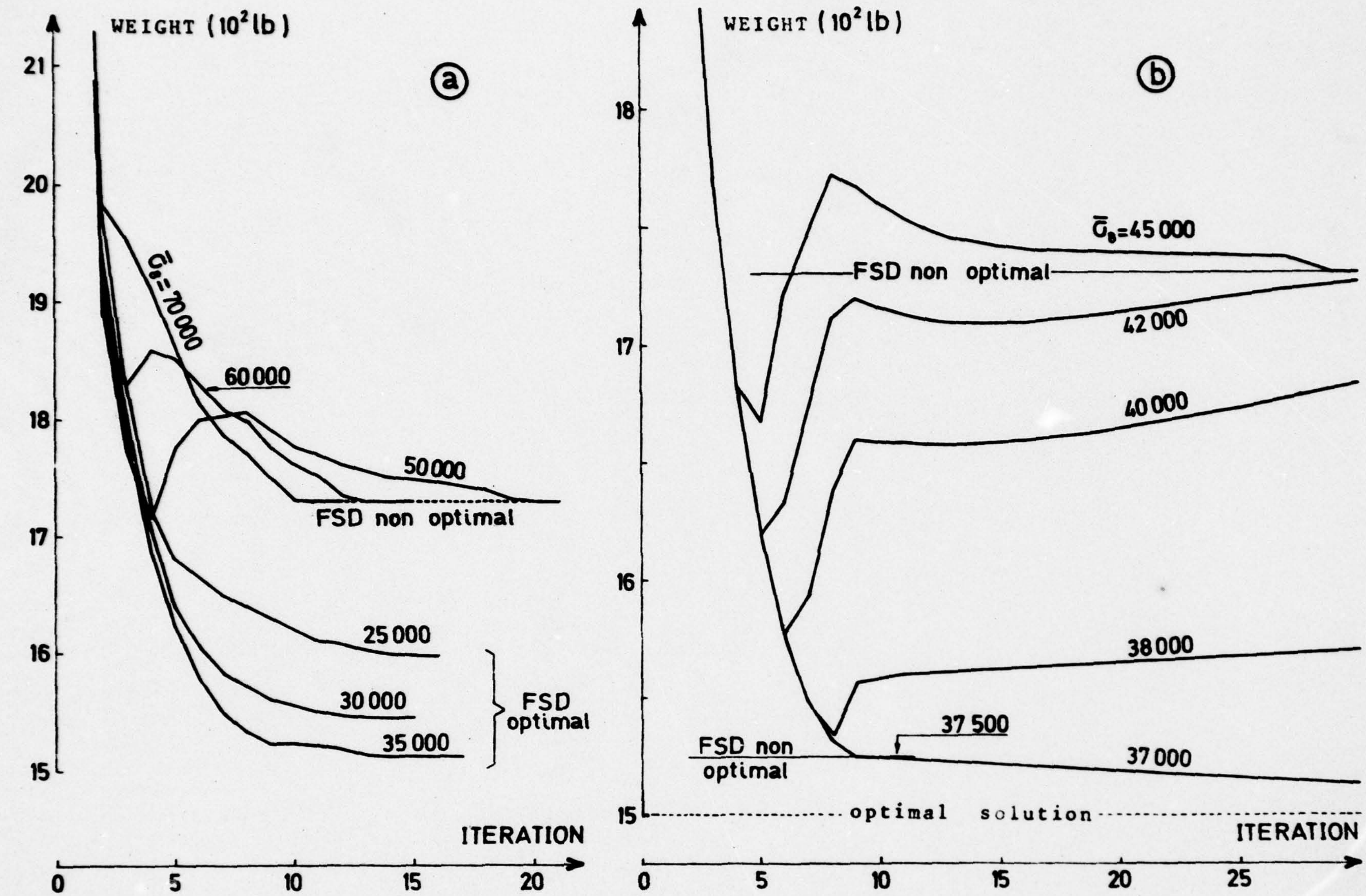


FIG. 4.6 : 10 BAR TRUSS - F.S.D. FOR VARIOUS LIMITS $\bar{\sigma}_8$

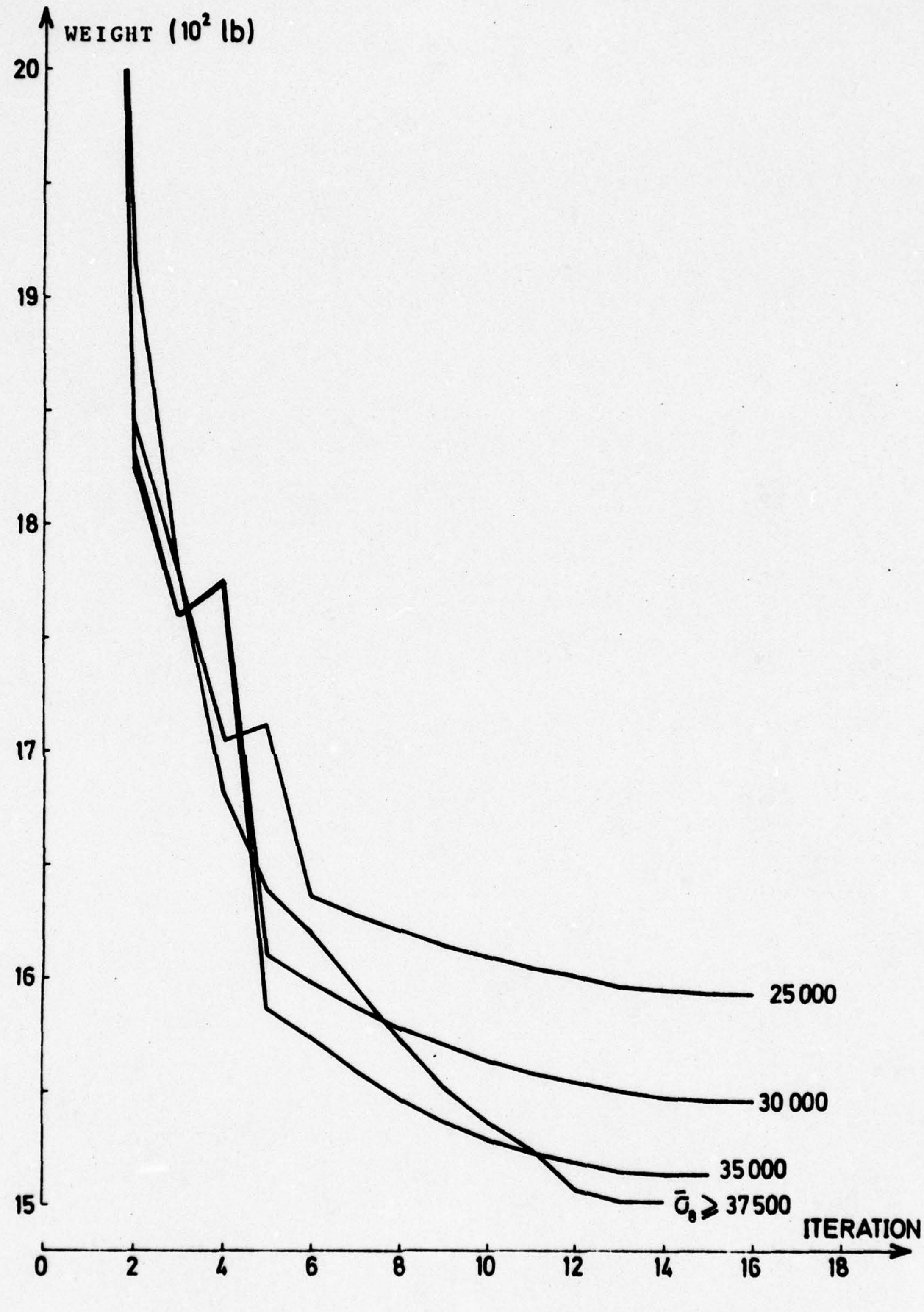


FIG. 4.7 : 10 BAR TRUSS - G.O.C. FOR VARIOUS $\bar{\sigma}_8$ LIMITS

10 BAR TRUSS WITH STRESS CONSTRAINTS

OPTIMUM SOLUTIONS

	$\bar{\sigma}_8 = 25000$		$\bar{\sigma}_8 = 30000$		$\bar{\sigma}_8 \geq 37500$	
bar	section	stress	section	stress	section	stress
1	8.0621	25000	8.0138	25000	7.9414	25000
2	3.9379	25000	3.9462	25000	3.9586	25000
3	0.1000	- 15533	0.1000	- 13462	0.1000	- 10355
4	0.1000	- 15533	0.1000	- 13462	0.1000	- 10355
5	7.9379	- 25000	7.9862	- 25000	8.0586	- 25000
6	0.1000	0	0.1000	- 10000	0.1000	- 25000
7	5.5690	25000	5.6373	25000	5.7397	25000
8	5.7447	- 25000	4.7304	- 30000	3.7160	- 37500
9	0.1000	21976	0.1000	19038	0.1000	14645
10	5.5690	- 25000	5.5807	- 25000	5.5983	- 25000
Weight	1593.18 (FSD)		1545.91 (FSD)		1500.82 (non FSD)	

TABLE 4.8

5. RELATIONS BETWEEN OPTIMALITY CRITERIA AND MATHEMATICAL PROGRAMMING APPROACHES

5.1. Interpretation of the generalized optimality criteria

In chapter 3, it has been established that consistent first order generalized optimality criteria (G.O.C.) could be defined by the application of the KUHN-TUCKER necessary conditions to an approximate problem, denoted problem A1 in (3.32), obtained from the exact problem R,by replacing the exact implicit constraints by their first order explicit approximations.
Mixing sero and first order approximations of the constraints leads to what has been named a hybrid optimality criterion (H.O.C.) corresponding to a problem A' defined in (4.1). The classical optimality criteria (C.O.C.) results from the introduction of additional approximations in the treatment of the constraints, in the selection of active and passive constraints and in the minimization algorithms themselves. These additional approximations were motivated by the desire to obtain simple redesign algorithms.

In chapter 4 it has been shown that the application of the dual methods furnishes an alternative way to obtain the solution of the problems A1 or A' which is rigorous and efficient.

The solution of the problems A1 or A' by primal methods of mathematical programming does appear to be rather inefficient due to the non linearity of the constraints even in their first order approximated form. However, as suggested by various authors [R4, J1, S8, F15] , the change of varibbles

$$x_i = \frac{1}{a_i} \tag{5.1}$$

makes the constraints very shallow. Indeed they are linear in the case of isostatic structures. The fact that the objective function $W(x_i)$ becomes non linear is not a handicap since it is known in an explicit form. In the new design space x_i, which is called the inverse design space, the approximate flexibility and stress constraints (3.32) are written

$$u_j = \sum_i C_{ij}\, x_i \leq \bar{u}_j \qquad j = 1, n_f \quad i = 1, n \tag{5.2}$$

$$\sigma_{k\ell} = \sum_i D_{ik\ell}\, x_i \leq \bar{\sigma}_k \qquad k = 1, n \quad \ell = 1, n_c \quad i = 1, n \tag{5.3}$$

The problem A1 is rewritten in the inverse design space and denoted problem L

problem

L

$$\min \quad W(x_i) = \sum_i \frac{\rho_i \ell_i}{x_i} \tag{5.4}$$

$$\text{with} \qquad \sum_i C_{ij}\, x_i \leq \bar{u}_j \tag{5.5}$$

$$\sum_i D_{ik\ell}\, x_i \leq \bar{\sigma}_k \tag{5.6}$$

$$x_i \geq \underline{x}_i \tag{5.7}$$

$$x_i \leq \bar{x}_i \tag{5.8}$$

When the first order approximations (5.6) of the stress constraints are totally or partially replaced by their zero order approximations $\tilde{x}_i$, derived from F.S.D., we denote

$$\bar{x}_i^* = \max \{ \bar{x}_i \, , \, \tilde{x}_i \} \tag{5.9}$$

The equivalent of the problem A' (4.1), which is denoted problem L' in the inverse design space, reads as follows problem

L'

$$\min \quad W(x_i) = \sum_i \frac{\rho_i \ell_i}{x_i} \tag{5.10}$$

$$\text{with } \sum_i C_{ij}\, x_i \leq \bar{u}_j \tag{5.11}$$

$$x_i \geq \underline{x}_i \tag{5.12}$$

$$x_i \leq \bar{x}_i^* \tag{5.13}$$

The filiation of the various problems and approximations is illustrated on figure 5.1. It is evident that an exact solution of the corresponding approximated problems L or A1 (or L' and A') must be identical if one recalls that these approximate problems are convex and hence, that there is no danger in converging to a local minimum.

This establishes the equivalence between a solution obtained by a G.O.C. and a solution obtained by any method solving exactly the corresponding problems A1 or L. The same conclusion holds between H.O.C. and problems A' or L'.

5.2. Approximation concepts in the mathematical programming approach

The most efficient optimization methods based on mathematical programming [SCHMIT, S18] use high quality explicit approximations of the constraints in a design space where they are as shallow as possible. In the present context, that is with stiffness matrices proportional to the design variables a_i, using the space of the inverse of the element sizes (5.1) is an obvious choice. In the spirit of the classical linearization methods, the constraints are developped in TAYLOR serie at the design point x_i^o where the structure has been analyzed

$$u_{j\ell} = u_{j\ell}^o + \sum_i \left(\frac{\partial u_{j\ell}}{\partial x_i}\right)^o (x_i - x_i^o) + O\left[\ (x_i - x_i^o)^2\ \right] \tag{5.14}$$

$$\sigma_{k\ell} = \sigma^o_{k\ell} + \sum_i \left(\frac{\partial\sigma_{k\ell}}{\partial x_i}\right)^o (x_i - x^o_i) + O\left[\,(x_i - x^o_i)^2\,\right]$$

It has been shown in (3.11) to (3.15) that the gradients of the constraints can be written as

$$\frac{\partial u_{j\ell}}{\partial a_i} = -\frac{C_{ij\ell}(a_i)}{a_i^2} \qquad \frac{\partial\sigma_{k\ell}}{\partial a_i} = -\frac{D_{ik\ell}(a_i)}{a_i^2} \tag{5.15}$$

with

$$C_{ij\ell} = (q^T_{ij}\, K_i\, q_{i\ell})a_i \qquad D_{ik\ell} = (q^T_{ik}\, K_i\, q_{i\ell})\, a_i$$

or in the inverse design space

$$\frac{\partial u_{j\ell}}{\partial x_i} = C_{ij\ell}\,(x_i) \qquad \frac{\partial\sigma_{k\ell}}{\partial x_i} = D_{ik\ell}(x_i) \tag{5.16}$$

Substitution in (5.14) and limiting the expressions to the terms linear in x_i yields

$$\tilde{u}_{j\ell} = u^o_{j\ell} - \sum_i C^o_{ij\ell}\, x^o_i + \sum_i C^o_{ij\ell}\, x_i$$

(5.17)

$$\tilde{\sigma}_{k\ell} = \sigma^o_{k\ell} - \sum_i D^o_{ik\ell}\, x^o_i + \sum_i D^o_{ik\ell}\, x_i$$

where the $\sim$ denotes the linearized form of the constraints. By definition of the coefficients $C_{ij\ell}$ and $D_{ik\ell}$, one has

$$u^o_{j\ell} = \sum_i C^o_{ij\ell}\, x^o_i \qquad \sigma^o_{k\ell} = \sum_i D^o_{ik\ell}\, x^o_i \tag{5.18}$$

and hence (5.17) reduces to

$$\tilde{u}_{j\ell} = \sum_i C^o_{ij\ell}\, x_i \qquad \tilde{\sigma}_{k\ell} = \sum_i D^o_{ik\ell}\, x_i \tag{5.19}$$

In classical linearization methods, the objective function as well as the constraints are linearized. However, as mentionned, the simple explicit form of the objective function makes its linearization useless. The linearized form of the original problem R appears to be

$$\left\{\begin{array}{ll} \min \quad W(x_i) = \dfrac{\rho_i \ell_i}{x_i} & (5.20) \\[2ex] \text{with } \tilde{u}_{j\ell} < \bar{u}_j & (5.21) \\[2ex] \qquad \tilde{\sigma}_{k\ell} < \bar{\sigma}_k & (5.22) \\[2ex] \qquad \underline{x}_i < x_i < \bar{x}_i & (5.23) \end{array}\right.$$

$$i=1,n \qquad j=1,n_f \qquad k=1,n \qquad \ell=1,n_c$$

which is identical to the definition of the problem L (5.4).

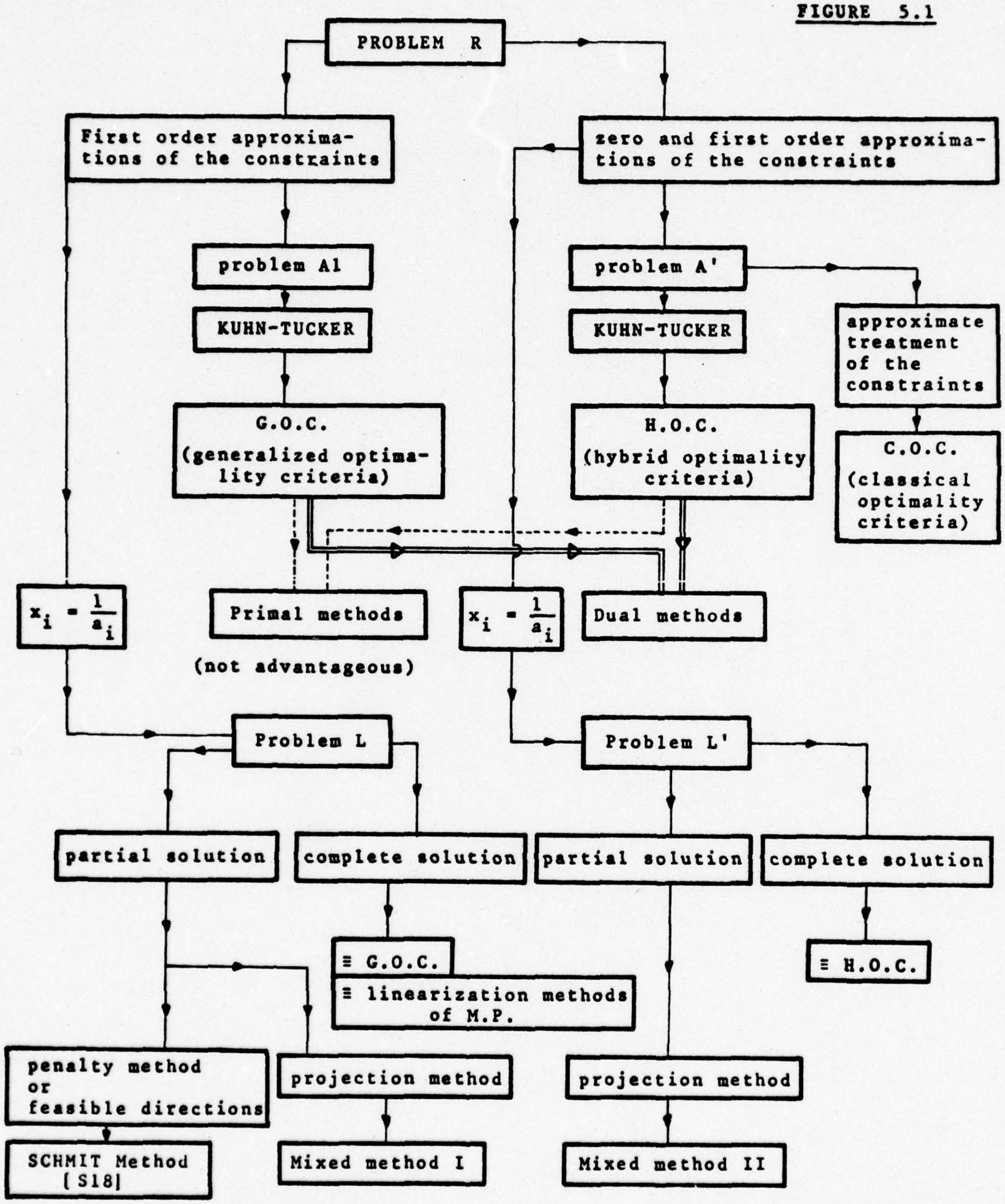
It is therefore evident that the solution of the problem R by a sequence of problems L can be interpreted equivalently as a G.O.C. approach or as a mathematical programming approach using a special form of the linearization method in the inverse design space.

It is well known in mathematical programming that linearization methods do not always converge and if they do, the convergence is not often monotonic. This is due to the fact that, at the optimum of the linearized problem, the constraints are violated to an extend that depends of their non linearity. This, in general, yields a sequence of non feasible design point. Eventually a restoration of a given point back to the boundary of the feasible domain could be achieved, by a scaling or any other means. In such an operation however,the weight might increase,which corresponds to starting an eventually diverging process. To avoid this difficulty a classical technique is to avoid to be driven too far away from the exact composite constraint surface. This is achieved easily by using a method that remains, at each step, inside or at the boundary of the linearized feasible domain. In such case, one simply stops the minimization after a limited number of steps, before reaching the minimum of the linearized problem. What is obviously a difficulty is to evaluate a priori the travel that can be achieved without violating "too much" the exact constraints.

Such a concept has been applied by SCHMIT [S18] with either an interior point penalty method or a feasible directions method. It will be applied in this work on the basis of the projection methods. The filiation of the various methods is illustrated in figure 5.1.

FILIATION OF THE PROBLEMS AND OF THE METHODS

FIGURE 5.1



6. PRIMAL PROJECTION ALGORITHMS FOR SOLVING THE PROBLEMS L OR L'

6.1. First order algorithm

The linear constraints (5.21) (5.22) are written in matrix form, using (5.19)

$$C^T x \geq - \bar{u} \qquad (6.1)$$

where C is an $(n \times m)$ matrix of constants $(- C^o_{ij\ell}$ and $- D^o_{ik\ell})$ evaluated at the design point for which the structure has been analyzed. Besides these m linear constraints, we have also 2n side constraints corresponding to (5.22) and (5.23). In a first order algorithm, the gradients of the objective function g_i and of the constraints are required. They are written

$$g_i = - \frac{\rho_i \ell_i}{x_i^2} \qquad i = 1, n \qquad (6.2)$$

while N denotes the matrix of the gradients of the active constraints. The method considered here is derived from the projected gradient method of ROSEN [R1] . Only the basic results contained in ROSEN's paper are recalled. The search direction is given by

$$z = - P\, g = - g + N\, r \qquad (6.3)$$

where P is the projection matrix

$$P = I - N\, (N^T N)^{-1}\, N^T \qquad (6.4)$$

and where r denotes the first order approximation of the lagrange multipliers

$$r = (N^T\, N)^{-1}\, N^T\, g \qquad (6.5)$$

This classical algorithm [F1] can not be applied in this form since the size of the matrix N can be as large as $(n \times n)$ which would require an unacceptable size of core storage in a computer program. It can be modified as follows.

Consider first the elimination of the side constraints (5.22), (5.23). The column of N corresponding to a given constraint $x_i = \underline{x}_i$ becoming active, contains only a factor +1. Let A be the matrix deduced from N by suppressing the line and column corresponding to the constraint and the variable x_i. The size of the projection matrix P reduces to (n-1) and we write

$$P_{n-1} = I_{n-1} - A\,(A^T A)^{-1} A^T \qquad (6.6)$$

To obtain more informations on the structure of the matrix P, let us rearrange the matrix N such that

$$N = \underbrace{\begin{bmatrix} A & 0 \\ & \\ B & I^* \end{bmatrix}}_{\displaystyle\underbrace{}_{\tilde m}\quad\underbrace{}_{\bar n}\;=\;q} \left.\begin{matrix} \}\ \tilde n \\ \\ \}\ \bar n \end{matrix}\right] n \qquad (6.7)$$

where $\tilde n$ denotes the number of free variables and $\bar n$ the number of fixed variables, that is, constrained to be constant during the iteration. The q active constraints are split in a group of $\tilde m$ principal (or linear) constraints and a group of $\bar n$ side constraints. The gradient of these latters are simply unit vectors (+1) or (-1) and the matrix I^* contains only (+1) or (-1) diagonal terms, depending whether the constraint is a lower or upper limit ($\underline{x}_i$ or $\bar x_i$). Using (6.7), it turns out that

$$(N^T N)^{-1} = \begin{bmatrix} (A^T A)^{-1} & -(A^T A)^{-1} B^{*T} \\ & \\ -B^*(A^T A)^{-1} & I_{\bar n} + B^*(A^T A)^{-1} B^{*T} \end{bmatrix} \qquad (6.8)$$

with $B^* = I^* B$, and finally

$$P = \begin{bmatrix} (I_{\tilde{n}} - A(A^TA)^{-1}A^T) & 0 \\ & \\ 0 & 0 \end{bmatrix} \tag{6.9}$$

depends only of the matrix A deduced from N by suppressing the lines and columns corresponding to the side constraints.

The projection relation (6.4) can now be transformed by using the same partitionning in the columns g and r

$$g = \begin{bmatrix} \tilde{g} \\ \\ \bar{g} \end{bmatrix} \begin{matrix} \} \tilde{n} \\ \\ \} \bar{n} \end{matrix} \qquad r = \begin{bmatrix} r_p \\ \\ r_s \end{bmatrix} \begin{matrix} \} \tilde{m} \\ \\ \} \bar{n} \end{matrix} \tag{6.10}$$

Using (6.8) yields

$$\begin{aligned} r_p &= (A^TA)^{-1} A^T \tilde{g} \\ r_s &= I^* \bar{g} - B^*(A^TA)^{-1} A^T\tilde{g} = I^*(\bar{g} - B\, r_p) \end{aligned} \tag{6.11}$$

and the search direction becomes

$$\begin{aligned} \tilde{z} &= - \tilde{g} + A\, r_p \\ \bar{z} &= 0 \end{aligned} \tag{6.12}$$

These modifications reduce sigaificantly the size of the core storage necessary to apply efficiently the algorithm,since the matrices required are only

$$\begin{aligned} &C && (n \times m) \\ &(A^TA) && (m \times m) \end{aligned} \tag{6.13}$$

where m is the number of linear (first approximated) constraints.

6.2. Second order algorithm

As the objective function is separable, a second order algorithm is well adapted. Detailed formulations of such algorithms can be found in [G3, G5, F9] . The Hessian matrix F is diagonal and its terms are given by

$$F_{ij} = \frac{\partial^2 W}{\partial x_i \partial x_j} = \frac{2\rho_i \ell_i}{x_i^3} \delta_{ij} \tag{6.14}$$

where δ_{ij} is the Kronecker symbol. Its inverse is therefore trivial and the generalized NEWTON method with oblique projections can be used [G3] . The second order approximation of the lagrange multipliers turns out to be

$$r = (N^T F^{-1} N)^{-1} N^T F^{-1} g \tag{6.15}$$

and the NEWTON search direction is

$$z = - \hat{P} F^{-1} g = F^{-1} (-g + Nr) \tag{6.16}$$

where $\hat{P}$ is the projection operator

$$\hat{P} = I - F^{-1} N (N^T F^{-1} N)^{-1} N^T \tag{6.17}$$

The computation of these general expressions is significantly facilitated if the same partitioning is used, as introduced in (6.7), not only for the matrix N of the gradient of the constraints, but also for the diagonal Hessian matrix F so that

$$F^{-1} = \begin{bmatrix} \tilde{F}^{-1} & 0 \\ 0 & \bar{F}^{-1} \end{bmatrix}$$

The projection matrix $\hat{P}$ turns out to be

$$\hat{P} = \begin{bmatrix} (I_{\tilde{n}} - \tilde{F}^{-1} A (A^T \tilde{F}^{-1} A)^{-1} A^T) & 0 \\ 0 & 0 \end{bmatrix} \qquad (6.18)$$

and depends only of the $\tilde{n}$ variables x_i that have not reached the upper or lower limits, $\bar{x}_i$ or $\underline{x}_i$. The lagrange multipliers take the simplified form

$$r_p = (A^T \tilde{F}^{-1} A)^{-1} A^T \tilde{F}^{-1} \tilde{g}$$
$$r_s = I^* (\bar{g} - B\, r_p) \qquad (6.19)$$

while the search direction becomes

$$\tilde{z} = \tilde{F}^{-1} (- \tilde{g} + A\, r_p)$$
$$\bar{z} = 0 \qquad (6.20)$$

This form requires a reduced amount of storage,that is given by (6.13) plus the array necessary to store the diagonal Hessian matrix. The convergence properties of a second order algorithm are expected to be significantly better than the first order algorithm, even when the comparison is made in terms of computing time rather than in number of steps. Examples are given in the following.

6.3. Auxiliary algorithms

A certain number of auxiliary algorithms are necessary to use either one of the basic algorithms described above.

Conjugate search directions

The basic first order algorithm of ROSEN is significantly improved if the search directions z are selected according to the concept of conjugate directions as introduced by FLETCHER [F4] and

MIELE [M3] . The algorithm is initiated by the simple projected gradient

$$z_o = - P_o g_o$$

The subsequent directions are extracted from

$$z_k = - P_k g_k + \beta_k z_{k-1} \tag{6.21}$$

with

$$\beta_k = \frac{||P_k g_k||^2}{||P_{k-1} g_{k-1}||^2}$$

The conjugation algorithm must be reinitiated after (n-q+1) iterations.

Restoration

At any time in the minimization, the current design point has to remain on the composite constraint surface.
A restoration is necessary due to the numerical round-off errors. In practice, it is applied each time the set of active constraints is changed, and each time the conjugation algorithm is reinitiated. The restoration consists in progressing along a direction that is normal to the plane tangent to the constraints at the current point. If $\tilde{x}_o$ denotes a point which suffers from an error, the corrected position $\tilde{x}_1$ is given by

$$\tilde{x}_1 = \tilde{x}_o - A(A^T A)^{-1} (A^T \tilde{x}_o + B^T \bar{x} + \bar{u}) \tag{6.22}$$

for the first order algorithm, and by

$$\tilde{x}_1 = \tilde{x}_o - \tilde{F}^{-1} A(A^T \tilde{F}^{-1} A)^{-1} (A^T \tilde{x}_o + B^T \bar{x} + \bar{u}) \tag{6.23}$$

for the NEWTON's algorithm . It is evident that the fixed variables $\bar{x}_o$ do not have to be restored.

Linear search algorithm

With the first order algorithm, a cubic interpolation method has been used [D1, F21] while in the second order algorithm , advantage has been taken of the fact that, in such methods, the optimal step is close to unity in the neighborhood of a minimum. The problem can be presented as finding the minimum of the function of the step length τ

$$f(\tau) = z^T \nabla W(\tau) = - \sum_i \frac{z_i \rho_i \ell_i}{(x_i - \tau z_i)^2} \qquad (6.24)$$

The TCHEBYCHEV method is used with the initial approximations $\tau_o = 1$, which would be exact if the objective function was quadratic. The first and second derivatives of (6.24) are easily obtained. This method is not applied in the projected gradient method because the initial choice of the step length τ_o reveals rather critical and difficult to obtain.

Strategy for selecting the active constraints

Some care has to be exercized in the selection of the active constraints to avoid the zigzaging behavior that is characteristic of projection methods. The approach used follows the suggestion of ROSEN [R1] . It consists in removing a constraint from the active set only if it leads to a sufficient gain in the decrease of the objective function. By the notion of marginal cost, the variation of the objective function can be deduced from the change in the value of the lagrange multiplier when the constraint becomes inactive [F10, G5] . The abandonment of the j^{th} constraint leads to a gain of the order of

$$\frac{1}{2} \frac{r_j^2}{v_j} \qquad (6.25)$$

where v_j is the j^{th} diagonal term of $(N^T N)^{-1}$ or $(N^T F^{-1} N)^{-1}$. Hence the strategy consists in determining among the constraints associated with negative r_j components, the one for which the gain obtained when becoming inactive exceeds γ times the decrease of the objective function that would be otherwise obtained. For a given constraint, it is explicitly written

$$z^T g = z^T F z < \frac{1}{\gamma} \frac{r_j^2}{2v_j} \tag{6.26}$$

γ being a positive constant. For first order algorithm the Hessian matrix is replaced by the unit matrix. The validity of this test depends of the quality of the approximation of lagrange multipliers. Therefore it is applied only if

$$z^T g < \varepsilon \tag{6.27}$$

where ε is a given tolerance. This insures that the accuracy on the multipliers is sufficient as (6.27) can only be verified close to stationary points.

6.4. Comparison of the various algorithms

First and second order projection algorithms have been considered for the solution of the problem L in the inverse design space x_i. In the original design space a_i, dual algorithms of the first and second order have been established for the problem Al in chapter 4. The solutions obtained by the various algorithms must coïncide as the linearized problem L is convex and therefore has a unique solution. Two special examples have been used to illustrate the relative performances of the 4 algorithms in competition. The comparison is made on the CPU time for programs written by the same author and with the same degree of generality. This insures that all the aspects of the computation are taken into account in the comparisons.

First the following explicit problem has been solved

$$\min \quad W = \sum_{1}^{1000} a_i$$

$$\text{with } u_1 = \sum_{1}^{950} \frac{1}{a_i} + 10^{-6} \sum_{951}^{1000} \frac{1}{a_i} \leq 1000 \qquad (6.28)$$

$$u_2 = \sum_{1}^{950} \frac{1}{a_i} - 10^{-6} \sum_{951}^{1000} \frac{1}{a_i} \leq 900$$

$$10^{-6} \leq a_i \leq 10^{+6}$$

The analytical solution of this problem is easy to find as

$$a_i^* = 1 \qquad \text{for} \qquad i = 1{,}950$$

$$a_i^* = 10^{-6} \qquad \text{for} \qquad i = 951{,}1000$$

and corresponds to a weight $W^* = 950.00005$.
The convergence of the 4 algorithms is illustrated on figure 6.1. The primal algorithms use of course the change of variables $x_i = 1/a_i$. The second order primal algorithm converges much more rapidly than the first order one which was stopped after 394 iterations. In general, it was observed that the first order algorithm requires a number of steps of the order of a multiple of the number of the design variables, while the second order algorithm converges in a number of steps inferior to the number of design variables.

Turning to the dual algorithms, the first observation is that both are much more efficient than the primal ones. This conclusion holds usually when

$$n >> m \qquad (6.29)$$

(here $n = 1000$, $m = 2$). The advantage of using the second order algorithm is still obvious but less significant (6 iterations instead of 3).

The test seems to be representative of the problems met in structural synthesis. The coefficients c_{ij} of the constraints are not all positive. As a consequence some of the design variables reach their lower limit. This characteristic reduces the speed of convergence of the primal algorithms, but not of the dual ones.

The 36 bar isostatic truss of figure 6.2 has been used as a second test. A detailed definition of the problem is given in table 6.3. 15 flexibility constraints are imposed

$$| w_j - w_k | < \varepsilon \qquad j = 1, 4 \quad k = 2, 5 \qquad (6.30)$$

and

$$w_j < \bar{w} \qquad j = 1, 5$$

among which only 5 are independent.
It is worth pointing out that the classical optimality criteria [B10, G13, T3] do not yield the optimum of this problem. As the structure is isostatic, only one analysis is sufficient.
The comparison of the performances of the 4 primal and dual algorithms is illustrated on figure 6.4. The test is different from the preceding one, by the fact that the number of constraints is of the order of that of design variables.
Even in this case the dual methods reveal competitive with the primal ones.

The conclusion can be drawn that the dual algorithms are to be recommended in general when the number of variables is larger than the number of constraints. With primal algorithms, the second order brings a significant advantage, while with dual algorithms the benefit is counterbalanced by the increased complexity of the computation. These conclusions have been verified on a large number of applications.

A significant difference still exists between primal and dual algorithms, which will be used at the advantage of the primal ones in the next chapter dealing with mixed methods. In the primal algorithms the design obtained after each iteration is a feasible design, with respect to the approximate constraints, and the weight is smaller than for the preceding iteration. It is therefore possible, and sometimes advantageous, to stop the solution of the problem L (or L') before reaching the minimum. With dual methods the design points obtained at each iteration are not feasible points and the corresponding weight is not, in general, decreasing, as it is the dual objective function which is maximized. Using the dual algorithms implies therefore to solve completely the problem A1 (or L) or the problem A' (or L').

