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ROBUSTNESS OF STABILITY CONDITIONS FOR LINEAR TIME-INVARIANT FE--ETC(U)  
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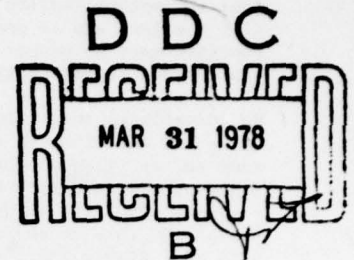
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ROBUSTNESS OF STABILITY CONDITIONS FOR  
LINEAR TIME-INVARIANT FEEDBACK SYSTEMSC. A. Desoer, F. M. Callier<sup>†</sup> and W. S. ChanDepartment of Electrical Engineering and Computer Sciences  
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## Abstract

The robustness of stability conditions for linear time-invariant feedback systems is examined assuming three different types of representations: state-space representation, coprime matrix fraction representation, and transfer function representation. ~~We stress~~ the importance of certain details of the representation used and, even more, the importance of making sure that the allowed perturbations be relevant to the physical situation under study.

relevance of some parameter perturbations.

In order to avoid repetitions, we start by defining some terms and some notations.

## II. Preliminary Definitions

$\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{R}(s)$  and  $\mathbb{R}[s]$  denote, respectively, the fields of real numbers, of complex numbers, of rational functions with real coefficients and the commutative ring of polynomials with real coefficients. The superscripts "n" and "nxm" (as in  $\mathbb{R}^n$ ,  $\mathbb{R}(s)^{nxm}$ ) denote the corresponding ordered n-tuples and nxm arrays.  
 $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$  denotes the closed right-half-plane.  $\mathbb{C}_- := \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$  denotes the open left-half-plane. Given any scalar rational function, we assume once and for all that it is written as  $n(s)/d(s)$  where the polynomials n and d are coprime and d is monic.

A continuous-time, linear, time-invariant, lumped, multi-input multi-output system is said to be exponentially stable (abbr. exp. stable) iff its transfer function  $G(s) \in \mathbb{R}(s)^{nxm}$  is proper (i.e. bounded at infinity) and  $G(s)$  has no  $\mathbb{C}_+$ -poles. For example, for the system shown on Fig. 1, this means that  $G: (u_1, u_2) \mapsto (e_1, e_2)$  has these properties.

## III. System Description

We consider the input-output stability problem of the continuous-time, linear, time-invariant, lumped, multi-input multi-output feedback system  $S: (u_1, u_2) \mapsto (e_1, e_2)$  described in frequency domain by (see Fig. 1)

$$u_1 = e_1 + G_2 e_2 \quad u_2 = e_2 - G_1 e_1 \quad (1)$$

where  $G_1, G_2 \in \mathbb{R}(s)^{nxn}$ , the  $u_i$ 's are the inputs and the  $e_i$ 's are the "errors". The transfer function of S is  $G: (u_1, u_2) \mapsto (e_1, e_2)$ .<sup>†</sup> Throughout this paper, we make the following assumption:

**Assumption:** The transfer functions  $G_1, G_2$  are proper (i.e. bounded at infinity) and

<sup>†</sup>We only need to consider  $(u_1, u_2) \mapsto (e_1, e_2)$  because the map  $(u_1, u_2) \mapsto (y_1, y_2)$  is exp. stable if and only if the map  $(u_1, u_2) \mapsto (e_1, e_2)$  is exp. stable [10].

## I. Introduction

Engineers design for production: therefore it is required that their nominal design as well as a very high portion of the systems produced — which suffer from element deviations and manufacturing tolerances — meet the specifications. Furthermore, the systems produced must meet the specifications not only as they leave the production line but also in the field — where they suffer from temperature effects, aging, weathering etc. — . Hence the interest in sensitivity and robustness. It is for these reasons that this subject has an extensive literature [e.g. 1,2].

In this paper, we consider the robustness of the stability conditions for a continuous-time, linear, time-invariant, lumped, multi-input multi-output feedback system. (See Fig. 1) If the feedback system is made of an interconnection of stable subsystems, it seems intuitively clear that under some reasonable conditions and under reasonable allowed perturbations the stability conditions are robust. But what if the subsystems are unstable? Might it not happen that due to perturbation some kind of pole-zero cancellation is destroyed? The purpose of this paper is to examine the conditions which, under several representations, guarantee robustness of the stability conditions. We will find that the nature of the representation and the nature of the allowed perturbations play a crucial role. This will also lead us to make some remarks on the

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$$\det[I+G_2(\infty) G_1(\infty)] \neq 0. \quad (2)$$

This assumption implies that the transfer function  $G$  of  $S$  exists and is proper. Note that a transfer function must be proper in order to have a state-space representation.

