

AD-A052 411

TEXAS A AND M UNIV COLLEGE STATION DEPT OF INDUSTRIA--ETC F/G 21/5  
REPLACEMENT PROCESS ANALYSIS, AN INTERIM REPORT ON REPLACEMENT --ETC(U)  
JAN 78 P HU, L L GEORGE AFOSR-78-3501

UNCLASSIFIED

AFOSR-TR-78-0679

NL

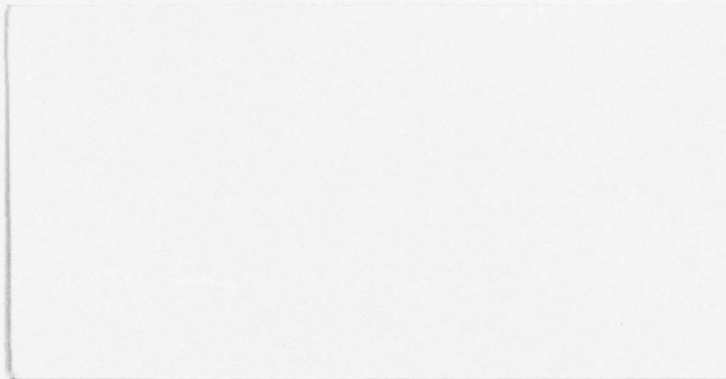
| OF |  
AD  
A052411



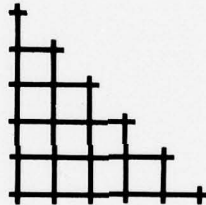
END  
DATE  
FILMED  
5-78  
DDC

2

AD A 052411



AD No.   
 DDC FILE COPY



DDC  
RECEIVED  
APR 10 1978  
F

TEXAS A&M UNIVERSITY  
*Industrial Engineering Department*  
COLLEGE STATION, TEXAS 77843

This document has been approved  
for public release and sale; its  
distribution is unlimited.

Approved for public release;  
distribution unlimited.

REPLACEMENT PROCESS ANALYSIS,  
AN INTERIM REPORT ON REPLACEMENT MODELS

Po-Wen Hu  
Laurence L. George

Department of Industrial Engineering  
Texas A&M University  
College Station, Texas 77843



January 30, 1978  
For Official Use Only  
Interim Report for Period 1 Oct. 1977-1 Feb. 1978  
Prepared for  
Air Force Office of Scientific Research  
Bolling AFB, D.C. 20332 78-3501

## TABLE OF CONTENTS

| SECTION  | PAGE |
|--|------|
| I. MODELS FOR AIRCRAFT ENGINE REPLACEMENT PROCESS .....                        | 1    |
| 1. Single Engine Replacement Model .....                                       | 1    |
| 2. Multiple Aircraft Engine Replacement Model .....                            | 2    |
| 3. Multiple Aircraft with Multiple Engines .....                               | 3    |
| II. MODELS FOR THE ENGINE AGE AT FAILURE DISTRIBUTION .....                    | 4    |
| 1. Competitive Risk Model .....  | 4    |
| 2. Phase Type Model .....  | 5    |
| 3. Time to Maintenance Model .....   | 6    |
| 4. A Concomitant Variables Model .....   | 8    |
| 5. A Markovian Model for the Age of Spare Engines.....                         | 9    |
| 6. Modeling the Removal Times of Conditional Maintenance<br>Engines .....      | 14   |
| III. MODELS FOR RANDOM FLYING HOURS PER QUARTER BY DIFFERENT<br>AIRCRAFT ..... | 24   |
| 1. Normal Distribution .....   | 26   |
| 2. Shared Flying Hour Program .....  | 26   |
| REFERENCES .....   | 27   |
| APPENDICES .....   | 31   |
| 1. Normal Approximations for the Number of Replacements .....                  | 28   |
| 2. Residual Life Distribution of Used Spare Components .....                   | 29   |
| 3. Residual Life of Used Components if the Age Distribution<br>is Known .....  | 30   |
| 4. Convergence of the Spare Engine Age Distribution .....                      | 31   |

FOREWORD

This report is the first interim report for AFOSR Grant No. 78-3501. It describes models for replacement processes with potential applications to gas turbine engine management. Subsequent reports will describe how the models will be used to forecast replacement requirements and to develop optimal policies for management of repairable high cost equipment.

|                                |   |
|--------------------------------|---|
| ACCESSION for                  |   |
| NTIS                           | White Section <input checked="" type="checkbox"/> |
| DDC                            | Buff Section <input type="checkbox"/>             |
| UNANNOUNCED                    | <input type="checkbox"/>                          |
| JUSTIFICATION                  |   |
| -----                          |   |
| DISTRICTION/AVAILABILITY CODES |   |
| SPECIAL                        |   |
| A                              |   |

## I. MODELS FOR AIRCRAFT ENGINE REPLACEMENT PROCESSES

A replacement process is a stochastic process similar to a renewal process except that the replacement components are not necessarily new. A renewal process is a sequence of independently, identically distributed (iid) nonnegative random variables which represent the component lives. The probability theory for renewal processes is well known and complete. In replacement processes, because the residual life of a used component depends on the past history of that component, the random variables of component lives may not be iid. The theory of renewal processes has to be extended to the non-iid situations to handle replacement processes.

The objective of replacement process analysis is to compute the number of replacements that will occur in satisfying a flying hour requirement of specified length given the ages of installed and spare engines. We are proposing several models which may be useful for this purpose.

### 1. Single Engine Replacement Model

Let  $A$  denote the age of the engine at installation. Start from time 0, assume  $A=a_1$ , if the engine failed after flying  $Y_1$  hours, then it is immediately repaired or overhauled. If it is repaired, the engine will go back to service with age  $A=a_1 + Y_1$ , if it is overhauled, the age will be  $A=0$ . (Assume the maintenance action takes no time on an operating time scale.) The process repeats again. The decision to repair or overhaul depends on two factors: if an engine reaches age MOT (maximum operating time) it will be overhauled, if age at failure is less than MOT, it may be repaired or overhauled.

Assume the new engine has failure age distribution

$$F(x) = P \{X \leq x\}$$

$$0 \leq x \leq \text{MOT},$$

and also assume the performance of an engine with age  $A$  is the same as another engine with the same age. Therefore, given an engine with age  $A$ , the conditional distribution of Residual Life  $Y$  is  $F(y|A) = P\{Y \leq y | X > A\}$ . Figure 1 shows the distribution of  $A$ 's at a sequence of "replacement points."

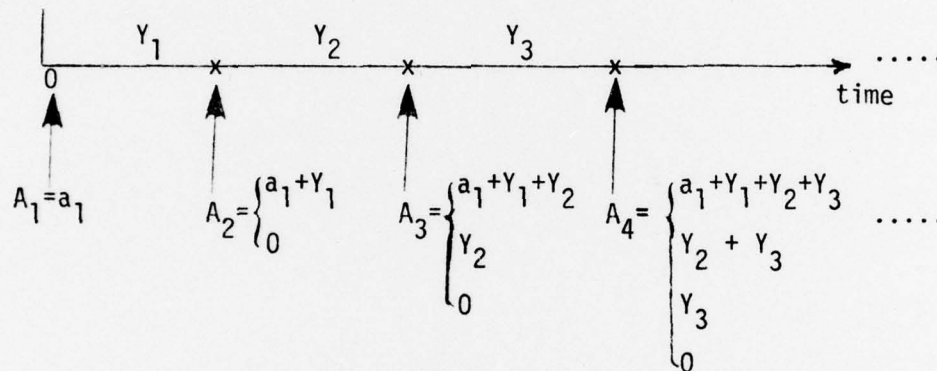


Figure 1 Ages at installation

Note that the distribution of  $Y_1$  is conditioned on  $A_1$ , i.e.  $F(y | A_1)$ . The distribution of  $Y_2$  is conditioned on  $A_2$  and in turn  $A_2$  is conditioned on  $Y_1$ , and so on. Therefore the replacement process  $Y_1, Y_2, Y_3, \dots, Y_n, \dots$  is a sequence of non-iid random variables. If the distribution of the sum of  $Y_1, Y_2, Y_3, \dots, Y_n$  can be obtained, the distribution of the number of replacements as a function of required operating time can be derived. If  $Y$  and  $A$  take values on positive integers, this model can be formulated as a Markov chain, if the Markov property is satisfied.