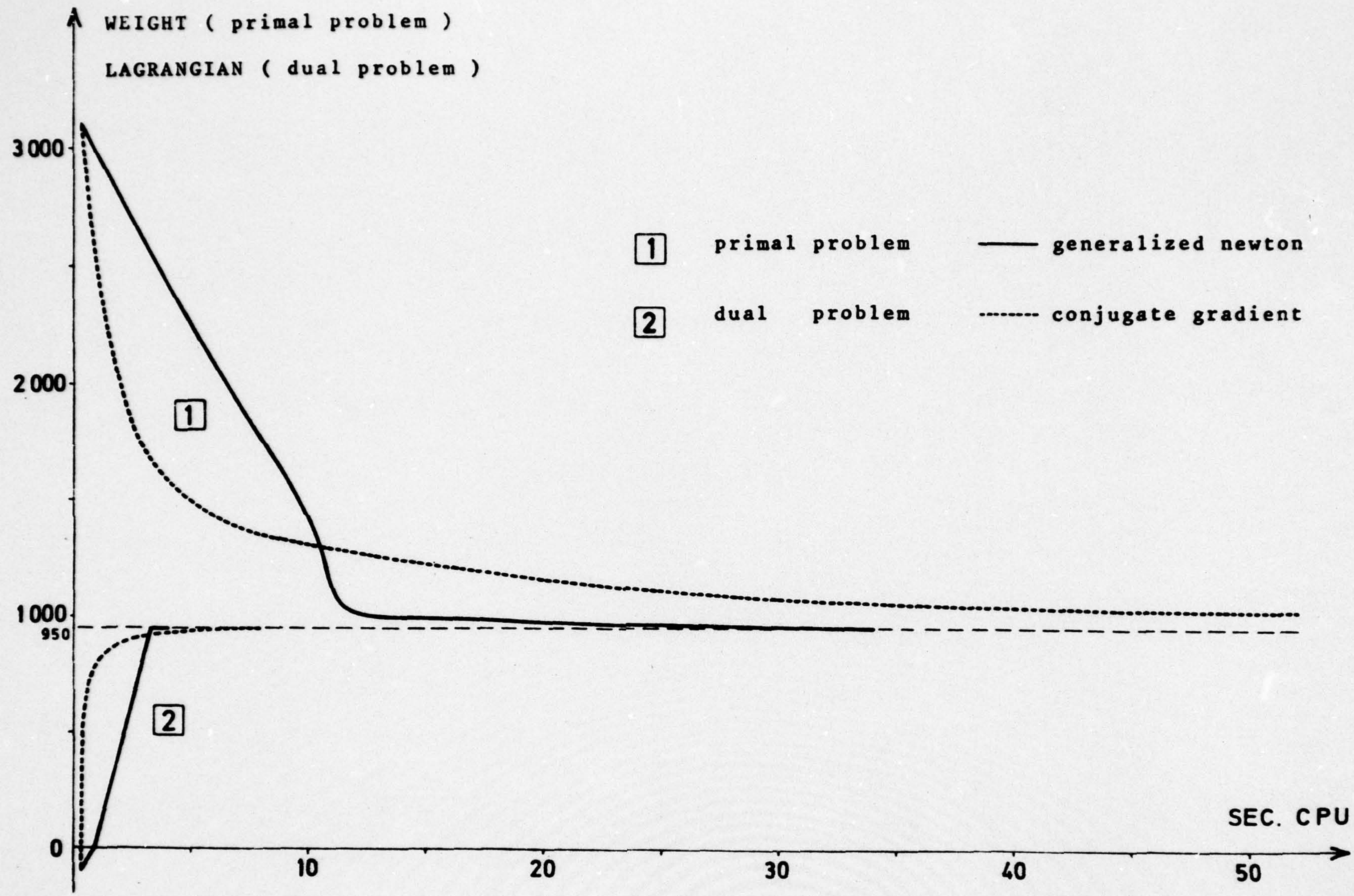


FIGURE 6.1 : EXPLICIT PROBLEM

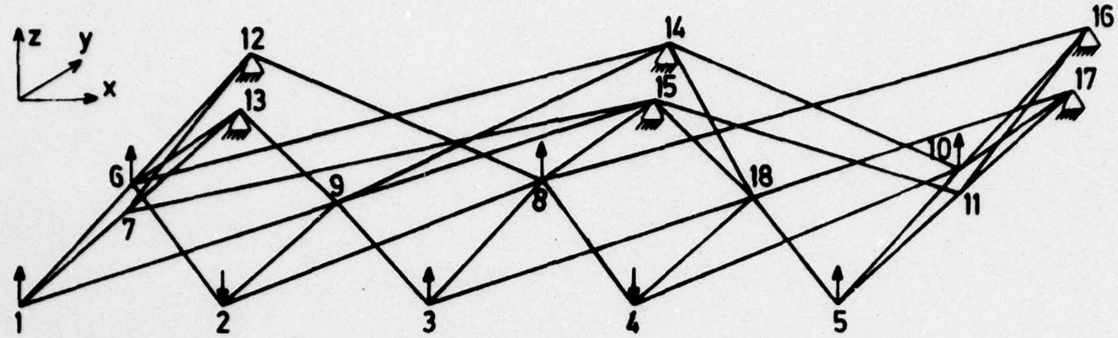


FIGURE 6.2 36 BAR ISOSTATIC TRUSS

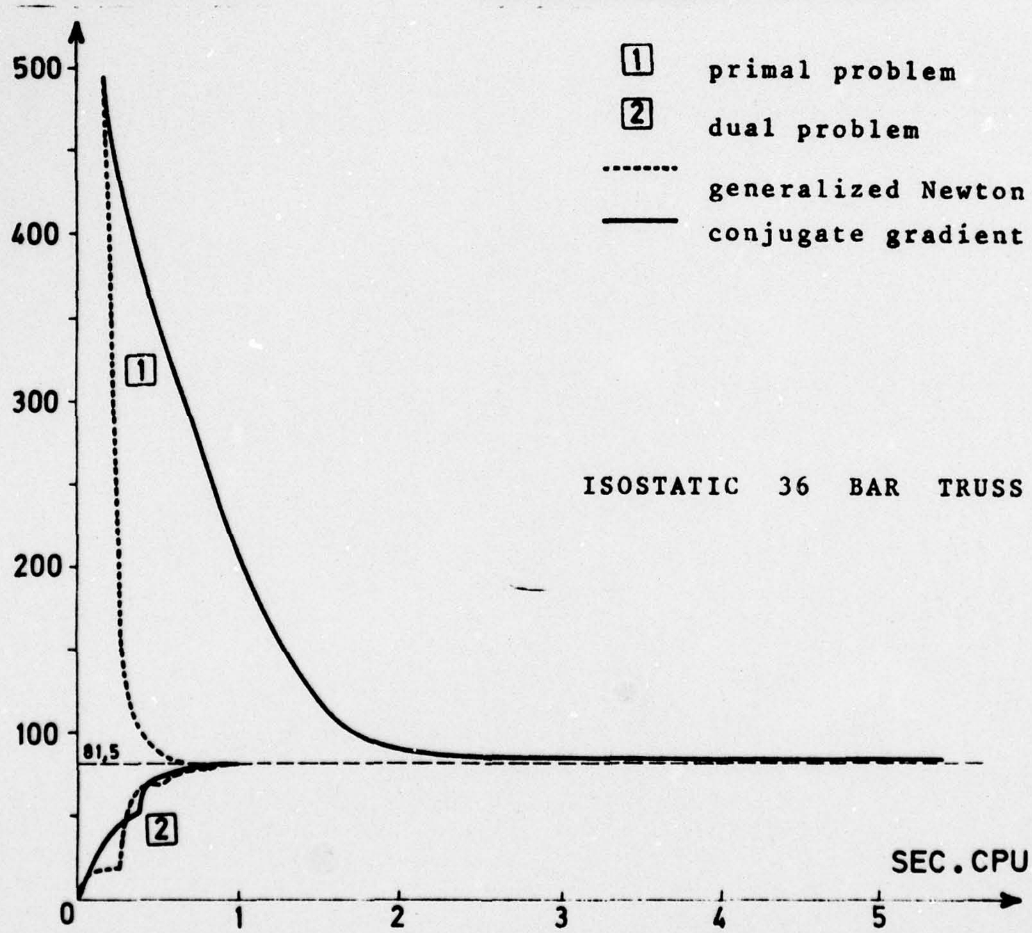


FIGURE 6.4

36 BAR ISOSTATIC TRUSS

aluminium

E = 7200 hb

ρ = 0.28 10^{-5} kg/mm^3

$\underline{a}$ = 0.5 mm^2

$\bar{\sigma}$ = 8 hb

$\bar{u}_z$ = 10 mm (at nodes 1, 2, 3, 4, 5)

ε = 0.5 mm

Node coordinates

Node	X	Y	Z	Node	X	Y	Z
1	25.00	41.20	0.00	10	1259.00	266.41	0.00
2	330.75	31.18	0.00	11	1259.00	264.43	- 32.37
3	636.50	21.18	0.00	12	25.00	548.00	0.00
4	947.75	11.00	0.00	13	25.00	516.01	- 58.30
5	1259.00	0.82	0.00	14	636.50	548.00	0.00
6	25.00	286.59	0.00	15	636.50	514.75	- 60.60
7	25.00	284.76	- 29.91	16	1259.00	548.00	0.00
8	636.50	276.59	0.00	17	1259.00	513.47	- 63.95
9	330.75	279.72	- 30.52	18	947.75	269.56	- 31.75

Loading

Node	Direction	Load (daN)	Node	Direction	Load (daN)
1	Z	10	5	Z	10
2	Z	- 20	6	Z	40
3	Z	30	8	Z	90
4	Z	- 20	10	Z	40

TABLE 6.3

7. MIXED METHODS

7.1. A projection method for the problem R

In chapter 5, it has been shown that the problem R can be replaced by a sequence of problems L. It was justified as a special form of the linearization methods. Such methods are successful in that case, due to the fact that the constraints are near to linear in the inverse design space. On the other hand the sequence of problems L corresponds to the application of a generalized optimality criterion. An equivalent solution is provided by the dual formulation of the solution of a sequence of problems A1.

Considering the quasi linear characteristic of the exact constraints in the inverse design space, one could contemplate a strict application of the projection methods directly to the problem R. Such methods have the fundamental advantage to lead to an, at least local, minimum of the exact problem R without any convergence problem, which is not always the case with linearization methods. Another advantage of the projection methods is that they generate a seqqence of admissible design points corresponding to a sequence of decreasing values of the objective function.

The application of the projection methods, like that of ROSEN [R2] , to the problem R (instead of the problem L considered in chapter 6) requires the following steps.

- At a given admissible design point, analyse the structure and compute the gradients of the objective function and of the constraints;
- Determine a search direction by a projection of the gradient of the objective function on the plane tangent to the active constraints;
- Determine a step length such that the function is minimized, but without violating the exact constraints, which requires a certain number of structural reanalyses;

- As the constraints are non linear, a restoration step has to be applied to bring the design point back on the composite constraint surface, which is a necessary condition for applying the projection method. This step requires also to reanalyse the structure in order to have an accurate estimation of the constraints;

- After restoration the design point is admissible and the process is repeated.

It is well known,and it has been illustrated in the applications of the projection primal methods to the problem L, that the number of steps increases in proportion to the number of design variables. As each iteration requires a certain number of structural analysis, the application of such methods can not be considered as such for large problems.
Two actions can be taken to reduce the cost of computation

- First, reduce the number of minimization steps;

- Second, reduce the number of structural analyses per step or for a number of steps.

The concepts used in establishing the projection algorithms for the problem L can obviously be applied here to the problem R. First,using the inverse design space insures to have nearly linear constraints, thus allowing larger steps. Second,the explicit first order approximations of the constraints can be used to compute the search directions and the step lengths.
Third,the restoration can be achieved by a simple scaling of the design. Such a procedure does not require a reanalysis and it has been shown in section 3.2 that it preserves the value of the constraints and of their gradients

$$\tilde{u}_j = u_j \qquad \frac{\partial \tilde{u}_j}{\partial x_i} = \frac{\partial u_j}{\partial x_i} \qquad \text{along } D(x_i^o) \quad (7.1)$$

Consequently the analysis after scaling is the only one which is necessary to reinitiate the process.

The algorithm can be summarized as follows

- Analyse the structure at an admissible point x_i^o;
- Compute the explicit first approximations of the constraints by using (3.9) and (3.20), that is, in the x_i space, linearize the constraints in the form (5.2), (5.3)

$$\tilde{u}_j = \sum_i C_{ij}^o \; x_i$$
$$\tilde{\sigma}_{k\ell} = \sum_i D_{ik\ell}^o \; x_i \qquad (7.2)$$

- Determine the search direction by an adequate projection of the gradient in the plane tangent to the active constraints and compute the step length. These operations are performed by using the linearized expressions (7.2) of the constraints and thus without reanalyzing the structure. It is expected that such linearized constraints do not lead to a significant error in the step length so that the algorithm still works normally;
- Restore the design on the exact constraint surface by a simple scaling by a factor f. It produces a new admissible design point

$$x_i^1 = \frac{x_i^o}{f} \qquad (7.3)$$

at which an analysis is achieved and the procedure repeated. The scaling factor is, according to (3.2)

$$f = \max\,(f_u \;, \; f_\sigma)$$

with (7.4)

$$f_u = \max_{j,\ell} \left(\frac{u_{j\ell}^o}{\bar{u}_j}\right) \qquad f_\sigma = \max_{k,\ell} \left(\frac{\sigma_{k\ell}^o}{\bar{\sigma}_k}\right)$$

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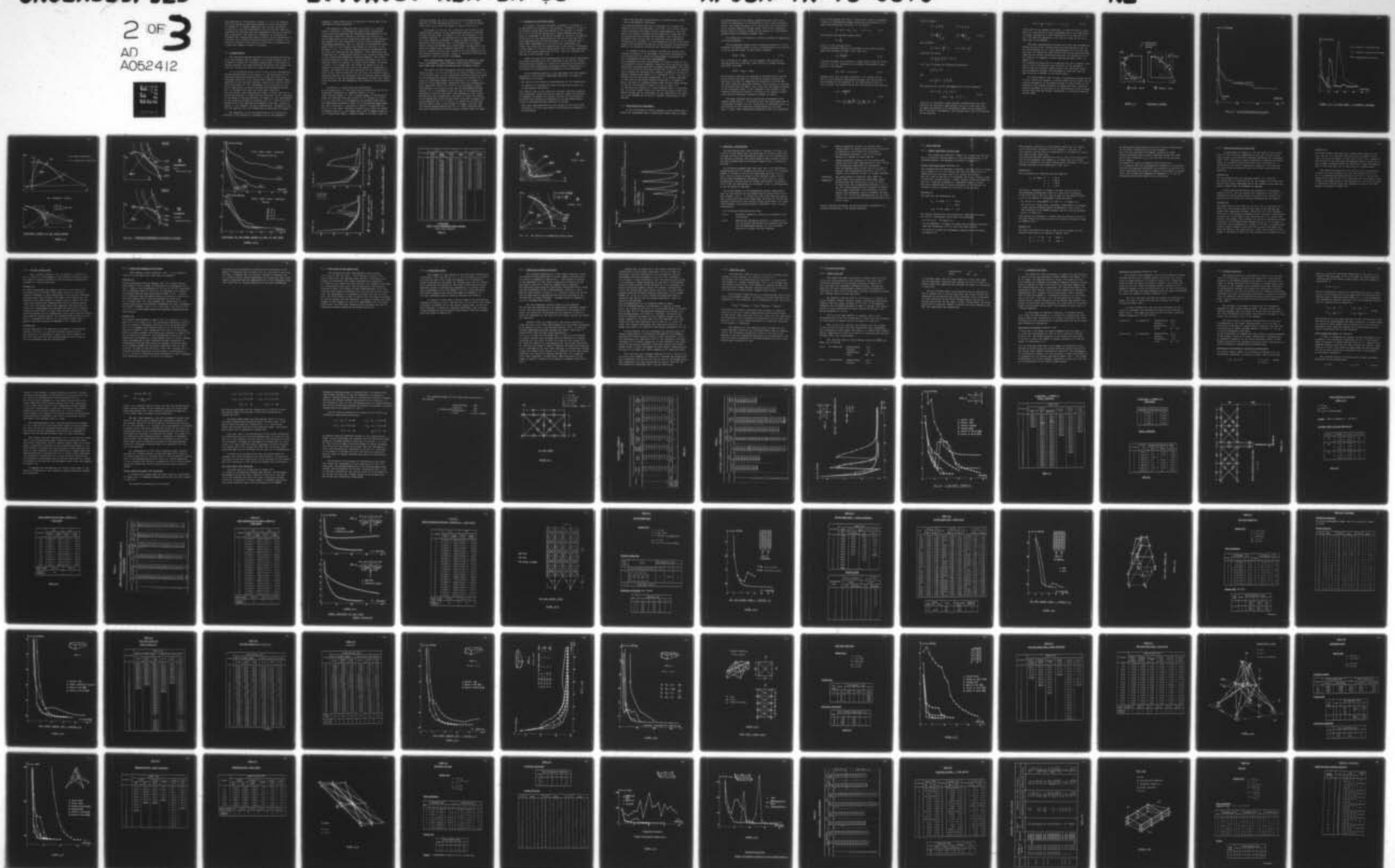


The algorithm is illustrated in figure 7.1. If it is compared with the projection algorithms proposed to solve the problem L in chapter 6, it appears immediatly that the only difference is that here, only one minimization step is achieved before reanalyzing the structure and thus redefining a new set of approximate constraints. This essential difference is sufficient to transform the linearization method, which is never guarantied to converge, into a primal projection method which is known to minimize the weight at each iteration and yield to convergence, if a sufficient number of steps is achieved.

7.2. A mixed method

The projection method outlined in the preceding section has still the property that is common to all primal methods, that is to require a number of iterations proportional to the number of design variables. The simplifications introduced have reduced the cost per iteration but have not changed this basic property.

It is illustrated on the 10 bar truss, in the configuration defined geometrically by the figure 4.5. The application of the projection algorithms described above, using the first order (conjugate gradient) or the second order (NEWTON's method) approximations of the objective function, leads to the results illustrated by figure 7.2. The convergence is characteristic of the strict applications of primal mathematical programming methods, that is extremely slow with the first order algorithm.
With the second order algorithm the number of steps, and thus of structural analyses, is significantly reduced, but still a multiple of the number of design variables. In figure 7.3, the performances of two classical optimality criteria [G13, T3] and of the consistent first order generalized optimality criterion, defined in chapter 3, are compared. The most sophisticated optimality criterion was applied via a dual formulation as described in chapter 4.

The comparison of the two applications of the projection methods, to the sequence of problems L and to the problem R, suggests a mixed method which is expected to retain some of the advantages of both approaches.

The simple, but essential, idea is to allow for a certain number, denoted $\bar{k}$, of minimization steps before reanalyzing the structure and updating the linearized approximations of the constraints. When $\bar{k}$ is limited to one step, the method is a strict application of the projection algorithms to the problem R and presents all the properties of strict primal mathematical programming approaches, that is, high cost but guarantied convergence. When $\bar{k}$ is not limited, the problem is identical to the problem L (5.4). It is solved completely before reanalyzing the structure and defining the next problem L. This method has been interpreted as a linearization method in the inverse design space, or as a generalized optimality criterion approach. The corresponding properties of fast, but uncertain convergence are to be expected. When $\bar{k}$ is limited to a given finite number, the method is mixed. It is evident that the larger the number $\bar{k}$ is, the cheaper the application will be, as the number of reanalyses is reduced. The reason of the high efficiency of the optimality criteria is now clearly related to the fact that the explicit approximations made for the constraints are not updated before reaching the solution of the corresponding approximate problem. This constitutes the power and the weakness of the optimality criteria approaches.

The figure 7.4 illustrates the mixed method.
At the design point x^o, the structure is analyzed and the set of linearized constraints is computed (figure 7.4 (a)).
After scaling (figure 7.4 (b)) a certain number of minimization steps are achieved along the surface $\tilde{u} = \bar{u}$, leading to a point x_1 or x_1'. The point x_1 corresponds to the minimum weight under the approximate constraint, while x^* is the minimum weight under the exact constraint. A scaling by a factor f brings back the point x_1 on the exact constraint surface in x_2, where the weight is smaller than in x_o, but larger than in x^*. A smaller step, or in a n-dimensional space, a smaller number $\bar{k}$ of steps, could be

selected leading, say in x_1', which is not the minimum weight under $\tilde{u} = \bar{u}$. The scaling by the factor $f' < f$ yields the admissible point x_2' at which the weight is smaller than in x_2.

The figure 7.5 (a) and (b) illustrates the reason why the linearization methods (or the generalized optimality criteria) diverge in some cases. In figure 7.5 (a) the constraint $u = \bar{u}$ is rather shallow. It is characteristic of weak hyperstaticity as, in an isostatic structure, it would be exactly linear. The solution of the problem L (that is $\bar{k}$ unlimited in the mixed method) yields the minimum at x_1 which after scaling comes in x_2 where the weight $W(x_2)$ is smaller than $W(x_o)$. In figure 7.5 (b) the non linearity of the exact constraint is more pronounced and the process diverges as $W(x_2) > W(x_o)$. A limitation of $\bar{k}$ would be necessary to keep the process converging.

This interpretation allows to consider the number of steps $\bar{k}$ as a convergence control parameter that should be assigned high values for economy and reduced when divergence occurs.

The effect of the $\bar{k}$ convergence control parameter is illustrated in Fig. 7.6 a & b for the 10 bar truss defined in Fig. 4.5. It is recalled that all the constraints, including therefore those on the stresses, are exactly linearized in this problem. The increase in convergence speed for an increase of the number of steps $\bar{k}$ is very significant, especially using the conjugate gradient algorithm. For $\bar{k} > 50$ (or $\bar{k} > 4$ in the NEWTON's method) the problems L are exactly solved and the performances become similar, as it corresponds, in fact, to that of the generalized optimality criterion, which has been presented in the chapter dealing with dual formulations. The table 7.7 gives the values of the weight and of the scaling factor using the conjugate gradient algorithm for $\bar{k} = 1, 5$ and 10. It shows that the scaling factors remain closer to unity when $\bar{k}$ is small, and also that the scaling remains moderate for $\bar{k} = 10$, which indicates that the exact constraints are not seriously violated after 10 steps.

7.3. Extension of the mixed method

In chapter 5 we have considered a sequence of problems L' to approximate the problem R. The problem L' defined in (5.10) in the inverse design space, corresponds to the problem A' defined in (4.1) in the direct design space. These problems have been shown to correspond to the hybrid optimality criteria, introduced in section 3.6, and characterized by the use of zero and first order approximations of the constraints. It is recalled that a first order approximation of all the stress constraints can reveal very costly for the analysis, due to the need to incorporate a large number of virtual loading cases. The zero order approximation of the stress constraints by F.S.D. does not introduce any increase in the analysis cost.

Indeed the zero order approximation for the stress constraints can reveal satisfactory if the conditions (3.38) are met. These conditions were proposed as a basis for selecting the potentially critical stress constraints requiring an exact linearization in the context of the generalized optimality criteria approach.

In the preceding section, it has been shown that the possible divergence of the various algorithms considered is related to essentially two factors

- the quality of the explicit approximations of the constraints;
- the number $\bar{k}$ of minimization steps achieved with a given set of approximate constraints.

This implies that, in the search for the most efficient minimization algorithms, these two factors should be taken into account. It suggests to consider the partial solution of the problem L' using the same concepts as developed in the preceding section. Doing so the method becomes mixed in two senses

- zero order approximations are used instead of the first order required by a strict application of the projection methods to the problem R;

- more than one step of minimization is achieved with a given set of approximate constraints.

The benefit expected from such a technique is to reduce the analysis cost by avoiding many, if not all, the virtual loading cases associated with higher quality approximations. The drawback is a possible reduction of the number $\bar{k}$ of steps that can be allowed and still avoids the divergence. The situation is evidently less critical than in using the various optimality criteria as the problem L' is not necessarily solved in the mixed method. The relations between the various methods (linearization, mixed or by optimality criteria) are illustrated in figure 5.1.

An example is given in figure 7.8 where the 10 bar truss defined in figure 4.5 is used again. The stress constraints are all treated by F.S.D. while the displacement constraints are still necessarily first order approximated.
The convergence has been investigated for various values of the $\bar{k}$ convergence control parameter, using the first and second order algorithms (conjugate gradient and generalized NEWTON). The effect of $\bar{k}$ is striking. For $\bar{k}$ unlimited ($\bar{k}_1 > 100$ or $\bar{k}_2 > 5$) the instability observed when applying the classical optimality criteria reappears (compare with figure 7.3). Limiting $\bar{k}$ to $\bar{k}_1 = 40$ or $\bar{k}_2 = 2$ brings back the monotonous convergence. It is however important to note that the generated design corresponds to a weight of 5076.7 lbs which is slightly heavier than the best known optimum of 5060.85 lbs obtained in figure 7.2 and in figure 7.6.
It indicates the presence of a local minimum close to the global one. As is well known, there is no way of insuring to reach the global minimum and the convergence of one algorithm or the other to a local minimum can be due to a specific peculiarity of the problem.

7.4. Controlling the convergence

If the $\bar{k}$ convergence control parameter allows effectively to insure the convergence when a sufficiently small value is taken, the determination of its highest admissible value, that still yields to convergence, is not possible a priori. When divergence occurs, the normal procedure is to restart the last minimization, with a reduced number of allowed steps. This is an a posteriori measure, which is closely related to the concept of "intermediate design vectors" introduced by KHOT [K10] .

The limitation of $\bar{k}$ would be a better procedure if applied as an a priori measure.

The intermediate design vector technique proposed in [K10] considers two design points a_o and a_1 corresponding to the results at two successive steps. If the process diverges

$$W(a_1) > W(a_o) \qquad (7.5)$$

The illustration of figure 7.8 (a) suggests that between the scaling lines $D(a_o)$ and $D(a_1)$ there exists an optimal scaling line $D(a^*)$ such that

$$W(a^*) < W(a_o) < W(a_1) \qquad (7.6)$$

One such point, denoted $\hat{a}$, is obtained by determining the intermediate scaling line $D(\hat{a})$ as a normalized average between $D(a_o)$ and $D(a_1)$, and next by computing the intersection of $D(\hat{a})$ with the exact composite constraint surface. This procedure is applied in the direct space by KHOT [K10] each time the weight increases in two successive steps. It was proposed in the context of the classical optimality criteria. It relies on the implicit assumptions that the composite constraint surface is convex.

The same technique can be applied with advantage in the inverse design space, in conjunction with the mixed method and is illustrated in figure 7.8 (b). Starting at an admissible point x_o, the solution of the linearized problem yields x_1 such that $W(x_1) < W(x_o)$. After scaling the admissible point x_2 is generated and it could happen that $W(x_2) > W(x_o)$ which leads to divergence. A family of intermediate design vectors can be generated in the space containing $D(x_o)$ and $D(x_1)$. It is written

$$\tilde{x} = x_o + \tau\,(x_1 - x_o) \qquad 0 < \tau < 1 \qquad (7.7)$$

and produces the restored design points

$$\hat{x} = \frac{\tilde{x}}{f}$$

where f is the scaling factor.
A simple procedure would be to transpose in the inverse design space the procedure of KHOT [K10] and write

$$\tilde{x} = \frac{x_o + x_1}{2} \qquad (7.8)$$

A better procedure is to achieve a linear search along the direction $(x_1 - x_o)$ for determining the optimum intermediate design vector. It is written

$$\min \quad \hat{W}(\tau) = f(\tau)\,W(\tau) \qquad (7.9)$$

Suppose first that the same constraint $u = \bar{u}$ remains active for all the $\tilde{x}$ possible intermediate designs. Use can be made of the knowledge of the gradients of the constraints at both x_o and x_1 points. The scaling factor and its derivative are given by

$$f(\tau) = \frac{u[\,x(\tau)\,]}{\bar{u}}$$

$$(7.10)$$

$$f'(\tau) = \frac{1}{\bar{u}} \sum_i \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial \tau} = \frac{1}{\bar{u}} \sum_i \frac{\partial u}{\partial x_i} (x_i^1 - x_i^o)$$

We have however

$$u^o = \sum_i C_i^o x_i^o \qquad\qquad u^1 = \sum_i C_i^1 x_i^1$$

$$C_i^o = \left.\frac{\partial u}{\partial x_i}\right|_{x^o} \qquad\qquad C_i^1 = \left.\frac{\partial u}{\partial x_i}\right|_{x^1} \tag{7.11}$$

and therefore

$$f_o = f(o) = \frac{u^o}{\bar{u}} = 1 \qquad\qquad f_1 = f(1) = \frac{u^1}{\bar{u}}$$

from which we derive

$$f_o' = \frac{1}{\bar{u}} \left(\sum_i C_i^o x_i^1 - u^o\right) = 0$$

as x^o and x^1 satisfy the linearized constraint

$$\sum_i C_i^o x_i = \bar{u}$$

and

$$f_1' = \frac{1}{\bar{u}} \left(u^1 - \sum_i C_i^1 x_i^o\right)$$

The function $f(\tau)$ can be approximated by a cubic polynomial

$$\begin{aligned} f(\tau) = {} & (f_1' - 2 f_1 + 2)\, \tau^3 \\ & + (3 f_1 - f_1' - 3)\, \tau^2 + 1 \end{aligned} \tag{7.12}$$

which is, in practice, a very accurate representation for such a function. For practical applications, one has to take care that the same constraints do not necessarily remain active between x^o and x^1 and that, consequently, the scaling factor has to be written as the envelope

$$f(\tau) = \max_j \ [\ f_j(\tau) \ ] \qquad j = 1, n_f \tag{7.13}$$

taken over the n_f imposed constraints. In addition when a combination of zero and first order approximations of the constraints are used, polynomials of lower degree have to be used for $f(\tau)$ in (7.12) when the gradients (7.11) are not known at one or at both ends of the segment.

This more sophisticated procedure for selecting intermediate design vectors in the inverse space has been applied to the 10 bar truss defined in 4.5. Cubic approximations of $f(\tau)$ have been used for the flexibility constraints, while linear ones have been taken for the stress constraints which are all treated by F.S.D. The results are compared to those obtained by KHOT [K10] in figure 7.9.
The proposed procedure had to be applied only once, between iterations 7 and 8, and reveals very efficient as the minimum of 5076.77 lbs was obtained rapidly without any additional search for intermediate designs. The advantage is important over the more intuitive procedure in the direct space.

It remains however that the use of intermediate design vectors is only a palliative measure. A better answer would be given by an a priori estimation of the maximum number of steps $\bar{k}$ that can be achieved in the coming iteration. It seems that such a forecast could be achieved on the basis of an extrapolation of the scaling factors affecting the various constraints. It certainly deserves further study.

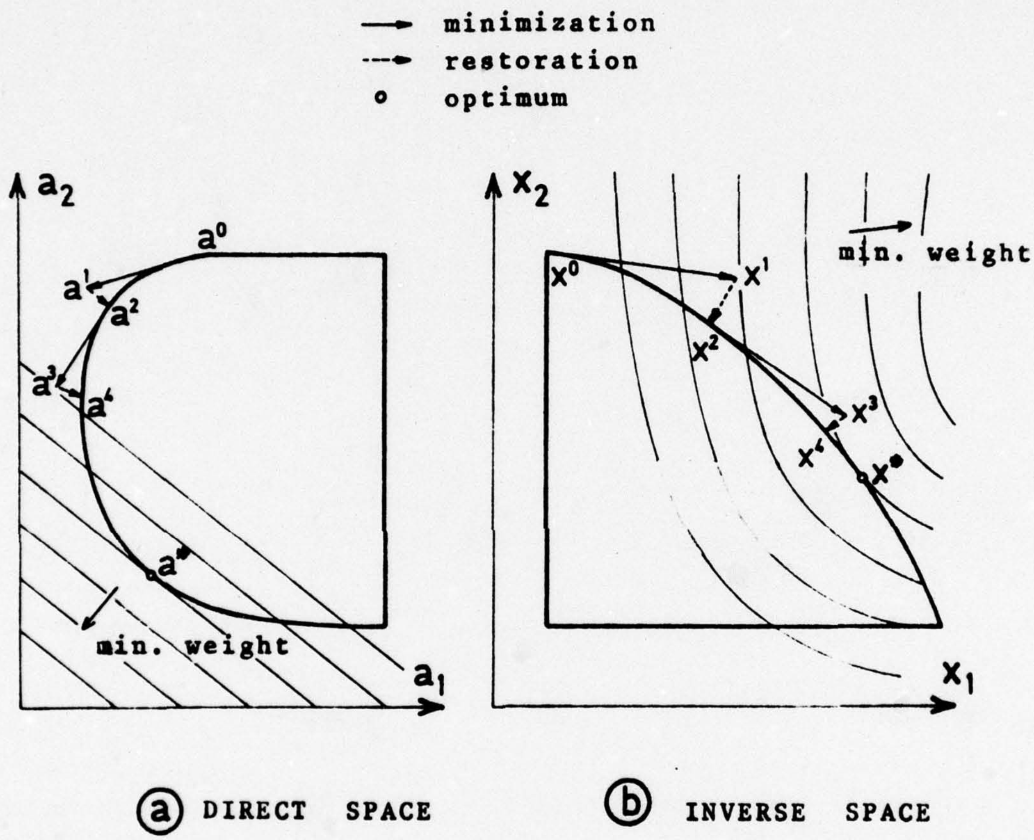


FIGURE 7.1 PROJECTION METHODS

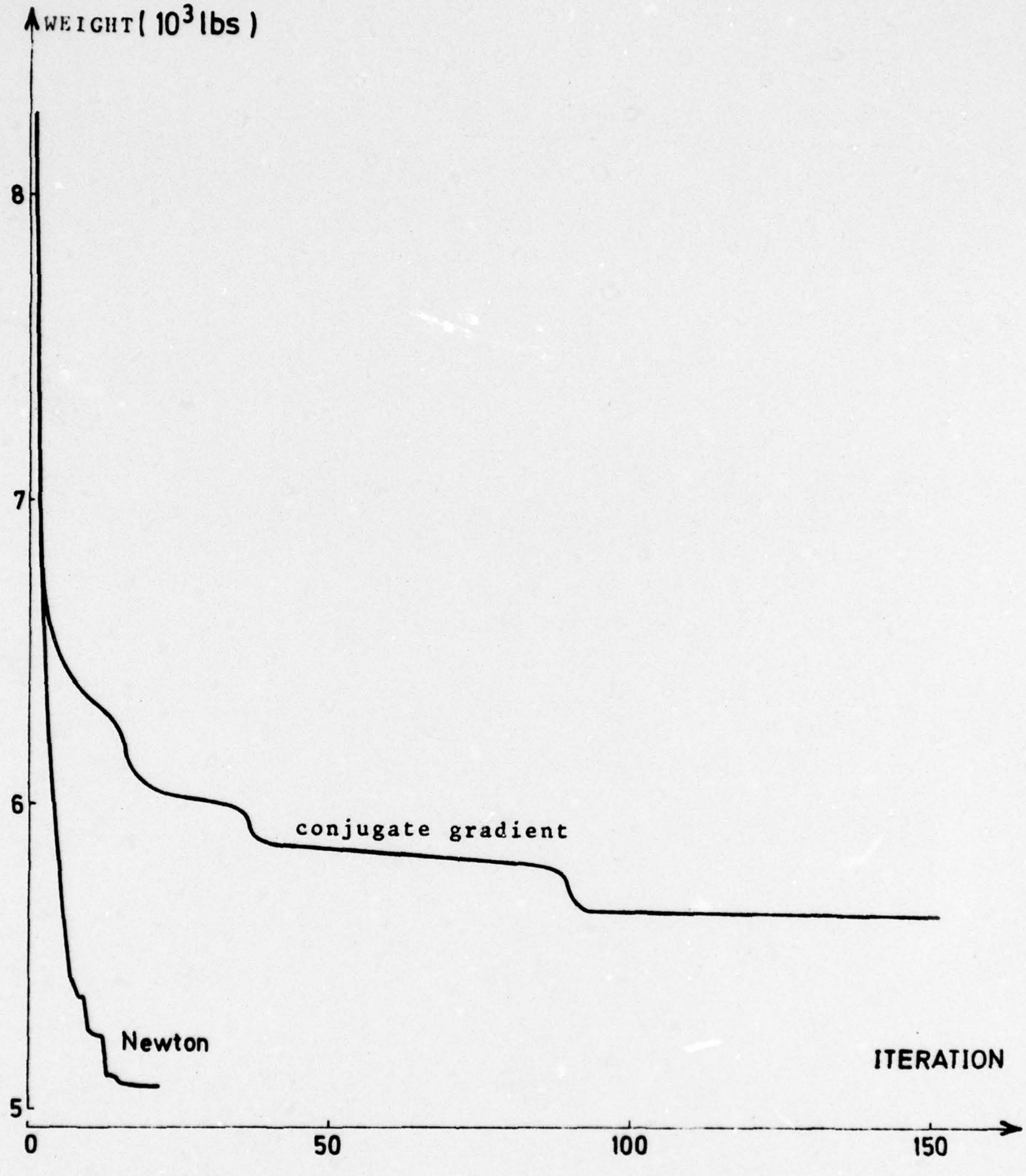


FIG. 7.2 : 10 BAR TRUSS-PROJECTION METHODS

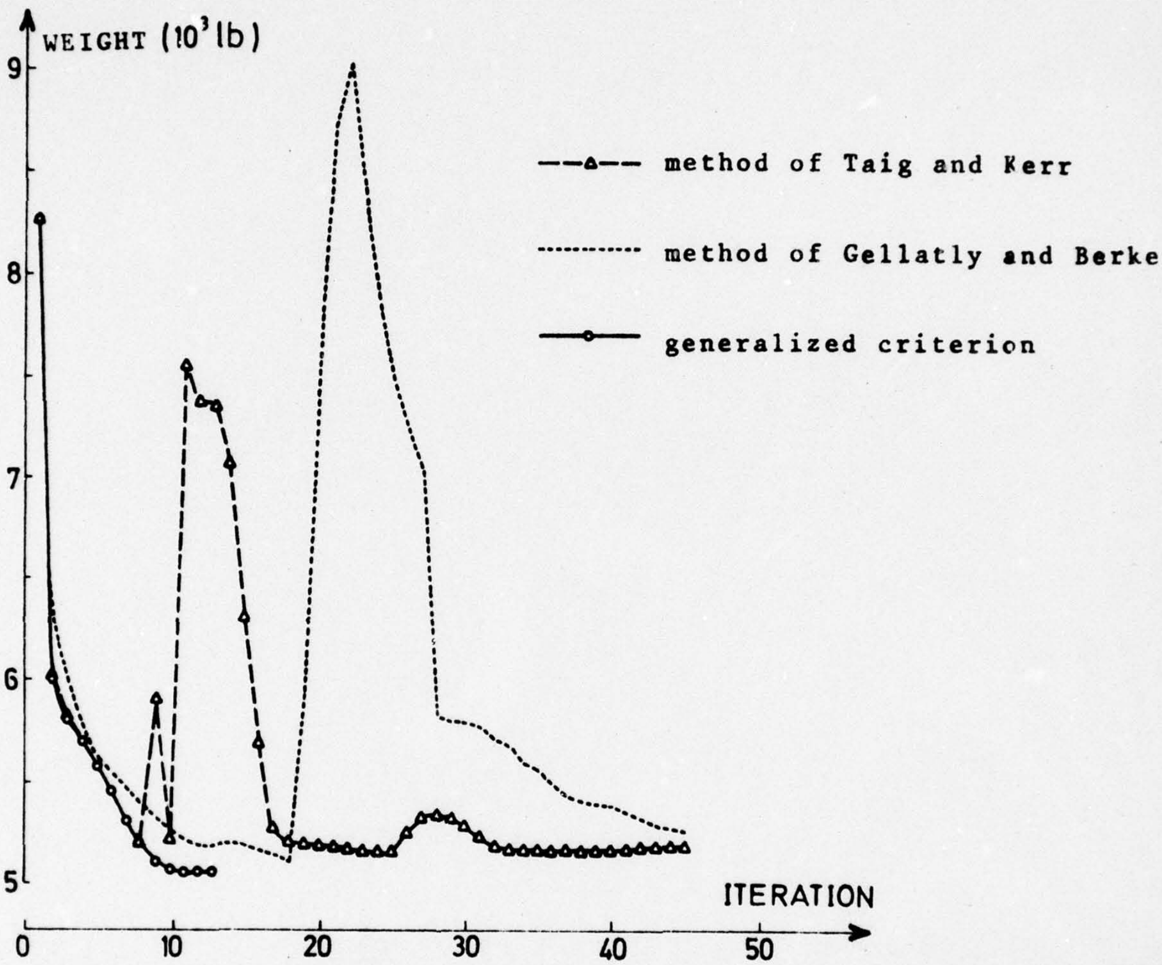
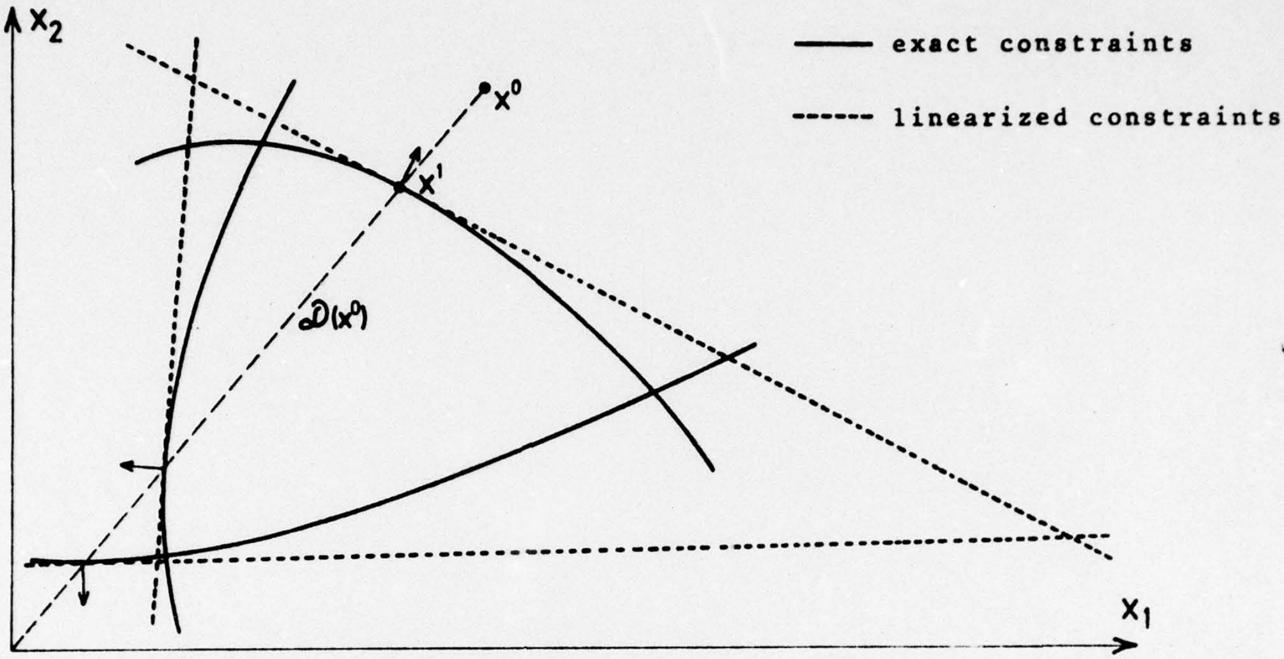
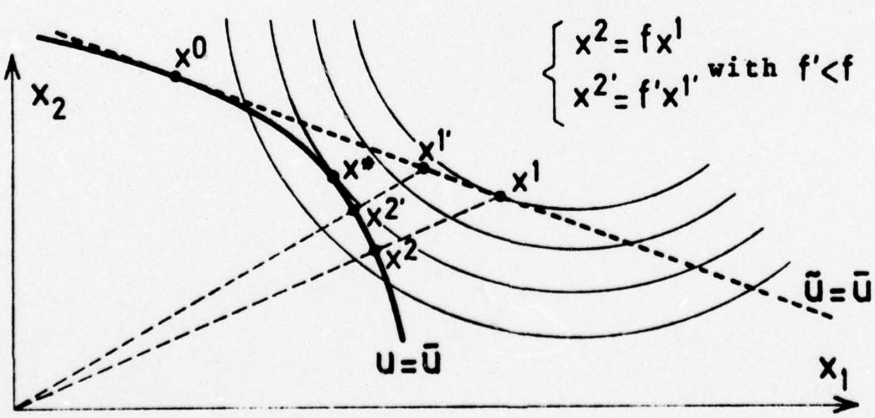