We investigate the following question: given that  $G_1, G_2$  are described in a specified way and given some set of allowed perturbations, are the input-output stability conditions for the system  $S$  robust under the allowed perturbations in  $G_1, G_2$ ? By this we mean: if the feedback system  $S$  is exp. stable for the nominal values of  $G_1, G_2$ , does it imply that it will remain exp. stable for all sufficiently small perturbations in  $G_1, G_2$ , selected from the allowed set? It will turn out that the answer depends very much on the representation of  $G_1, G_2$  and the set of allowed perturbations.

#### IV. A Preliminary Lemma

All the argumentation below hinges on a lemma that essentially says "small perturbations in the coefficients of an algebraic equation cause small perturbations in its zeros." More precisely, we state a well-known lemma.

**Lemma** [3, Thm 9.17.4]. Let  $D(z_i; \epsilon)$  denotes the open disc in  $\mathbb{C}$  centered on  $z_i$  and with radius  $\epsilon$ . Consider the polynomial  $p$  defined by

$$p(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n \quad (3)$$

where  $a_i \in \mathbb{R}$ ,  $\forall i$ , and without loss of generality,  $a_0 > 0$ . Let the polynomial  $p$  have  $q$  pairwise distinct zeros:  $z_1, z_2, \dots, z_q$  with respective multiplicities  $m_i$ , (hence  $\sum m_i = n$ ). Then for all  $\epsilon > 0$ , there is an  $\eta(\epsilon) > 0$  such that for all  $\delta a_i$  satisfying

$$|\delta a_i| < \eta \quad i = 0, 1, \dots, n \quad (4)$$

the perturbed polynomial

$$(a_0 + \delta a_0) s^n + \dots + (a_{n-1} + \delta a_{n-1}) s + (a_n + \delta a_n) \quad (5)$$

still has  $m_i$  zeros in  $D(z_i; \epsilon)$  for  $i = 1, 2, \dots, q$ .

**Comments:** (i) If all zeros of  $p$  were simple, this lemma would be a direct consequence of the implicit function theorem; the point is that the continuous dependence of the zeros is still valid in the case of multiple zeros. (ii) It is crucial to observe that the degree of  $p$  was not affected by the perturbations: indeed suppose that instead of (5) we had

$$\tilde{p}(s) = \delta a_{-1} s^{n+1} + (a_0 + \delta a_0) s^n + \dots + (a_n + \delta a_n) \quad (6)$$

as the perturbed polynomial; then, for  $\eta > 0$  sufficiently small, if  $|\delta a_i| < \eta$  for  $i = -1, 0, 1, \dots, n$ ,  $\tilde{p}$  would have  $n+1$  zeros,  $n$  of them in the discs  $D(z_i; \epsilon)$  and one approximately equal to  $-(a_0 + \delta a_0)/\delta a_{-1}$ . (This approximate zero is the leading term of a sequence of successive approximation which converges for  $\eta$  small).

This additional zero is very large: its sign is positive or negative according to  $\delta a_{-1} < 0$  or  $\delta a_{-1} > 0$ , respectively. Similarly, if there had been several additional terms of degrees larger than  $n$ , there would have been several such zeros with very large magnitude and whose location in the  $s$ -plane depends on the magnitudes and signs of the  $\delta a_i$ 's. Note that when the perturbed problem is of the form (6), we are essentially dealing with a singular perturbation problem; see e.g. [4, 5].