## 2. Multiple Aircraft Engine Replacement Model

Assume at time 0 there are initially  $K$  aircraft (single-engined) at a base with a spare engine inventory which consists of engines of ages  $a_1, a_2, a_3, \dots$  with distribution  $G(a, 0)$ . During the process, if an engine fails, it is immediately replaced by a spare engine

randomly selected from the spare inventory. The removed engine will go back to the spare inventory after being repaired or overhauled. Thus the entire process is a multi-channel replacement process or superposition of replacement processes. The random variables  $Y_1, Y_2, Y_3 \dots$  for each channel not only depend on the past history of its own channel, but also depend on the past history of the other channels. The age distribution  $G(a,t)$  will depend on the past history of all channels and there may not exist a stationary distribution  $\lim_{t \rightarrow \infty} G(a,t)$ .

Thus the problem of this model is to find the distribution of  $G(a,t)$  and obtain the residual life distribution from  $G(a,t)$ . The total engine replacement requirement will be found from the superposition of several non-independent individual replacement processes. If the number of aircraft and inventory size is large, we may be able to assume each replacement process is independent of the other, the age distribution is independent of the past, and all times between failures are identically distributed. In this case renewal theory will be applied.

### 3. Multiple Aircraft with Multiple Engines

If aircraft are equipped with multiple engines, the residual life of each engine will not only depend on its own past history but also on the other engines of the same aircraft. Therefore the age distribution at installation may not result from randomly selecting engines out of the inventory.

The complexity of this model may make the problem solvable only by approximation methods. We have not examined practical applications sufficiently to propose models of this situation.



## II. MODELS FOR ENGINE AGE AT FAILURE DISTRIBUTION

The analysis of the models in section I depends on a suitable engine age at failure model which will satisfy the following needs:

- a. the model should represent the actual operating conditions of the engine,
- b. all relevant engine performance data should be fully utilized by this model,
- c. the model should be flexible even in the situation of changing maintenance policies and engine configurations, and
- d. the model should be manageable in the computation of engine replacement requirements.

Several potentially useful models for engine ages at removal are described in this section.

### 1. "Competitive Risk" Model

This model is described in [1] and takes into account the significant probability of removals at inspection times. The distribution function of time to first failure is assumed to be of the form

$$F(t) = 1 - (1 - F_1(t))_{i=1}^{i(t)} (1 - p_i)$$

where  $F_1(t)$  is the distribution function of a continuous nonnegative random variable representing usage failure times, the  $p_i$  are the probabilities a failure is detected at inspection times  $t_1, t_2, \dots, MOT$  and  $i(t)$  is the index of the last inspection prior to or at time  $t$ . The maximum likelihood estimator for this model is derived in [1] and [2] for right censored samples. The structure of this model may make it easier for computation of replacement requirements using analytical approximation methods or simulation. If the stationary

spare engine age distribution can be obtained (by simulation or analytical methods), the residual life distribution can be derived from the conditional life distribution  $F(t|a)$ . By using the central limit theorem, the number of replacements can be approximated. The normal approximation is illustrated in Appendix I.

## 2. Phase Type Model

One of the major difficulties in analytical and computational studies of replacement processes lies in the increasing complexity which is introduced by conditioning on the past behavior of the processes. Each unconditioning requires one integration. The phase type model discussed in [3] will simplify the computation if the problems are formulated according to [3].

Assume the life of the engine proceeds in several stages. Each stage represents the state of the life variable. Now consider a  $(m+1)$  state Markov chain with integer state space  $\{1, 2, \dots, m, m+1\}$  whose transition matrix  $P$  is of the form

$$P = \begin{bmatrix} T & \underline{I}^0 \\ 0 & 1 \end{bmatrix} \dots \dots \dots (1)$$

where  $T$  is a  $m \times m$  matrix and  $\underline{I}^0$  is a column vector with  $m$  components. State  $m+1$  is an absorbing state, namely the failure state. The probability of absorption into the state  $m+1$ , starting from any state, is assumed to be 1.

A probability density  $\{V_k\}$  on the positive integers is of phase type, if and only if there exists a finite matrix  $P$  of type (1) and a vector  $\underline{\alpha}$  of initial probabilities, such that  $\{V_k\}$  is the density of the time till absorption.

Suppose that each time an absorption into state  $m+1$  occurs in the chain, we instantaneously restart the chain by choosing the initial state according to  $\underline{\alpha}$ . Any restarting of the chain is called a replacement. The density of time between replacements is given by  $\{V_k\}$ . The convolution of  $\{V_k\}$ 's will give the distribution of the time till the  $n^{\text{th}}$  removal. Some of the classical theory on finite Markov chains can be applied to this model. The recursive computation method in [3] will reduce the computation effort to an algorithm.

The problems with this model are the discretized failure time variable, the estimation of the transition probability matrix  $T$  and vector  $\underline{I}^0$ , and the initial probability vector  $\underline{\alpha}$  is not time homogeneous for a typical spare engine inventory. We are willing to assume that under stable conditions, the spare engine age distribution has some limiting distribution  $\lim_{t \rightarrow \infty} G(a,t)$  but we would like to use the actual ages  $a_1, a_2, \dots, a_3$  of the current stock of spare engines in forecasting replacement requirements.

Nevertheless, the phase type model has the potential to incorporate information in addition to engine age into the state space. The state space could be the Cartesian product of a partition of the engine life span  $[0, MOT]$  crossed with the number of prior repairs.

### 3. Time to Maintenance Models

These models are appropriate for engines which use the "on condition maintenance" policy, fix it when it breaks. Such engines have no periodic inspections and no MOT. To eliminate safety threatening failures, life limits on critical components are specified in terms of operating times or cycles or both. An engine is removed

for repair at failure or at expiration of a life limit. No overhaul is done but replacement of life limited components rejuvenates engines to some extent.

The models are similar to the competitive risk model formulated for engines with periodic inspections and an MOT except the inspections do not occur at fixed ages and the engines are never overhauled. The random variable represented by these models is the time between removals. It is bounded above by the shortest time limit on the life limited components. However, an engine may undergo a scheduled removal for some other life limited component prior to the upper bound. For example, suppose an engine with two life limited components with limits 1000 and 1500 hours. The second scheduled removal occurs at 1500 hours assuming no unscheduled removals occur and no opportunistic replacement of the second component is done at the 1000 hour removal.

Superimposed on the sequence of scheduled removals for replacement of life limited components will be the unscheduled removals caused by engine failures. There will be a distribution function dependent on time since last removal but not necessarily dependent on the age of the engine. Perhaps it will be sufficient to characterize the time between failures distribution by the number of prior removals.

The form of the distribution function of times between removals will be

$$F(t) = F_1(t|l_1, l_2, \dots, l_k) F_2(t)$$

where  $F_1(t|l_1, l_2, \dots, l_k)$  is the probability distribution of time between scheduled removals and  $F_2(t)$  is for times between failures.

The  $l_i$  are the time limits on life limited components.  $F_1(t|l_1, l_2, \dots, l_k)$  incorporates ages of life limited components because this distribution may be practically degenerate making forecasting of scheduled removals very accurate.

#### 4. "Concomitant Variables" Model

A model based on regression is suggested in Cox [4] when several measurements of performance are recorded as the concomitant variables. It is assumed that the failure rate function at time  $t$  for a given set of concomitant variables  $\underline{z} = (z_1, z_2, z_3, \dots, z_p)$  is given by

$$\lambda(t, \underline{z}) = \lambda_0(t) \exp(\underline{z}\underline{\beta})$$

where  $\underline{z}$  is a row vector of measurements of age, number of prior removals, etc. and  $\underline{\beta}$  is a column vector of corresponding parameters for those variables. The function  $\lambda_0(t)$  is the failure rate function when all concomitant variables are zero. If the failure rate function is continuous, we can express the relation between life distribution  $F(t)$  and failure rate function as follows:

$$1 - F(t) = \exp \left[ - \int_0^{\infty} \lambda_0(t) e^{\underline{z}\underline{\beta}} dt \right]$$

which is equivalent to

$$\ln \left[ \frac{-\ln(1-F(t))}{\int_0^{\infty} \lambda_0(t) dt} \right] = \underline{z}\underline{\beta}.$$

Since  $\lambda_0(t) = -\ln(1-F(t))$  when the vector  $\underline{z} = \underline{0}$ , we can interpret  $\lambda_0(t)$  as the failure rate of new engine on which all the concomitant variables are zero. By properly selecting the variables in  $\underline{z}$  vector, the vector  $\underline{\beta}$  can be estimated by regression. Marginal likelihoods of the parameters  $\underline{\beta}$  are obtained in [4] and [5].