FIGURE 7.3. : 10 BAR TRUSS - OPTIMALITY CRITERIA



THE LINEARIZED PROBLEM



<u>CONVERGENCE CONTROL IN THE MIXED METHOD</u>

<u>FIGURE 7.4</u>

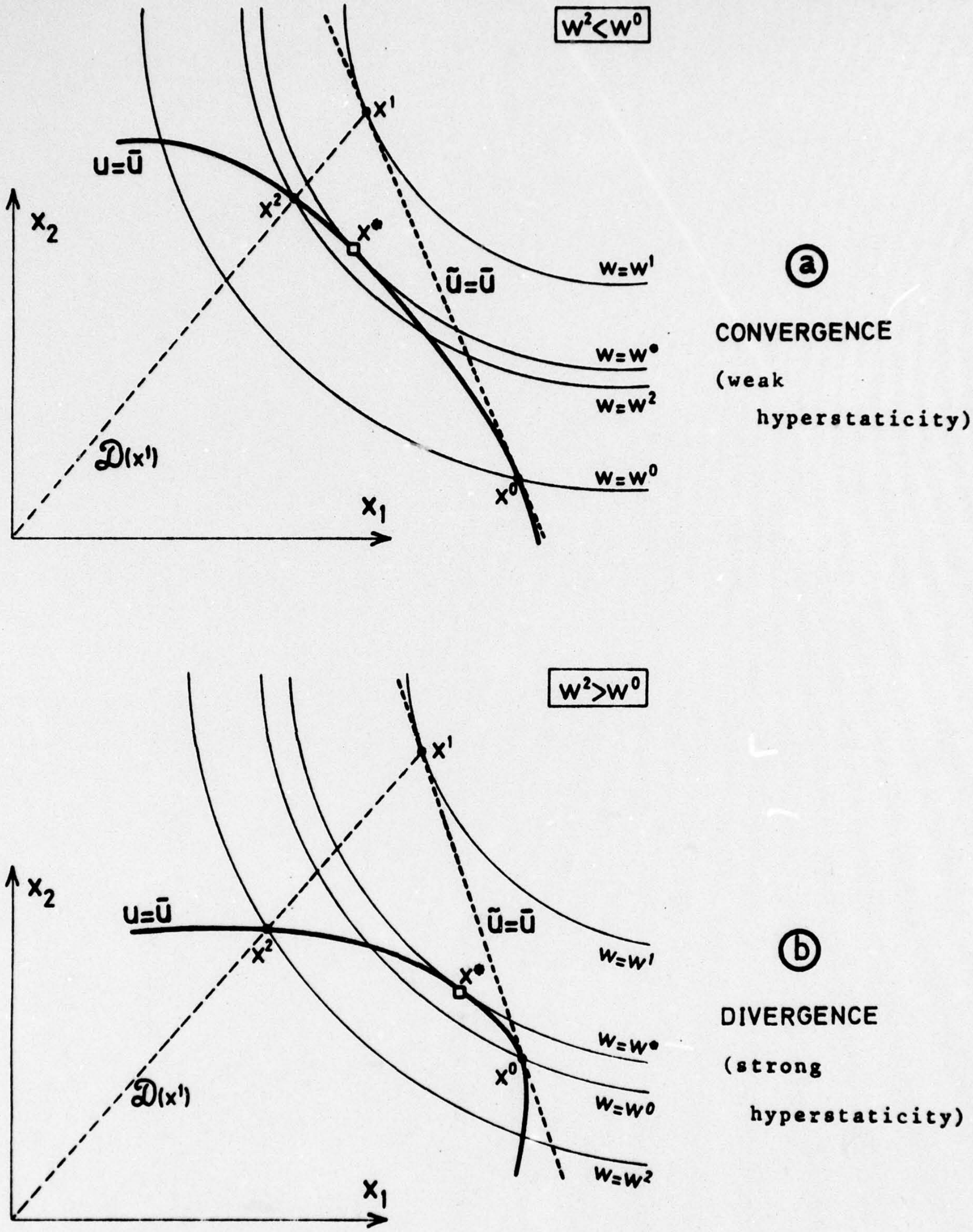
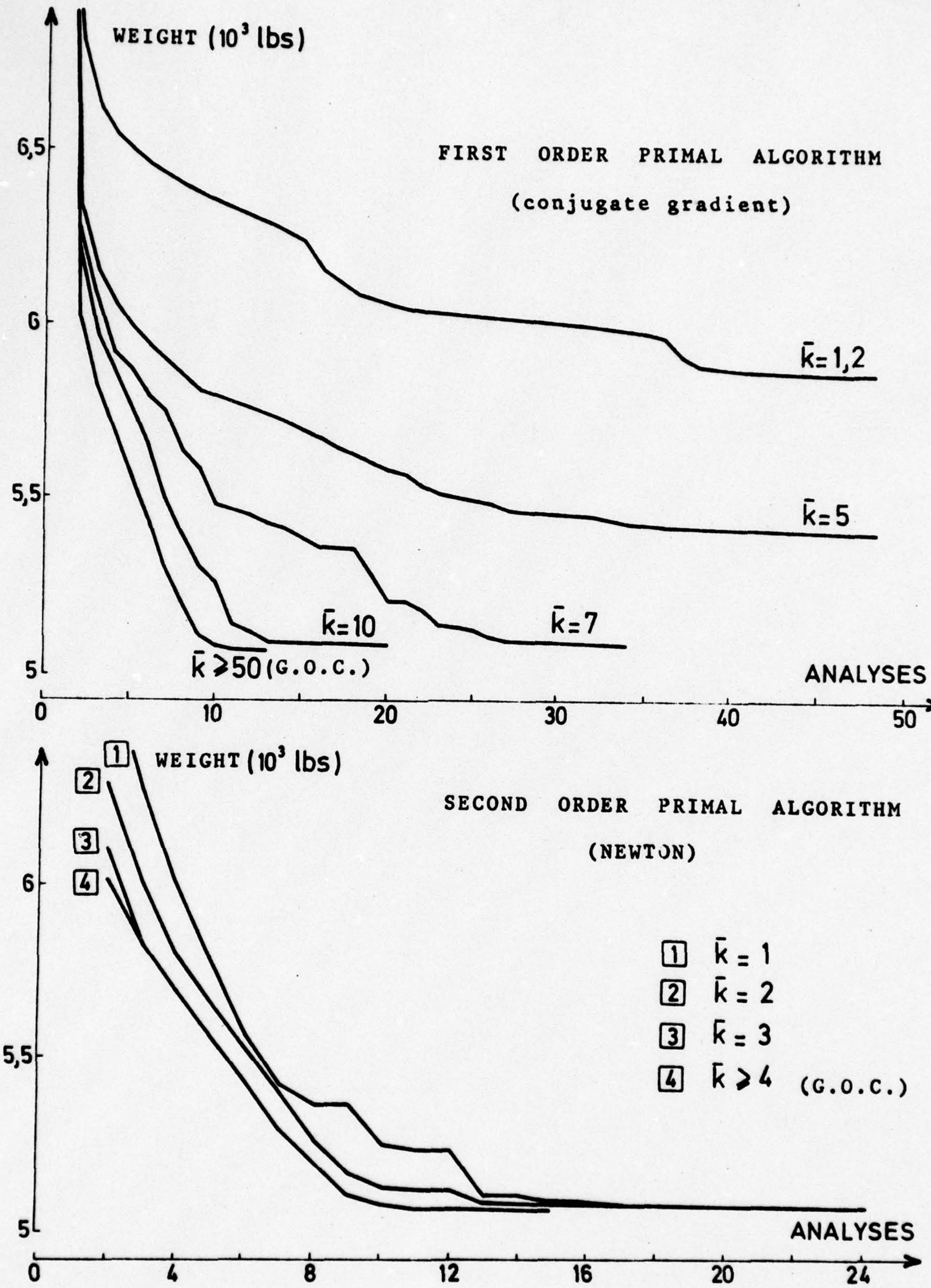
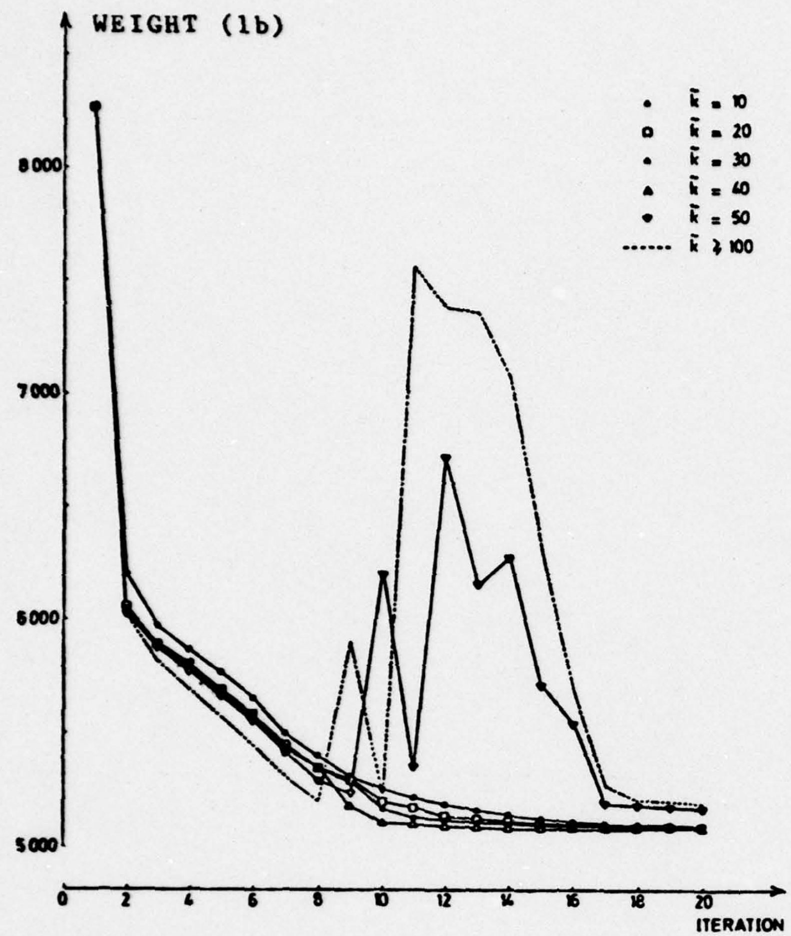


FIG. 7.5 : CONVERGENCE PROPERTIES OF OPTIMALITY CRITERIA

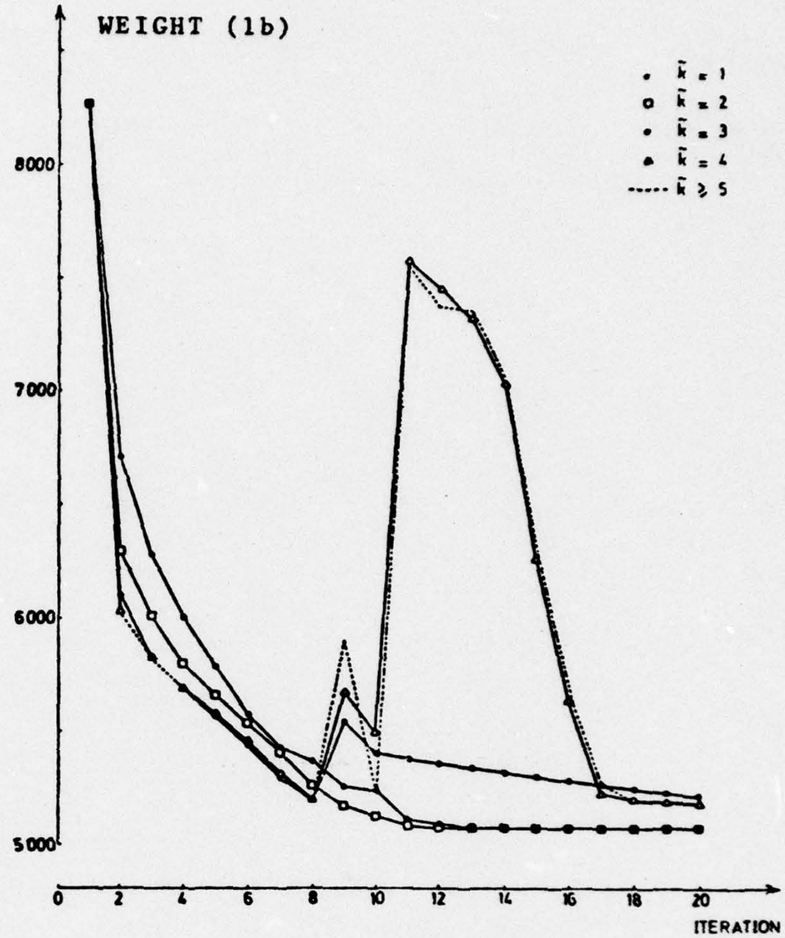


APPLICATION OF THE MIXED METHOD TO THE 10 BAR TRUSS

FIGURE 7.6.A.



(a) FIRST ORDER PRIMAL ALGORITHM
(conjugate gradient)



(b) SECOND ORDER PRIMAL ALGORITHM
(Newton)

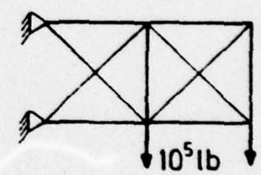


FIGURE 7.6.B. 10 BAR TRUSS - THE MIXED METHOD CONCEPT

iteration	$\bar{k}$ = 1		$\bar{k}$ = 5		$\bar{k}$ = 10	
	WEIGHT (lbs)	SCALE FACTOR	WEIGHT (lbs)	SCALE FACTOR	WEIGHT (lbs)	SCALE FACTOR
1	8266	1.0	8266	1.0	8266	1.0
2	6321	0.9996	6337	0.9924	6203	0.9902
3	6626	0.9993	6153	0.9907	5970	0.9364
4	6541	0.9993	6060	0.9973	5866	0.9923
5	6497	0.9995	5989	0.9974	5766	0.9925
6	6456	0.9994	5939	0.9985	5651	0.9925
7	6425	0.9996	5894	0.9995	5492	0.9364
8	6396	0.9996	5858	0.9988	5396	0.9920
9	6373	0.9997	5810	0.9972	5306	0.9975
10	6349	0.9997	5791	0.9993	5259	1.0293
11	6329	0.9997	5772	0.9999	5133	0.9995
12	6307	0.9998	5756	0.9999	5113	1.0035
13	6283	0.9997	5736	0.9999	5036	1.0000
14	6260	0.9997	5719	0.9999	5085	1.0000
15	6239	0.9997	5693	0.9996	5085	1.0000
16	6157	0.9972	5675	0.9999	5081	1.0000
17	6121	0.9995	5641	0.9987	5077	1.0000
18	6086	0.9982	5625	0.9999	5077	1.0000
19	6073	0.9999	5597	0.9998		
20	6053	0.9996	5578	0.9998		
21	6041	0.9999	5567	0.9992		
22	6036	0.9998	5530	0.9994		
23	6031	1.0000	5510	0.9996		
24	6027	0.9999	5504	0.9995		
25	6023	1.0000	5491	0.9992		
26	6019	0.9999	5485	0.9993		
27	6015	1.0000	5464	0.9987		
28	6011	0.9999	5458	0.9998		
29	6006	1.0000	5455	0.9999		
30	6002	0.9999	5453	1.0000		
31	5998	1.0000	5450	1.0000		
32	5993	0.9999	5448	1.0000		
33	5987	1.0000	5436	1.0000		
34	5982	0.9999	5427	0.9998		
35	5972	0.9999	5421	1.0000		

10 BAR TRUSS

EFFECT OF THE $\bar{K}$ CONVERGENCE CONTROL PARAMETER

(First order algorithm)

Table 7.7

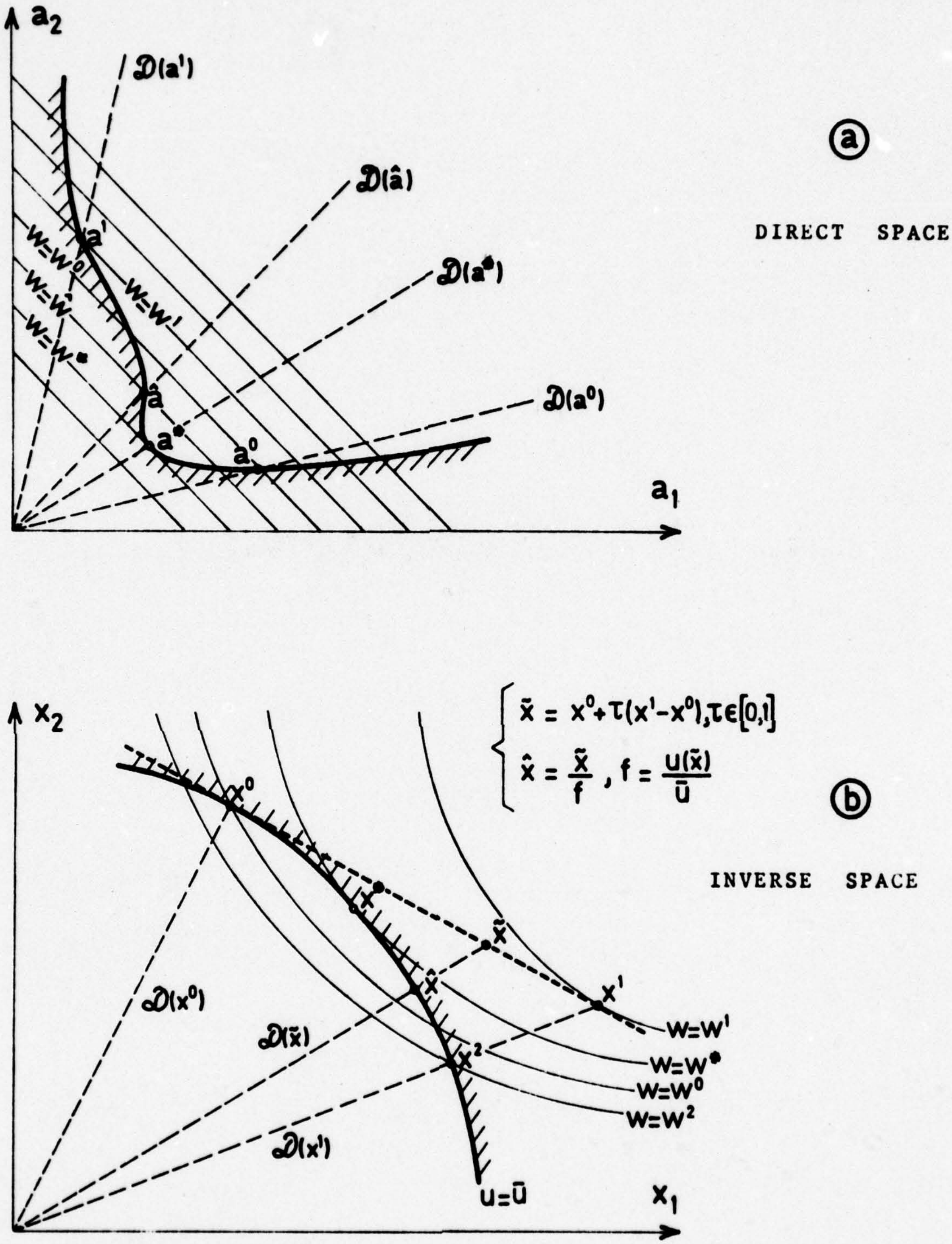
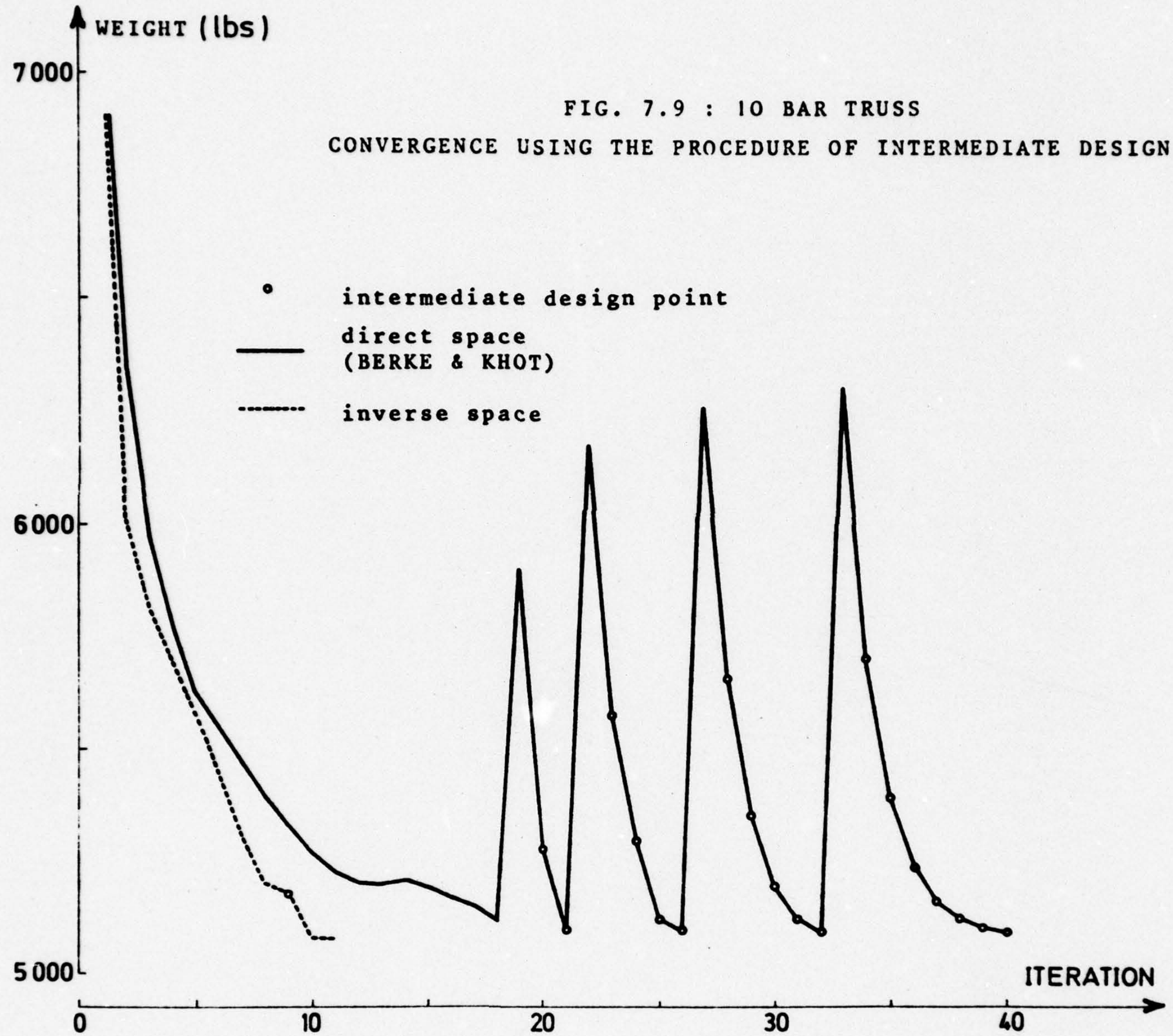


FIG. 7.8 : THE CONCEPT OF INTERMEDIATE DESIGN POINTS

FIG. 7.9 : 10 BAR TRUSS

CONVERGENCE USING THE PROCEDURE OF INTERMEDIATE DESIGN POINTS



8. NUMERICAL APPLICATIONS

The applications have been divided in 3 groups. The first one concerns trusses,which have the advantage of yielding rather pure problems for which the optimization results are not influenced by the modeling technique. The second group of applications contains box beam problems and the third,more complex, near to industrial problems.

Two different programs have been used. One is an experimental program using only bar elements and working in core. It is denoted OPTBAR. It allows to use all the algorithms described in this work, that are based on primal or dual methods, using first or second order minimization techniques. It also allows to select automatically the most critical stress constraints, according to test (3.38), by specifying two tolerances ε_1 and ε_2 corresponding to the two parts of the test.

The second program is the optimization module, named OPTIM, that has been written for the SAMCEF general purpose finite element program [S19] . It allows also for primal and dual approaches, using first and second algorithms, but does not include the automatic selection of the critical stress constraints. For comparison purposes, the envelope method of GELLATLY [G13] is also incorporated.

The following abbreviations are used

- C.O.C   Classical optimality criteria, as proposed in the litterature;
- G.O.C   Generalized optimality criteria. It implies that all the active constraints are linearized and that the corresponding problem L (or A1) is solved exactly by a primal (or dual) method;

- H.O.C Hybrid optimality criteria. It implies that a limited number of stress constraints are linearized in addition to the flexibility constraints. The approximate problem L' (or A') is solved exactly by a primal (or dual) method;

- S.O.C Simple optimality criteria. All the stress constraints are zero order approximated by F.S.D. while the flexibility constraints are linearized. This corresponds often to what the C.O.C claim to achieve. The corresponding problem A' (or L') is however solved exactly by dual (or primal) methods;

- MM1($\bar{k}$=xx)
- MM2($\bar{k}$=xx) The mixed method is used with first (MM1) or second (MM2) order projection algorithms. These abbreviations can follow G.O.C, H.O.C or S.O.C. It means that the corresponding problems (L or L') are not solved completely and therefore the method is no longer equivalent to an optimality criteria method. The convergence control parameter has been fixed to $\bar{k}$ = xx and the approximations of the constraints are those defined by the abbreviations G.O.C, H.O.C, S.O.C.

Unless otherwise specified, the starting point corresponds to a uniform distribution of the design variables.

8.1. Truss problems

8.1.1. Simple cantilever 10 bar truss

This simple and classical example has already been mentionned to illustrate various aspects of the methods proposed in the preceding chapter. The geometry is defined in figure 8.1.

Stress constraints only (Problem 1.A)

This problem has been examined in chapter 4 and the results illustrated by figure 4.6. It is recalled that when the stress limit is 25000 psi in all the bars, the F.S.D. yields the optimum.
When the limit in bar nbr 8 is increased, the F.S.D. reveals unable to yield the optimum from $\bar{\sigma}_8$ = 36,800. Between 36,800 and 45,000 psi the F.S.D. does not converge in a reasonable number of steps.
When the linearization of stress constraints is introduced, the optimum is easily obtained for any value of the $\bar{\sigma}_8$ limit.

Problem 1.B

The loads and the constraints are

$$F_y \quad \text{at nodes} \quad 2 = -100 \text{ K}$$

$$3 = -100 \text{ K}$$

$$|u_y| \quad \text{at all nodes} < 2.\text{in}$$

The results obtained for this problem are summarized in tables 8.2 and 8.3. Convergence curves have been presented

- for the G.O.C in figure 7.3;
- for the MM2($\bar{k}$=2) in figure 7.6 where certain results extracted from the litterature on C.O.C have also been plotted;
- for the S.O.C using the intermediate design vector technique, in figure 7.9.

The essential difficulty in this problem comes from the element nbr 6. The stress constraint in this element is the only one that reveals critical and its linearization is necessary to obtain the lowest weight (5061 lbs).
This solution has been obtained by a H.O.C, that is by using F.S.D. for all the elements, but element 6. It seems that this design can not be reached by the other methods which converge to the local optimum existing in the neighborhood (5076 lbs).

Problem 1.C

The constraints are identical, but the loads are

$$F_y \text{ at nodes } \begin{array}{lcl} 2 & = & -150 \text{ K} \\ 3 & = & -150 \text{ K} \\ 4 & = & +\ 50 \text{ K} \\ 5 & = & +\ 50 \text{ K} \end{array}$$

The stress constraint in element 6 is more critical than in the preceding problem, which leads to a more severe divergence of the C.O.C and of the S.O.C and in fact for any method relying on F.S.D in element 6. Convergence curves are given

- for the S.O.C, using MM2($\overline{k}$ = 1,2 and ∞), in figure 8.4;
- for the H.O.C, using stress constraint linearization in element 6 only, for the envelope method of GELLATLY [G13] which is also available in the SAMCEF program, and for various reference works, in figure 8.5.

The results are presented in tabular form in tables 8.6 and 8.7. The best results are those of the H.O.C closely followed by those of SCHMIT [S18] .

Problem 1.D

The loads are applied in nodes 2 and 3, as in problem 1.B, but equality constraints are imposed at nodes 4 and 5

$$u_y = -2. \text{ in} \quad \text{at} \quad \text{node } 4$$

$$u_y = -1. \text{ in} \quad \text{at} \quad \text{node } 5$$

The problem has been solved by the H.O.C using a linearization of the stress constraint,only in the element 3.
Note that the procedure of automatic solution of these stress constraints that require linearization,has been used and the two tolerances ε_1 ,ε_2 for the test (3.38) are given on the figure.
The results are given in digital form in table 8.8.
It is observed that the convergence is obtained in 8 iterations and that the equality constraints are satisfied at the last iteration only,while the stress in element 3 has reached its upper limit. This problem was suggested by VENKAYYA [V7] but no reference solution was found in the litterature.

8.1.2. Double cantilever 22 bar truss

As mentionned in chapter 3, the FSD leads to poor results when all the structural components do not have the same stress limit, which is the case when different materials are considered. BERKE [B15] proposed a special problem to illustrate the difficulty. It is made of twice the 10 bar truss of section 8.1.1, connected by two bars as indicated on figure 8.9. The data and the constraints are given in table 8.10. Three cases have been examined.

Problem 2.A

All the elements have the same mass density (0.1 lbs/in^3).
The stress limit is the same for all the elements ($\bar{\sigma}$ = 25000 psi) but for the 2 bars connecting the 10 bar trusses to the load, for which a stress limit of 500000psi is imposed.

In this case the FSD yields the optimum in 46 iterations. Using the G.O.C. does not improve the convergence. The table 8.11 presents the design obtained, while the table 8.12 contains the evolution of the weight for various configurations and methods.

Problem 2.B

The stress limit in the left part of the truss is increased up to 50000 psi. The optimum design in this case is not FSD and the stress ratio algorithm diverges. Using the H.O.C. with automatic selection of the elements requiring stress constraint linearization according to (3.38) leads to the linearization of 10 stress constraints. This number could probably be reduced by using more severe tolerances. The optimum design obtained by the H.O.C. is given in table 8.13 and corresponds to a weight of 654 lbs. It differs drastically from the heavier (1044 lbs) FSD in which the larger sections are given to the wrong side of truss. The evolution of the weight is recorded in table 8.12 and illustrated in figure 8.14 (a).

Problem 2.C

This case differs from case B by a higher mass density in the left part ($\rho = 0.4$ lbs/in^3) which also has the higher stress limit. The data are given in table 8.10. In the F.S.D. approach, the sections in the left part are assigned smaller values due to their higher allowable stress. So the F.S.D. brings the material back to the left side, but it does this for a wrong reason !

The results obtained by S.O.C. and by H.O.C. are presented in table 8.12 and in figure 8.14 (b). The optimum is a F.S.D. but it is obtained much more rapidly by the H.O.C. than by the simple F.S.D. used in the S.O.C. Note that all the elements reach either the maximum stress or the minimum size and that the design is completely reversed from that obtained in case B. The final design is detailed in table 8.15.

8.1.3. 200 bar planar truss

This classical problem [V2] is defined in figure 8.16 and in table 8.17. The symmetry of the problem is used to reduce the number of design variables to 105. Each design variable corresponds to a group of two bars.

Problem 3.A

No stress constraints are imposed. The S.O.C, H.O.C and G.O.C. are identical in this case. They differ from C.O.C. by their rigorous treatment, by dual formulation, of the explicit problems generated after each analysis. The figure 8.18 presents the convergence of the weight,given in digital form in table 8.19. This table also contains the computing time for the envelope method [G13] and the second order dual algorithm. They have been obtained for a programming that presents the same level of sophistication in both cases. It shows that the more general and rigorous dual methods are not significantly more costly than the simple and intuitive envelope method. The optimum design obtained by the H.O.C. is presented in table 8.20. Its weight (29037 lbs) is smaller than that reported by VENKAYYA [V2] (31020 lbs).

Problem 3.B

Stress constraints ($\bar{\sigma} < 10000$ psi) are added to the preceding problem. The S.O.C. is now less efficient than the H.O.C.
The results are illustrated in figure 8.21 and in tables 8.19 and 8.20. Note that the final design weights 29163 lbs and is not very different from the preceding one.

8.1.4. Wing carry through box (63 bars)

This problem is also classical [B8] . It is defined in figure 8.22 and table 8.23. Two cases are examined.

Problem 4.A

Stress constraints only are imposed. The S.O.C.,which reduces here simply to F.S.D.,leads to instability in convergence. The H.O.C. leads to the linearization of 58 out of the 126 constraints and is therefore close to the G.O.C. It converges quite rapidly to a design that has also been found by SHMIT [S18] . Figure 8.24 illustrates the convergence,which is detailed in table 8.25. The final design (4975.1 lbs) is presented in table 8.26.
It is evident that, for this problem, the linearization of the stress constraints is an essential ingredient for fast convergence. The method of SCHMIT [S18] is, in this case, very close to the G.O.C. as it solves almost exactly the problem L generated after each analysis.

Problem 4.B

The relative displacement at nodes 1 and 2 is limited to 1.0 in. Again the S.O.C, using F.S.D. for the stress constraints, exhibits poor and unstable convergence (6159 lbs in 50 iterations) while the G.O.C. converges very rapidly (5120 lbs in 12 iterations) just as the SCHMIT [S18] algorithm (5121 lbs in 13 iterations) which is closely related. The results are displayed in figure 8.27 and in tables 8.25 and 8.26. The performances of the G.O.C. is obtained at the expense of a large number (126) of virtual loading cases corresponding to each linearized stress constraints. An investigation has been done using the automatic selection option of the program OPTBAR for the stress constraints that are most critical, as based on the criterion (3.38). The tolerance for the selection has been made more and more severe, corresponding to 0, 6, 7, 10 and 27 linearized stress constraints so that a progressive transition from S.O.C. to G.O.C. is achieved.