#### V. Robustness Results

**Case I:** The transfer functions  $G_1$  and  $G_2$  are specified by minimal state-space representations

It is well known that if  $[A, B, C, D]$  is a minimal state-space representation of the proper transfer function  $G$ , then  $G$  is exp. stable if and only if  $\det(sI - A)$  has all its zeros in  $\mathbb{C}_-$ . For  $i = 1, 2$ , let  $[A_i, B_i, C_i, D_i]$  be a minimal state-space representation of the proper transfer function  $G_i$  with the state  $x_i \in \mathbb{R}^{n_i}$ . Let  $[A, B, C, D]$  be the state-space representation of the feedback system  $S: (u_1, u_2) \mapsto (e_1, e_2)$  with the state  $(x_1, x_2)$ . We know that [6] (i)  $[A, B, C, D]$  is a minimal state-space representation of  $S$ , (ii)

$$A = \begin{bmatrix} A_1 - B_1(I + D_2 D_1)^{-1} D_2 C_1 & -B_1(I + D_2 D_1)^{-1} C_2 \\ B_2(I + D_1 D_2)^{-1} C_1 & A_2 - B_2(I + D_1 D_2)^{-1} D_1 C_2 \end{bmatrix} \quad (7)$$

and (iii)

$$\det(sI - A) = \det(sI - A_1) \cdot \det(sI - A_2) \cdot \det(I + G_2 G_1)(s) \quad (8)$$

Now the feedback system  $S$  is exp. stable if and only if  $\det(sI - A)$  has all its zeros in  $\mathbb{C}_-$ ; furthermore for any perturbation  $[\delta A_i, \delta B_i, \delta C_i, \delta D_i]$  in the constant matrices  $[A_i, B_i, C_i, D_i]$ ,  $i = 1, 2$ , the degree of  $\det(sI - A)$  remains equal to  $n_1 + n_2$ . In view of (7), we obtain from the lemma above a well-known result:

##### Robustness Result I.1.

If the given feedback system  $S$  is exp. stable at the nominal data point, then for any sufficiently small perturbation  $[\delta A_i, \delta B_i, \delta C_i, \delta D_i]$ ,  $i = 1, 2$ , the resulting perturbed system is still exp. stable.  $\square$

**Remark I.1:** In many applications, neither measurements nor system-component element values directly specify the  $[A_i, B_i, C_i, D_i]$ ; therefore, engineers should ask whether the perturbations  $[\delta A_i, \delta B_i, \delta C_i, \delta D_i]$  of this analysis cover all the possible perturbations expected in the contemplated physical environment.

**Remark I.2:** Note that for sufficiently small allowed perturbations,  $[A_i + \delta A_i, B_i + \delta B_i, C_i + \delta C_i, D_i + \delta D_i]$  will also be a minimal state-space





do not increase the degree of any scalar polynomial.

### III.1: Single-input single-output subsystems

Consider the case where  $G_i = n_i/d_i$ ,  $i = 1, 2$ , where  $n_i, d_i$  are coprime scalar polynomials, i.e., they are coprime elements of  $\mathbb{R}[s]$ . Clearly the characteristic polynomial of the feedback system  $S$  is  $(d_1 d_2 + n_1 n_2)$ . Since  $d_1, d_2$  are scalar polynomials, they are both row-proper and column-proper. Thus by Robustness Result II.2, if the given feedback system is exp. stable at the nominal data point, for any sufficiently small allowed perturbation (for Case III), the resulting perturbed system is still exp. stable.

### III.2: Multi-input multi-output subsystems

Recall the notations introduced in Case I and

$$\det(sI-A) = \det(sI-A_1) \cdot \det(sI-A_2) \cdot \det(I+G_2 G_1)(s) \quad (8)$$

Let

$$p_{0+} = \text{number of } \mathbb{C}_+ \text{-zeros of } \det(sI-A_1) \cdot \det(sI-A_2), \text{ counting multiplicities.} \quad (11)$$

Then recall the Graphical Stability Conditions: [12,13] the feedback system  $S: (u_1, u_2) \mapsto (e_1, e_2)$  is exp. stable if and only if the Nyquist diagram of  $s \mapsto \det[I + G_2(s)G_1(s)]$  — for the contour  $C$  which is duly indented to the left at all  $j\omega$ -axis poles of  $\det[I + G_2(s)G_1(s)]$  — does not go through the origin and does encircle the origin  $p_{0+}$  times in the counterclockwise sense.