The advantage of this model is the potential to incorporate many engine performance variables. But on the other hand, the concomitant variables should be identified before the model can be constructed. The regression model can also be used to test the relevancy of the variables. However, the resulting life model is not convenient for analytical replacement computation, even for renewal processes. Since the life is a function of  $z$ , for used engines with different  $z$  values, the convolution of the life variables will become fairly complicated.

This model may be applied to construct the "phase type model" to simplify the computation (subsection 2).

5. A Markovian Model for the Age of Spare Engines

The age of the installed and spare engines in a fleet has an effect on the number of replacements required to achieve a flying hour program. If the engines are old, the number of replacements will be larger than if the fleet consists of a large proportion of new or recently overhauled engines. The current actuarial system takes into account the ages of installed engines but does not use available information on the ages of spare engines. This is not because it can't be done but because the choice of the sequence of spare engines to install is left to the discretion of the base maintenance managers and is consequently unknown to the AFLC personnel computing replacement requirements. A random sequence for installation could be used, but the loss of accuracy for choosing the wrong sequence has not been estimated. Furthermore, there aren't very many spare engines with known ages in serviceable status at base level at the time replacement requirements are computed. Many of the spare engines used during the 30 quarter duration of the

flying hour program are engines that fail, are repaired, and return to service during the 30 quarters.

In order to incorporate spare engine ages into replacement requirements computations, we need to be able to represent the cumulative distribution function of the ages of the serviceable spare engines at any time in the future (as well as the number of spares which is a consequence of the replacement requirements). The representation would appear like an empirical distribution function given the number of spare engines  $s$  in stock, (figure 2 is an example).

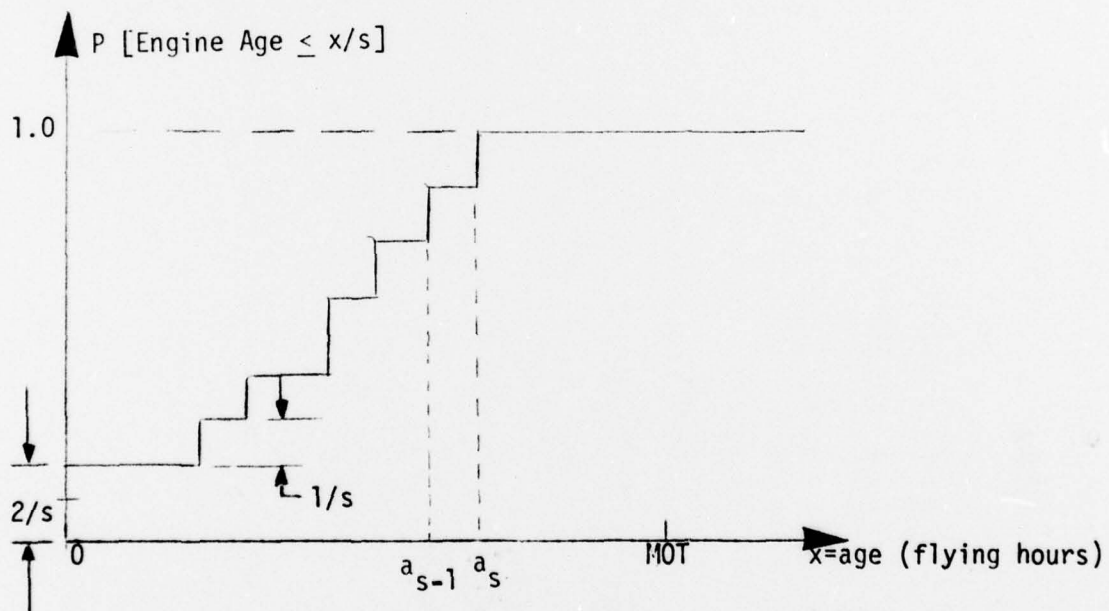


Figure 2 The Age Distribution of Spare Engines

The age of a spare engine depends on the age of the engine at its last previous removal and its removal cause. Engines with ages exceeding maximum operating time (MOT) are overhauled and their age after overhaul when they return to serviceable spares status is assumed to be zero. There is also an age dependent probability an engine removed with age  $< \text{MOT}$  will also be overhauled. Otherwise when an engine is returned to serviceable status, its age is its age at the last previous removal, and, in the actuarial replacement requirements computation [8], such an engine is assumed to be as good as any other engine with the same age regardless of repair history.

A probability model to describe the age of spare engines is as follows. Let  $X_n$  denote the age of an engine after its  $n^{\text{th}}$  removal. The sequence  $\{(X_n, n); n=0,1,\dots\}$  is a stochastic process which will be a discrete or continuous state Markov process with state space  $R^+ \times N^+$ , (nonnegative reals  $\times$  nonnegative integers).

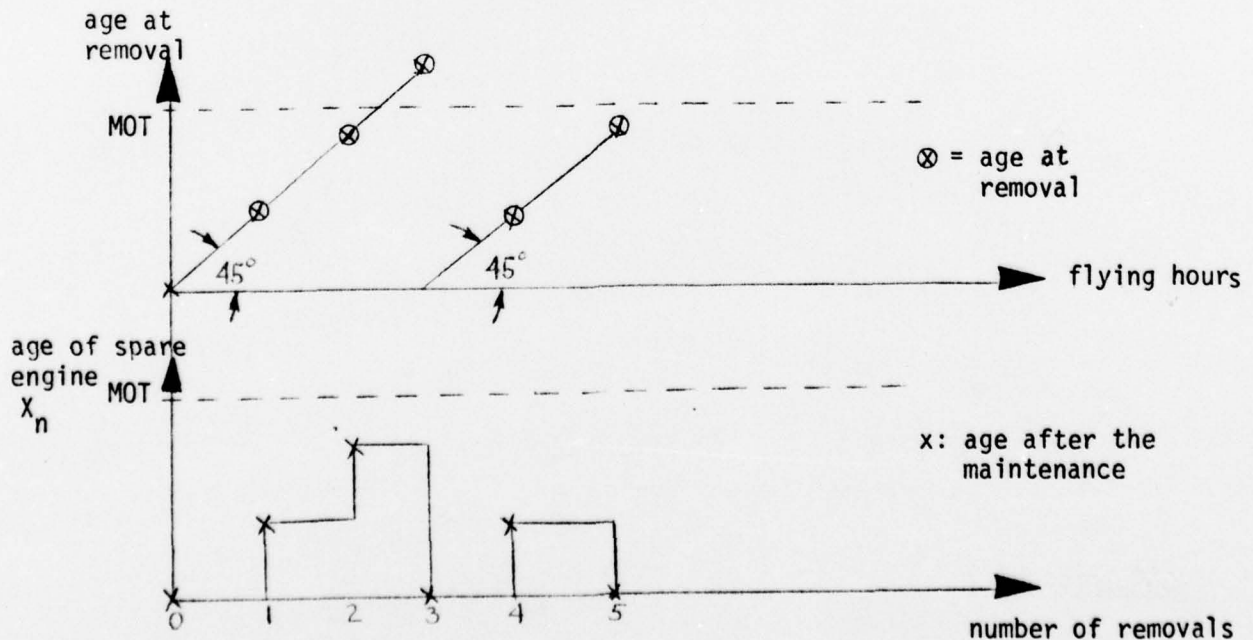


Figure 3 Relationship of age at removal and age of spare engine



Let  $P(a,n)$  denote the probability an engine will be overhauled if its age at  $n^{\text{th}}$  removal is  $a$ . The stochastic process  $X_n$  is defined by the relation

$$X_{n+1} = \begin{cases} 0 & \text{if } X_n + T \geq \text{MOT} \\ 0 & \text{if } X_n + T < \text{MOT} \text{ with probability } P(X_n + T, n+1) \\ X_n + T & \text{if } X_n + T < \text{MOT} \text{ with prob. } 1 - P(X_n + T, n+1). \end{cases}$$

The random variable  $T$  is the time between removals and its distribution may depend on  $X_n$  and  $n$  but on no prior history of the engine. (See figure 3 for a sample realization.) The distribution of  $X_n$  is related to the distribution of spare engine ages given the number of spare engines  $s$ . In particular,  $\lim_{n \rightarrow \infty} X_n$  represents a typical engine in steady state.