The results are illustrated in figure 8.28 as a function of the number of reanalyses and in figure 8.29 as a function of the total CPU time. It appears that the economic optimum is obtained when the 10 most critical stress constraints are linearized. This investigation indicates that the generalized use of the selection criterion (3.38) for the stress constraints is strongly recommended.

8.1.5. Four level 72 bar space truss

This problem has been investigated by many authors [T3, G13, V3, B8, S8, S18] . It is defined in figure 8.30 and in table 8.31. Two loading cases and four flexibility constraints are specified. The stress constraints are uniform at $\bar{\sigma}$ = 25000 psi but reveal not very critical. The optimum is obtained in 5 iterations by the S.O.C. and is identical to the solution obtained by TAIG [T3], which is quite understandable as in both cases the same explicit problem A is solved exactly at each iteration. The convergence is illustrated in figure 8.32 and in table 8.33. The final design is given in table 8.34. The results obtained by various authors are also presented. It appears clearly that this example is well adapted to the F.S.D. approximation of the stress constraints. Any additional sophistication increases the cost of computation without improving the global efficiency.

8.1.6. Transmission tower

This example is very popular in the structural optimization litterature [F27, G12, M9, V2, D5] . It is defined by the figure 8.35 and the table 8.36. Using the symmetry of the problem 25 design variables are necessary. Two loading cases and two flexibility constraints are specified. The stress limitations are given different values in traction and compression. The stress limits used by GELLATLY [G13] and SCHMIT [S8, S18] have been used. They differ slightly from those mentionned by VENKAYYA [V3] but this difference does not seem to influence the results.

In figure 8.37 and table 8.38 are compared the performances obtained by using the S.O.C. and the H.O.C. In the H.O.C. the stress constraint in element 20 for the second loading case, is the only one linearized. It leads to a moderate improvement. All the algorithms proposed in the litterature yield almost identical designs, which are given in table 8.39.

8.1.7. Simplified airbrake structure

This problem reproduces on a very simple equivalent truss, the essential difficulties met on a more complex airbrake structure which is examined in section 8.3. The problem is reduced to the hyperstatic 57 bar truss illustrated in figure 8.40 and precised in table 8.41. Only one loading case is applied. The difficulty comes from the flexibility constraints which impose that the difference of the w displacements, along the z axis, between three couples of nodes 1, 2, 3 should be less than 0.5 mm. This is equivalent to specifying that the trailing edge of the airbrake has to remain straight. A maximum stress is imposed as well as a minimum cross section.

All the optimality criteria failed on this problem. The solution was obtained only by using stress constraint linearization and the mixed method with limitation of the number $\bar{k}$ of steps. The results are illustrated by the figures 8.43 and 8.44 (note the difference in the scale for the weight) and by the tables 8.45 and 8.46.

Consider first, in figure 8.43, the results obtained by using the F.S.D. zero order approximation for the stress constraints. As an example of C.O.C., the envelope method [G13] is used and diverges immediatly, yielding designs that remain more than 10 times heavier than the best found subsequently.
The S.O.C. is used via the primal ($\bar{k} = \infty$) and the dual methods. It reveals unstable, but converges after 20 analyses to a design weighting 20.5 kg, which is still far from the best design (15.6 kg). Using the same F.S.D. approximation for the stress constraints but limiting the number of steps $\bar{k}$ improves the speed and stability of the convergence, but still does converge to the same non optimum design. The number $\bar{k}$ was limited to 57 using the first order primal algorithm. The results are denoted SOC-MM1 ($\bar{k} = 57$). Using the first order algorithm, it seems that a good estimation of the number of steps $\bar{k}$ is given by the number of elements, which is 57 here.

Consider now, in figure 8.44, the results obtained with linearization of the stress constraints. If all of them are linearized, the exact solution of the corresponding explicit approximate problem corresponds to a G.O.C. The convergence of a sequence of such problems is still non monotonic,but yields in 20 analyses a design weighting 15.6 kg which is the minimum found for this problem. During the next 10 to 15 iterations the results oscillate slightly around this design. Linearization of all the constraints being costly, the H.O.C. has been investigated using the automatic selection algorithm of the program OPTBAR ($\varepsilon_1 = 0.8$, $\varepsilon_2 = 0.02$). Indeed the algorithm leads to linearize only one of the five stress constraints which are active at the optimum.
The corresponding H.O.C. is therefore cheap to apply.
If $\bar{k}$ is unlimited however, the behavior is similar to using F.S.D. as in figure 8.43 (S.O.C.). If the mixed method is applied (H.O.C.-MM1, $\bar{k} = 57$) the algorithm converges very rapidly to a near optimum design (17.32 kg) in 7 analyses. During the next 10 to 15 analyses the design is only slightly udapted and the weight reduces to 15.57 kg. After the 7^{th} reanalyses, the limitation of $\bar{k}$ has been suppressed (H.O.C.-MM1, $\bar{k} = \infty$ after 7^{th} step). This change reintroduces two large oscillations of the weight but leads to an identical final design weighting 15.57 kg.

In this example, the 15.6 kg design was obtained only with stress linearization. It illustrates the need for an automatic algorithm for detecting the critical stress constraints. If all the constraints are linearized (G.O.C.) the optimality criterion converges to the lighter design but is very expensive. The combination of one stress constraint linearization and of the mixed method yields a faster convergence at much cheaper computing cost, as shown in the table 8.47 which summarized the computing times.

Note also that the 2 designs obtained by S.O.C. and by H.O.C. (or G.O.C.) are compared in figure 8.46. Although the weights are quite different (20.5 kg and 15.6 kg) as well as the design variables themselves, the same constraints are active. They are the two flexibility constraints and 5 stress constraints.

8.1.8. Computing times

The execution times for most of the truss problems investigated in this chapter, are recorded in table 8.47.
The methods and the program used are indicated. It is recalled that OPTBAR is a simple, experimental, all in core program used for research essentially, while SAMCEF is a general purpose program with much larger analysis capability,but which is evidently less efficient in small scale, academic applications.

The execution times are split in preprocessing, which exists only for the SAMCEF program, in analysis,including eventually the computation of the gradients, and in minimization time. The time is

$$t_{total} = t_{preproc.} + n_{iter.}(t_{analysis} + t_{minim.}).$$

This splitting allows to appreciate the increase of computing time due to the use of more sophisticated minimization methods.
In some cases, especially with the G.O.C. where all the constraints are linearized, this increase in minimization time is clearly prohibitive, as it is not counterbalanced by a sufficient reduction of the number of iterations.

The comparison of the computing times obtained for the various methods reinforces the conclusion drawn from the examination of the convergence properties, which is, that the H.O.C. appears as the best compromise,especially if the selection of the critical stress constraints is achieved automatically and if the mixed method limiting $\overline{k}$ is used.

8.2. Box beam structures

8.2.1. Simple wing box

The simple box beam illustrated in figure 8.48 is a classical test case proposed in the litterature [G12, G13, S18] . A detailed statement of the problem is given in table 8.49. Two load cases are considered. Maximum allowable stresses are given for the bars and membranes elements which are grouped as indicated in table 8.49, so that only 16 design variables are involved. Minimum values are also specified.

By symmetry, only one half of the structure is modeled in finite elements. Simple, constant strain, conforming displacement elements have been used, yielding to 45 D.O.F. In the membrane elements, the von MISES equivalent stress is limited. The linearization of these constraints has been achieved according to the method presented in (3.35).

The S.O.C has been applied. It reduces here to the straightforward F.S.D. The convergence is rather fast and monotonic, as illustrated in figure 8.50 and in tables 8.51 and 8.52.

When the G.O.C is applied, one realizes that the optimum is not a F.S.D and that, with the linearized stress constraints, the convergence is much faster and leads to a significantly lighter design. Comparing the results in table 8.52 indicates that the two designs differ also significantly.

The computing time for this problem, using the SAMCEF program, are as follows

S.O.C	20 iterations	preprocessing	12.0	
		analysis	8.8	
		minimization	0.5	
		total	198	sec.
G.O.C	10 iterations	preprocessing	14.1	
		analysis	36.9	

minimization	0.8	
total	390	sec.

It indicates again that the large number of virtual load cases and the additional computing cost induced by the G.O.C lead to almost double the computing time although the number of iterations is just half.

The H.O.C should bring a more efficient way of reaching the 200 lbs design, but it was not used here. The reason is that the comparison of the results with those presented in the litterature is rather difficult as the finite element models used by the various authors are not the same. In such a simple model the differences due to using one element or another are quite important and make the comparisons non significant.

8.2.2. A composite box beam

The simple box beam illustrated in figure 8.53 and detailed in table 8.54 has been proposed by KHOT, VENKAYYA and BERKE [K15] as an example of composite material design. The upper and lower skin elements are supposed to be made of composite material (boron-epoxy as defined in table 8.54) while the transverse members and the reinforcing bars are in aluminium. The transverse members are idealized by shear panels, while the skin elements are obtained by superposition of 4 constant strain displacement finite elements. Each of them corresponds to a different orientation of the fibers, that is 0°, 90°, + 45° and - 45°. The superposition of these 4 layers yields a unique definition of the design. The material properties in each layer are orthotropic and correspond to a unidirectional reinforcement of the resin. The tickness of each layer is a design variable.

In the design of composite structure, a frequent problem is to limit the displacement or to tailor the flexibilities according to given laws. Consequently, no stress limitations have been introduced in this problem which is governed entirely by two different flexibility constraints.

Limitation in bending (Problem 2.2.A)

In this case 4 tip loads are applied symmetrically at nodes 1, 2, 3, 4 and the displacements at each of the 36 nodes are limited (table 8.54). The double symmetry of the problem has been used. It allows to reduce the number of design variables to 45 and the number of constraints to 9.

The S.O.C has been used and, as the number of flexibility constraints is rather large, the primal method has been selected with the second order (NEWTON) minimization algorithm. The convergence is illustrated in figure 8.55 and tabulated in table 8.56 where it is compared with the results obtained by KHOT [K15] . The slight differences in the final designs, presented in table 8.57, are due to the differences in the finite element models.

Limitation in torsion (Problem 2.2.B)

The loading is now asymmetric and induces primarily bending deformations but also a limited amount of torsion. The limitations in deflections are 14 inches at nodes 1 and 2 where the loads are 1000 lbs and of 15 inches at nodes 3 and 4 where the loads are 975 lbs. These limitations require evidently a non symmetric design. As the tip deflections are the only one limited, the shear webs keep the same thickness along the span. This allows to reduce the number of design variables to 30.

The S.O.C has been used and the results are presented in figure 8.55 and in tables 8.58 and 8.59 together with those obtained by KHOT [K15] .

In both problems (2.A and 2.B) the convergence of the process is monotonic and significantly faster than reported by KHOT [K15] . The computing times are the following, using the SAMCEF program,still on IBM 370/158.

problem 2.A	10 iterations	preprocessing	34.3	
		analysis	57.3	
		minimization	1.1	
		total	617	sec.
problem 2.B	10 iterations	preprocessing	34.9	
		analysis	27.4	
		minimization	0.9	
		total	317	sec.

8.3. Airbrake structure

The airbrake structure illustrated in figure 8.60 has been used in an exercise to determine the capability of the methods proposed in the SAMCEF program, when applied to a problem with a number of design variables and of degrees of freedom larger than those met in the classical tests extracted from the litterature on optimization. Moreover the flexibility constraints revealed rather difficult to take into account in the classical algorithms, although they could be frequently encountered in practical problems. Only the essential results obtained for this problem are presented here. A more detailed description has been given by FLEURY in [F18, F29] .

The structure illustrated in figure 8.60 is classically designed in light alloy aluminium sheet. The main (front) spar and the secondary spar are joined by twelve ribs and covered by two skins reinforced by stringers. The airbrake is hinged at three points and actuated at one, in the midspan.

The loads consist in pressure distributions on both faces, corresponding to two flight configurations. In one of them, a flexibility constraint is imposed which stipulates that the trailing edge has to remain straight within a tolerance $\varepsilon = 0.5$ mm to insure aerodynamic efficiency.
Stress constraints are imposed and take different values depending of the material used and of other technological considerations. For membrane elements, the von Mises equivalent stress is used. Minimum thicknesses are assigned which also differ from places to places depending of manufacturing considerations.

Reduction of the number of flexibility constraints

In a finite element model, as illustrated in figure 8.61, the flexibility constraint, as initially stated, has to be written

$$| w_j - w_k | < \varepsilon \qquad \begin{array}{l} j = 1, N-1 \\ k = j+1, N \end{array} \qquad (8.31)$$

were w_j, w_k are the transverse deflections at any node j or k along the trailing edge. If N is the number of nodes along the trailing edge in the finite element model, the total number of constraints is

$$N\ (N - 1)\ /\ 2$$

which is prohibitive, when it is considered that 10 to 12 nodes are to be used along the trailing edge. After having solved the problem using a simplified model where N = 5, it was recognized and confirmed by physical intuition that the constraints (8.3.1) could be replaced by the reduced set

$$\begin{aligned} |\, w_1 - w_M \,| &< \varepsilon & \qquad |\, w_N - w_M \,| &< \varepsilon \\ |\, w_1 - w_{M^*} \,| &< \varepsilon & \qquad |\, w_N - w_{M^*} \,| &< \varepsilon \end{aligned} \qquad (8.3.2)$$

where M and M^* are two nodes selected so that the distances (1-M) and (M^*-N) are approximately the same. Eventually a certain number of couples of nodes (M-M^*) could be selected. It was verified a posteriori that, using the reduced set of constraints, all the initial constraints (8.3.1) are satisfied.

<u>First simplified model</u> (27 elements)

In order to determine the behavior of the algorithms in the presence of the flexibility constraints (8.3.1), a very simple model illustrated in figure 8.62 was built up. It consists in 26 membrane elements, plus one bar used to model the control, which leads to 43 D.O.F. Five nodes are taken along the trailing edge and the 10 exact flexibility constraints (8.3.1) are imposed, but no stress constraint.

This problem reveals the obvious need to limit the global deflection of the trailing edge.

$$w_j < \bar{w} \qquad\qquad j = 1,\ N \qquad\qquad (8.3.3)$$

Indeed, if one consider a design yielding a perfectly straight deflected trailing edge, the application of scaling factor does not affect the satisfaction of the constraints (8.3.1).
Thus the process converges to a zero weight design with a very larger deflection. This happens because the stress constraint are not taken into account in this simplified problem. However it was felt inconsistent to leave the limitation of the global deflection to the stress constraints, as it is not known a priori whether they become active before reaching an unacceptable large deflection. Therefore a set of five constraints (8.3.3) was added to the problem, with $\overline{w}$ = 10 mm.

The envelope method [G13] has been applied as an example of C.O.C. Next the S.O.C has been used, which is here equivalent to a G.O.C as all the constraints are flexibility constraints, and thus are exactly linearized. The results are presented in figure 8.63 and revealed extremely poor (if any) convergence.

This implies that the constraints (8.3.1) are very non linear, even in the inverse design space. Using the mixed method (MM1), even with $\overline{k}$ being small, did not improve significantly the convergence. Looking more carefully at the optimization process, in the mixed method, it appears that the difficulty arises from the fact that, at each iteration, the algorithm selects a design where the maximum deflection constraint (8.3.3) is active. After reanalysis, the differential constraints (8.3.1) are seriously violated and the subsequent scaling,which has to satisfy them,leads to increase the weight in such way that the divergence starts (see the comments of figure 7.5).

It suggests the introduction of a kind of move limit in the form of a limitation to the increase of global deflection between two iterations.

$$w_j < \min (w^* , \bar{w}) \qquad j = 1, N$$

with

$$w^* = \delta \max_{j=1,N} (w_j)$$

where δ is a constant close to unity. In this way the deflection cannot exceed δ times the value obtained after the current analysis and scaling cycle. The constant δ defines the amplitude of the "move" that is allowed in the deflection.

The move limit parameter δ and the convergence control parameter $\bar{k}$ can now be use to improve the convergence behavior of the problem. Taking $\bar{k} = 20$ in the MM1 for 20 iterations and then $\bar{k} = 10$, and allowing $\delta = 1.1$ brings a nice convergence as indicated in figure 8.64, which also contains the history of the global deflection, and that of the trailing edge deflection. Note that 40 iterations are necessary to stabilize the weight. The change of $\bar{k}$ at iteration 21 was introduced after a beginning of divergence was detected. This weight should not be compared to that of the final model since no attention was payed to assigning minimum values for the design variables which are consistent with the full scale problem.

The investigation on this first simplified model revealed the need of a careful control of both the number $\bar{k}$ of steps in the mixed method and of the move limit δ. The optimality criteria, even in the sophisticated form of the G.O.C, failed completely. It also revealed that the set of constraints (8.3.1) can be replaced by the reduced set (8.3.2) with only one couple of points $(M.M^*)$.

Second simplified model (125 elements)

A second, still simple model was used, which is illustrated in figure 8.65. 42 membrane elements and 83 bars are used leading to 103 D.O.F.

The flexibility constraints are written

$$| w_1 - w_3 | < 0.5 \text{ mm} \qquad | w_1 - w_4 | < 0.5 \text{ mm}$$

$$| w_3 - w_6 | < 0.5 \text{ mm} \qquad | w_4 - w_6 | < 0.5 \text{ mm}$$

$$w_1 < 10 \text{ mm} \qquad w_6 < 10 \text{ mm}$$

The stress constraints and the minimum sizes of the full scale problem are taken into account. The stress constraints are treated by F.S.D.

The mixed method (MM1) has been applied with $\bar{k} = 40$ up to the sixth iteration and unlimited after. The move limit on the global deflection was still defined by $\delta = 1.1$. The initial design was obtained from the first simplified model. The convergence curves are illustrated in figure 8.66.

Note that, due to the introduction of the stress constraints the deflection does not reach the maximum value of 10 mm but flattens at 3.6 mm. However the move limit is still necessary to avoid the divergence. Although the convergence is rather fast (9 iterations) it is felt that even faster convergence could be obtained by a larger value of $\bar{k}$, around 100. Note that the second order algorithm (MM2) was not available at the time of the investigation.

It should also be pointed out that the weight cannot be compared with those obtained by the other models as no attention was payed to the exact equivalence in the statement of the problem and, more precisely, of the stress and minimum size constraints.

The final model (627 elements)

The final model is illustrated in figure 8.67.
It is made of 378 membrane elements and 249 bars. Second degree displacement fields are used, leading to 2300 D.O.F. The sizes of all the elements have been left free, yielding 627 design variables. In practice a certain number of elements should have been given identical sizes. This is achieved in the SAMCEF

program by defining groups of elements,which are assigned the same design variables. It has the advantage of reducing the number of design variables and of yielding more realistic designs. However, the present problem being essentially an exercise, it was decided to solve it with the maximum number of design variables. The results obtained by grouping the elements according to various design options are reported in [F29] .

The 627 stress constraints are treated by F.S.D while the flexibility constraints are written

$$| w_1 - w_6 | < 0.5 \text{ mm} \qquad | w_1 - w_7 | < 0.5 \text{ mm}$$

$$| w_{12} - w_6 | < 0.5 \text{ mm} \qquad | w_{12} - w_7 | < 0.5 \text{ mm}$$

$$w_1 < 10 \text{ mm} \qquad w_{12} < 10 \text{ mm}$$

As expected from the experience gained on the simplified problems, a good convergence was obtained with the mixed method MM1, using the conjugate gradient algorithm, by setting $\delta = 1.1$ and $\bar{k} = 500$, that is slightly below the number of design variables. The choice of $\bar{k}$ was guided by the fact that the convergence of first order minimization algorithms is achieved in a multiple of the number of unknowns and hence the same ratio between $\bar{k}$ and the number of design variables was kept as in the previous problems.

The evolution of the weight is illustrated by the figure 8.68. Again the convergence is very satisfactory in 10 to 12 iterations. The maximum deflection of the trailing edge is not reached. The satisfaction of the exact constraints (8.3.1) can be verified a posteriori. The initial scale up of the weight is due to the fact that the original design of the airbrake did not satisfy the flexibility constraints.

The computing times for this final application are, on IBM 370/158

	preprocessing	282.	
13 iterations	analysis	813.	
	minimization	196	
→	total	13 203	seconds.

DATA

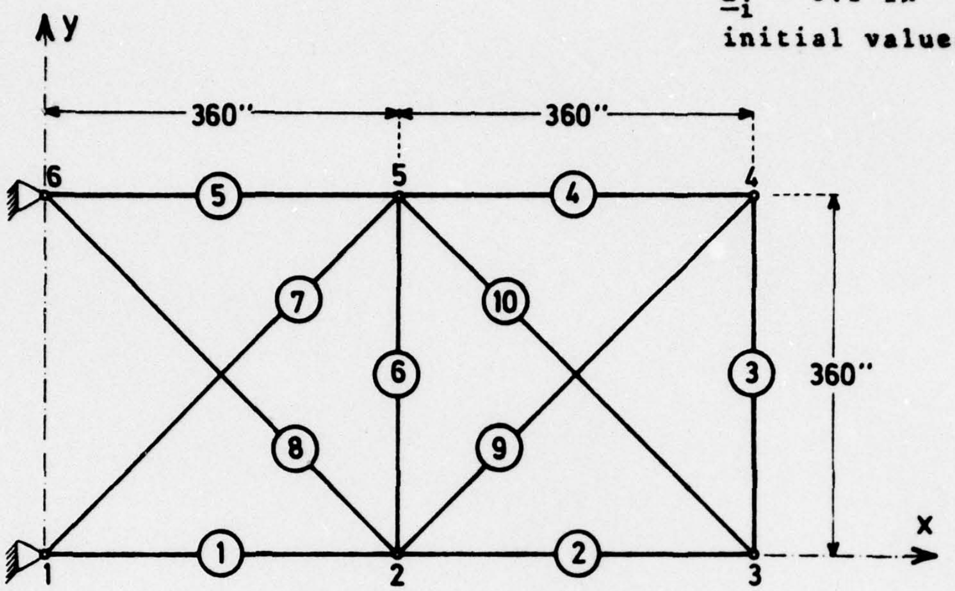
$E = 10^7$ psi

$\rho = 0.1$ lb/in^3

$\bar{\sigma}_k = 25000$ psi

$\underline{a}_i = 0.1$ in^2

initial values : $a_i^o = 20.$ in^2



10 BAR TRUSS

FIGURE 8.1

10 BAR TRUSS - PROBLEM 1.B

FINAL DESIGNS

element	Cross sections (in^2)							
	SAMCEF $\bar{k}$ limited	SAMCEF (HOC)	SAMCEF (SOC), Taig & Kerr [T3]	Gellatly & Berke [G13]	Venkayya [V3]	Berke & Khot [B3]	Schmit & Farshi [S3]	Schmit & Miura [S18]
1	23.941	23.200	24.100	20.027	23.408	23.536	24.260	23.76
2	14.733	15.223	15.025	15.598	14.904	14.915	14.260	14.59
3	0.100	0.551	0.563	0.241	0.101	0.527	0.100	0.100
4	0.100	0.100	0.662	0.100	0.123	0.100	0.100	0.100
5	30.730	30.522	31.160	31.352	30.416	30.360	33.432	30.67
6	0.100	0.100	0.100	0.133	0.101	0.100	0.100	0.100
7	20.951	21.036	21.301	22.206	21.084	21.231	20.740	21.07
8	8.540	7.457	7.993	8.347	8.696	7.477	8.338	8.573
9	0.100	0.100	0.936	0.100	0.136	0.100	0.100	0.100
10	20.336	21.523	21.249	22.060	21.077	21.092	19.690	20.96
WEIGHT (lbs)	5076.67	5060.35	5199.11	5112.17	5034.9	5061.36	5089.0	5076.35
nbr of analyses	13 à 16	14	8	13	25	23	24	13

TABLE 8.2

TABLE 8.3

10 BAR TRUSS - PROBLEM 1.B - WEIGHT CONVERGENCE

iteration	WEIGHT (lbs)							
	SAMCEF			Gellatly & Berke [G13]	Venkayya [V3]	Berke & Khot [B8]	Schmit & Farshi [S8]	Schmit & Miura [S18]
	MM2($\bar{k}$=2)	HOC	SOC					
1	8266.1	8266.1	8266.1	8266	8266.1	8266.14	12846.7	7852.9
2	6295.0	6016.2	6016.2	6356	6281.7	5934.36	8733.4	6650.8
3	6009.4	5817.9	5817.9	5980	6065.7	5736.00	9144.6	6161.4
4	5797.6	5688.6	5683.5	5779	5984.5	5675.75	8332.5	5892.6
5	5653.5	5572.8	5572.0	5625	5963.1	5558.76	7243.0	5656.3
6	5537.9	5445.0	5445.0	5547	5920.1	5430.56	6749.6	5426.8
7	5408.4	5302.0	5302.0	5470	5831.6	5285.55	6507.9	5290.8
8	5263.0	5199.1	5199.1	5392	5848.1	5210.61	6384.3	5153.3
9	5172.1	5107.2	5900.6	5323	5819.7	5298.11	6339.8	5110.3
10	5127.6	5075.2	5218.3	5266	5795.9	5173.84	6314.9	5087.2
11	5084.7	5065.3	7555.6	5225	5776.4	8996.24	5998.7	5031.1
12	5076.9	5061.3	7376.9	5200	5760.7	5311.34	5750.1	5076.9
13	5076.7	5060.9	7354.8	5195	5748.2	5113.35	5734.6	
14		5060.9	7065.3	5206	5738.3	5724.31	5705.6	
15			6303.4	5191	5730.7	5104.45	5468.8	
16			5637.0	5169	5724.7	5095.08	5315.8	
17			5265.6	5147	5720.2	9714.69	5306.2	
18			5200.5	5112	5716.7	6146.62	5215.3	
19			5194.2	5897	5713.7	5062.02	5162.9	
20			5135.4	7640	5712.2	9263.72	5135.3	
21			5176.0	8703	5502.9	6696.64	5107.0	
22			5167.4	9029	5343.8	5793.53	5094.1	
23			5160.1	8371	5221.5	5404.56	5089.0	
24			5154.9	7862	5127.0	5223.08		
25			5155.4	7508	5034.9	5135.32		
26			5253.3	7239		5092.16		
27			5317.7	7050		5070.75		
28			5339.3	5318		5061.36		
29			5323.5	5793		9254.35		
30			...	...		...		

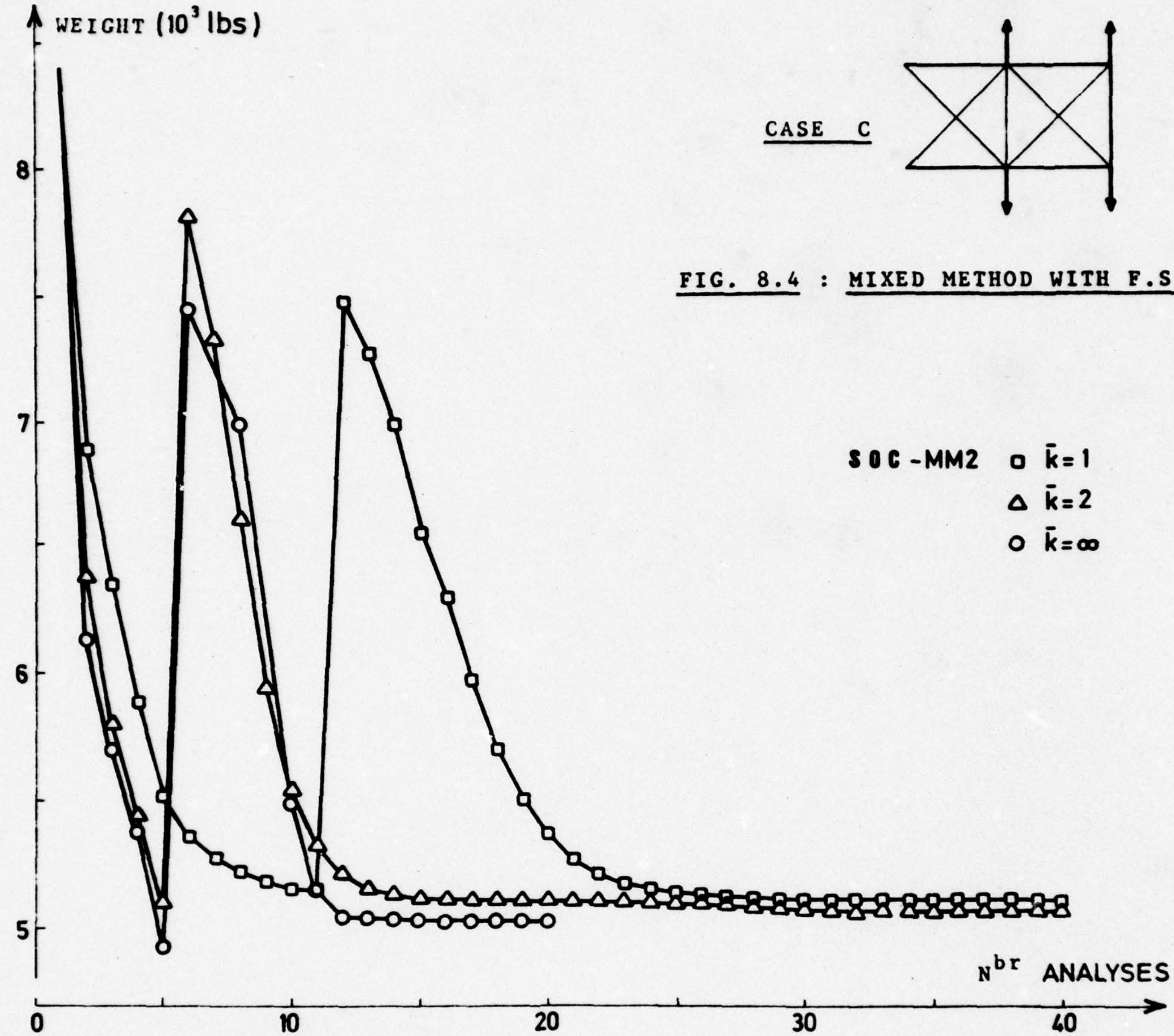


FIG. 8.4 : MIXED METHOD WITH F.S.D.

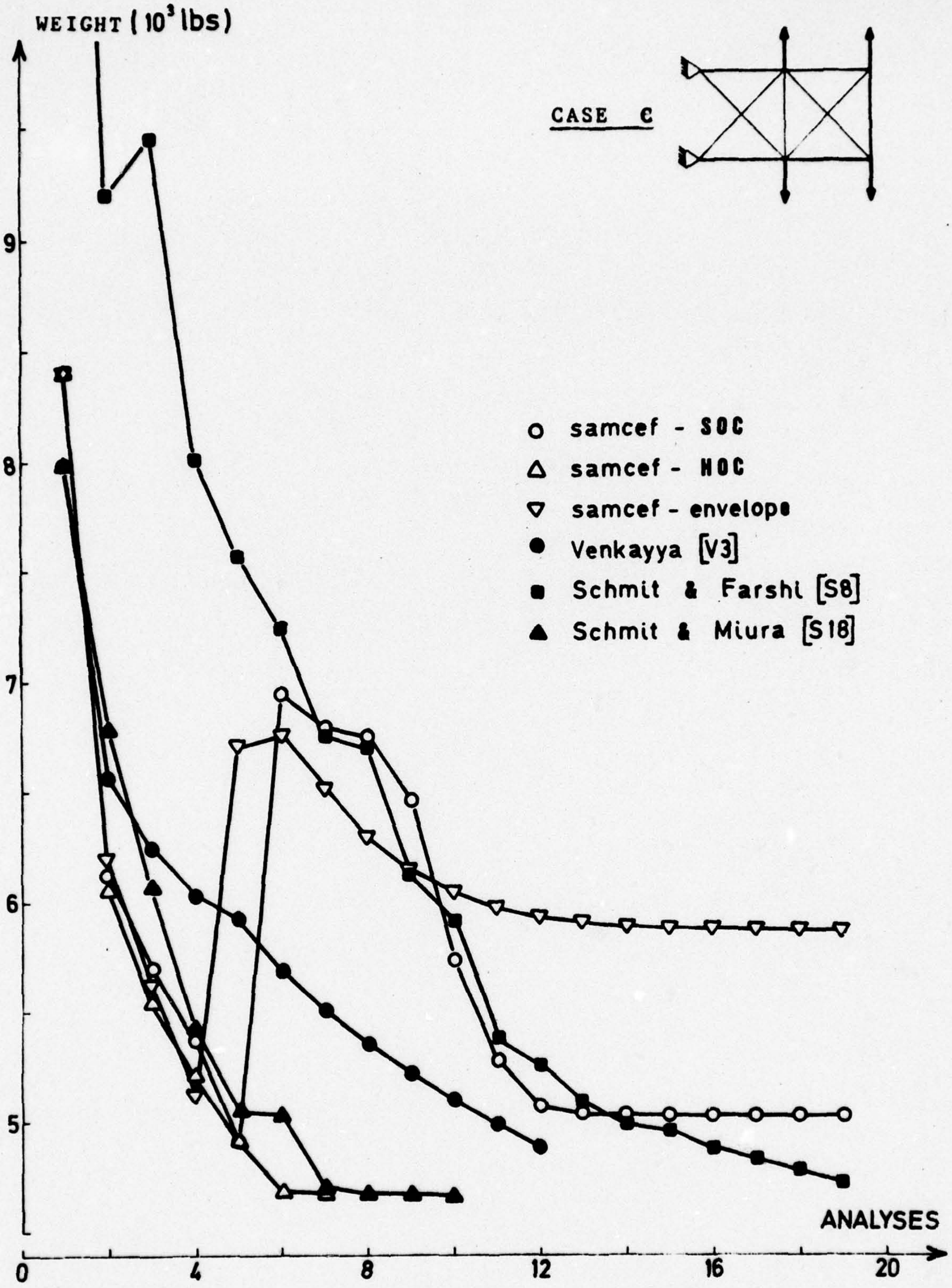


FIG. 8.5 : 10 BAR TRUSS - PROBLEM 1C

10 BAR TRUSS - PROBLEM 1.C

WEIGHT CONVERGENCE

iteration	Weight (lbs)					
	SAMCEF			Venkayya [V3]	Schmit & Farshi [S8]	Schmit & Miura [S13]
	HOC	SOC	envelope			
1	8417.7	8417.7	8417.7	8417.7	13315.7	7933.3
2	6072.4	6126.7	6200.5	6565.2	9204.9	6782.8
3	5553.7	5696.3	5635.1	6242.8	9455.0	6061.7
4	5217.8	5372.2	5134.1	6031.6	8009.2	5427.2
5	4925.8	4916.0	6719.2	5935.4	7665.2	5054.3
6	4684.0	6942.1	6764.4	5636.3	7240.9	5034.7
7	4676.9	6787.7	6517.8	5505.2	6755.6	4700.1
8		6749.4	6301.3	5354.9	6694.1	4680.3
9		6473.4	6147.1	5220.2	6143.7	4677.1
10		5744.0	6045.9	5099.0	5915.3	4677.0
11		5286.0	5981.8	4991.4	5377.8	
12		5079.0	5941.7	4895.6	5269.7	
13		5040.5	5916.6		5096.7	
14		5037.1	5901.0		4936.3	
15		5033.6	5891.3		4964.1	
16		5030.9	5885.5		4832.1	
17		5029.3	5882.1		4826.8	
18		5028.3	5880.2		4736.6	
19		5027.9	5879.4		4722.3	
20		5027.8	5879.2		4706.1	
21		5028.1	5879.4		4636.5	
22		5028.2	5879.8		4691.8	
23		5028.1	5880.4			
24		5028.0	5881.0			
25		5028.0	5881.6			
26		5027.9	5882.2			
27		5027.9	5882.7			
28		5027.9	5883.2			
29		5027.9	5883.7			
30		5027.9	5884.2			

TABLE 8.7

10 BAR TRUSS - PROBLEM 1.D

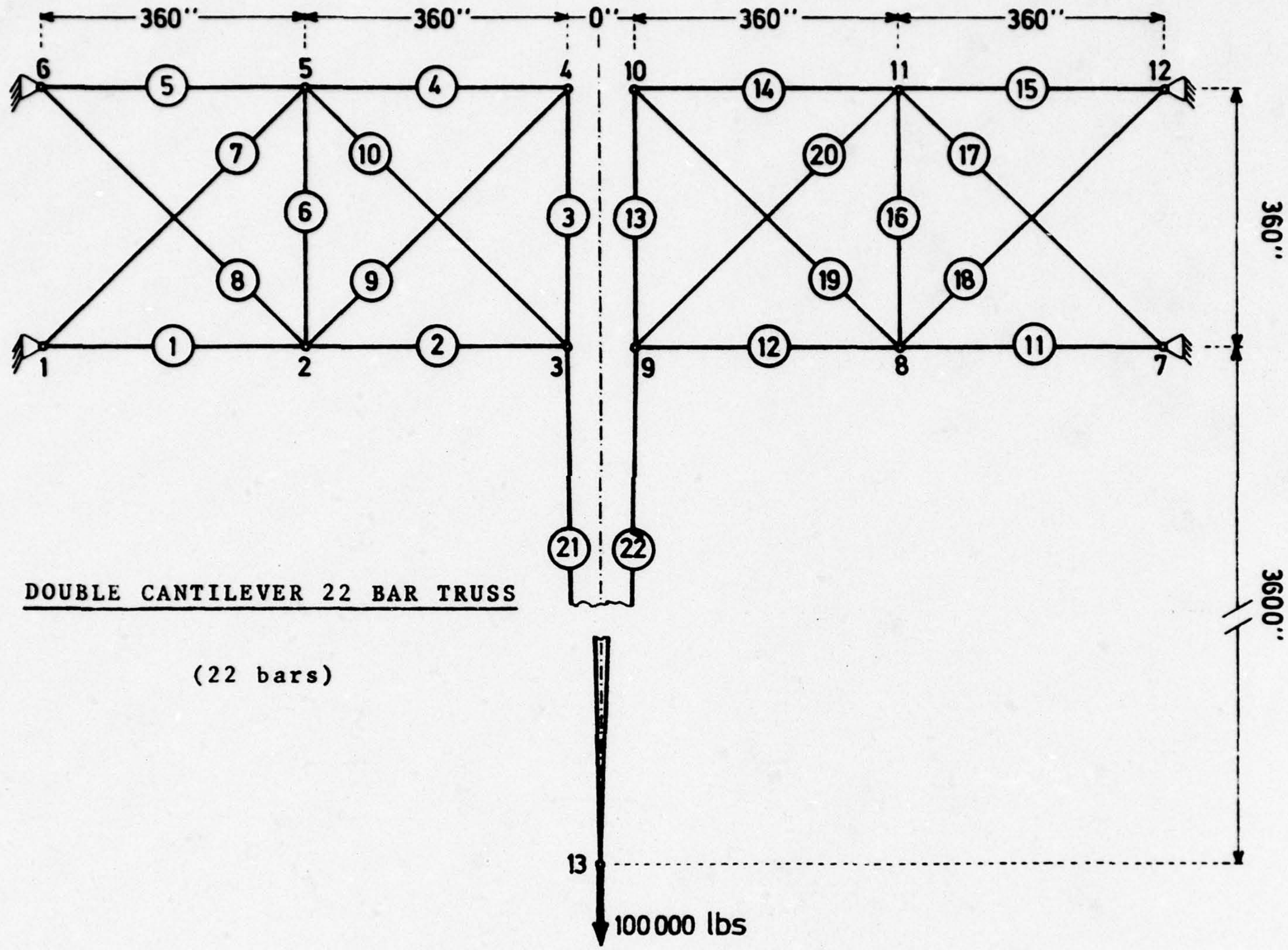
FINAL DESIGN

element	section	element	section
1	21.579	6	0.100
2	3.434	7	14.545
3	0.100	8	12.695
4	1.401	9	1.932
5	22.663	10	11.928