Now suppose that for the nominal parameter values in  $G_1$  and  $G_2$  the Nyquist diagram satisfies the stability conditions above. Consider the effect on the Nyquist diagram of small allowed parameter perturbations in  $G_1$  and  $G_2$ . Now, (a) for each  $s \in \mathbb{C}$ , except at poles,  $\det[I + G_2(s)G_1(s)]$  is a continuous function of all the numerator- and denominator-coefficients of  $G_1$  and  $G_2$ ; (b) the Nyquist diagram of the contour  $C$  is a compact curve in  $\mathbb{C}$ , therefore for any sufficiently small allowed parameter perturbation, the Nyquist diagram will still avoid the origin and encircle it  $p_{0+}$  times; (c) for sufficiently small allowed perturbations and for a fixed contour  $C$  (indented to the left), any  $j\omega$ -axis pole of  $G_1$  and/or  $G_2$  that is perturbed will still remain in  $\mathbb{C}_+$ , the open set in  $\mathbb{C}$  enclosed by the indented contour  $C$ .

Recall that the order of zero,  $\lambda$ , of  $\det(sI-A_1)$  is equal to the McMillan degree of  $G_1$  at its pole  $\lambda$  — which we denote by  $\Delta(G_1; \lambda)$  [14,15]. Thus

$$p_{0+} = \sum_{i=1,2} \sum_{\lambda} \Delta(G_i; \lambda)$$

where the sums are taken over all the  $\mathbb{C}_+$ -poles  $\lambda$  of  $G_1$  and  $G_2$ , or equivalently, over all the poles  $\lambda$  in  $\mathbb{C}_+$  of  $G_1$  and  $G_2$ . With this in mind we see that some sufficiently small allowed perturbation may change the required number of encirclements in only one way: namely to have

$$p_{0+} \neq \sum_{i=1,2} \sum_{\lambda} \Delta(G_i + \delta G_i; \lambda)$$

where the sums are taken over all the poles  $\lambda$  in  $\mathbb{C}_+$  of perturbed transfer functions  $G_1 + \delta G_1$  and  $G_2 + \delta G_2$ .

For example, consider

$$G_1(s) = \frac{1}{s-1} \begin{bmatrix} 2 & 1 \\ 3 & 1.5 \end{bmatrix}, \quad G_1(s) + \delta G_1(s) = \frac{1}{s-1} \begin{bmatrix} 2+\alpha & 1 \\ 3 & 1.5 \end{bmatrix}$$

then  $\Delta(G_1; 1) = \text{rank} \begin{bmatrix} 2 & 1 \\ 3 & 1.5 \end{bmatrix} = 1$ , [15, p.115], but, for any  $\alpha \neq 0$ ,  $\Delta(G_1 + \delta G_1; 1) = 2$ .

Therefore we must formulate our result as follows:

### Robustness Result III.1.

If the given feedback system  $S$  is exp. stable at the nominal data point, then for any sufficiently small allowed perturbations  $\delta G_1, \delta G_2$  which satisfy

$$\sum_{i=1,2} \sum_{\lambda} \Delta(G_i; \lambda) = \sum_{i=1,2} \sum_{\lambda} \Delta(G_i + \delta G_i; \lambda)$$

where the sums are taken over all the poles  $\lambda$  in  $\mathbb{C}_+$  of the corresponding transfer functions, the resulting perturbed system is still exp. stable.  $\square$

In particular, robustness of stability conditions for the feedback system  $S$  follows if  $G_1$  and  $G_2$  are both exp. stable.

In the following discussion, we restrict ourselves to simple  $\mathbb{C}_+$ -poles of  $G_1$  and  $G_2$  because there always exists some allowed perturbation which splits a multiple pole of  $G_i$  ( $i=1$  and/or  $2$ ) in many ways. To see this suppose that the  $(1,1)$  element of  $G_1$  has a third-order pole at  $s=-1$ , thus its denominator,  $d_{11}$ , has the form

$$d_{11}(s) = (s+1)^3 p_{11}(s) \quad \text{with } p_{11}(1) \neq 0$$

This triple pole can be split into three simple poles, for example

$$d_{11}(s) + \delta d_{11}(s) = [(s+1)^3 + \epsilon^3] p_{11}(s)$$

or into a double pole and a simple pole

$$d_{11}(s) + \delta d_{11}(s) = (s+1)^2 [(s+1) + \epsilon] p_{11}(s),$$

etc. More generally,  $d_{11}$  has multiple zero(s) if and only if its discriminant  $\Delta_{11} = 0$  [16].

Since  $\Delta_{11}$  is a polynomial in the coefficients of  $d_{11}$ , it is clear that multiple zeros are not generic [2].