The distribution of  $\lim_{n \rightarrow \infty} X_n$  can be converted into a discrete distribution with  $s$  jumps of equal height to represent the steady state age distribution of  $s$  spare engines. One procedure is shown in figure 4. Three engines have age zero and the remaining 8 engines have ages spread over  $(0, \text{MOT})$ . The bias of this discretization will have to be investigated.

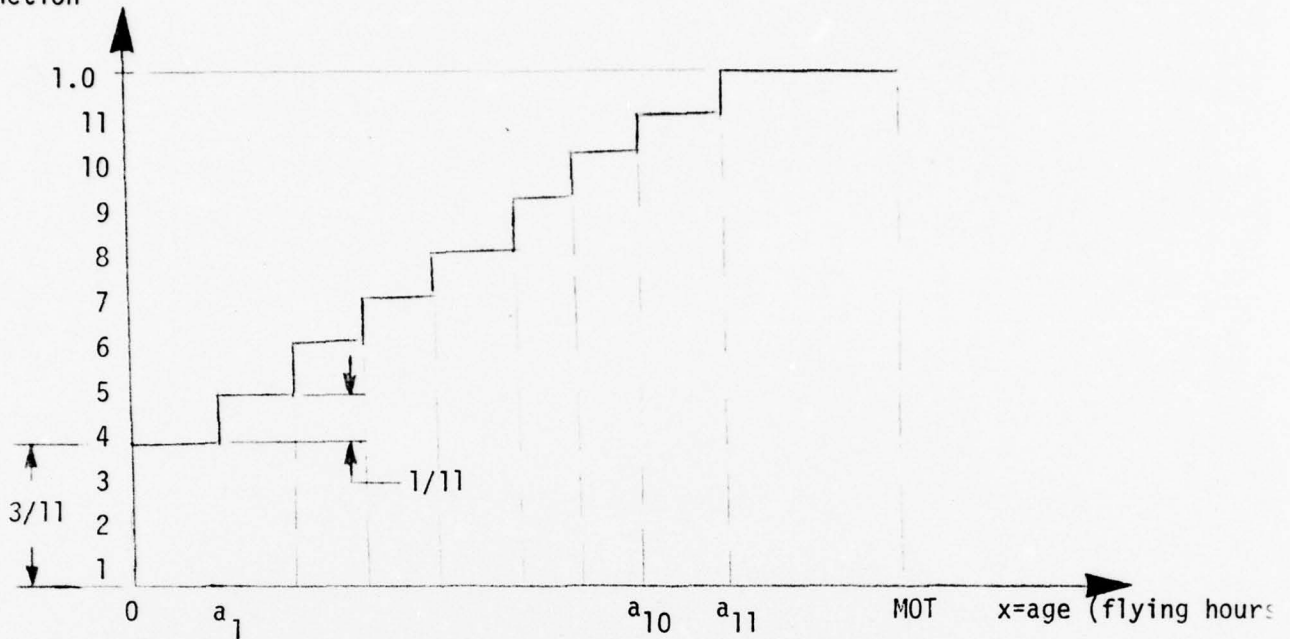
cumulative  
distribution  
function

Figure 4 Converting the cdf of  $\lim_{n \rightarrow \infty} X_n$  to an Age Distribution of 11 spare engines

If the state space is discretized to  $(N^+)^2$ , the distribution of  $X_n$  can be computed by classical Markov chain theory from the transition probability matrix.

In a simpler case, where the distribution of random variable  $T$  only depends on  $X_n$  and  $P(a, n) = P(a)$  (i.e. assume the overhaul probability is only age dependent), the transition probability matrix can be estimated from the available actuarial data.

Appendix 1 shows an approximation procedure for replacement requirements which depends on knowledge of the age distribution of the entire stream of future spare engines. That is the reason for our concern with some representation of ages of spare engines. Markovian models are particularly attractive because continuous state continuous time Markov processes can be approximated by diffusion processes which have normally distributed transition probabilities. Diffusion processes have representations as (sometimes) tractable differential equations for probability densities.

## 6. Modeling the Removal Times of Conditional Maintenance Engines

The current actuarial model of engine ages at removal [8] has a fixed upper limit called the maximum operating time (MOT). Engines which reach this age are removed, overhauled, and their age is reset to zero. This policy results in removal and overhaul of engines that may have continued to operate beyond MOT. The conditional maintenance policy has been adopted as an alternative for several engine types and will probably spread throughout the fleet. Under this policy, an engine is never removed at MOT for an overhaul but may continue to age indefinitely unless its performance calls for repair.

Of course engine removals do occur, but the reasons for removal are different.

1. Removals occur at unscheduled times for failures of engine components or accessories.
2. Removals occur for safety reasons to replace critical, life limited components when they reach their time or cycle limit, whichever occurs first.
3. Removals occur for external causes such as foreign object damage (FOD).

Other causes for removal occur but either they do not cause a demand on the spare engine inventory or else they are unpredictable, directed removals.

A competitive risk model [7] hypothesizes engine removal is caused by the occurrence of the first of several causes.

The model can incorporate right censoring readily because censorship is another competing cause that results in removal of an engine from the sample. This is handy to represent engines that survive removal to a fixed calendar time such as the end of a quarter. The competitive

risk model lends itself to actuarial computation of engine removals because the actuarial failure rates for competing removal causes add.

Assume the failure rate functions for  $n$  competing removal causes are  $r_1(t), \dots, r_n(t)$ . Then the cumulative distribution function of age at removal is

$$F(x) = 1 - \exp\left[-\int_0^x \sum_{i=1}^n r_i(t) dt\right]$$

(This is true if the causes act independently, but is also true for some models even if the causes aren't independent [7].)

One failure rate function can be used to represent each of the three causes of removal listed above. In turn, each of these three failure rate functions may be the sum of several failure rate functions for each of several competing causes of engine removal.

For instance, there are 75 life limited components in the F100PW100 engine. The life limits are chosen so that the probability of failure prior to the life limit is acceptably small. Furthermore, the failure process is a function of operating time (or flying hours) and low cycle fatigue, and the life limits are in terms of hours and engine cycles. Whichever limit occurs first causes engine removal. Unfortunately low cycle fatigue is not accumulated in direct proportion to operating flying time (figure 5) so there is some probability the life limit in cycles is reached prior to the time limit. We have three competing causes for removal of an engine for each life limited component:

1. The hour limit is reached,  $L_j$
2. the cycle limit is reached,  $C_j$  and
3. failure occurs prior to reaching either life limit.

We expect the failure rate function for a life limited component to appear as shown in figure 6 and the corresponding actuarial failure rates to be as shown below the figure. The actuarial failure rate is the area under the failure rate function between the end points of the actuarial age intervals.