WEIGHT CONVERGENCE

iteration	Weight (lbs)	imposed displacements		stress (psi) element 3
		node 4	node 5	
1	7963.06	2.0	0.8824	- 2114
2	5141.56	2.0	0.9362	- 14223
3	4955.10	1.834	1.0	- 18792
4	4319.40	2.0	0.9475	- 21470
5	4371.69	2.0	0.8835	- 15979
6	4069.00	2.0	0.9925	- 24407
7	4051.33	1.999	0.9995	- 25000
8	4043.96	2.0	1.0	- 25000

TABLE 8.8



DOUBLE CANTILEVER 22 BAR TRUSS

(22 bars)

FIGURE 8.9

DOUBLE CANTILEVER 22 BAR TRUSS

GENERAL DATA

$E = 10^7$ psi

$\underline{a} = 0.001$ in^2

$a^o = 1.0$ in (initial design)

Loading : node 13, component Y, - 100 000 lbs

ALLOWABLE STRESS LIMITS AND MASS DENSITY

problem	elements	$\bar{\sigma}$ (psi)	ρ (lb/in^3)
2A	1-20	25000	0.1
	21,22	500000	0.1
2B	1-10	50000	0.1
	11-20	25000	0.1
	21,22	500000	0.1
2C	1-10	50000	0.4
	11-20	25000	0.1
	21,22	500000	0.1

TABLE 8.10

DOUBLE CANTILEVER 22 BAR TRUSS - PROBLEM 2.A

FINAL DESIGN

elements	FSD		GOC	
	section (in^2)	Stress (psi)	section (in^2)	Stress (psi)
1,11	2.00069	- 25000	2.01023	- 25000
2,12	1.99937	- 25000	1.99038	- 24990
3,13	0.00100	15818	0.01191	20783
4,14	0.00100	15818	0.01191	20783
5,15	3.99932	25000	3.99094	24993
6,16	0.00100	- 1374	0.00100	- 8131
7,17	2.32745	- 25000	2.81506	- 24990
8,18	0.00100	24313	0.01731	20896
9,19	0.00100	- 22370	0.01684	- 20738
10,20	2.32753	25000	2.81553	24990
21,22	0.10000	500000	0.10002	499993
Weight (lbs)	1224.185		1226.031	
number of analyses	50		30	

TABLE 8.11

TABLE 8.12

DOUBLE CANTILEVER 22 BAR TRUSS - PROBLEMS 2A, 2B, 2C

WEIGHT CONVERGENCE

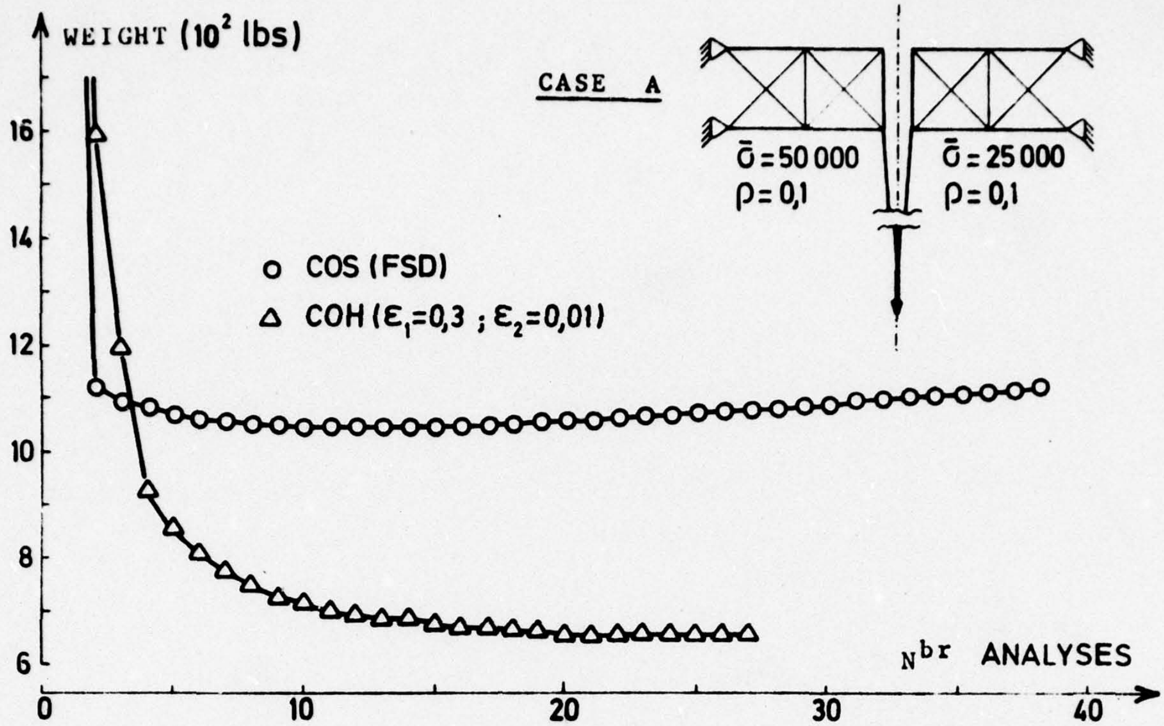
iteration	Weight (lbs)					
	2A - σ,ρ uniform		2B - σ non uniform		2C - σ,ρ non uniform	
	FSD	GOC	FSD	HOC	FSD	HOC
1	4696.3	4696.3	4696.3	4696.76	8488.8	8483.8
2	1437.6	1422.5	1120.0	1591.02	2153.5	2463.1
3	1399.7	1363.7	1096.7	1195.26	2036.3	1970.2
4	1371.6	1349.0	1030.7	925.96	2030.7	1811.8
5	1350.6	1333.0	1070.2	853.13	1935.9	1696.7
6	1334.1	1319.2	1062.9	809.17	1948.1	1600.5
7	1320.1	1307.5	1057.5	775.29	1914.5	1521.1
8	1307.7	1297.1	1053.3	749.39	1883.6	1457.5
9	1296.3	1287.9	1050.0	723.56	1855.2	1407.2
10	1287.1	1279.3	1047.5	712.49	1828.6	1367.7
11	1278.4	1272.5	1045.3	699.93	1804.2	1336.9
12	1270.7	1265.9	1044.3	690.10	1781.3	1313.1
13	1263.9	1260.1	1044.4	682.40	1760.0	1294.6
14	1253.0	1255.0	1044.7	685.90	1740.2	1301.5
15	1252.9	1250.5	1045.4	676.00	1721.7	1278.5
16	1248.5	1246.6	1046.7	668.91	1704.4	1262.9
17	1244.7	1243.2	1048.4	666.96	1688.1	1257.6
18	1241.5	1240.3	1050.5	663.33	1672.3	1249.0
19	1238.7	1237.7	1052.9	661.34	1653.4	1246.6
20	1236.4	1235.6	1055.6	655.00	1644.7	1236.7
21	1234.4	1233.3	1058.4	654.18	1631.7	1230.5
22	1232.3	1232.2	1061.5	654.14	1619.3	1230.3
23	1231.3	1230.9	1064.3	654.10	1607.4	1230.2
24	1230.2	1229.3	1068.1	654.08	1596.0	1230.1
25	1229.2	1223.9	1071.5	654.07	1585.0	1230.1
26	1228.3	1223.1	1075.0	654.05	1574.4	1230.0
27	1227.6	1227.4	1078.6	654.05	1564.1	1230.0
...	...	...	...		...	
50	1224.2	1224.2	1147.3		1391.4	

TABLE 8.13

DOUBLE CANTILEVER 22 BAR TRUSS - PROBLEM 2.B

FINAL DESIGN

elements	FSD		HOC	
	section (in^2)	stress (psi)	section (in^2)	stress (psi)
1	0.90980	- 46524	1.99069	- 50000
2	0.72350	- 48127	1.98937	- 50000
3	0.09297	40284	0.00100	31661
4	0.09297	40284	0.00100	31661
5	1.53960	47654	3.97931	50000
6	0.00103	- 16756	0.00100	- 2875
7	1.02267	- 48127	2.81331	- 50000
8	0.13199	40314	0.00100	48563
9	0.13148	- 40284	0.00100	- 44776
10	1.02318	48127	2.81339	50000
11	2.78963	- 24167	0.02069	- 25000
12	2.21850	- 25000	0.01937	- 25000
13	0.28538	20930	0.00100	15313
14	0.28538	20930	0.00100	15813
15	4.72201	24754	0.03931	25000
16	0.00103	- 8720	0.00100	- 1374
17	3.13690	- 25000	0.02731	- 25000
18	0.40411	20934	0.00100	24313
19	0.40358	- 20930	0.00100	- 22370
20	3.13744	25000	0.02739	25000
21	0.08165	472340	0.20729	430000
22	0.12519	490719	0.00100	500000
Weight (lbs)	1044.42		654.049 (non FSD)	
number of analyses.	13		27	



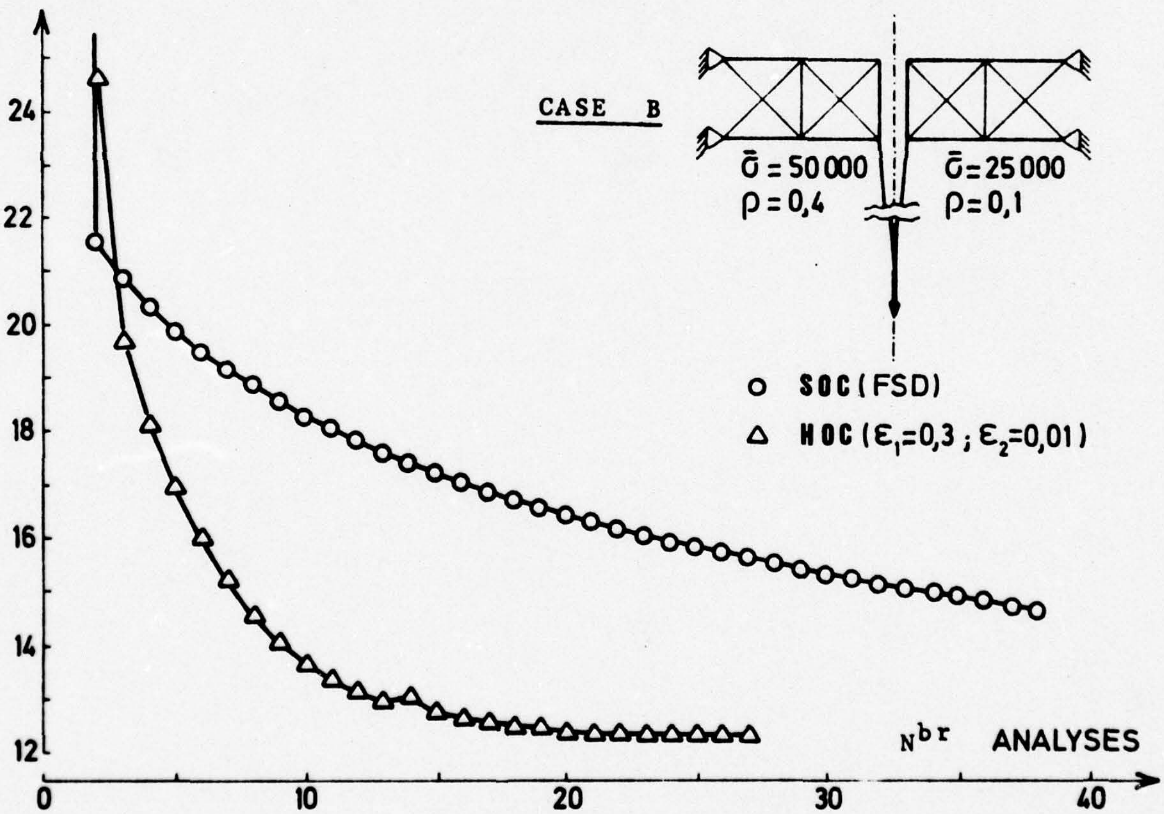


FIGURE 8.14

DOUBLE CANTILEVER 22 BAR TRUSS

WEIGHT CONVERGENCE

TABLE 8.15

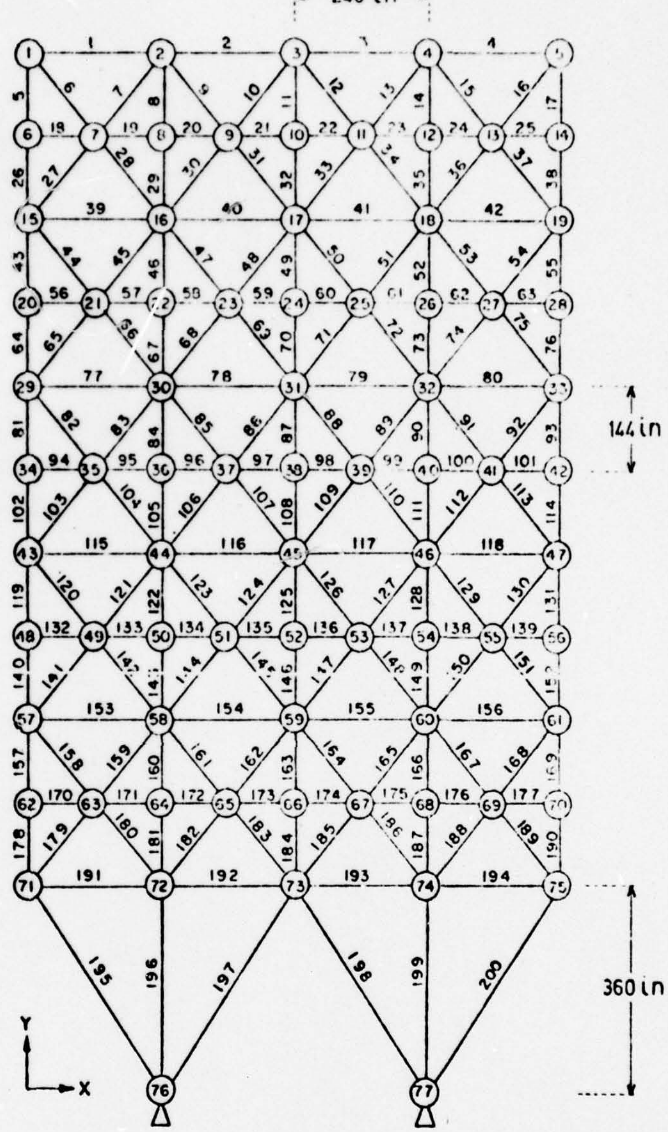
DOUBLE CANTILEVER 22 BAR TRUSS - PROBLEM 2.C - FINAL DESIGN

elements	FSD		HOC	
	section (in^2)	stress (psi)	section (in^2)	stress (psi)
1	0.27941	- 48151	0.01029	- 50000
2	0.27813	- 48144	0.00397	- 50000
3	0.00101	30461	0.00100	31635
4	0.00101	30461	0.00100	31635
5	0.55632	48146	0.01351	50000
6	0.00101	- 2645	0.00100	- 2748
7	0.39326	- 48144	0.01260	- 50000
8	0.00101	46819	0.00100	48626
9	0.00101	- 43078	0.00100	- 44739
10	0.39333	48144	0.01268	50000
11	3.46404	- 24999	3.93149	- 25000
12	3.46255	- 25000	3.93017	- 25000
13	0.00101	16175	0.00100	16477
14	0.00101	16175	0.00100	16477
15	6.92560	25000	7.96091	25000
16	0.00101	- 3100	0.00100	- 4553
17	4.89656	- 25000	5.62873	- 25000
18	0.00117	23451	0.00100	24313
19	0.00101	- 22875	0.00100	- 22370
20	4.89679	25000	5.62331	25000
21	0.02787	481476	0.00100	480000
22	0.17316	499993	0.19904	500000
Weight (lbs)	1391.37		1230.00 (FSD)	
number of analyses	50		27	

200 bars

150 d.o.f.

105 design variables



200 BAR PLANAR TRUSS

FIGURE 8.16

TABLE 8.17

200 BAR PLANAR TRUSS

General data :

$E = 3\ 10^7$ psi

$\rho = 0.283$ lb/in^3

$\bar{\sigma} = 10000$ psi (in problem 3.B)

$\underline{a} = 0.1$ in^2

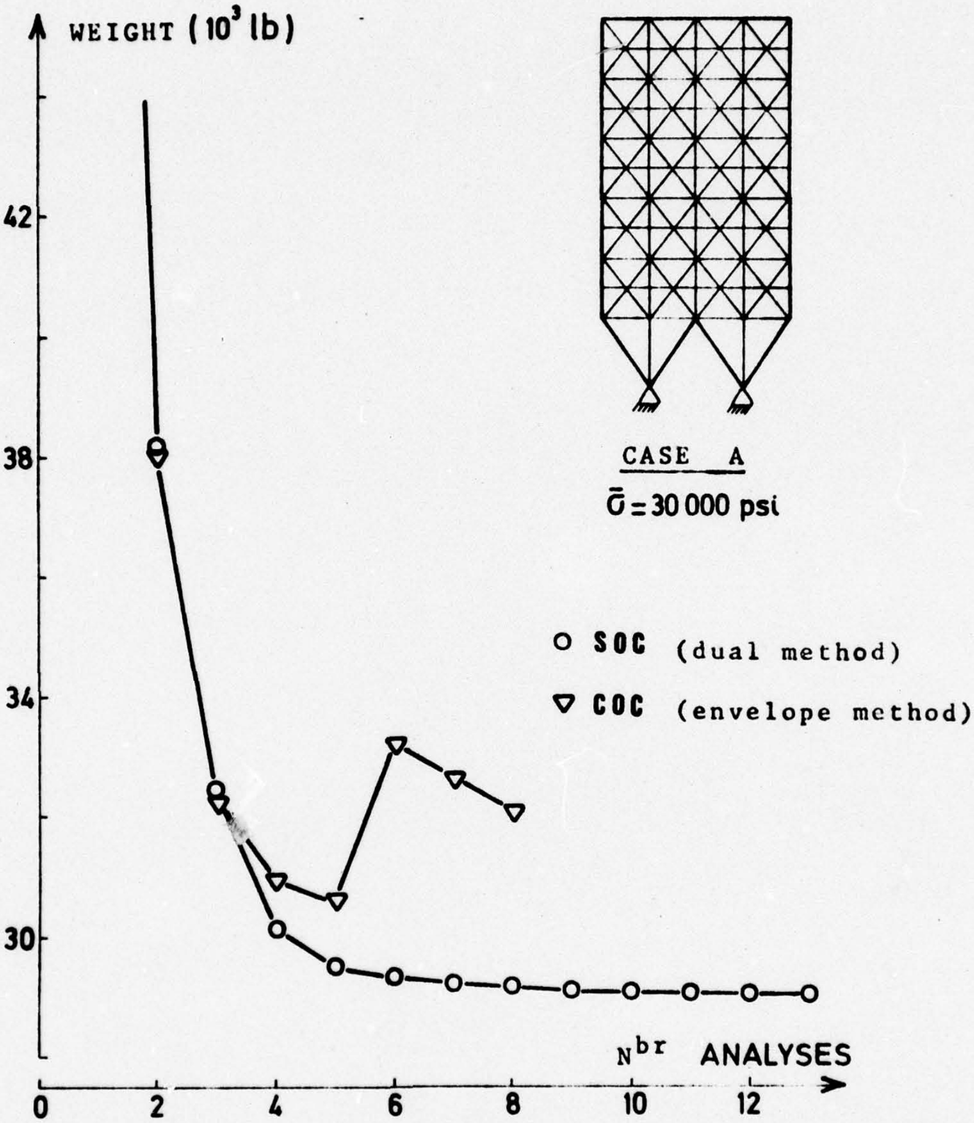
$a^o = 10.0$ in^2 (initial design)

Multiple loading case

Load cases	nodes	load components (lbs)		
		x	y	z
1	1,6,15,20,29,34,43,48,57,62,71	1000	0	0
2	1 , 2 , 3 , 4 , 5 , 6 , 8 , 10 , 12 , 14 , 15 , 16 , 17 , 18 , 19 , 20 , 22 , 24 , 26 , 28 , ... 71 , 72 , 73 , 74 , 75	0	- 10000	0
3	load cases 1 plus 2			

Flexibility constraints (max. values)

node	components (in)		
	x	y	z
1	0.5	0.5	/
2	0.5	0.5	/
3	0.5	0.5	/
4	0.5	0.5	/
5	0.5	0.5	/



200 BAR PLANAR TRUSS - PROBLEM 3.A

FIGURE 8.18

TABLE 8.19

200 BAR PLANAR TRUSS - WEIGHT CONVERGENCE

iteration	Weight (lbs)			
	case A : $\overline{\sigma} = \infty$		case B : $\overline{\sigma}$ = 10000 psi	
	SOC	envelope	SOC	HOC
1	76082	76082	144770	144770
2	38169	38043	42872	42356
3	32449	32249	40351	36102
4	30134	30934	35262	30353
5	29513	30595	29546	29547
6	29348	33216	29343	29340
7	29252	32612	30375	29244
8	29176	32077	30116	29163
9	29117		29495	
10	29090		29293	
11	29071			
12	29053			
13	29037			

EXECUTION TIMES

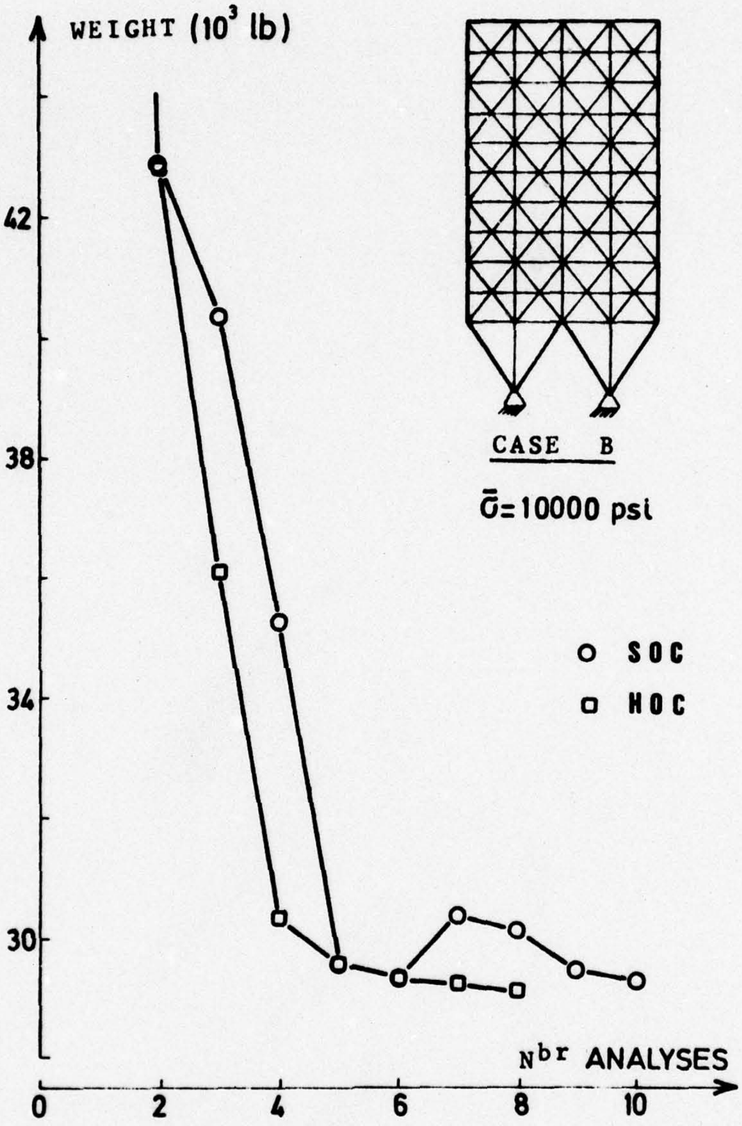
optimization cycle	Nbr of iterations		CPU (sec) IBM 370-158	
	method		method	
	dual	envelope	dual	envelope
1	8	3	2.2	1.2
2	4	3	1.4	1.3
3	6	3	1.9	1.4
4	6	3	1.9	1.5
5	3	3	1.5	1.5
6	4	3	1.6	1.8
7	5	3	2.0	1.8
8	4		1.9	
9	3		1.9	
10	3		1.3	
11	3		1.3	
12	3		1.3	

TABLE 8.20

200 BAR PLANAR TRUSS - FINAL DESIGN

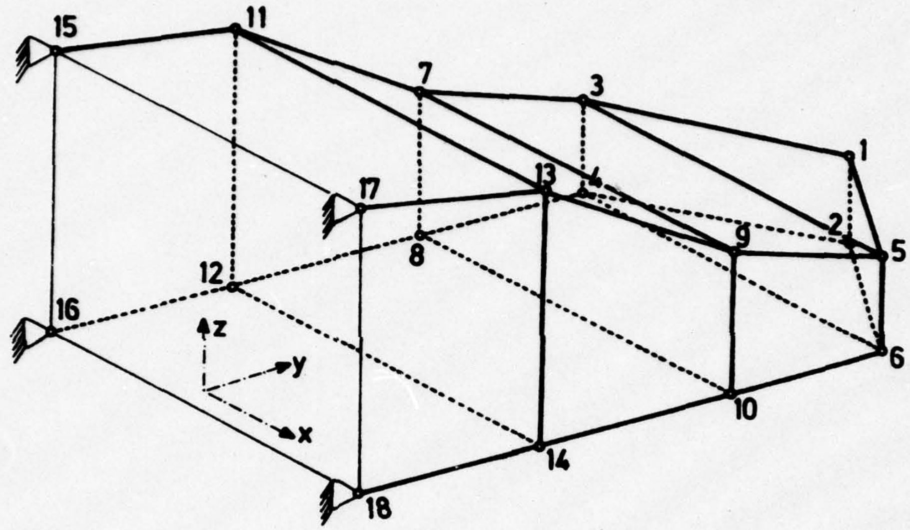
bars	section (in^2)		bars	section (in^2)		bars	section (in^2)	
	case A	case B		case A	case B		case A	case B
1,4	0.147	0.159	66,74	0.741	1.139	133,138	0.177	0.274
2,3	0.100	0.100	67,73	4.975	4.870	134,137	0.177	0.274
5,17	4.333	4.872	68,72	0.251	0.216	135,136	0.243	0.315
6,16	0.230	0.327	69,71	0.316	0.327	140,152	10.046	9.613
7,15	0.100	0.100	70	5.525	5.520	141,151	0.193	0.270
8,14	2.241	2.021	77,80	0.841	1.019	142,150	2.705	2.909
9,13	0.100	0.100	78,79	0.373	0.242	143,149	9.365	10.514
10,12	0.103	0.115	81,93	9.273	8.940	144,148	0.673	0.615
11	2.777	2.795	82,92	1.441	1.735	145,147	0.599	0.593
18,25	0.100	0.100	83,91	0.100	0.100	146	7.470	7.334
19,24	0.100	0.100	84,90	6.527	6.327	153,156	2.460	2.495
20,23	0.100	0.100	85,89	0.443	0.441	154,155	0.766	0.642
21,22	0.100	0.100	86,88	0.403	0.367	157,169	7.639	7.045
26,38	6.863	6.965	87	6.009	5.945	158,168	4.010	4.145
27,37	0.100	0.100	94,101	0.100	0.100	159,167	0.100	0.100
28,36	0.318	0.437	95,100	0.150	0.196	160,166	13.133	13.937
29,35	3.187	2.371	96,99	0.150	0.196	161,165	1.647	1.460
30,34	0.133	0.133	97,98	0.205	0.227	162,164	0.591	0.722
31,33	0.144	0.125	102,114	10.266	9.954	163	7.753	7.554
32	3.981	4.009	103,113	0.169	0.208	170,177	0.100	0.100
39,42	0.332	0.578	104,112	1.586	1.913	171,176	0.130	0.256
40,41	0.210	0.256	105,111	7.097	7.402	172,175	0.130	0.256
43,55	7.960	7.806	106,110	0.415	0.389	173,174	0.169	0.274
44,54	0.640	1.015	107,109	0.470	0.470	178,190	8.329	7.743
45,53	0.100	0.100	108	6.626	6.560	179,189	0.133	0.231
46,52	4.323	4.244	115,118	1.511	1.571	180,188	4.109	4.342
47,51	0.291	0.308	116,117	0.523	0.394	181,187	13.646	14.464
48,50	0.234	0.210	119,131	9.230	8.733	182,186	0.629	0.801
49	4.754	4.749	120,130	2.548	2.675	183,185	1.702	1.557
56,63	0.100	0.100	121,129	0.100	0.100	184	8.240	3.540
57,62	0.100	0.111	122,128	9.326	9.962	191,194	5.845	5.343
58,61	0.100	0.111	123,127	0.473	0.543	192,193	3.549	3.015
59,60	0.100	0.147	124,126	0.654	0.576	195,200	10.657	9.367
64,76	9.239	9.124	125	6.931	6.846	196,199	17.723	18.313
65,75	0.100	0.122	132,139	0.100	0.100	197,198	7.745	7.380

case	stress limit (psi)	method	Final weight (lbs)	number of analyses
A	/	SOC	29037	13
B	10000	HOC	29163	8



200 BAR PLANAR TRUSS - PROBLEM 3.B

FIGURE 8.21



WING CARRY THROUGH BOX

FIGURE 8.22

TABLE 3.23

WING CARRY THROUGH BOX

General data :

$E = 1.6\ 10^7$ psi

$\rho = 0.16\ lb/in^3$

$\bar{\sigma} = 100000$ psi

$\underline{a} = 0.01\ in^2$

$a^o = 50.0\ in^2$

Nodal coordinates

nodes	coordinates (in)			nodes	coordinates (in)		
	x	y	z		x	y	z
1	0.0	140.0	20.0	10	30.0	80.0	- 3.0
2	0.0	140.0	0.0	11	- 30.0	40.0	55.0
3	- 30.0	120.0	21.0	12	- 30.0	40.0	- 5.0
4	- 30.0	120.0	- 1.0	13	30.0	40.0	55.0
5	30.0	120.0	21.0	14	30.0	40.0	- 5.0
6	30.0	120.0	- 1.0	15	- 30.0	0.0	60.0
7	- 30.0	80.0	30.0	16	- 30.0	0.0	- 7.0
8	- 30.0	80.0	- 3.0	17	30.0	0.0	60.0
9	30.0	80.0	30.0	18	30.0	0.0	- 7.0

Loading cases (multiple)

Load case	node	load components (Kips)		
		x	y	z
1	1	2500	- 5000	2500
	2	- 2500	5000	2500
2	1	5000	- 2500	2500
	2	- 5000	2500	2500

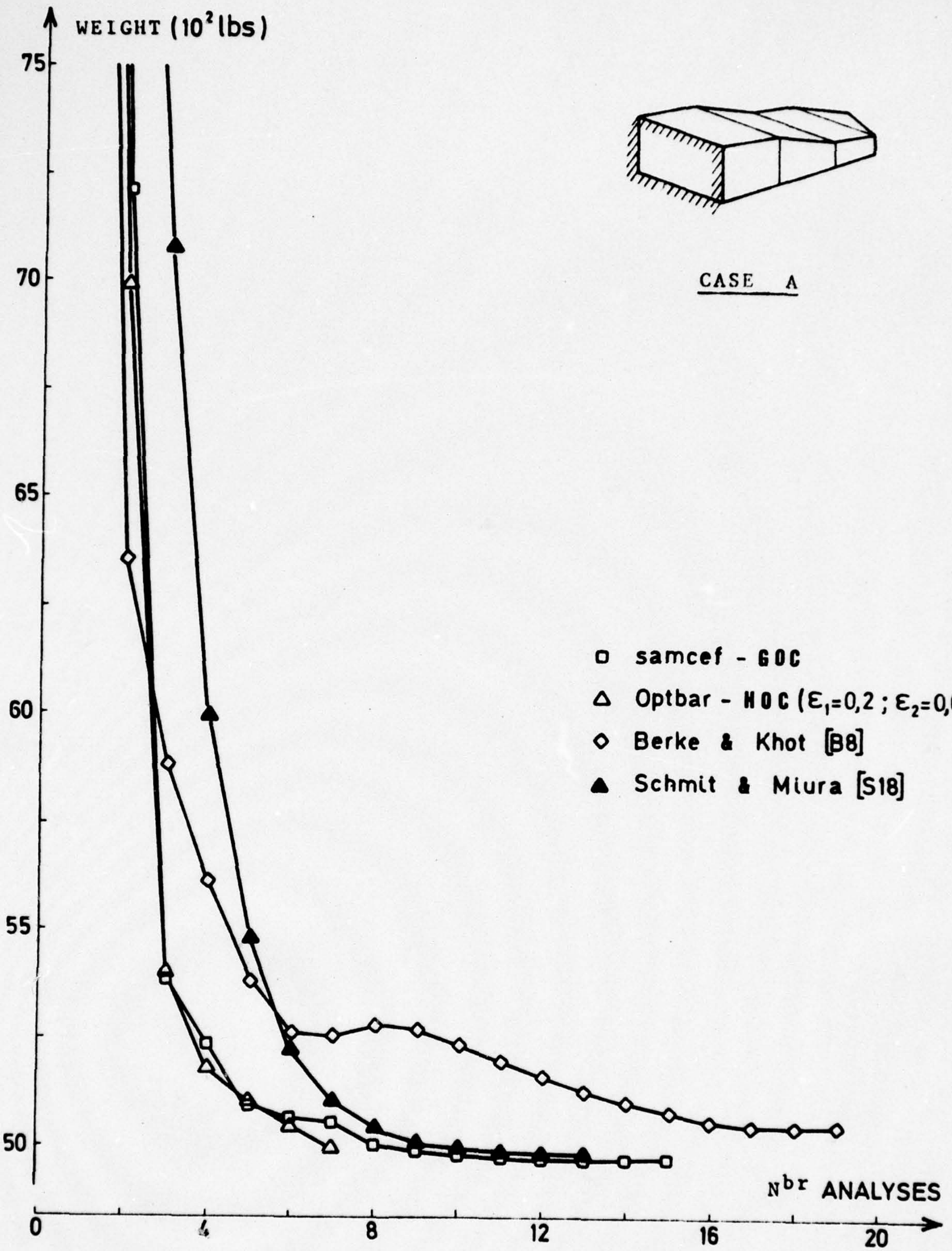
(Continue)

Flexibility constraints

The relative displacement of nodes 1 and 2 in x direction is limited to 1.0 in.