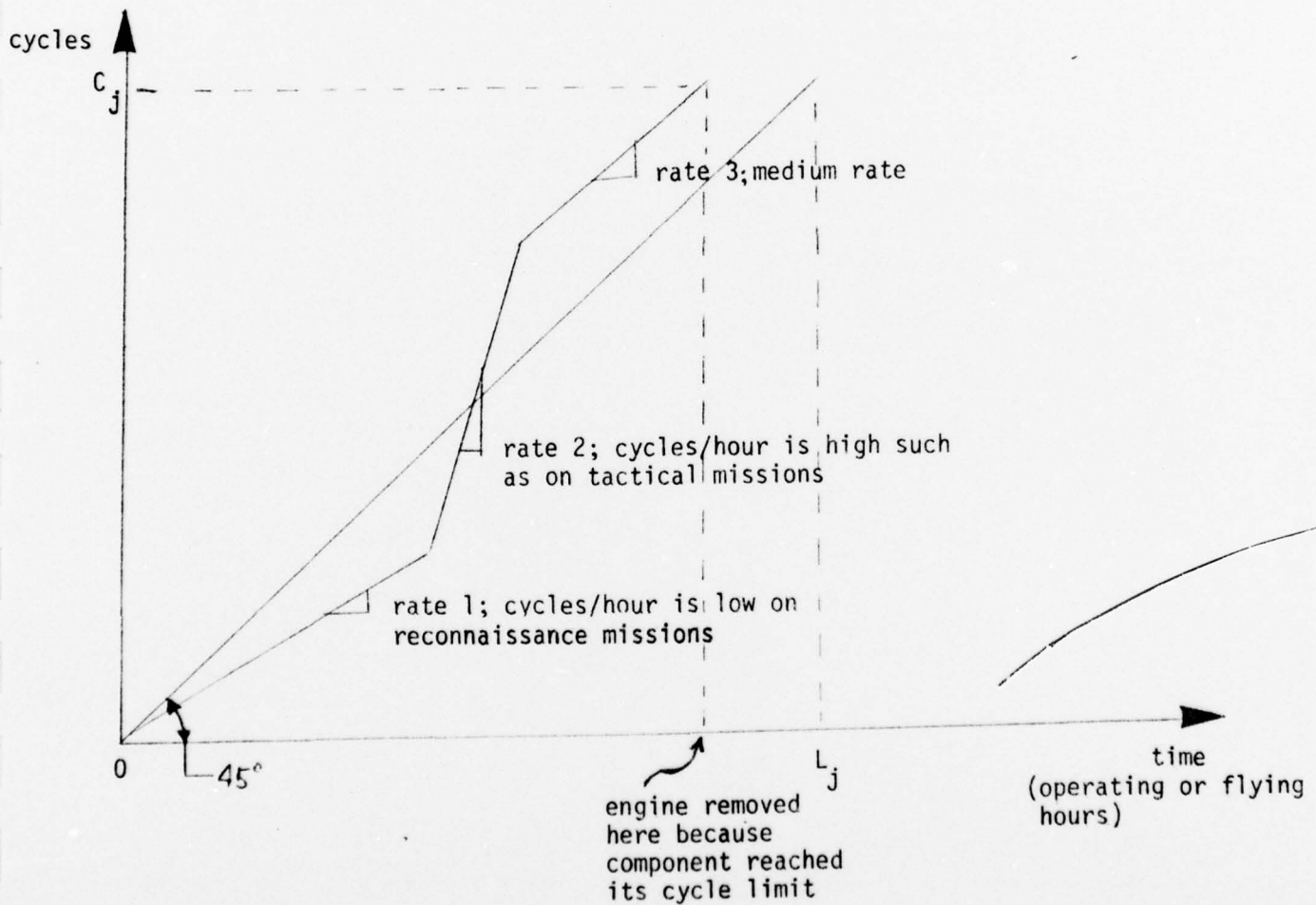


Figure 5 The Random Accumulation of Low Cycle Fatigue as a Function of Time

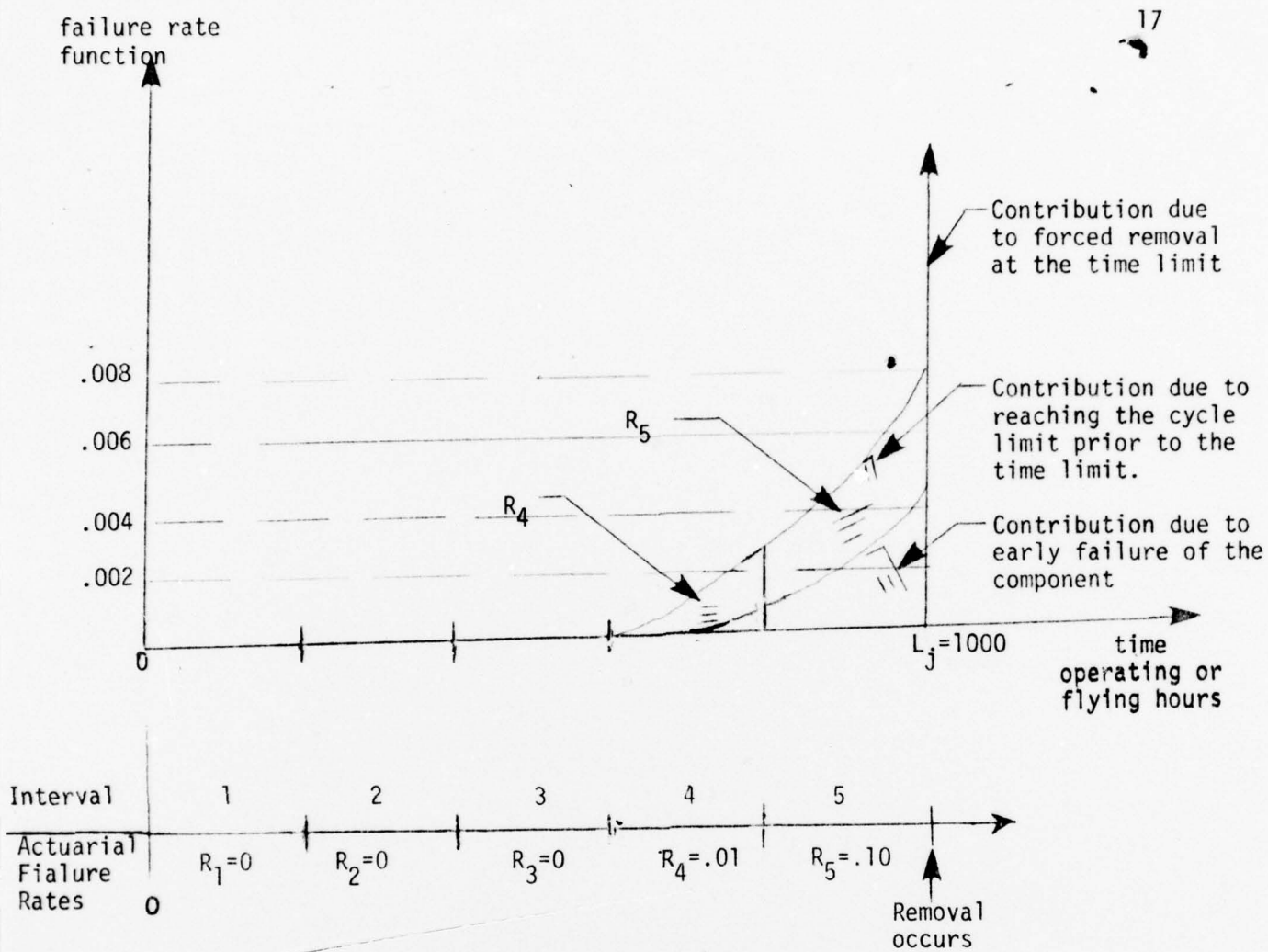


Figure 6 The Failure Rates for a Life Limited Component

The relation between the accumulation of cycles and hours may be linearly proportional, but the proportionality factor is random.

Let  $N_c(T)$  be the number of cycles accumulated during  $T$  flying hours, and  $K$  is the random proportionality factor; then

$$N_c(T) = KT$$

and engine removal for replacement may be represented as occurring at time

$$\inf \{t \mid (N_c(t) \geq C) \cup (T \geq t)\}$$

The probability  $K$  takes on its several values can be estimated as the proportion of aircraft flying each type of mission, and the values of the proportionality factor are the average rate at which cycles are accumulated for each type of mission.

The other causes of engine removal can also be modeled by failure rates or a failure rate function. External causes probably operate at a rate independent of engine age and so may be assumed constant, or at least dependent only on operating time since last removal, but not on the age of an engine. Removals due to failure of external accessories or non-life limited components are a major proportion of total removals. The failure rates for this cause may be estimated by the actuarial method with the time origin as the time of the last removal or the time of the last major (depot) repair. If the time origin is taken at age zero, the number of actuarial age intervals is unbounded.

For modular engines, there can be sets of failure rates for unscheduled removals for each module as well as for the whole engine and its external accessories. The origin for modules might be set at each major module repair such as replacement of a life limited component, and the time origin for the engine removals might be any removal and repair.

This procedure of selecting the time origin means that the random variable time between removals has the upper bound which is the smallest time limit on the life limited components. Consequently the actuarial methods can be used, and the smallest time limit is probably smaller than current MOT's allowing the use of fewer actuarial age intervals without loss of accuracy.

However, the procedure for making the actuarial computations must be modified because some components are not renewed at a removal and because the mechanism causing unscheduled removals of modules is not reset to zero at minor repairs on the module or if a removal had no effect on the module. Table 1 shows the modified actuarial computation. Two examples illustrate the computations procedures.

Example 1 Suppose an engine has two life limited components with failure rates  $R_{1j}$  and  $R_{2j}$  and ages 400 hours and 0 hours respectively (as shown in Table 2). Assume  $R_{0j}$  is a set external cause failure rates. The ranges of the indexes  $j$  for different components depends on their time limits as in Table 1. To adjust  $R_{ij}$  for used components, we shift the  $R_{ij}$  column upward to its corresponding interval (as shown in  $R'_{1j}$  column of Table 2). Thus the combined failure rate  $R_j$  is the sum of  $R_{0j}$ ,  $R'_{1j}$ , and  $R_{2j}$ . The number of removals and the number of survivors are computed in the same manner as in the actuarial method. In order to compute the replacement requirements for a given flying program, we should know the maintenance policy. For instance, the outputs from Table 1 are 0.01 removals and 0.99 survivors for flying one pass of 200 hours. The 0.99 survivors can continue through the computation, but the 0.01 removal will need a replacement for the next pass. Notice the cause of this 0.01 removal is the external cause. Whether the component #1 and #2 will be renewed at the time of an externally caused removal depends on the maintenance policy.



Example 2 Table 3 is an example of an engine composed of 2 modules with 3 life limited components. The computation method is similar to example 1 except the  $R_{ij}$  matrix is expanded and the 3 life limited component failure rates are adjusted to the component ages at last repair.

The general structure of this computation is a multivariate Markov chain where the state is a vector of times or ages since each component was new. The Markov chain vector advances from its initial conditions, the ages of the components at the time of engine installation, actuarial age interval by actuarial age interval until one of the time limits is reached or until one of the competing causes results in engine removal. The problem with this representation is specifying the initial distribution over all possible initial engine states because initial engine state depends on maintenance policy.

This discretized representation of engine ages has several virtues. It is based on the current actuarial system so it will be familiar to Air Force engine managers. The discretization allows us to derive replacement requirement results for the discrete case and then take a limit as the width of the actuarial age intervals goes to zero to prove continuous time results. Last, the whole actuarial structure is based on Markovian assumptions so we may be able to borrow on the theory of Markov processes which is rich in maintenance applications and loaded with computation simplifications. The vice of the discrete representation is that practically every engine will require its own actuarial table. Only engines with the same component ages at last repair could be incorporated into a single actuarial table. We have not broached this complication to AFLC actuaries yet.



| Interval <sub>j</sub> | time<br>(flying hours) | Engine<br>External<br>Causes<br>R <sub>0j</sub> | Life Limit Component            |                                 |                                 | Combined<br>failure<br>rate<br>R <sub>j</sub> =R <sub>0j</sub> +R <sub>1j</sub> +R <sub>2j</sub> | #<br>Engines | Expected<br>#<br>Removals | Expected<br>#<br>Survivals |
|-----------------------|------------------------|---|---------------------------------|---------------------------------|---------------------------------|--|--------------|---------------------------|----------------------------|
|                       |                        |   | Component #1<br>R <sub>1j</sub> | Component #1<br>R <sub>1j</sub> | Component #2<br>R <sub>2j</sub> |  |              |                           |                            |
| 1                     | 0-200                  | 0.01  | 0                               | 0                               | 0                               | 0.01   | 1            | 0.01                      | 0.99                       |
| 2                     | 200-400                | 0.01  | 0                               | 0.01                            | 0                               | 0.01+0.01=0.02   |              |                           |                            |
| 3                     | 400-600                | 0.01  | 0                               | 0.10                            | 0                               | 0.01+0.10=0.11   |              |                           |                            |
| 4                     | 600-800                | 0.01  | 0.01                            | ∞                               | 0.01                            | ∞  |              |                           |                            |
| 5                     | 800-1000               | 0.01  | 0.10                            | ∞                               | 0.05                            | ∞  |              |                           |                            |
| 6                     | 1000-1200              | 0.01  | ∞                               | ∞                               | 0.12                            | ∞  |              |                           |                            |
| 7                     | 1200-1400              | 0.01  | ∞                               | ∞                               | ∞                               | ∞  |              |                           |                            |

Table 2. Computation of Actuarial Failure Rate in Example 1.

| Interval j | Flying Hours | Engine External Cause<br>$R_{oj}$ | Module External cause |                      | Life Limit Component |           |          |               |           |          |             |           |          | Combined Failure Rate<br>$R_j = \sum R_{ij}$ | Initial # Engine | Expected # Removals | Expected # Survivals |   |       |       |
|------------|--------------|-----------------------------------|-----------------------|----------------------|----------------------|-----------|----------|---------------|-----------|----------|-------------|-----------|----------|--|------------------|---------------------|----------------------|---|-------|-------|
|            |              |                                   | Module A<br>$R_{Aj}$  | Module B<br>$R_{Bj}$ | #1<br>Age=400        |           |          | #2<br>Age=400 |           |          | #3<br>Age=0 |           |          |  |                  |                     |                      |   |       |       |
|            |              |                                   |                       |                      | $R_{1j}$             | $R'_{1j}$ | $R_{1j}$ | $R_{2j}$      | $R'_{2j}$ | $R_{2j}$ | $R_{3j}$    | $R'_{3j}$ | $R_{3j}$ |  |                  |                     |                      |   |       |       |
| 1          | 0-200        | .01                               | .0002                 | .0002                | 0                    | 0         | 0        | 0             | 0         | 0        | 0           | 0         | 0        | 0  | 0                | 0                   | 0                    | 1 | .0104 | .9896 |
| 2          | 200-400      | .01                               | .0001                 | .0001                | 0                    | 0         | 0        | 0             | .1        | 0        | 0           | .1        | 0        | 0  | 0                | 0                   | 0                    |   |       |       |
| 3          | 400-600      | .01                               | .0002                 | .0002                | 0                    | .1        | 0        | 0             | .3        | 0        | 0           | .3        | 0        | 0  | 0                | 0                   | 0                    |   |       |       |
| 4          | 600-800      | .01                               | .0002                 | .0002                | 0                    | .3        | .1       | ∞             | ∞         | .1       | ∞           | .1        | .1       | ∞  | ∞                | ∞                   | ∞                    |   |       |       |
| 5          | 800-1000     | .01                               | .0002                 | .0002                | .1                   | ∞         | .3       | ∞             | .3        | ∞        | ∞           | ∞         | ∞        | ∞  | ∞                | ∞                   | ∞                    |   |       |       |
| 6          | 1000-1200    |                                   |                       |                      | .3                   | ∞         |          |               |           |          |             |           |          |  |                  |                     |                      |   |       |       |
| 7          | 1200-1400    |                                   |                       |                      | ∞                    |           |          |               |           |          |             |           |          |  |                  |                     |                      |   |       |       |

Table 3. Computation of Actuarial Failure Rate in Example 2

### III. MODELS FOR RANDOM FLYING HOURS PER QUARTER BY DIFFERENT AIRCRAFT

At the time the proposal was submitted we proposed analysis of replacement processes, renewal processes where the replacement items aren't new. We were also interested in the replacement requirements for the superposition of replacement processes. All of the properties for superposition of renewal processes are based on the assumption that each component renewal process operates for exactly the same amount of time. This is not true for our application, predicting engine replacement requirements for a fleet of aircraft, because not all aircraft fly the same share of the flying hour program. We are concerned about the effect of this observation on the superposition of renewal and replacement processes. Consequently we have formulated models to describe unequal flying hours achieved by a fleet of aircraft in accomplishing a fixed total flying hour program.