Element definition

element	nodes		element	nodes		element	nodes	
1	1	3	22	12	14	43	1	4
2	2	4	23	1	2	44	5	2
3	1	5	24	3	4	45	3	2
4	2	6	25	5	6	46	5	10
5	7	3	26	7	8	47	3	8
6	8	4	27	9	10	48	9	6
7	9	5	28	11	12	49	7	4
8	10	6	29	13	14	50	9	14
9	11	7	30	3	9	51	7	12
10	12	8	31	4	10	52	13	10
11	13	9	32	5	7	53	11	8
12	14	10	33	6	8	54	13	18
13	15	11	34	7	13	55	11	16
14	16	12	35	8	14	56	17	14
15	17	13	36	9	11	57	15	12
16	18	14	37	10	12	58	5	4
17	3	5	38	11	17	59	3	6
18	4	6	39	12	18	60	9	8
19	7	9	40	13	15	61	7	10
20	8	10	41	14	16	62	13	12
21	11	13	42	1	6	63	11	14



WING CARRY THROUGH BOX - PROBLEM 4.A

FIGURE 8.24

TABLE 8.25

WING CARRY THROUGH BOX

WEIGHT CONVERGENCE

iteration	Weight (lbs)					
	case A : no flex. constr.			case B: with flex. const.		
	SAMCEF	Schmit [S13]	Berke [B8]	SAMCEF	Schmit [S13]	Berke [B3]
1	30214.3	14264.3	30214.0	30214.3	13022.3	30214.0
2	7211.6	9352.6	6360.0	7680.1	9550.6	7577.0
3	5389.7	7079.7	5886.0	6590.6	7544.2	6334.0
4	5230.3	5997.6	5615.0	6398.0	6806.3	6923.0
5	5090.2	5433.4	5385.0	6269.9	6456.7	6301.0
6	5063.5	5230.2	5262.0	6246.4	6284.9	6609.0
7	5050.7	5104.6	5255.0	6199.1	6201.5	6473.0
8	5012.8	5042.2	5284.0	6158.5	6160.4	6388.0
9	4992.6	5009.3	5272.0	6126.2	6159.9	6333.0
10	4984.5	4992.5	5239.3	6123.0	6140.6	6292.5
11	4980.7	4983.1	5201.1	6121.0	6123.8	6262.6
12	4979.0	4932.2	5164.1	6119.8	6120.9	6240.5
13	4977.9	4978.4	5131.8	6119.0		6230.7
14	4976.6		5104.6	6118.4		6215.7
15	4975.1		5082.3	6117.9		6220.1
16			5064.1	6117.6		6253.7
17			5049.3	6117.5		6236.3
18			5049.7			6300.4
19			5051.5			6301.7
20			5053.2			6296.0
21			5054.8			6236.1
22			5056.2			6273.9
23			5057.4			6261.1
24			5058.4			6248.4
25			5059.2			6236.5
26			5059.8			6225.7
27			5060.4			6216.2
28			5060.3			6207.7
29			5060.0			6200.4
30			5059.7			6194.1
...			...			...
...			...			...
...			...			...
45			5042.8			6160.9
46			5041.1			6160.5
47			5039.5			6160.2
48			5037.9			6159.8
49			5036.2			6159.6
50			5034.5			6159.3

TABLE 8.26

WING CARRY THROUGH BOX - FINAL DESIGN

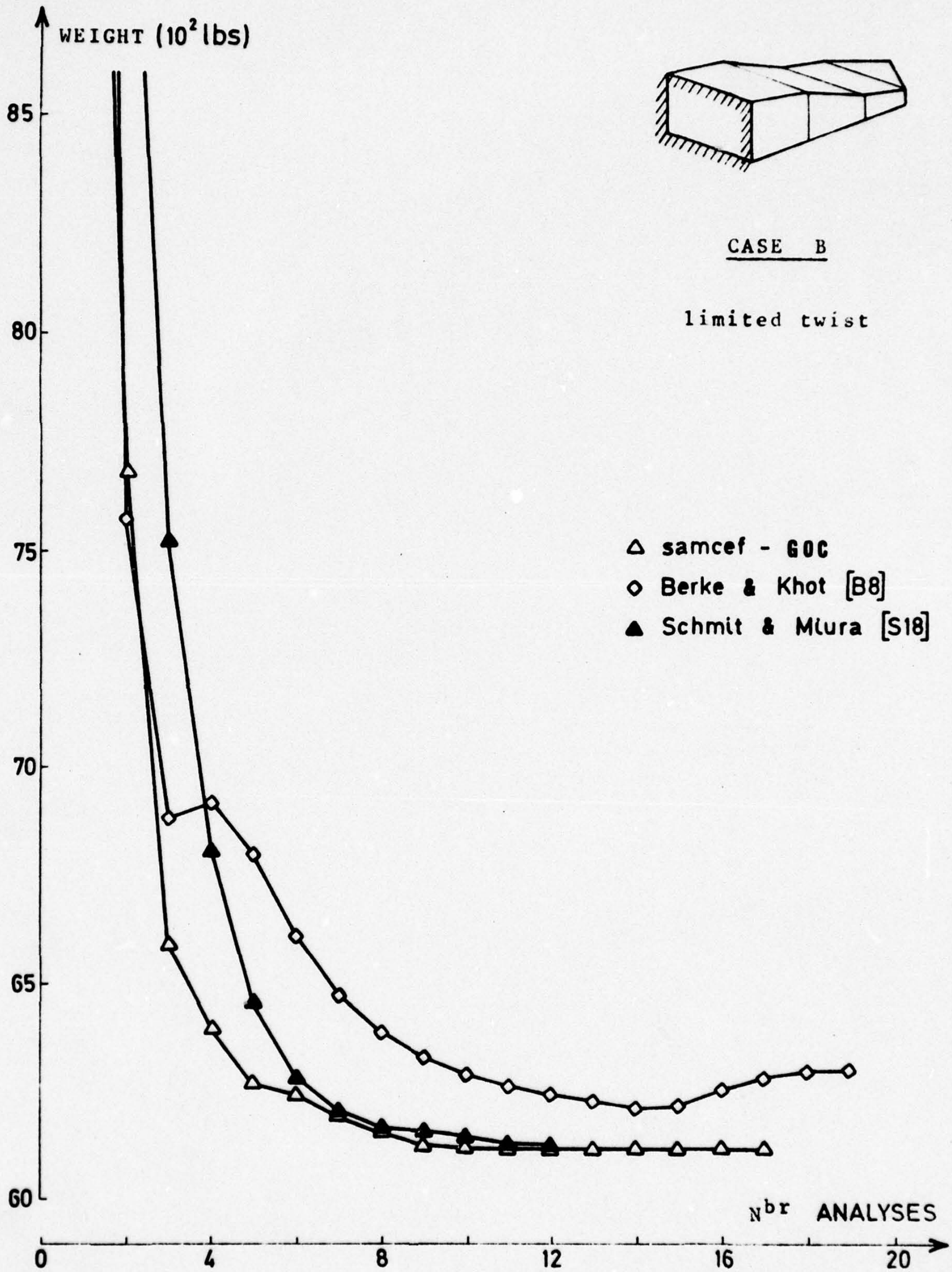
element	cross sections (in^2)					
	case A: no flex. constr.			case B : with flex. constr.		
	SAMCEF	Schmit [S13]	Berke [B3]	SAMCEF	Schmit [S13]	Berke [B3]
1	38.282	38.28	38.78	37.629	37.55	36.86
2	35.918	35.93	36.40	36.336	36.49	36.90
3	51.689	51.69	52.38	52.535	52.66	53.33
4	54.487	54.49	55.04	53.302	53.76	53.31
5	24.994	24.98	25.44	23.672	23.79	24.13
6	28.448	28.46	28.69	28.965	28.95	27.32
7	17.624	17.64	17.73	17.255	17.26	17.35
8	20.547	20.52	20.75	21.281	21.40	22.00
9	25.173	25.21	25.32	26.100	26.06	23.42
10	26.844	26.82	27.49	25.126	25.15	25.95
11	7.526	7.535	7.62	8.794	8.734	9.44
12	8.772	8.801	8.82	9.007	8.966	9.82
13	24.203	24.21	24.62	23.422	23.43	22.37
14	20.630	20.63	20.98	19.594	19.57	18.59
15	4.097	4.123	4.16	5.172	5.165	5.79
16	2.454	2.495	2.38	2.948	2.956	4.47
17	37.051	37.07	37.53	37.121	37.07	36.39
18	37.133	37.14	37.65	37.345	37.30	37.52
19	0.010	0.010	0.01	0.010	0.010	0.01
20	0.010	0.010	0.01	0.010	0.010	0.01
21	0.189	0.151	0.07	0.099	0.218	0.15
22	0.103	0.067	0.01	0.061	0.170	0.01
23	0.132	0.137	0.08	0.010	0.010	0.01
24	1.100	1.035	1.22	0.010	0.010	0.18
25	0.052	0.065	0.07	0.010	0.010	0.01
26	2.592	2.574	2.83	4.042	4.191	6.11
27	0.797	0.804	0.31	0.911	0.935	0.01
28	4.607	4.532	4.37	3.251	3.235	4.43
29	0.641	0.670	0.51	0.010	0.010	1.15
30	2.630	2.651	2.69	8.263	7.861	6.94
31	2.568	2.580	2.70	8.064	7.799	9.76
32	5.845	5.829	5.89	8.394	9.300	11.03
33	5.845	5.839	5.32	8.982	9.229	8.09
34	6.139	6.122	6.19	9.342	9.769	11.59
35	5.345	5.839	5.32	8.983	9.230	8.09

(continue)

TABLE 8.26

(continued)

element	cross sections (in^2)					
	case A: no flex. constr.			case B: with flex. constr.		
	SAMCEF	Schmit [S18]	Berke [B3]	SAMCEF	Schmit [S13]	Berke [B3]
36	2.762	2.783	2.32	3.679	3.257	7.30
37	2.568	2.579	2.70	8.066	7.301	9.77
38	2.704	2.705	2.71	8.183	7.333	6.93
39	2.604	2.603	2.70	3.084	7.352	9.77
40	5.731	5.736	5.33	8.934	9.134	10.92
41	5.314	5.321	5.30	8.968	9.131	3.09
42	16.437	16.45	16.60	25.475	25.23	24.61
43	13.784	18.80	18.99	27.318	27.07	24.54
44	10.986	11.01	11.25	19.047	19.35	19.78
45	13.381	13.40	13.66	20.088	21.18	21.63
46	11.434	11.42	11.60	16.598	16.81	17.19
47	5.948	5.961	6.03	12.462	12.65	12.98
48	12.148	12.16	12.24	18.732	13.55	18.09
49	14.266	14.25	14.40	20.089	19.91	19.67
50	7.223	7.240	7.26	6.666	6.650	6.73
51	7.451	7.416	7.35	5.686	5.753	7.49
52	5.519	5.501	5.62	8.102	3.123	9.50
53	0.497	0.566	0.01	3.336	3.642	0.01
54	3.639	3.639	3.67	5.946	5.986	6.79
55	9.686	9.631	9.97	11.966	11.93	8.90
56	4.392	4.375	4.54	5.377	5.343	3.30
57	0.245	0.310	0.03	1.593	1.691	3.38
58	0.127	0.051	0.01	0.010	0.010	0.01
59	0.129	0.125	0.01	0.010	0.010	0.01
60	0.010	0.010	0.01	0.010	0.010	0.01
61	0.010	0.010	0.01	0.010	0.010	0.01
62	0.010	0.010	0.01	0.010	0.010	0.01
63	0.010	0.015	0.01	0.010	0.010	0.01
Weight (lbs)	4975.1	4976.0	5034.5	6117.5	6120.9	6159.3
nbr. of analyses	15	14	50	17	13	50



WING CARRY THROUGH BOX - PROBLEM 4.B

FIGURE 8.27

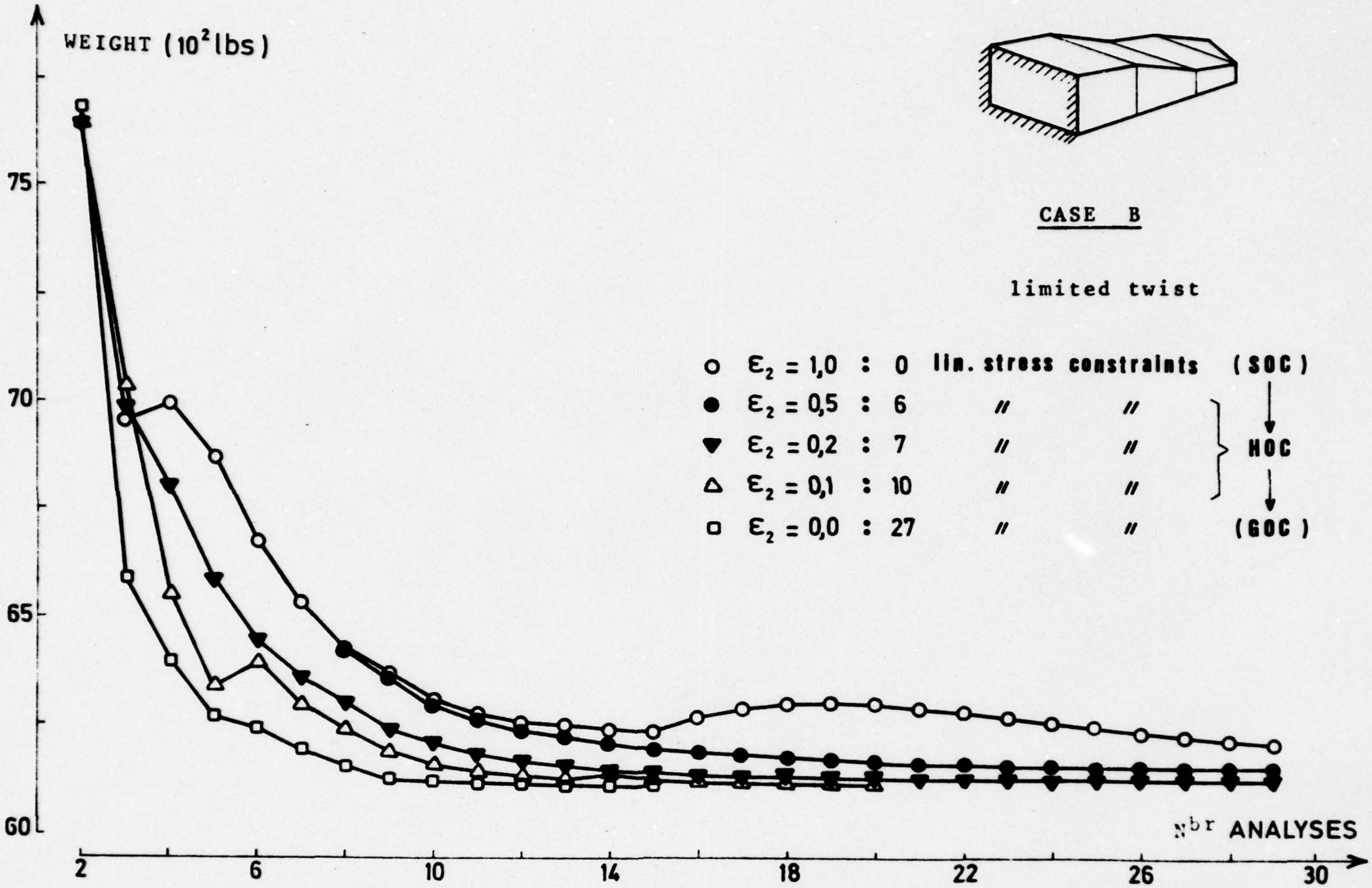

WEIGHT (10^2 lbs)
75
70
65
60
2
6
10
14
18
22
26
30
N^{br} ANALYSES
CASE B
limited twist
ε_2 = 1,0 : 0 lin. stress constraints (SOC)
ε_2 = 0,5 : 6 " "
ε_2 = 0,2 : 7 " "
ε_2 = 0,1 : 10 " "
ε_2 = 0,0 : 27 " "
HOC
(GOC)

FIGURE 8.28

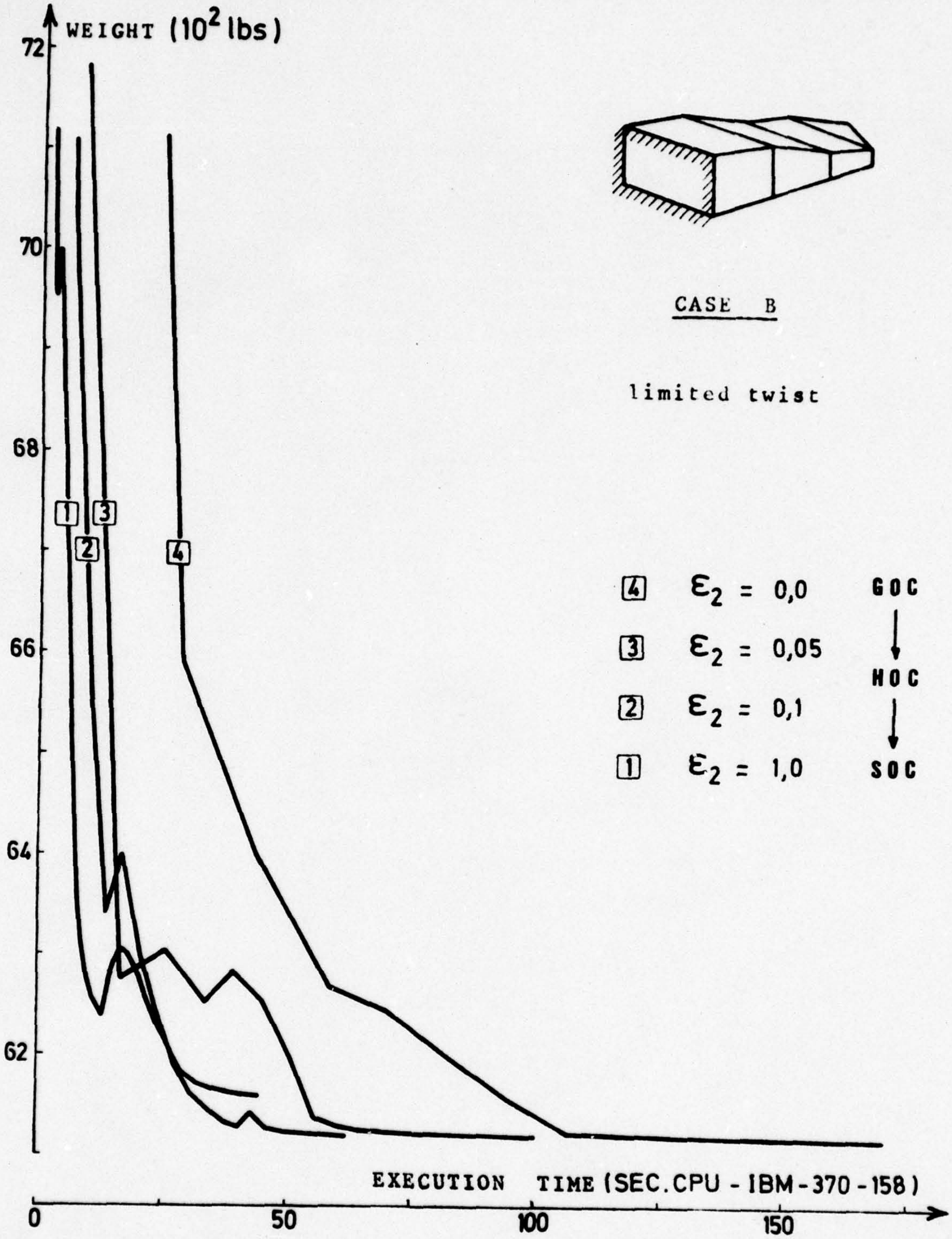
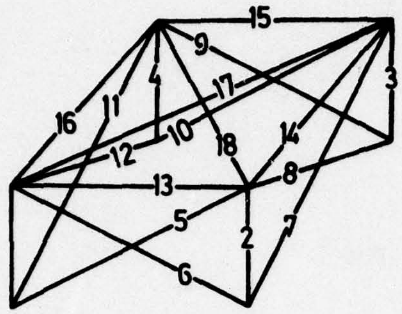
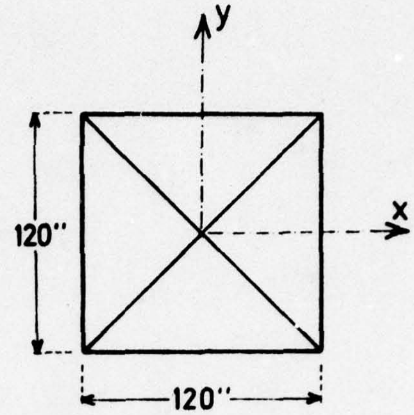

WEIGHT (10² lbs)
72
70
68
66
64
62
1
2
3
4
CASE B
limited twist
4 ε₂ = 0,0 GOC
3 ε₂ = 0,05
2 ε₂ = 0,1 HOC
1 ε₂ = 1,0 SOC
EXECUTION TIME (SEC.CPU - IBM-370-158)
0
50
100
150

FIGURE 8.29

element numbering

(first level)





72 bars

64 D.O.F.

16 design variables

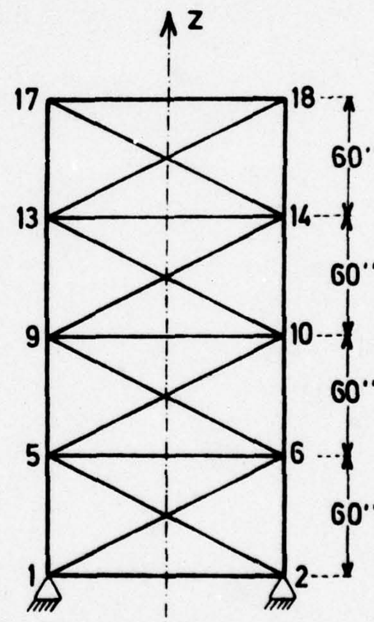


FIGURE 8.30

FOUR LEVEL SPACE TRUSS

FOUR LEVEL SPACE TRUSS

General data :

$E = 10^7$ psi

$\rho = 0.1$ lb/in^3

$\bar{\sigma} = 25000$ psi

$\underline{a} = 0.1$ in^2

$a^o = 1.0$ in^2

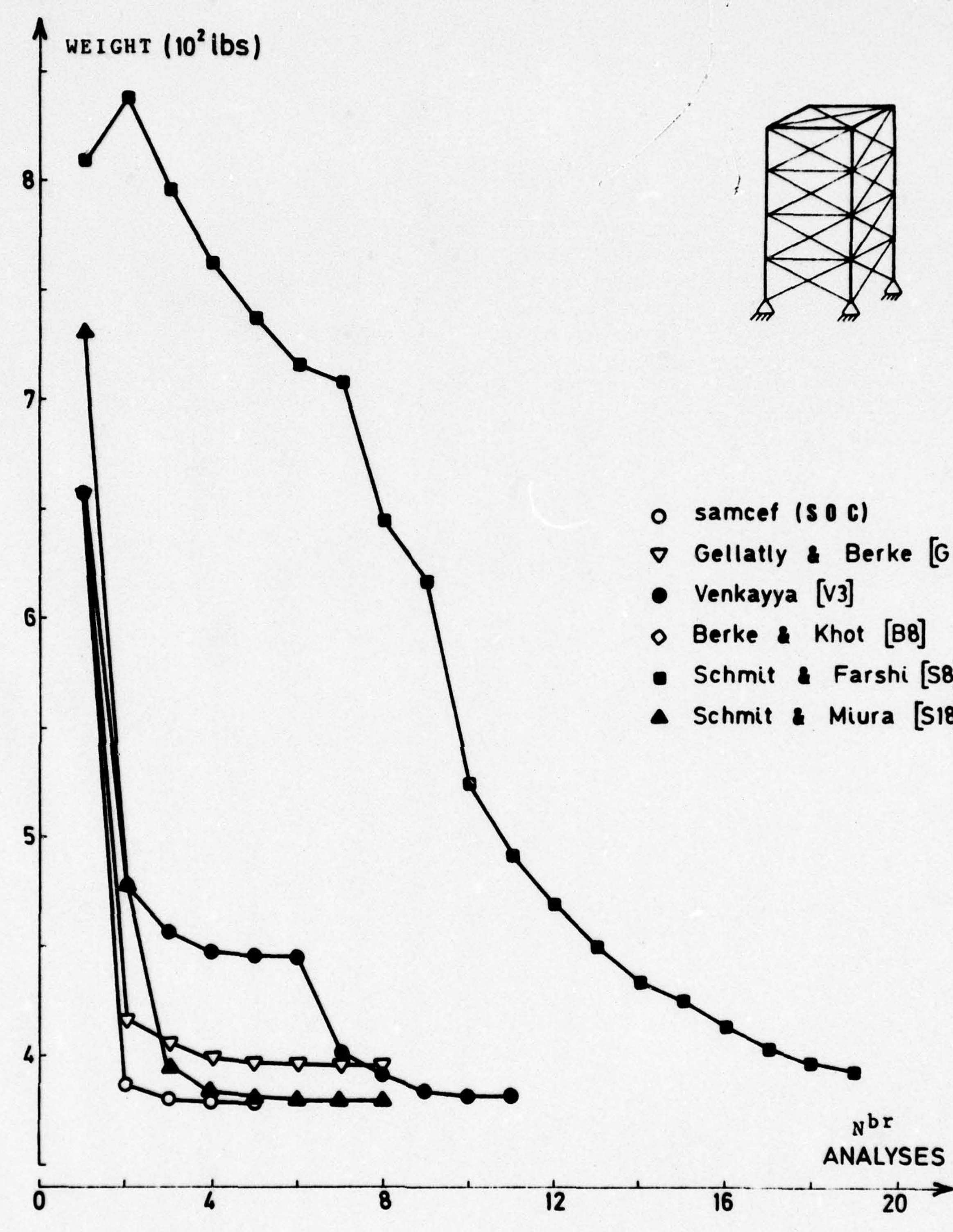
loading cases

load cases	node	load component (lbs)		
		x	y	z
1	17	5000	5000	- 5000
2	17	0	0	- 5000
	18	0	0	- 5000
	19	0	0	- 5000
	20	0	0	- 5000

Flexibility constraints

node	max. allowable displacements (in)		
	x	y	z
17	0.25	0.25	/
18	0.25	0.25	/
19	0.25	0.25	/
20	0.25	0.25	/

TABLE 8.31



WEIGHT (10^2 lbs)
8
7
6
5
4
samcef (S O C)
Gellatly & Berke [G13]
Venkayya [V3]
Berke & Khot [B8]
Schmit & Farshi [S8]
Schmit & Miura [S18]
N^{br} ANALYSES
0
4
8
12
16
20

FIGURE 8.32

TABLE 8.33

FOUR LEVEL SPACE TRUSS - WEIGHT CONVERGENCE

iteration	Weight (lbs)					
	SAMCEF, Taig & Kerr[T3]	Gellatly & Berke [G13]	Venkayya [V3]	Berke & Khot [B8]	Schmit & Farshi [S3]	Schmit & Miura [S13]
1	656.77	656.77	656.8	656.77	309.12	731.15
2	387.25	416.07	478.6	387.01	838.09	477.95
3	379.78	406.21	455.0	379.67	796.16	397.43
4	379.68	399.06	446.9	379.37	763.61	383.27
5	379.66	396.32	445.5		736.69	380.47
6		396.25	445.4		716.63	379.86
7		396.02	401.7		708.77	379.63
8		395.97	391.5		645.07	379.64
9			333.6		616.97	
10			331.6		525.29	
11			331.2		491.96	
12					468.69	
13					450.22	
14					433.77	
15					423.94	
16					413.65	
17					404.08	
18					397.43	
19					393.88	
20					338.14	
21					388.63	

TABLE 8.34

FOUR LEVEL SPACE TRUSS - FINAL DESIGN

elements	Cross sections (in^2)					
	SAMCEF, Taig & Kerr[T3]	Gellatly & Berke [G13]	Venkayya [V3]	Berke & Khot [B8]	Schmit & Farshi [S8]	Schmit & Miura [S18]
1-4	1.8973	1.4636	1.818	1.8931	2.0734	1.385
5-12	0.5158	0.5207	0.523	0.5171	0.5034	0.5125
13-16	0.1000	0.1000	0.100	0.1000	0.1000	0.1000
17,18	0.1000	0.1000	0.100	0.1000	0.1000	0.1000
19-22	1.2801	1.0235	1.246	1.2793	1.1067	1.267
23-30	0.5148	0.5421	0.524	0.5149	0.5792	0.5118
31-34	0.1000	0.1000	0.100	0.1000	0.1000	0.1000
35,36	0.1000	0.1000	0.100	0.1000	0.1000	0.1000
37-40	0.5067	0.5521	0.611	0.5032	0.2643	0.5233
41-48	0.5200	0.6034	0.532	0.5196	0.5480	0.5173
49-52	0.1000	0.1000	0.100	0.1000	0.1000	0.1000
53,54	0.1000	0.1000	0.100	0.1000	0.1500	0.1000
55-58	0.1571	0.1492	0.161	0.1571	0.1535	0.1565
59-66	0.5356	0.7733	0.557	0.5335	0.5936	0.5458
67-70	0.4099	0.4534	0.377	0.4156	0.3414	0.4195
71,72	0.5690	0.3417	0.506	0.5510	0.6076	0.5699
Weight (lbs)	379.66	395.97	381.2	379.67	338.63	379.64
number of analyses	5	8	11	4	22	9

TRANSMISSION TOWER

25 bars

18 D.O.F.

8 design variables

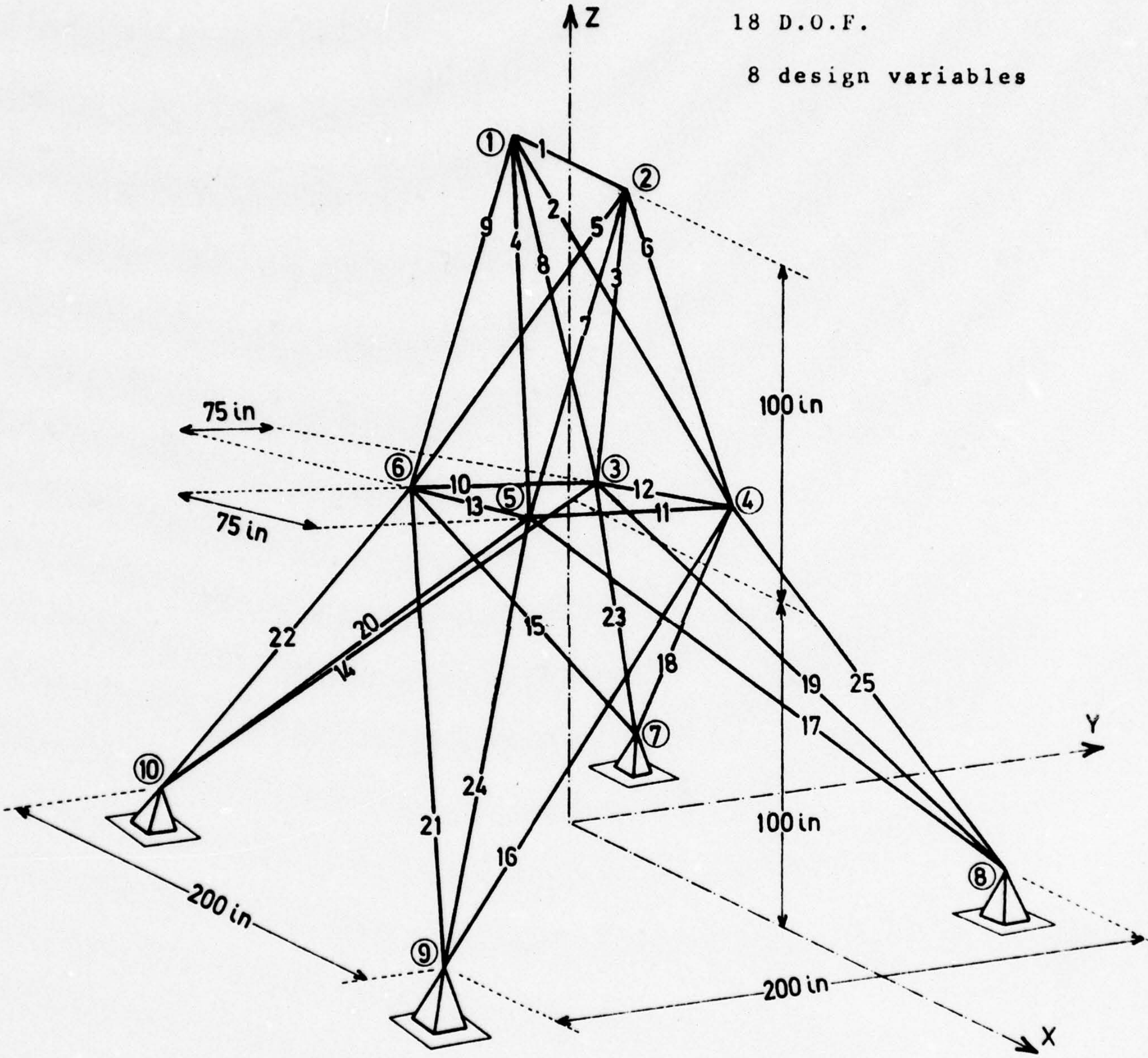


FIGURE 8.35

TABLE 8.36

TRANSMISSION TOWER

General data :

$E = 10^7$ psi

$\rho = 0.1$ lb/in^3

$\underline{a} = 0.01$ in^2

$a^o = 2.0$ in^2

Allowable stresses

elements	stress limits (psi)		elements	stress limits (psi)	
	traction	compression		traction	compression
1	40000	- 35092	12,13	40000	- 35092
2-5	40000	- 11590	14-17	40000	- 6759
6-9	40000	- 17305	18-21	40000	- 6959
10,11	40000	- 35092	22-25	40000	- 11082

Loading cases

load case	node	load component (lbs)		
		x	y	z
1	1	1000	10000	- 5000
	2	0	10000	- 5000
	3	500	0	0
	6	500	0	0
2	5	0	20000	- 5000
	6	0	- 20000	- 5000

Flexibility constraints

node	max. displacement (in)		
	x	y	z
1	0.35	0.35	/
2	0.35	0.35	/

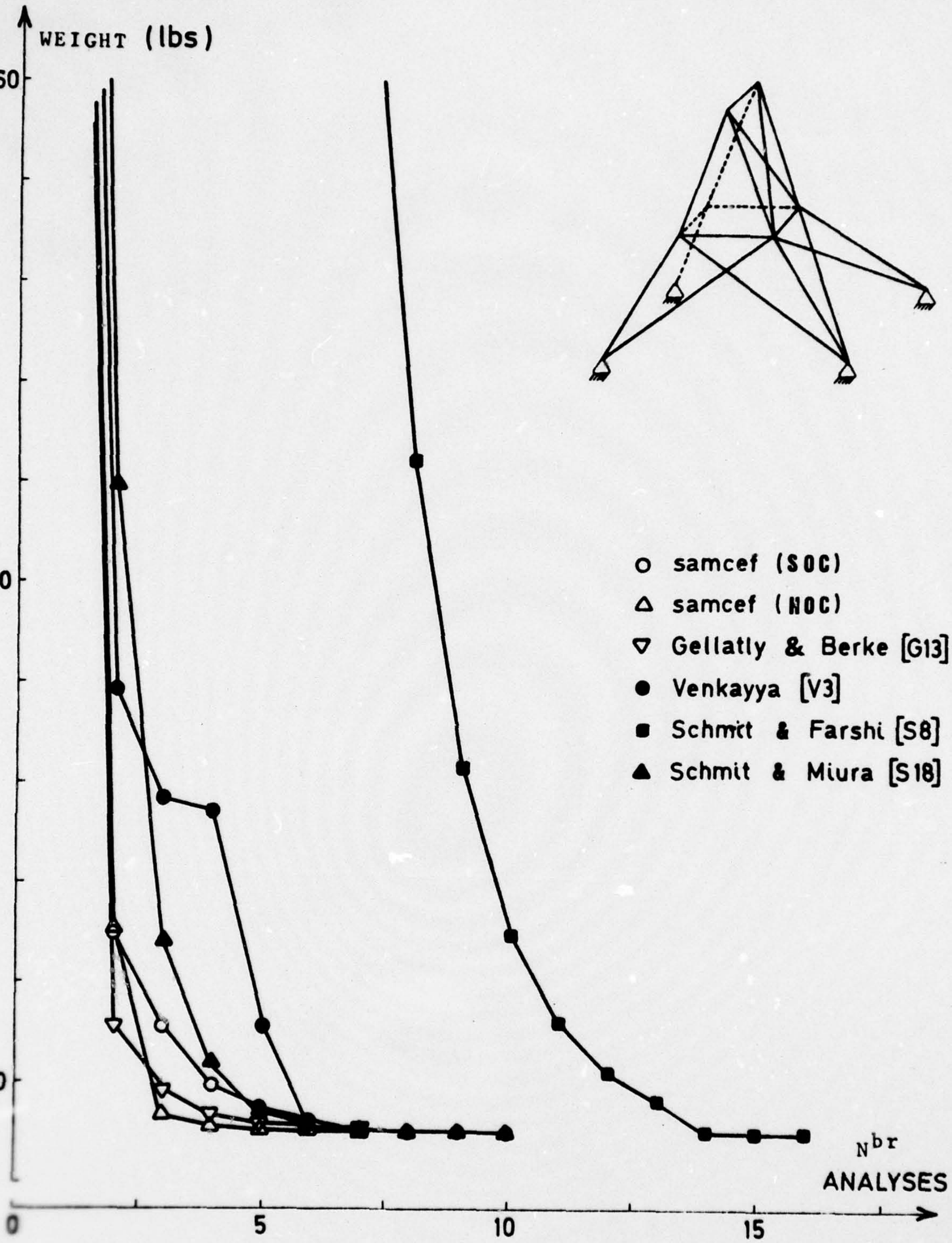

WEIGHT (lbs)
650
600
550
0
5
10
15
N br
ANALYSES
samcef (SOC)
samcef (NOC)
Gellatly & Berke [G13]
Venkayya [V3]
Schmit & Farshi [S8]
Schmit & Miura [S18]

FIGURE 8.37

TABLE 8.38

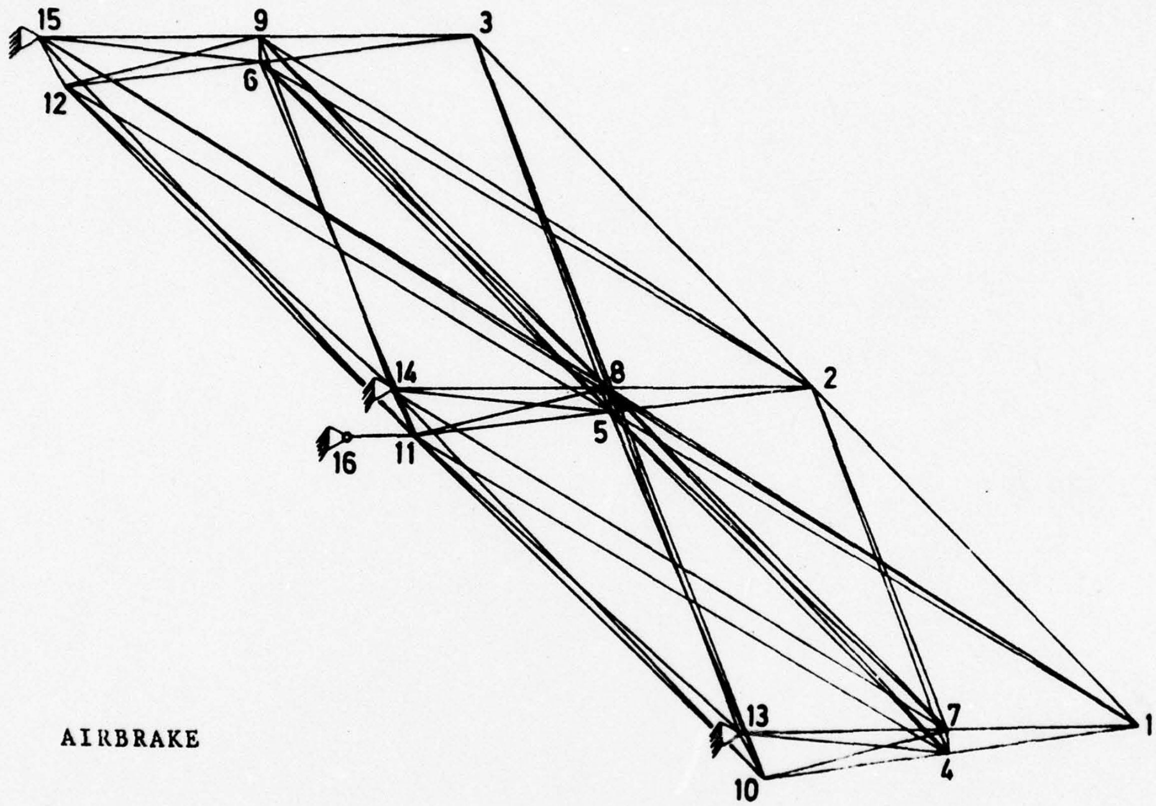
TRANSMISSION TOWER - WEIGHT CONVERGENCE

iteration	Weight (lbs)					
	SAMCEF		Gellatly & Berke [G13]	Venkayya [V3]	Schmit & Farshi [S3]	Schmit & Miura [S13]
	SOC	HOC				
1	734.38	734.38	734.38	734.4	1060.9	783.70
2	565.02	565.32	555.72	589.2	1073.1	609.72
3	555.83	546.82	549.08	578.3	1019.0	564.42
4	549.70	545.76	546.54	577.3	906.53	552.07
5	547.49	545.40	545.92	555.6	864.06	547.36
6	546.08	545.23	545.45	545.5	748.64	546.02
7	545.42	545.16	545.36		666.68	545.39
8	545.31				614.49	545.22
9	545.24				581.75	545.17
10	545.21				564.95	
11					556.13	
12					551.07	
13					548.39	
14					545.22	
15					545.23	
16					545.23	