Most studies on the prediction of the engine replacement requirements formulate the requirement as a function of total flying hours assigned to a fleet. This is based on two assumptions, the total flying hours assigned to a fleet are assumed to be met exactly and all aircraft share an equal amount of the flying hours assigned to a fleet. In fact, based on past experience, neither assumption holds precisely.

The randomness caused by violation of either assumption will have the following effects:

- a. The total expected requirements in a quarter may no longer be accurate,
- b. The long term replacement requirements based on the known flying hours will have large variance than expected, and
- c. The different usage rate near the end of a quarter may affect the engine life distribution and consequently will affect the accuracy of the requirements forecast.

For more satisfactory results on the forecast of replacement requirements, it is necessary to investigate the distribution of the actual flying hours on each aircraft and the distribution of actual flying hours accomplished by a fleet. Several models are proposed here for this purpose:

1. Normal Distribution

Let  $T_A$  denote the flying hours assigned to a fleet and  $T$  denote the actual total flying hours accomplished by a fleet. Assume that events  $\{T > T_A\}$  and  $\{T < T_A\}$  are possible, then the normal distribution may describe the actual flying hours per aircraft per quarter. This has been suggested by one study of Air Force fleets. (We do not have the reference to this study.)

If there are  $n$  aircraft in a fleet and  $X_1, X_2, X_3, \dots, X_n$  denote the actual flying hours on each aircraft, then  $T = X_1 + X_2 + X_3 + \dots + X_n$ . Suppose  $X_1, X_2, \dots, X_n$  are independently and identically distributed with mean  $\frac{T_A}{n}$  and variance  $\frac{\sigma^2}{n}$ , then by central limit theorem, if  $n$  is large,  $T$  has approximately a normal distribution with mean  $T_A$  and variance  $\sigma^2$ .

The problems with this model will be:

- (1) Test the assumptions used in this model, and
- (2) Estimate the parameters of the distribution.

2. Shared Flying Hour Program

Suppose the allocation of flying hours to each aircraft follows this procedure. At the beginning of a quarter, the total amount of flying hours  $T_A$  are shared equally by each aircraft. During the quarter, assume some accidents or aircraft maintenance actions occur, and an aircraft is removed from the fleet and its remaining flying hours are equally reallocated to all remaining aircraft. If another

removal occurs, the remaining flying hours on that aircraft are again equally reallocated to the others. If the removal rate is small, then at the end of each quarter, most of the aircraft will have an equal amount of flying hours (which are greater than the original assignment) except a few aircraft will have unequal hours.

If there is an upper limit on the flying hours assigned to each aircraft during a quarter, excess flying hours will not be flown. Thus the flying hours assigned to this fleet may not be met exactly. (One such limit is the number of calendar hours in a quarter,  $90 \times 24 = 2160$  hours.)

This model may represent the actual flying hour assignment process even though additions to the fleet have not been represented. If it is easy to represent removals from the aircraft fleet, it shouldn't be too hard to also represent additions to the fleet and the consequent reduction in flying hours per aircraft. This model reflects the fact that the flying hours are accumulated at a faster rate near the end of each quarter. This change of the accumulation rate can also be incorporated in the model.

## REFERENCES

1. George, L. L., "Integrated Aerospace Engine Management: Foundations in Estimation and Prediction of Engine Removals," ARL-TR 75-0137, Aerospace Research Laboratories, Wright-Patterson AFB, Ohio, 1975.
2. George, L. L., "Estimation of Engine Removal Times and Prediction of Replacement Requirement," AFFDL-TR-76-130, Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio, 1976.
3. Neuts, M. F., 'Probability Distribution of Phase Type,' Department of Statistics, Division of Mathematical Sciences, Mimeograph Series #374, Purdue University, 1974.
4. Cox, D. R., "Regression Models and Life Tables," Journal of the Royal Statistical Society, Series B V34, pp 187-220.
5. Kalbfleisch, J. D and Prentice, R. L., "Marginal Likelihoods Based on Cox's Regression and Life Model," Biometrika 60: 267-268, 1973.
6. Gnedenko, B. V., Theory of Probability, second edition, Chelsea, New York, 1962.
7. Gail, M., "A Review and Critique of Some Models Used in Competitive Risk Analysis," Biometrics, Vol. 31, pp 209-222, March 1975.
8. "Procedures for Determining Aircraft Engine (Propulsion Unit) Failure Rates, Actuarial Engine Life, and Forecasting Monthly Engine Changes by The Actuarial Method," T.O. 00-25-128, changed 29 Nov. 1965.



## REFERENCES

1. George, L. L., "Integrated Aerospace Engine Management: Foundations in Estimation and Prediction of Engine Removals," ARL-TR 75-0137, Aerospace Research Laboratories, Wright-Patterson AFB, Ohio, 1975.
2. George, L. L., "Estimation of Engine Removal Times and Prediction of Replacement Requirement," AFFDL-TR-76-130, Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio, 1976.
3. Neuts, M. F., 'Probability Distribution of Phase Type,' Department of Statistics, Division of Mathematical Sciences, Mimeograph Series #374, Purdue University, 1974.
4. Cox, D. R., "Regression Models and Life Tables," Journal of the Royal Statistical Society, Series B V34, pp 187-220.
5. Kalbfleisch, J. D and Prentice, R. L., "Marginal Likelihoods Based on Cox's Regression and Life Model," Biometrika 60: 267-268, 1973.
6. Gnedenko, B. V., Theory of Probability, second edition, Chelsea, New York, 1962.
7. Gail, M., "A Review and Critique of Some Models Used in Competitive Risk Analysis," Biometrics, Vol. 31, pp 209-222, March 1975.
8. "Procedures for Determining Aircraft Engine (Propulsion Unit) Failure Rates, Actuarial Engine Life, and Forecasting Monthly Engine Changes by The Actuarial Method," T.O. 00-25-128, changed 29 Nov. 1965.

## APPENDIX I

1. Normal Approximation for the Number of Replacements

Consider a replacement process to be a sequence of independent but not identically distributed random variables  $\{Y_i; i=1,2,\dots\}$ . Suppose the  $Y_i$  variables are conditional random variables all drawn from the same unconditional distribution  $H(y)$  but given the  $Y_i$  value is greater than some age  $a_i$  so  $Y_i$  comes from the distribution  $H(y|a) = P[Y < y | Y > a]$ . The objective is to obtain an approximation for the number of replacements in the sequence  $\{Y_i\}$  during an interval from zero to  $x$ ,  $N(x)$ .

The total operating time till  $n^{\text{th}}$  replacement is  $\sum_{k=1}^n Y_k$ .

Since  $\{\sum_{k=1}^n Y_k < x\}$  is equivalent to  $\{N(x) > n\}$ , the distribution of number of replacements is

$$P\{N(x) > n\} = P\left\{\sum_{k=1}^n Y_k < x\right\} \quad \dots (1)$$

The distribution of  $\sum_{k=1}^n Y_k$  will provide the information to compute the distribution of  $N(x)$ .

Since the  $Y_k$ 's are not identically distributed, the central limit theorem of the sum of i.i.d. random variables cannot be applied directly. But by Lindeberg Condition, Gnedenko [6] page 289, the sum  $\sum_{k=1}^n Y_k$  is asymptotically normally distributed, and so, too, may be  $N(x)$ .

$$\text{Let } M_k = E(Y_k)$$

$$B_n^2 = \text{Var}\left(\sum_{k=1}^n Y_k\right)$$

$\tau$ : a positive constant

if the condition

$$\lim_{n \rightarrow \infty} \frac{1}{B_n^2} \sum_{k=1}^n \int_{|Y_k - M_k| > \tau B_n} (Y_k - M_k)^2 d H_k(Y_k) = 0 \quad \dots (2)$$

exists, then

$$P \left[ \frac{1}{B_n} \sum_{k=1}^n (Y_k - M_k) < x \right] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-z^2/2) dz \quad \dots (3)$$

By some algebraic manipulation,

$$\begin{aligned} & P \left\{ \frac{1}{B_n} \sum_{k=1}^n (Y_k - M_k) < x \right\} \\ &= P \left\{ \sum_{k=1}^n (Y_k - M_k) < B_n x \right\} \\ &= P \left\{ \sum_{k=1}^n Y_k < B_n x + \sum_{k=1}^n M_k \right\} \end{aligned}$$

$$\text{from (1), } P \left\{ N(B_n x + \sum_{k=1}^n M_k) > n \right\} \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-z^2/2) dz \quad \dots (4)$$

This implies that the distribution of number of replacements in the time interval  $[0, B_n x + \sum_{k=1}^n M_k)$  can be approximated by a normal distribution if the condition (2) holds. If

- (a) the variances of  $Y_k$ 's are finite, and
- (b)  $Y_1, Y_2, \dots, Y_k \dots$  will converge to a random variable  $Y$  when  $k$  increases, then the integral of the conditions (2) will converge to a finite value while  $B_n$  increases indefinitely.

Since the  $Y_k$ 's are bounded by the maximum component life (in aircraft applications), the variances of  $Y_k$  are finite. The only condition left to be proved is the convergence of  $Y_k$ . If the convergence of  $Y_k$  is sure, the replacement process in the steady state is a renewal process! This convergence is shown in the following subsections.

## 2. Residual Life Distribution of Used spare components

Suppose a brand new spare component has life distribution Function  $F(x) = P\{T \leq t\}$ .  $T$  is a random variable representing the life of the component.

If the spare component is put into service when its age is  $a$ , the distribution of time till failure is no longer  $F(t)$  unless  $a=0$ . The remaining life

time is called residual life of the used spare component. It depends on the age  $a$ .

The conditional distribution of residual life can be derived as follows. Let  $t$  denote the residual life. The probability that a component of age  $a$  will survive until  $a+t$  is

$$\begin{aligned} F(t|a) &= P_r\{a < T \leq a+t | T > a\} \\ &= \frac{P_r\{a < T \leq a+t\}}{P\{T > a\}} \\ &= \frac{F(a+t) - F(a)}{1-F(a)} \quad \dots (4) \end{aligned}$$

Thus if  $a$  is a known fixed constant, the distribution of residual life can be computed using (4).

3. Residual life distribution of a used spare component if the age distribution of spare components is known.

Let the random variable  $A$  represent the age of the spare component with distribution

$$G(a) = P\{A \leq a\}, \quad g(a) \text{ is the probability density function.}$$

Define the unconditional residual life distribution  $H(t) = P\{T \leq t\}$

$$\begin{aligned} \text{Since } H(t) &= E_A[F(t|A)] \\ &= \int_0^{\infty} F(t|a)g(a)da \end{aligned}$$

$$\text{from (4)} \rightarrow = \int_0^{\infty} \frac{F(a+t) - F(a)}{1-F(a)} g(a)da \quad \dots (5)$$

If  $F$  &  $G$  are completely specified, the equation (5) can be evaluated to obtain the residual life distribution. If  $F$  is not exponential, the evaluation of the integral will become completed. If  $F$  is piecewise exponential, as in the "competitive risk" model of section II.1, for example,

$$F(x) = 1 - e^{-\lambda x} \prod_{i=1}^i (1 - p_i)$$

$$= \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x < x < 1000 \\ 1 - e^{-\lambda x} (1 - 0.25) & 1000 \leq x < 2000 \\ 1 - e^{-\lambda x} (1 - 0.25)^2 & 2000 \leq x < 3000 \\ 1 - e^{-\lambda x} (1 - 0.25)^3 & 3000 \leq x < 4000 \\ 1 & 4000 \leq x \end{cases}$$

Then for some age distribution  $G$ .

$$\begin{aligned} H(t) &= \int_0^{\text{MOT}} \frac{F(a+t) - F(a)}{1 - F(a)} g(a) da \\ &= \int_0^{1000} \frac{F(a+t) - F(a)}{1 - F(a)} g(a) da + \int_{1000}^{2000} \frac{F(a+t) - F(a)}{1 - F(a)} g(a) da + \dots + \int_{3000}^{4000} \dots \end{aligned}$$

In case the integral cannot be evaluated explicitly, numerical integration may be used. Therefore  $H(t)$  can be determined as long as  $G(a)$  is known.

#### 4. Convergence of spare engine age distribution

Since the non-identical residual life distributions  $H_1(Y_1), H_2(Y_2), \dots, H_k(Y_k)$  are produced from non-identical age distribution  $G_1, G_2, \dots, G_k$ , if  $G_k$ 's converge to a distribution  $G$ ,  $H_k(Y_k)$  will also converge to a distribution  $H(Y)$ .

The distribution of  $G_k$ 's depend on the past history of the replacement process, maintenance policies, and parts provisioning. The convergence of the  $G_k$  will be first tested on the simulation model. Analytical proof will be also investigated. If the analytical proof can not be obtained, the distribution of  $G$  can still be estimated from the simulation and eq (5) can be evaluated numerically. A model for  $G$  is in subsection II.5.

|  |  |   |  |
|--|--|---|--|
| <p>18 19 REPORT DOCUMENTATION PAGE</p>   |  | <p>READ INSTRUCTIONS<br/>BEFORE COMPLETING FORM</p>   |  |
| <p>REPORT NUMBER<br/>AFOSR TR-78-0679</p>  |  | <p>GOVT ACCESSION NO. 2. RESIDENT'S OR REG. NUMBER<br/>Rept. for 2 Oct 77 - 2 Feb 78</p>      |  |
| <p>4. TITLE (and Subtitle)<br/>REPLACEMENT PROCESSES ANALYSIS, AN INTERIM<br/>REPORT ON REPLACEMENT MODELS.</p>  |  | <p>5. TYPE OF REPORT &amp; PERIOD COVERED<br/>Interim</p>                                     |  |
| <p>7. AUTHOR(s)<br/>Po-Wen/Hu, Laurence L./George</p>  |  | <p>8. CONTRACT OR GRANT NUMBER(s)<br/>AFOSR-78-3501</p>                                       |  |
| <p>9. PERFORMING ORGANIZATION NAME AND ADDRESS<br/>Department of Industrial Engineering<br/>Texas A&amp;M University<br/>College Station, Texas 77843</p>  |  | <p>10. PROGRAM ELEMENT, PROJECT, TASK<br/>AREA &amp; WORK UNIT NUMBERS<br/>61102F 2304/A5</p> |  |
| <p>11. CONTROLLING OFFICE NAME AND ADDRESS<br/>Air Force Office of Scientific Research/NM<br/>Bolling AFB, Dc 20332</p>  |  | <p>12. REPORT DATE<br/>January 30, 1978</p>   |  |
| <p>14. MONITORING AGENCY NAME &amp; ADDRESS (if different from Controlling Office)</p>   |  | <p>13. NUMBER OF PAGES<br/>32</p>   |  |
|  |  | <p>15. SECURITY CLASS. (of this report)<br/>UNCLASSIFIED</p>                                  |  |
|  |  | <p>15a. DECLASSIFICATION/DOWNGRADING<br/>SCHEDULE</p>   |  |
| <p>16. DISTRIBUTION STATEMENT (of this Report)<br/>Approved for public release; distribution unlimited</p>   |  |   |  |
| <p>17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)</p>  |  |   |  |
| <p>18. SUPPLEMENTARY NOTES</p>   |  |   |  |
| <p>19. KEY WORDS (Continue on reverse side if necessary and identify by block number)<br/>replacement processes, renewal processes, failure, residual life,<br/>removals, requirements</p>   |  |   |  |
| <p>20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br/>This report describes models for the aircraft engine replacement processes<br/>and models for the probability distribution of engine ages at failure.<br/>Potential uses of these models for predicting replacement requirements<br/>are discussed. Also models for the randomness of flying hours on different<br/>aircraft are proposed for the study of the effect on replacement models<br/>which assume each aircraft flies an equal share of a flying hour program.</p> |  |   |  |

20. Abstract

The first section briefly describes the problem of computing replacement requirements for a fleet of aircraft. The second section describes several models of engine lives, ages at replacement, or times between replacement. These models are fundamental to the computation of replacement requirements because replacement requirements are computed based on engine life data. The third section describes models for a problem that is not usually recognized either theoretically or in practice; not all aircraft fly the same share of a flying hour program.

The future work is to incorporate the models of engine lives in section 2 and the models for aircraft flying times in section 3 into a procedure for computing replacement requirements.

UNCLASSIFIED