TABLE 8.39

TRANSMISSION TOWER - FINAL DESIGN

elements	Cross sections (in^2)					
	SAMCEF		Gellatly & Berke [G13]	Venkayya [V3]	Schmit & Farshi [S8]	Schmit & Miura [S13]
	SOC	HOC				
1	0.0100	0.0100	0.0100	0.028	0.010	0.010
2-5	1.9761	1.9863	2.0069	1.942	1.964	1.935
6-9	3.0108	2.9946	2.9631	3.031	3.033	2.996
10,11	0.0100	0.0100	0.0100	0.010	0.010	0.010
12,13	0.0100	0.0100	0.0100	0.010	0.010	0.010
14-17	0.6848	0.6840	0.6876	0.693	0.670	0.634
18-21	1.6786	1.6770	1.6784	1.678	1.630	1.677
21-25	2.6565	2.6618	2.6638	2.627	2.670	2.662
Weight (lbs)	545.21	545.16	545.36	545.49	545.23	545.17
number of analyses	10	7	7	6	17	10



AIRBRAKE

57 bars

42 D.O.F.

FIGURE 8.40

TABLE 8.41

SIMPLIFIED AIR BRAKE

General data :

$E = 7200$ hb

$\rho = 2.3 \; 10^{-6}$ kg/mm^3

$\bar{\sigma} = 3$ hb

$\underline{a} = 1.0$ mm^2

$a^o = 100$ mm^2

Nodal coordinates

node	coordinates (mm)			node	coordinates (mm)		
	x	y	z		x	y	z
1	25.0	41.18	0.0	9	1259.0	266.41	0.0
2	636.5	21.18	0.0	10	25.0	516.01	- 58.30
3	1259.0	0.82	0.0	11	636.5	514.75	- 60.60
4	25.0	284.76	- 29.91	12	1259.0	513.47	- 62.5
5	636.5	274.69	- 31.13	13	25.0	548.00	0.0
6	1259.0	264.43	- 32.37	14	636.5	548.00	0.0
7	25.0	286.59	0.0	15	1259.0	548.00	0.0
8	636.5	276.59	0.0	16	636.5	600.00	- 60.60

Loading case

node	Load component (daN)		
	x	y	z
1	0	0	220
3	0	0	220

Support : displacements at nodes 13, 14, 15, 16 equal zero

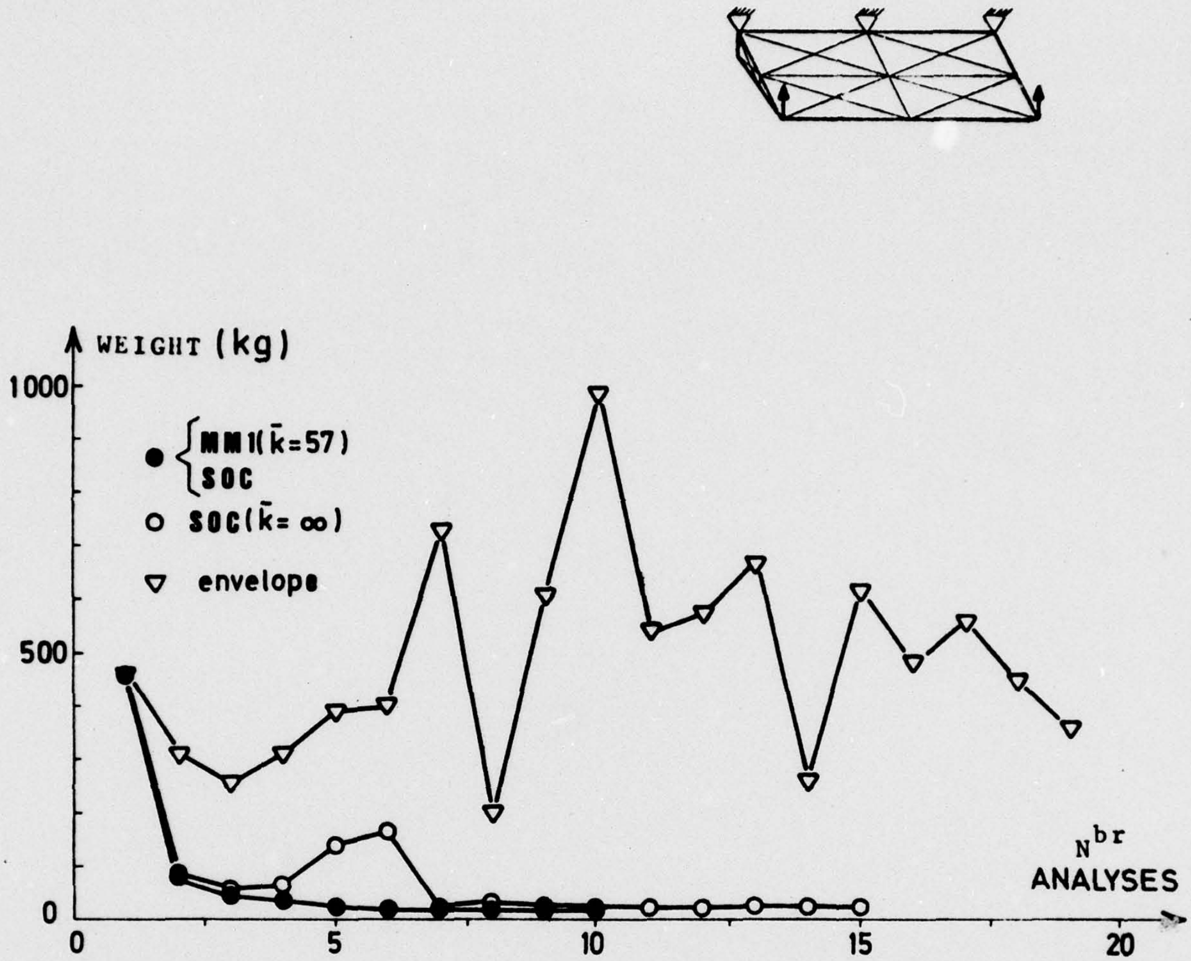
TABLE 8.42

Flexibility constraints

nodes	max. difference in deflection		
	x	y	z
1-2	/	/	0.5
3-2	/	/	0.5
1-3	/	/	0.5

Element definition

element	nodes		element	nodes		element	nodes	
1	1	7	20	4	11	39	8	9
2	1	4	21	5	10	40	5	6
3	4	7	22	10	11	41	6	8
4	7	13	23	10	14	42	5	9
5	4	10	24	11	13	43	8	15
6	7	10	25	2	8	44	9	14
7	4	13	26	2	5	45	5	12
8	10	13	27	5	8	46	6	11
9	1	2	28	8	14	47	11	12
10	1	8	29	5	11	48	12	14
11	2	7	30	8	11	49	11	15
12	1	5	31	5	14	50	3	9
13	2	4	32	11	14	51	3	6
14	7	8	33	11	16	52	6	9
15	4	5	34	2	3	53	9	15
16	4	8	35	2	9	54	6	12
17	5	7	36	3	8	55	9	12
18	7	14	37	2	6	56	6	15
19	8	13	38	3	5	57	12	15



Simplified Airbrake

Weight convergence using F.S.D.

FIGURE 8.43

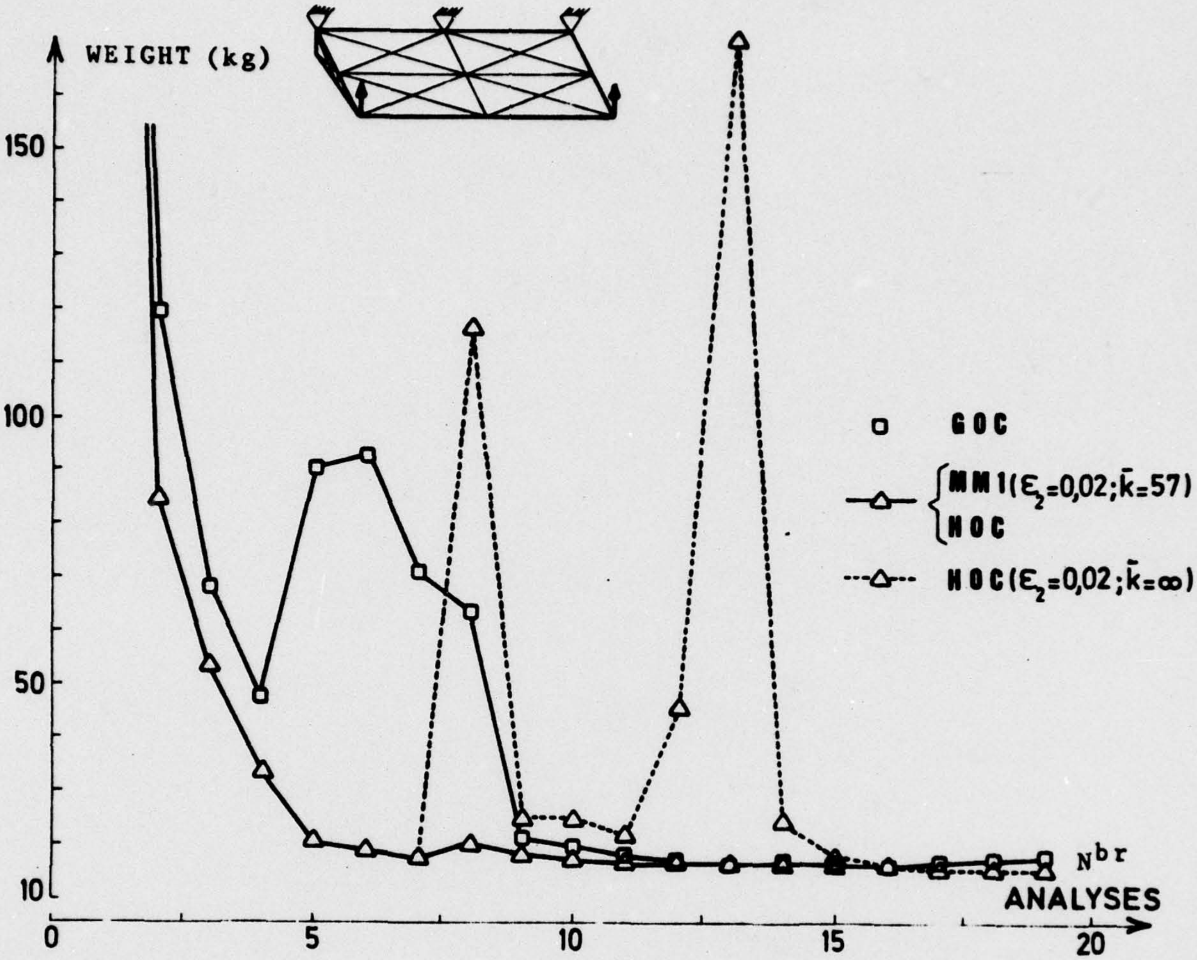


FIGURE 8.44

Simplified Airbrake

Weight convergence using first order approximations

TABLE 8.45

SIMPLIFIED AIR BRAKE - WEIGHT CONVERGENCE

iteration	COC (envelope)		SOC		GOC		HOC (ε_2 = 0.02)		
	weight (kg)	scale factor	weight (kg)	scale factor	weight (kg)	scale factor	weight (kg)	scale factor	
1	463.92	64.708	463.92	64.708	463.92	64.708	463.92	64.708	$\bar{k}$ = 57 (MM1)
2	310.09	1.140	85.69	3.990	119.32	4.779	84.29	3.563	
3	257.23	1.998	54.26	3.165	68.21	2.941	53.63	2.062	
4	310.36	1.577	62.47	3.198	47.34	2.406	33.19	1.516	
5	391.57	3.371	139.01	7.064	90.47	5.046	20.68	1.037	
6	400.09	1.435	164.34	8.444	92.39	5.490	13.32	1.033	
7	730.17	4.430	22.04	1.138	70.33	4.314	17.32	1.002	
8	200.96	0.530	30.10	1.684	63.43	3.926	116.91	7.415	$\bar{k} = \infty$ (pure HOC)
9	601.96	5.505	25.33	1.380	21.05	1.313	24.29	1.729	
10	981.42	2.317	21.50	1.129	19.42	1.218	24.34	1.726	
11	539.08	2.182	21.40	1.093	17.47	1.102	21.18	1.354	
12	572.24	1.507	21.09	1.054	16.03	1.017	45.72	3.008	
13	662.60	3.204	20.88	1.029	16.14	1.028	170.67	14.192	
14	257.33	0.649	20.66	1.012	16.33	1.043	24.31	1.569	
15	612.19	4.131	20.36	1.019	16.28	1.042	15.71	1.141	
16	478.71	1.531	20.72	1.011	16.21	1.039	16.00	1.029	
17	556.45	3.477	20.52	1.001	16.67	1.070	15.61	1.003	
18	447.64	1.372	<u>20.52</u>	1.001	17.01	1.092	15.57	1.000	
19	353.70	1.983			17.70	1.137	<u>15.57</u>	1.000	
20	203.16	0.763			17.65	1.135			
21	...	...			15.66	1.007			
22					15.71	1.011			
23					15.75	1.013			
24					15.73	1.015			
25					15.82	1.013			
					...	...			

TABLE 8.46

SIMPLIFIED AIR BRAKE - FINAL DESIGNS

element	section (mm^2)		element	section (mm^2)		element	section (mm^2)	
	SOC	HOC		SOC	HOC		SOC	HOC
1	367.18	1010.48	20	1156.18	1341.36	39	1.00	1.00
2	877.25	1016.70	21	1.00	1.00	40	1.19	1.00
3	1.00	1.00	22	1.00	1.00	41	1.00	7.64
4	864.16	1006.72	23	1.00	1.00	42	68.20	1.00
5	1.00	2.59	24	1.00	1.00	43	1.00	1.00
6	1.23	2.59	25	39.30	28.26	44	79.61	1.00
7	1.00	1.00	26	197.75	227.61	45	81.69	52.51
8	1.00	1.00	27	4.22	1.00	46	1169.11	1354.52
9	1.00	3.16	28	62.50	1.00	47	1.00	1.00
10	890.03	1.00	29	262.04	232.51	48	75.53	43.54
11	1.00	1.00	30	20.07	8.44	49	1.00	1.00
12	891.94	4.23	31	1.00	1.00	50	914.76	1016.74
13	1169.00	1349.89	32	63.03	62.91	51	926.11	1025.11
14	1.00	1.00	33	478.73	478.33	52	1.00	1.00
15	1.00	1.00	34	1.00	1.00	53	846.80	1012.06
16	9.65	16.66	35	1.00	1.00	54	44.04	20.70
17	1.00	1.00	36	922.38	59.26	55	13.78	1.00
18	1.00	1.00	37	1162.33	1339.56	56	1.00	1.00
19	1.00	50.80	38	921.93	59.33	57	6.42	5.12

method	deflections (mm)			weight (kg)	nbr of analyses
	node 1	node 2	node 3		
SOC	8.387	7.887	8.386	20.518	20
HOC	8.393	7.393	8.393	15.568	19

COMPUTING TIME - I.B.M. 370/158

SECTION	PROBLEMS and METHODS		PROGRAM	NUMBER OF ITERATIONS	EXECUTION CPU TIME (seconds)			
					PREPROCESSING	ANALYSIS	MINIMIZATION	TOTAL
8.1.1	1.B	S.O.C	SAMCEF	30	5.0	5.0	0.2	161.
	1.B	H.O.C	SAMCEF	7	6.0	7.1	0.5	59.
	1.B	S.O.C-MM2	OPTBAR	13	-	0.05	0.2	3.1
	1.C	H.O.C	OPTBAR	14	-	0.10	0.5	7.9
8.1.2	2.A	S.O.C	OPTBAR	50	-	0.07	0.09	8.1
	2.A	G.O.C	SAMCEF	30	8.7	9.1	0.7	302.
	2.B	G.O.C	SAMCEF	30	9.4	13.4	1.1	458.
	2.B	H.O.C	OPTBAR	27	-	1.6	0.93	28.5
8.1.3	3.A	S.O.C	SAMCEF	13	26.5	58.1	1.8	803.
	3.B	H.O.C	SAMCEF	8	27.9	63.6	2.5	555.
8.1.4	4.A	S.O.C	OPTBAR	50	-	0.65	0.10	37.4
	4.A	H.O.C	OPTBAR	8	-	1.17	34.8	253.
	4.A	G.O.C	OPTBAR	15	-	1.65	72.5	1040.3
	4.A	H.O.C	SAMCEF	8	11.9	102.8	34.8	1078.
	4.B	S.O.C	OPTBAR	50	-	0.75	0.15	44.9
	4.B	H.O.C	OPTBAR	20	-	0.89	2.36	62.6
	4.B	G.O.C	OPTBAR	17	-	1.03	11.28	198.0
	4.B	G.O.C	SAMCEF	12	10.9	52.2	9.1	737.
8.1.5	5.	S.O.C	SAMCEF	5	10.0	10.5	0.5	65.
8.1.6	6.	S.O.C	SAMCEF	10	6.5	8.0	0.4	90.
		H.O.C	SAMCEF	7	9.0	8.2	0.6	70.
8.1.7	7.	S.O.C	SAMCEF	20	10.8	12.2	1.3	280.
		S.O.C	OPTBAR	20	-	0.54	0.93	28.9
		G.O.C	OPTBAR	20	-	0.65	1.79	47.0
		H.O.C-MM1	OPTBAR	7	-	0.61	2.63	20.1

TABLE 8.47

WING BOX

10 bars

20 quadrilateral membranes

2 triangular membranes

16 design variables

45 D.O.F.

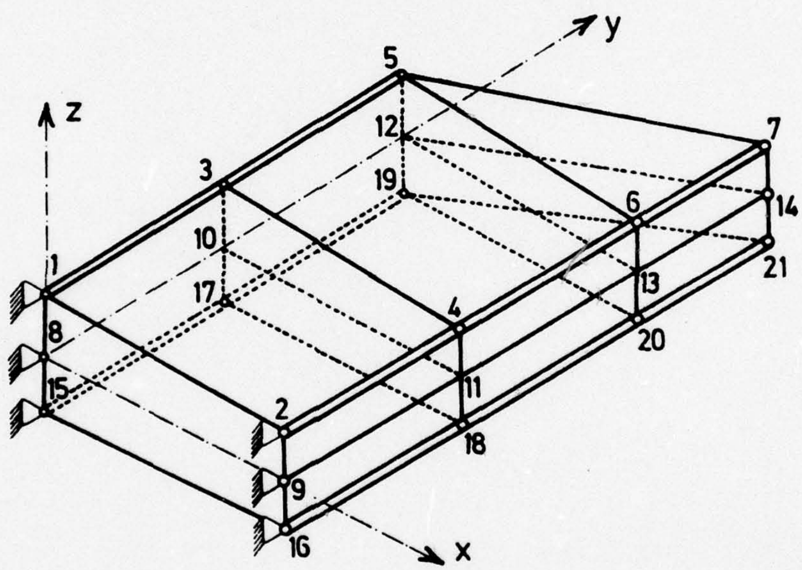


FIGURE 8.48

TABLE 8.49

WING BOX

General data :

$E = 10^7$ psi

$\nu = 0.3$

$\rho = 0.1$ lb/in^3

$\bar{\sigma} = 10000$ psi

$\underline{a} = \begin{cases} 0.1 \text{ in}^2 \\ 0.02 \text{ in} \end{cases}$

$a^o = \begin{cases} 0.1 \text{ in}^2 \\ 0.1 \text{ in} \end{cases}$

Nodal coordinates

supports at nodes 1, 2, 8, 9, 15, 16

node	coordinate (in)			node	coordinate (in)			node	coordinate (in)		
	x	y	z		x	y	z		x	y	z
1	0	0	10	8	0	0	0	15	0	0	- 10
2	100	0	8	9	100	0	0	16	100	0	- 8
3	0	70	10	10	0	70	0	17	0	70	- 10
4	100	70	8	11	100	70	0	18	100	70	- 8
5	0	140	10	12	0	140	0	19	0	140	- 10
6	100	140	8	13	100	140	0	20	100	140	- 8
7	100	190	8	14	100	190	0	21	100	190	- 8

Loading

load case	node	load component (lbs)		
		x	y	z
1	7	0	0	10000
2	5	0	0	20000

(continue)

TABLE 8.49 (Continued)

ELEMENT AND DESIGN VARIABLE DEFINITION

design variable	elements	nodes 1	nodes 2	nodes 3	nodes 4	element type
1	1	1	3			bar
	2	15	17			
2	3	1	3	10	8	membrane
	4	8	10	17	15	
3	5	2	4			bar
	6	16	18			
4	7	2	4	11	9	membrane
	8	9	11	18	16	
5	9	1	3	4	2	membrane
	10	15	17	18	16	
6	11	3	4	11	10	membrane
	12	10	11	18	17	
7	13	3	5			bar
	14	17	19			
8	15	3	5	12	10	membrane
	16	10	12	19	17	
9	17	4	6			bar
	18	18	20			
10	19	3	6	13	11	membrane
	20	11	13	20	18	
11	21	3	5	6	4	membrane
	22	17	19	20	18	
12	23	5	6	13	12	membrane
	24	12	13	20	19	
13	25	6	7			bar
	26	20	21			
14	27	6	7	14	13	membrane
	28	13	14	21	20	
15	29	5	7	6		membrane
	30	19	21	20		
16	31	5	7	14	12	membrane
	32	12	14	21	19	

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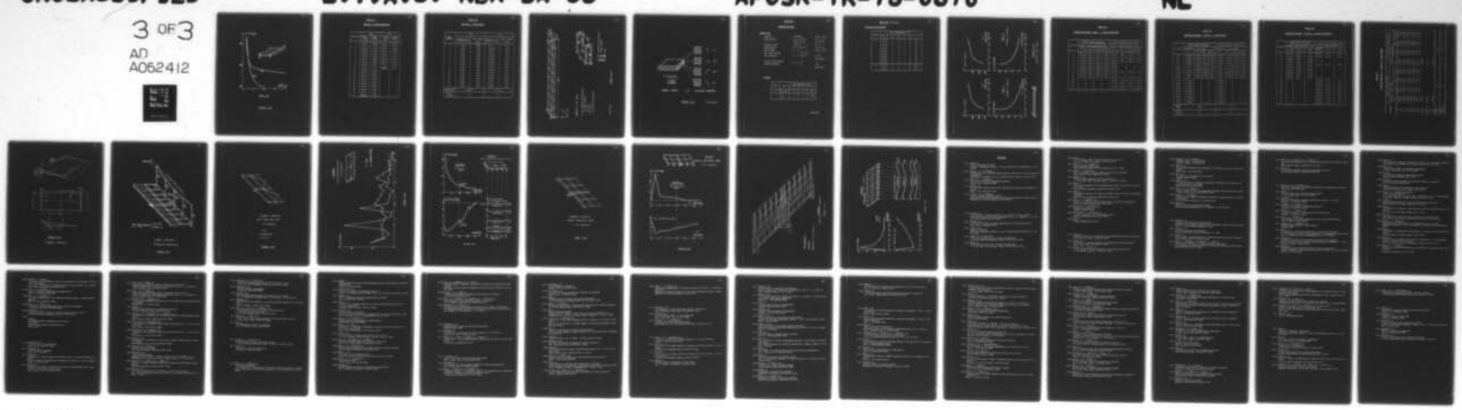

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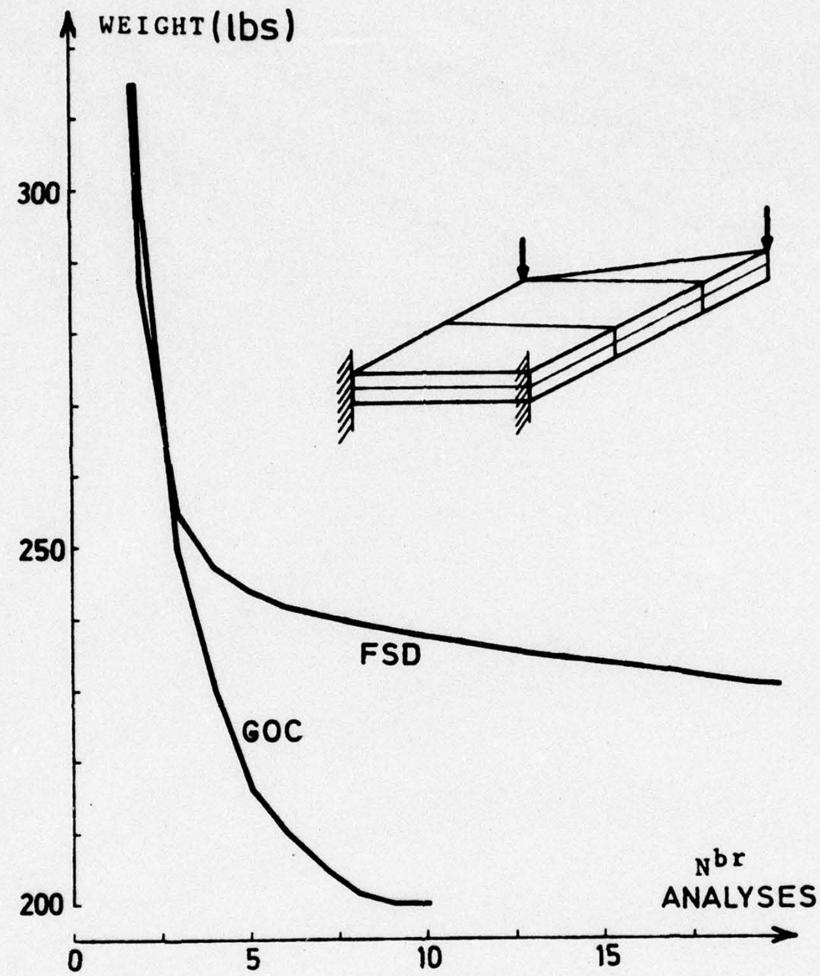


WING BOX

FIGURE 8.50

TABLE 8.51

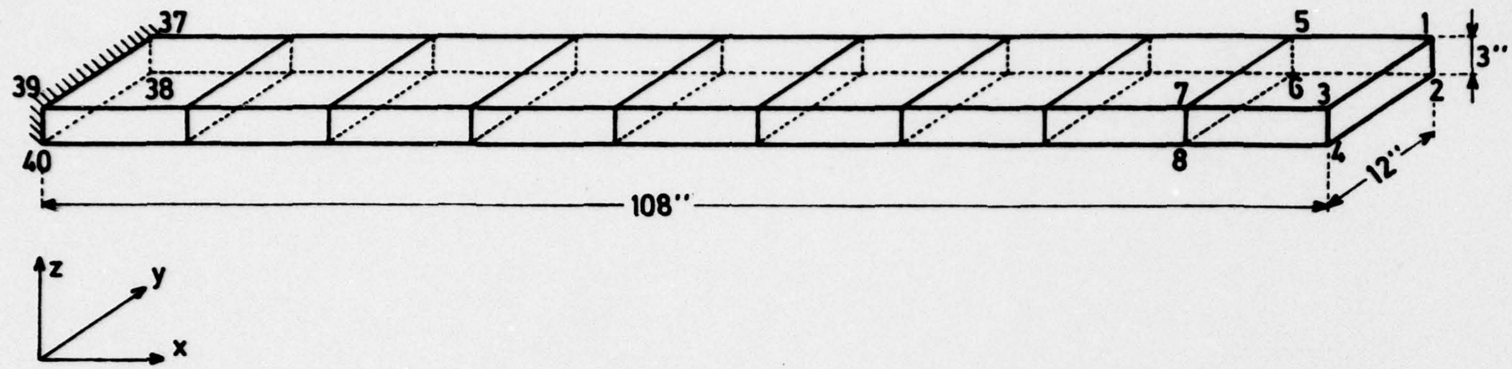
WING BOX - WEIGHT CONVERGENCE

iteration	FSD		GOC	
	weight (lbs)	scale factor	weight (lbs)	scale factor
1	723.97	1.6046	723.97	1.6046
2	288.32	1.1091	301.44	0.9913
3	254.60	1.1166	249.57	1.0146
4	247.42	1.1570	230.29	1.0284
5	243.95	1.1822	216.48	1.0052
6	241.96	1.1993	210.21	1.0038
7	240.47	1.2089	205.22	1.0023
8	239.41	1.2147	201.24	1.0020
9	238.30	1.2173	200.07	1.0001
10	237.38	1.2172	200.03	1.0000
11	236.50	1.2145		
12	235.67	1.2074		
13	234.92	1.1917		
14	234.37	1.1784		
15	233.75	1.1643		
16	233.08	1.1494		
17	232.37	1.1339		
18	231.65	1.1183		
19	230.94	1.1031		
20	230.27	1.0886		

TABLE 8.52

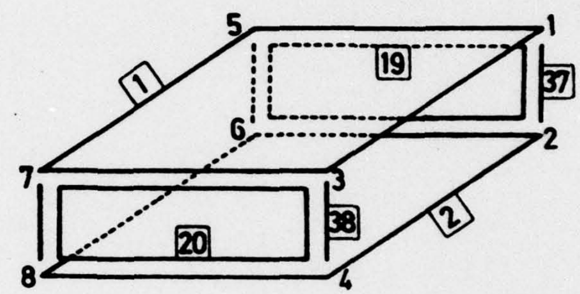
WING BOX - FINAL DESIGN

design variable	FSD		GOC	
	section (in^2) thickness (in)	max. stress. (psi)	section (in^2) thickness (in)	max. stress. (psi)
1	1.735	10000	0.10000	10000
2	0.213	9139	0.28409	7700
3	0.195	- 9467	0.10000	- 9503
4	0.119	9175	0.13936	3267
5	0.022	8426	0.02000	8341
6	0.024	9224	0.02000	9235
7	0.109	- 3619	0.10000	- 4105
8	0.130	9193	0.16670	10000
9	0.109	5450	0.10000	5975
10	0.078	9165	0.09005	3631
11	0.041	9214	0.03278	10000
12	0.022	4937	0.02000	6106
13	0.109	- 4802	0.10000	- 5254
14	0.065	9174	0.06235	10000
15	0.022	9030	0.02000	9662
16	0.049	9200	0.04222	10000
weight (lbs)	230.27		200.03	
nbr of analyses	20		10	



COMPOSITE BOX BEAM

4 × 18 composite membranes
18 shear panels
18 bars
108 D.O.F.
45 design variables



$\boxed{1}$, $\boxed{2}$ composite membranes
$\boxed{19}$, $\boxed{20}$ shear panels
$\boxed{37}$, $\boxed{38}$ bars

(continue)

FIGURE 8.53

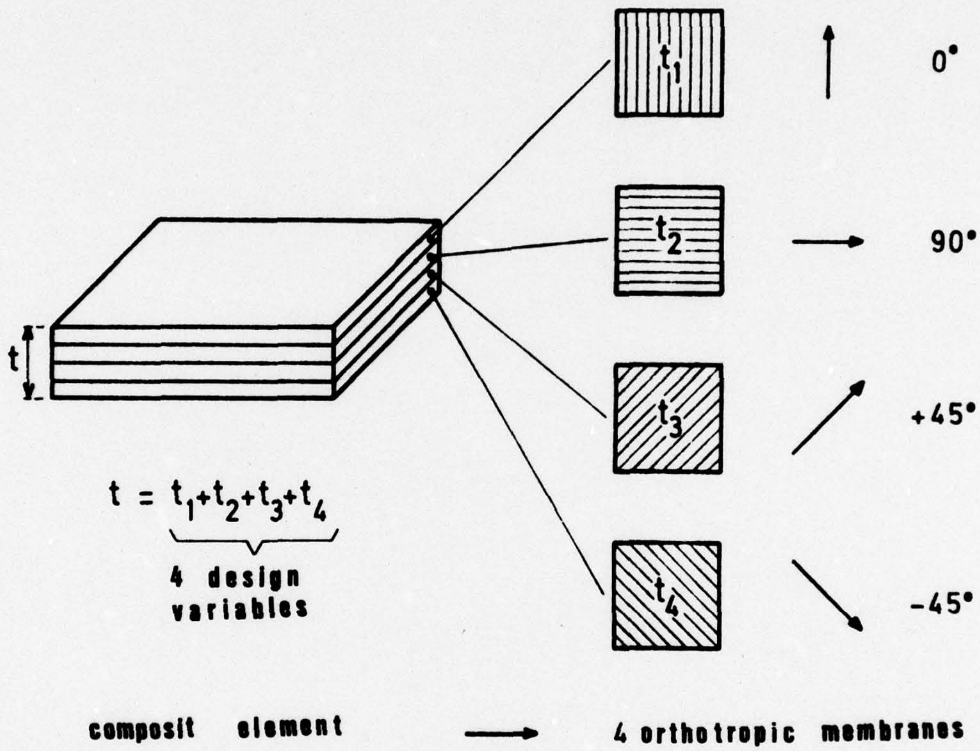


FIGURE 8.53 (continued)

TABLE 8.54

COMPOSITE BOX BEAM

General data

Material	: aluminium	boron - epoxy
Young modulus	: $E_{11} = 1.05\ 10^6$	$3.0\ 10^7$ psi
	$E_{22} = 1.05\ 10^6$	$2.7\ 10^6$ psi
Poisson's ratio	: $\nu_{12} = 0.3$	0.21
Shear modulus	: $G = 4.033\ 10^6$	$0.7\ 10^6$ psi
Mass density	: $\rho = 0.1$	0.0725 lb/in^3
Min. cross section	: $\underline{a} = \{$ 0.005 in^2	/
Min. thickness	: 0.005 in	0.005 in (per layer)
Initial cross section	: $a^o = \{$ 1.0	/
Initial thickness	: 0.1	0.025 (per layer)

Loading

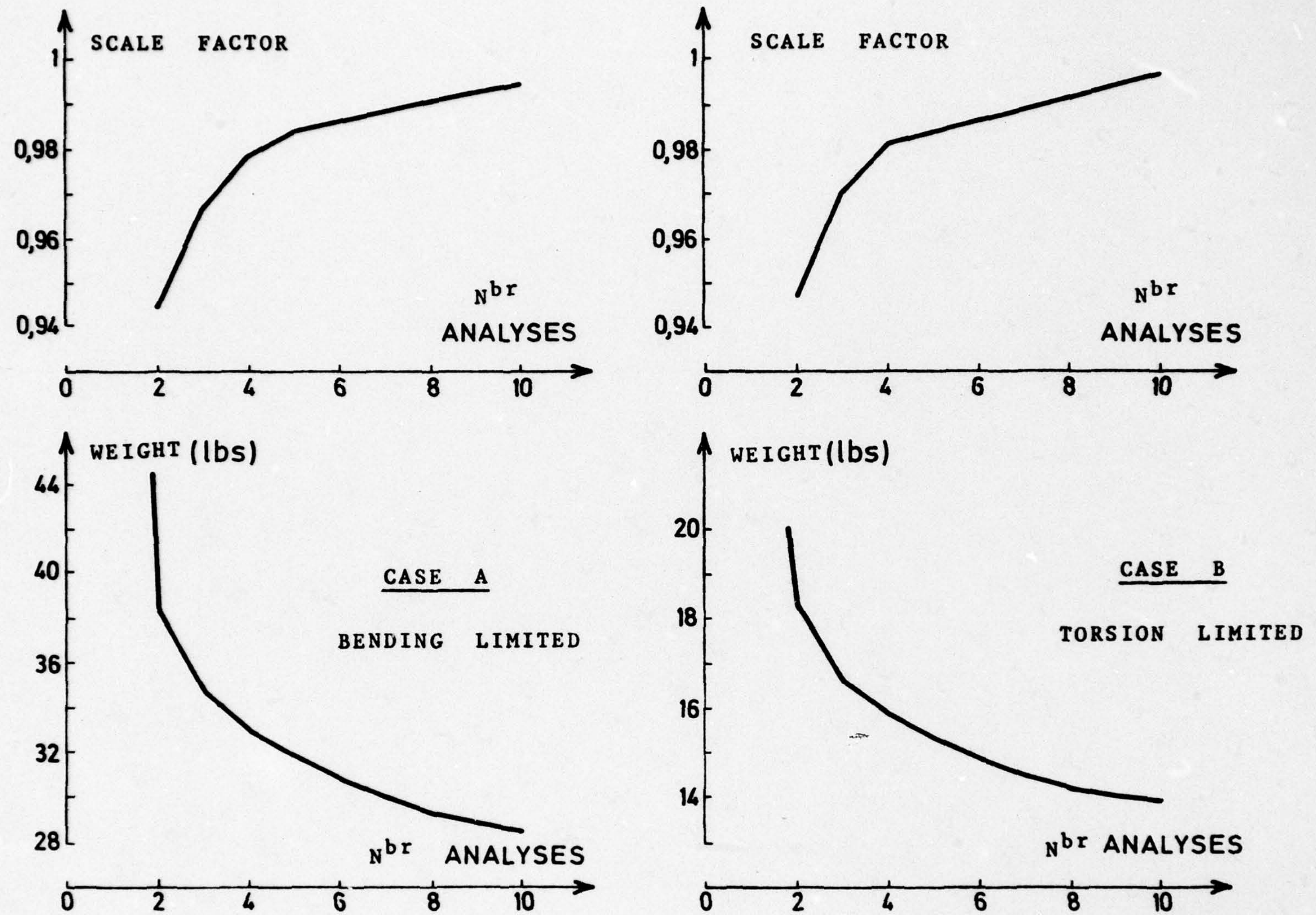
problem	node	lead component (lbs)		
		x	y	z
A	1,2,3,4	0	0	- 1000
B	1,2	0	0	- 1000
	3,4	0	0	- 975

(continue)

TABLE 8.54 (Continued)

Flexibility constraints

problem	node	max. displacement (in)		
		x	y	z
A	1-4	/	/	7.0
	5-8	/	/	6.0
	9-12	/	/	4.0
	13-16	/	/	3.0
	17-20	/	/	2.0
	21-24	/	/	1.5
	25-28	/	/	0.8
	29-32	/	/	0.3
	33-36	/	/	0.1
B	1,2	/	/	14.0
	3,4	/	/	15.0



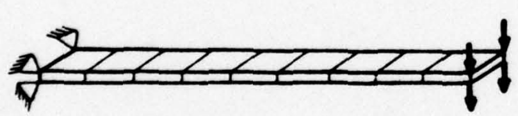


FIGURE 8.55

COMPOSITE BOX BEAM

TABLE 3.56

COMPOSITE BOX BEAM - CASE A - WEIGHT CONVERGENCE

iteration	SAMCEF (primal algorithm)				Khot, Venkayya & Berke [K15]			
	weight (lbs)	nodal deflection (in)			weight (lbs)	nodal deflection (in)		
		1-4	9-12	29-32		1-4	9-12	29-32
1	135.6	4.313	2.902	0.300	186.09	4.440	2.930	0.300
2	38.28	6.970	3.973	0.300	50.59	6.262	3.571	0.300
3	34.73	6.995	3.989	0.300	40.01	6.139	3.514	0.300
4	32.90	7.000	3.997	0.300	34.65	6.085	3.492	0.300
5	31.76	6.990	3.993	0.300	32.43			
6	30.78	6.996	3.994	0.300	31.94			
7	29.97	6.997	3.995	0.300	31.71			
8	29.33	6.998	3.996	0.300	31.55	6.219	3.561	0.300
9	28.83	7.000	3.996	0.300	31.39			
10	28.45	7.000	3.997	0.300	31.23	6.256	3.532	0.300
20					29.48	6.634	3.324	0.300
30					28.73	6.927	3.967	0.300
40					28.62	6.973	4.000	0.299
50					28.62	6.974	4.000	0.299

TABLE 8.57

COMPOSITE BOX BEAM - CASE A - FINAL DESIGNS

elements	SAMCEF (primal algorithm)				Khot, Venkayya & Berke [K15]			
	thickness (in)	% per layer			thickness (in)	% per layer		
		0°	90°	± 45°		0°	90°	± 45°
1,2	0.01939	0.2500	0.2500	0.5000	0.02197	0.2896	0.2368	0.4736
3,4	0.03382	0.5590	0.1470	0.2940	0.04023	0.6123	0.1292	0.2584
5,6	0.05919	0.7480	0.0840	0.1680	0.06322	0.7533	0.0822	0.1644
7,8	0.08944	0.8332	0.0556	0.1112	0.09112	0.8283	0.0571	0.1142
9,10	0.1215	0.8471	0.0409	0.1120	0.1218	0.8719	0.0427	0.0854
11,12	0.1533	0.8545	0.0324	0.1131	0.1543	0.8989	0.0337	0.0674
13,14	0.1850	0.8594	0.0269	0.1137	0.1865	0.9163	0.0279	0.0553
15,16	0.2458	0.8647	0.0202	0.1151	0.2442	0.9361	0.0213	0.0426
17,18	0.3271	0.8703	0.0152	0.1145	0.3193	0.9512	0.0163	0.0326
19,20	0.01917				0.02111			
21,22	0.01917				0.02130			
23,24	0.03536				0.03059			
25,26	0.03536				0.03234			
27-30	0.03536				0.03669			
31,32	0.03536				0.03675			
33,34	0.07556				0.07325			
35,36	0.07556				0.07344			
37-54	0.0050				0.00999			
weight (lbs)	28.45				28.62			
nbr of analyses	10				50			

TABLE 3.58

COMPOSITE BOX BEAM - CASE B - WEIGHT CONVERGENCE

iteration	SAMCEF (dual algorithm)			Khot, Venkayya & Berke [K15]		
	weight (lbs)	nodal deflection (in)		weight (lbs)	nodal deflection (in)	
		1,2	3,4		1,2	3,4
1	42.74	14.00	13.03	59.57	14.00	13.33
2	18.31	13.85	15.00	22.05	14.00	13.33
3	16.62	13.93	15.00			
4	15.90	13.97	15.00			
5	15.34	13.97	15.00	14.70	14.00	13.54
6	14.83	13.98	15.00			
7	14.51	13.99	15.00			
8	14.24	13.99	15.00			
9	14.05	13.99	15.00			
10	13.94	14.00	15.00	14.76	14.00	13.33
20				14.66	14.00	14.30
30				14.62	14.00	14.63
40				14.60	14.00	14.82
50				14.59	14.00	14.91
60				14.58	14.00	14.96
70				14.58	14.00	14.93
80				14.58	14.00	14.99
90				14.58	14.00	14.99
100				14.58	14.00	14.99

TABLE 8.59

COMPOSITE BOX BEAM - CASE B - FINAL DESIGN

elements	SAMCEF (dual algorithm)				Khot, Venkayya & Berke [K15]			
	thickness (in)	% per layer			thickness (in)	% per layer		
		0°	90°	± 45°		0°	90°	± 45°
1,2	0.01994	0.2500	0.2500	0.5000	0.01950	0.2500	0.2500	0.5000
3,4	0.02933	0.4901	0.1699	0.3393	0.03177	0.5237	0.1537	0.3174
5,6	0.04268	0.6496	0.1163	0.2336	0.04503	0.6613	0.1128	0.2256
7,8	0.05604	0.7332	0.0889	0.1779	0.05836	0.7370	0.0876	0.1752
9,10	0.06932	0.7843	0.0719	0.1438	0.07165	0.7851	0.0716	0.1432
11,12	0.08215	0.8180	0.0607	0.1213	0.08493	0.8134	0.0605	0.1210
13,14	0.09445	0.8417	0.0523	0.1055	0.09820	0.8423	0.0523	0.1046
15,16	0.1067	0.8599	0.0467	0.1045	0.1116	0.8614	0.0461	0.0922
17,18	0.1191	0.8744	0.0419	0.1053	0.1243	0.8755	0.0414	0.0823
19,21,... ...,35	0.02030				0.02056			
20,22,... ...,36	0.00634				0.00694			
37-54	0.00493				0.00500			
weight (lbs)	13.94				14.53			
nbr of analyses	10				190			

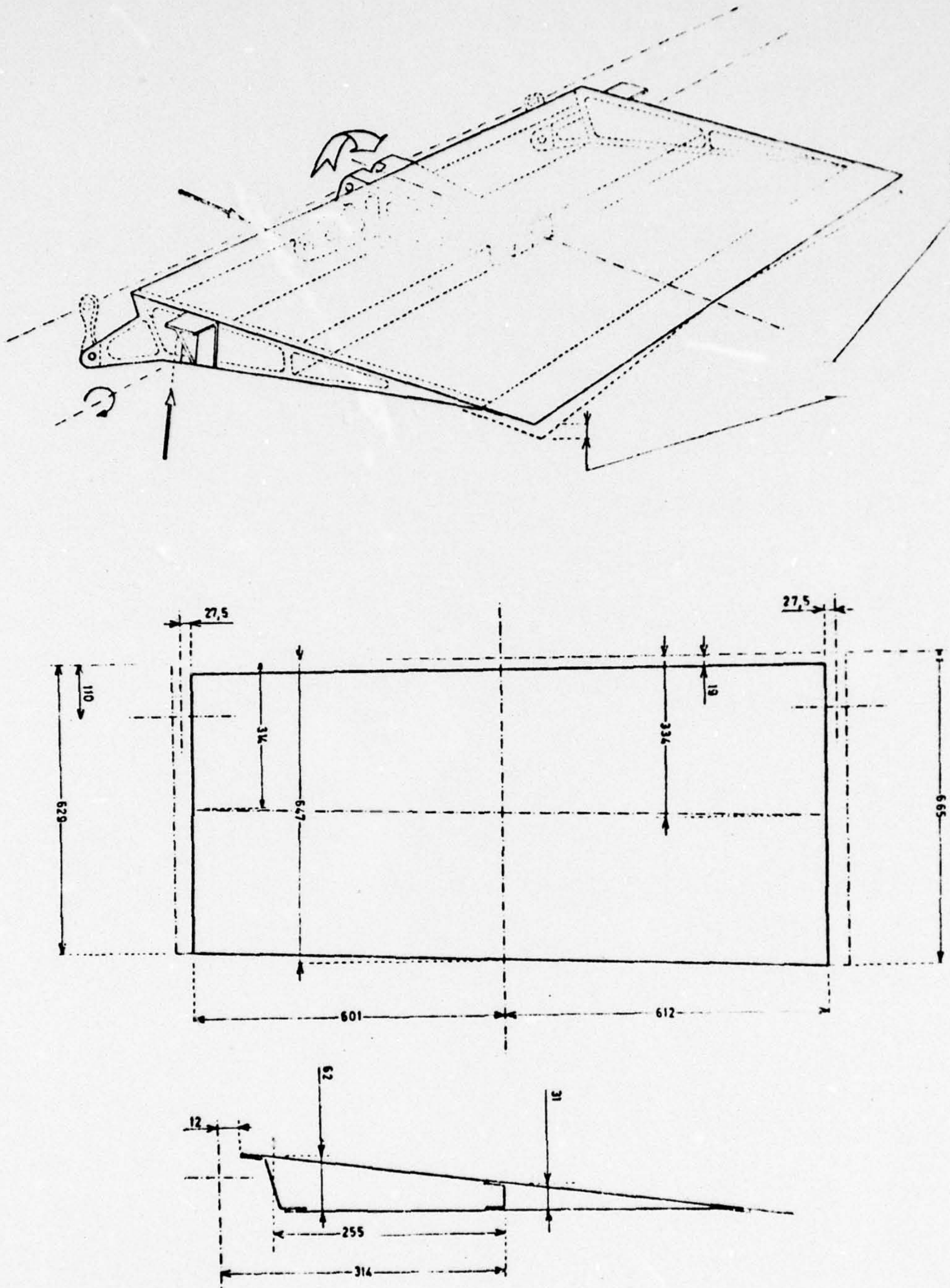
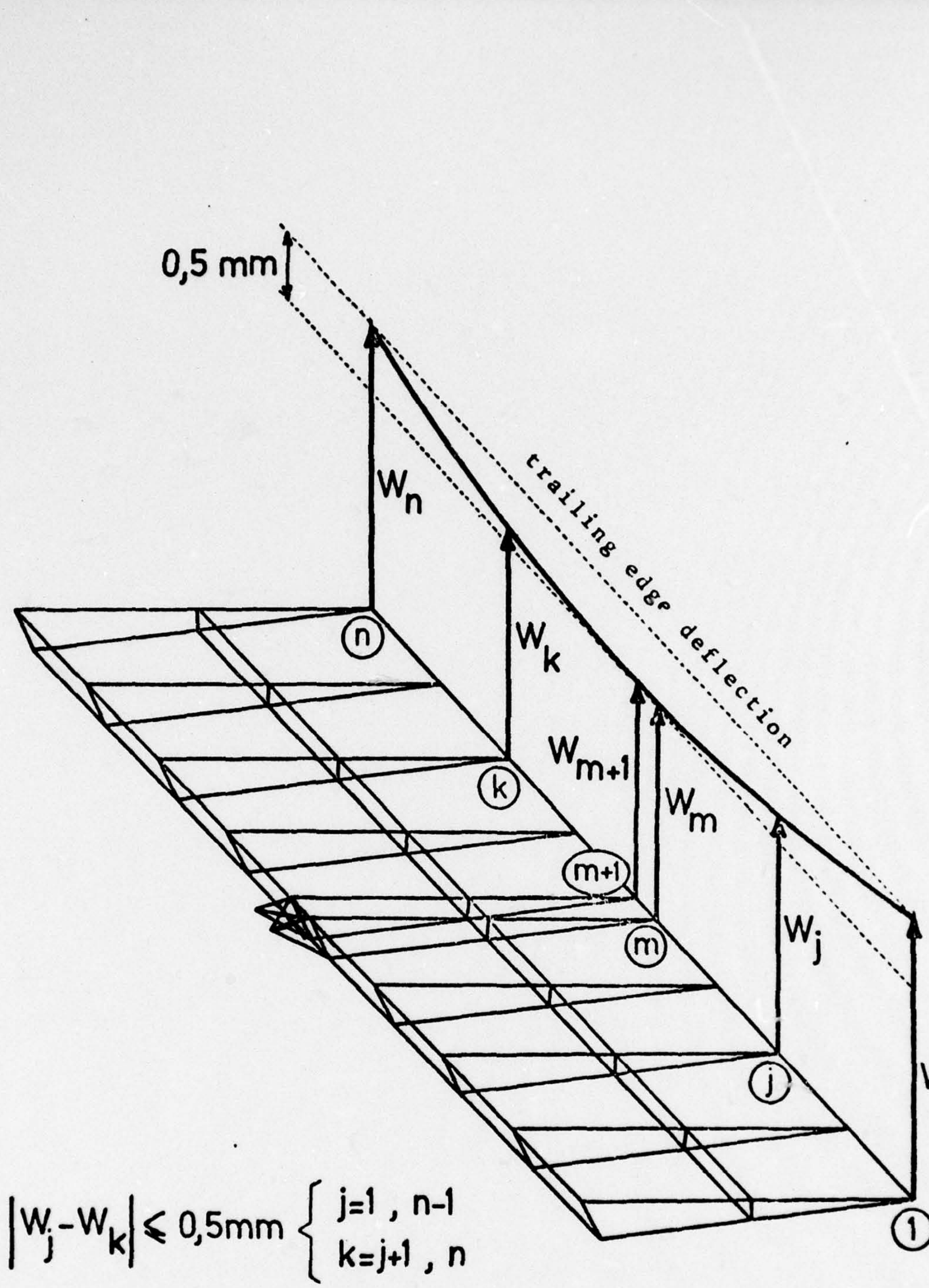


FIGURE 8.60

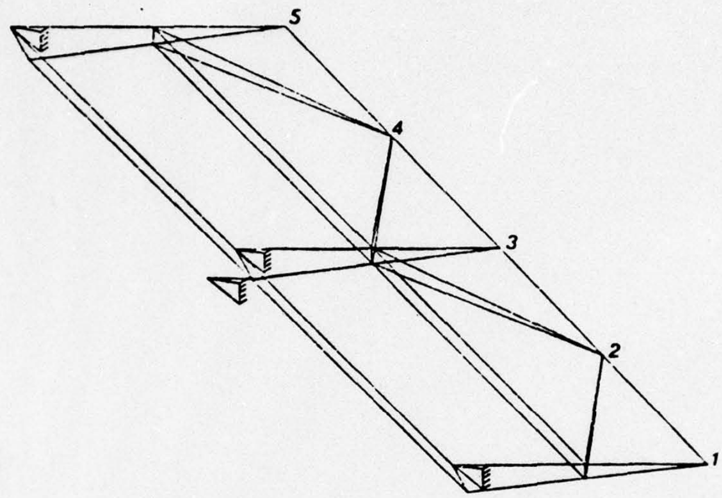
AIRBRAKE STRUCTURE



AIRBRAKE STRUCTURE

FLEXIBILITY CONSTRAINTS

<u>FIGURE 8.61</u>



AIRBRAKE STRUCTURE

First simplified model

(27 elements)

1 bar

26 membranes

43 D.O.F.

FIGURE 8.62

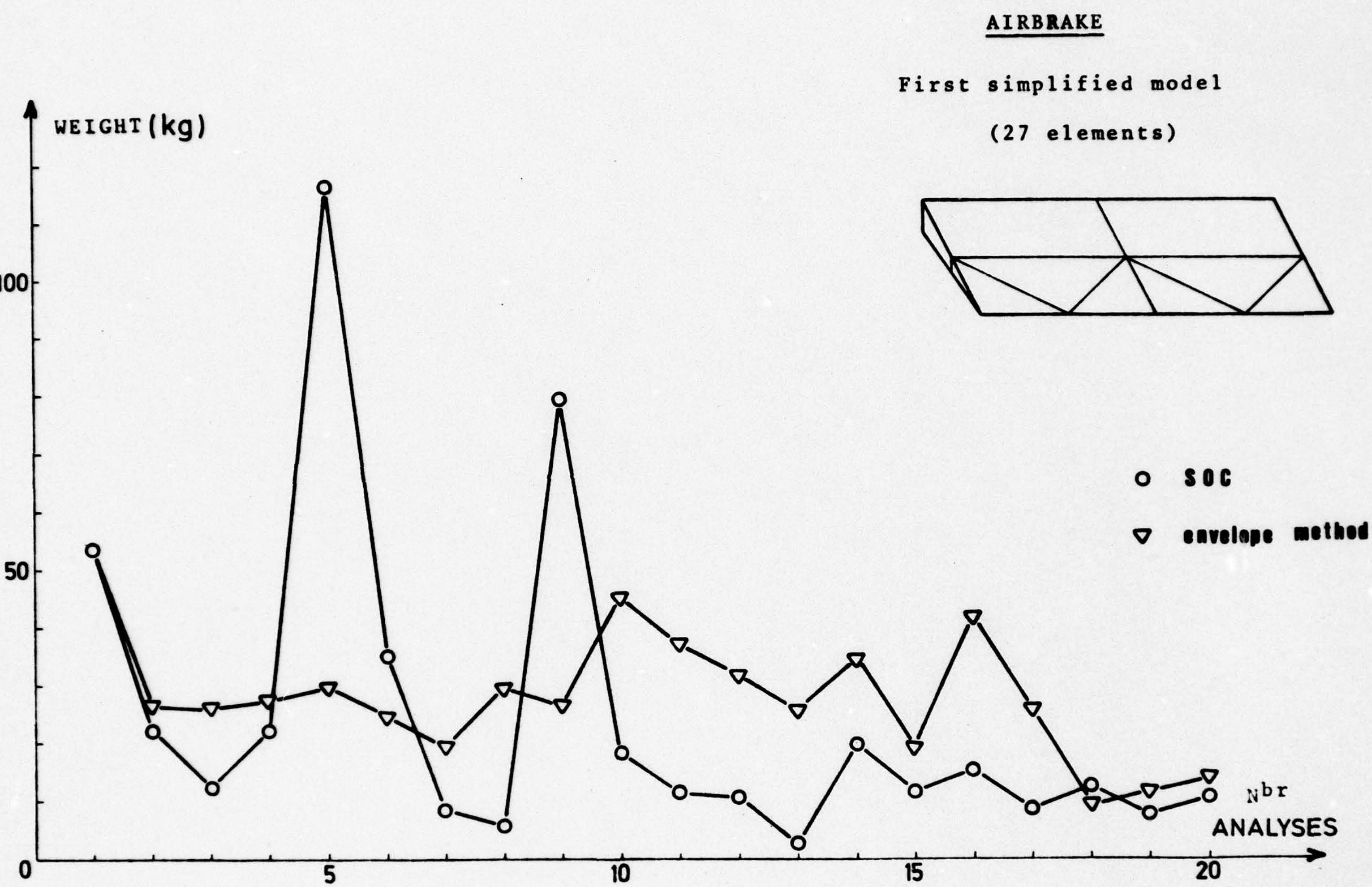

AIRBRAKE
First simplified model
(27 elements)
WEIGHT (kg)
100
50
0
5
10
15
20
SOC
envelope method
N^{br} ANALYSES

FIGURE 8.63

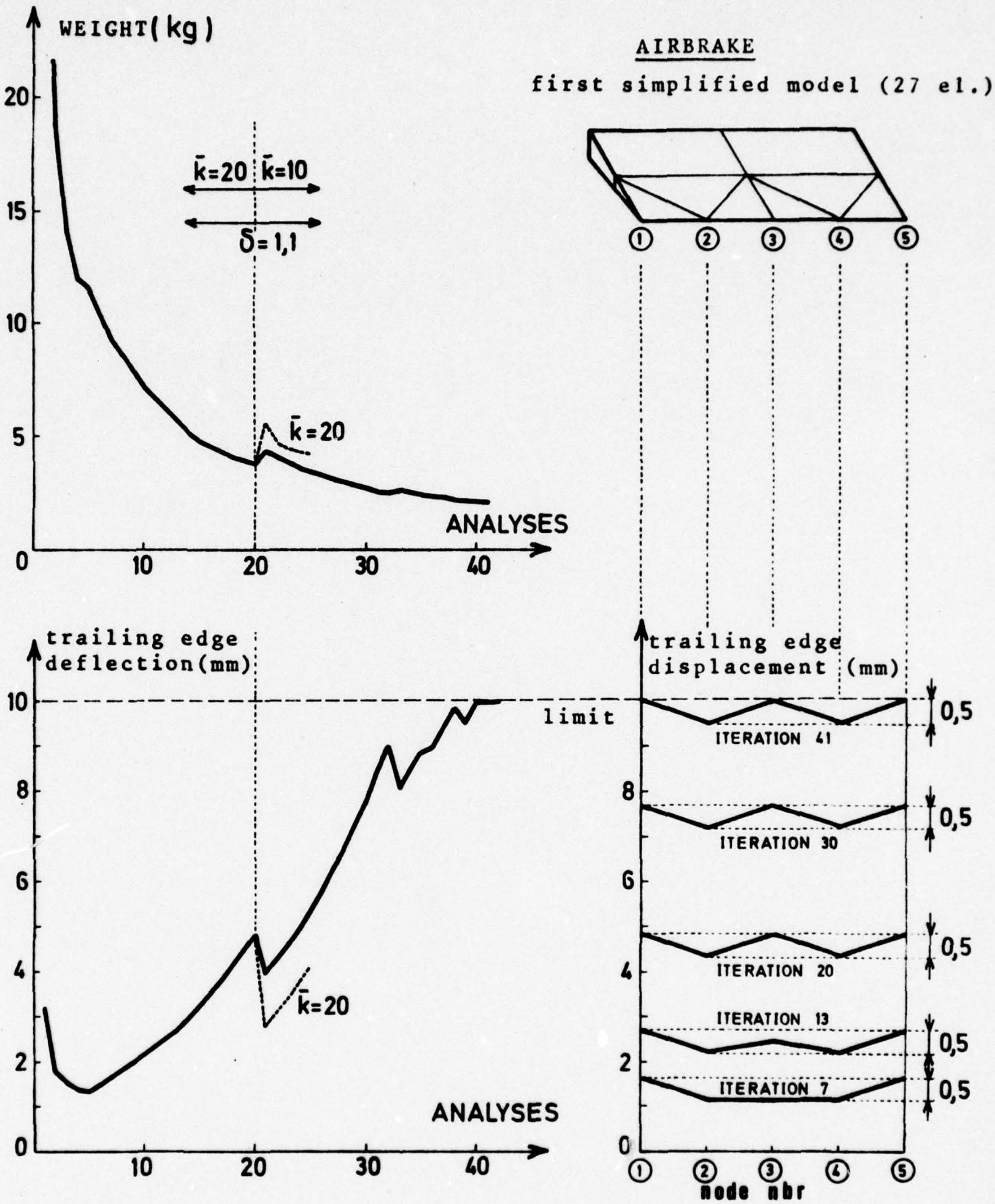
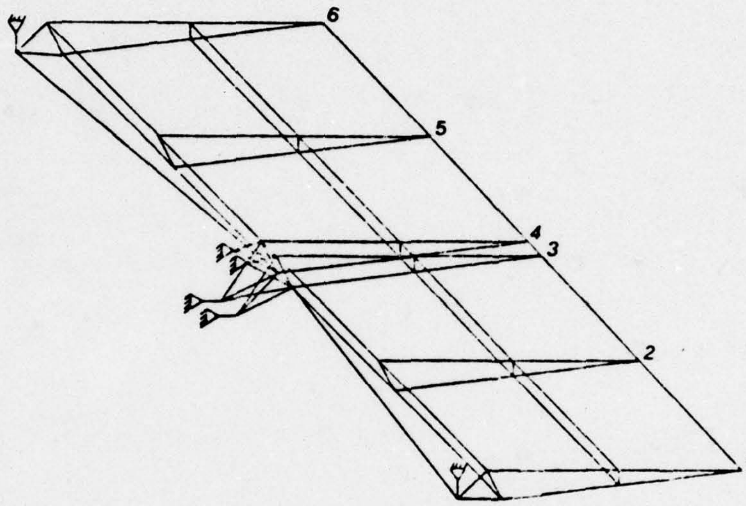

WEIGHT(kg)
20
15
10
5
0
$\bar{k}$=20 $\bar{k}$=10
δ=1,1
$\bar{k}$=20
ANALYSES
10
20
30
40
AIRBRAKE
first simplified model (27 el.)
① ② ③ ④ ⑤
trailing edge deflection(mm)
10
8
6
4
2
0
limit
$\bar{k}$=20
ANALYSES
10
20
30
40
trailing edge displacement (mm)
ITERATION 41
ITERATION 30
ITERATION 20
ITERATION 13
ITERATION 7
0,5
0,5
0,5
0,5
0,5
node nbr

FIGURE 8.64

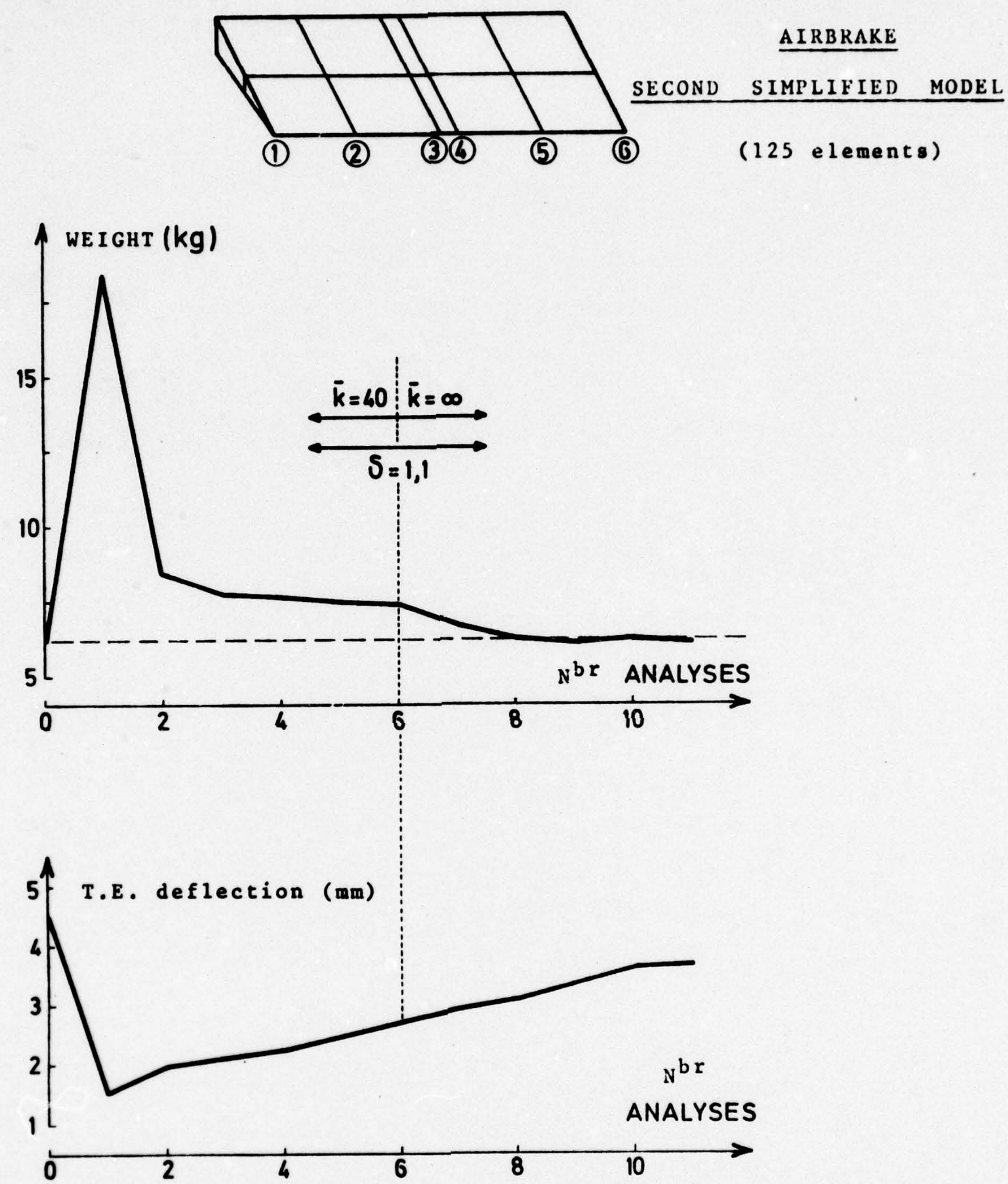


AIRBRAKE STRUCTURE

SECOND SIMPLIFIED MODEL

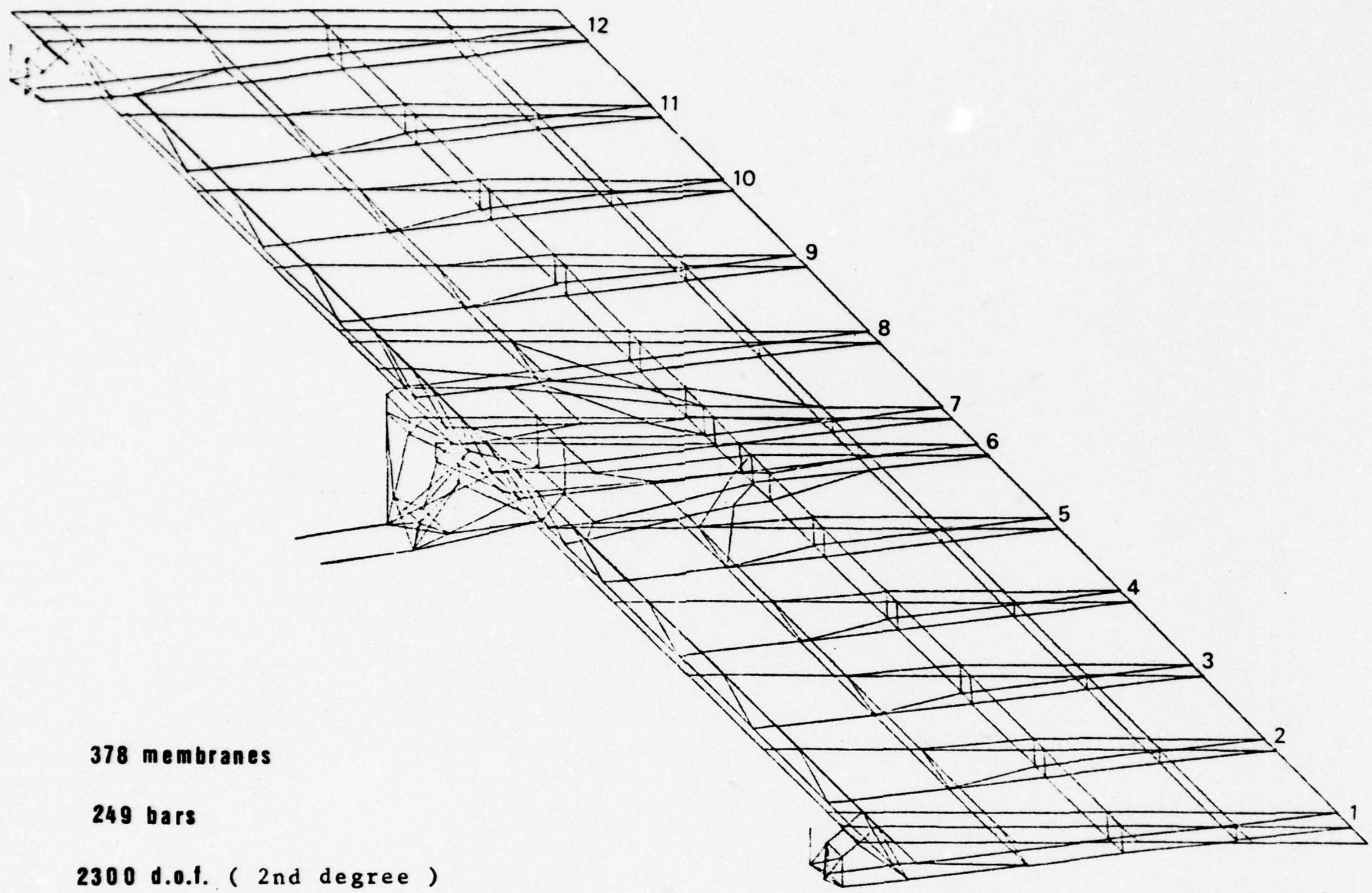
(125 elements)

Figure 8.65



1
2
3
4
5
6
AIRBRAKE
SECOND SIMPLIFIED MODEL
(125 elements)
WEIGHT (kg)
15
10
5
$\bar{k}=40$ $\bar{k}=\infty$
$\delta = 1,1$
N^{br} ANALYSES
0
2
4
6
8
10
T.E. deflection (mm)
5
4
3
2
1
N^{br}
ANALYSES
0
2
4
6
8
10

Figure 8.66



378 membranes

249 bars

2300 d.o.f. (2nd degree)

AIBRAKE
Final model (627 elements)

FIGURE 8.67

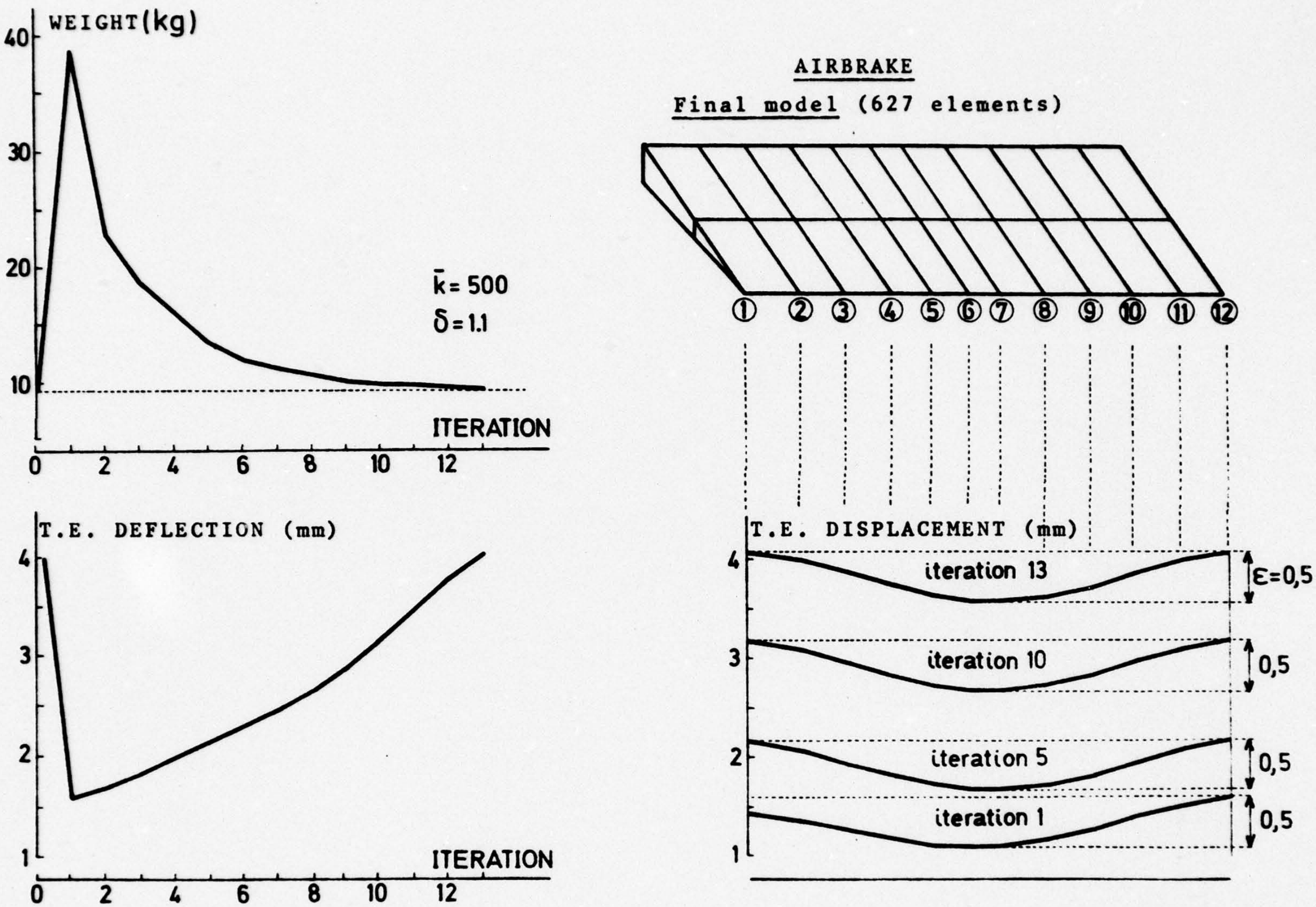

WEIGHT(kg)
40
30
20
10
0
2
4
6
8
10
12
ITERATION
k̄=500
δ=1.1
T.E. DEFLECTION (mm)
4
3
2
1
ITERATION
AIRBRAKE
Final model (627 elements)
1 2 3 4 5 6 7 8 9 10 11 12
T.E. DISPLACEMENT (mm)
iteration 13
iteration 10
iteration 5
iteration 1
ε=0,5
0,5
0,5
0,5

FIGURE 8.68

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