Consider now the effect of an arbitrary small allowed perturbation on the McMillan degree of  $G_1$  at  $\lambda_k$ , where  $\lambda_k$  is a simple pole with  $\text{Re } \lambda_k \geq 0$ . Usually only some of the  $n^2$  elements of  $G_1$  have a pole at  $\lambda_k$  and for any sufficiently small allowed perturbation, if the  $(i,k)$  element of  $G_1$  has no pole at  $\lambda_k$ , then the  $(i,k)$  element of  $G_1 + \delta G_1$  will still have no pole in a small neighborhood of  $\lambda_k$ . Consequently, only the nonzero elements in the residue matrix  $R_{ko}$  of  $G_1$  at  $\lambda_k$  are affected by the allowed perturbations. Thus  $R_{ko}$  is a structured matrix in the sense of Shields and Pearson [17,18] i.e., it has a fixed pattern of zero elements. Let  $v_1$  be the number of coefficients which specify  $G_1$ . The generic rank [17,18] of the structured matrix  $R_{ko}$  is defined to be the maximal rank that  $R_{ko}$  achieves as a function of these  $v_1$  parameters.  $R_{ko}$  does not achieve its generic rank only for parameter values in some proper, closed, nowhere-dense variety  $V \subset \mathbb{R}^{v_1}$ . Consequently, if the rank of the nominal  $R_{ko}$  is less than its generic rank, then, for some arbitrarily small allowed perturbation, the rank of  $R_{ko}$  will jump to its generic rank. Thus we have the

#### Robustness Result III.2.

Suppose that all  $\mathbb{C}_+$ -poles of  $G_1$  and of  $G_2$  are simple and that for each of these poles the nominal residue matrix has a rank equal to its generic rank; if the given feedback system is exp. stable at the nominal data point, then for any sufficiently small allowed perturbation, the resulting perturbed system is still exp. stable.

Furthermore, if the nominal residue matrix of some  $\mathbb{C}_+$ -pole of either  $G_1$  or  $G_2$ , has a rank less than its generic rank, and if the nominal system  $S$  is exp. stable, then for some arbitrarily small allowed perturbation, the resulting perturbed system is unstable.  $\square$

**Remark III.1:** The allowed perturbations considered in Case III appear quite reasonable. However, we should be on guard that they might include perturbations that have no physical meaning for the case at hand. For example, this could occur if  $G_1$ , instead of being specified by a collection of  $n^2$  rational functions — the entries of the matrix  $G_1$  — were specified by a block diagram delineating the interconnections between the subsystems constituting  $G_1$ . In that case the appropriate perturbations to consider are not arbitrary perturbations in all the coefficients in the  $n^2$  rational function specifying  $G_1$  but rather perturbations in the parameters specifying the subsystems constituting  $G_1$ . This is illustrated by the following example.

**Example.** Suppose that  $G_1$  consists of a collection of subsystems in series and in parallel (no feedback!) and that only one subsystem has an unstable pole, say at  $p$ , with  $\text{Re } p \geq 0$ . The contribution of that subsystem to  $G_1$  is exhibited

on Fig. 2: the  $i$ th scalar input of  $G_1$  can only affect the scalar input  $v$  of the unstable subsystem through the gain  $\beta_1$ ; similarly the unstable subsystem output  $z$  is also scalar and is fanned out to the  $i$ th output of  $G_1$  through gain  $\gamma_i$ . Clearly the residue of  $G_1$  at  $p$  is the dyad (i.e. a rank-one matrix!)  $R = \gamma\beta^T$  where  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)^T$   $\beta^T = (\beta_1, \beta_2, \dots, \beta_n)$ . Now any small perturbation in the physical parameters  $(\gamma_i, \beta_i)$ ,  $i = 1, 2, \dots, n$  and  $p$  will not change the rank of  $R$  because the dyadic structure of  $R$  is dictated by the nature of the interconnection of the subsystems constituting  $G_1$ , and not by the numerical value of the parameters. Another way of viewing this fact is to say: the only meaningful physical parameters which determine  $R$  are the  $2n$  scalars  $\gamma_1, \gamma_2, \dots, \gamma_n, \beta_1, \beta_2, \dots, \beta_n$ . To take the abstract mathematical point of view that  $R$  is an array of  $n^2$  real numbers, and therefore  $n^2$  independent perturbations of its parameters are appropriate is mathematical fiction, and not an analysis of the physical system under consideration.  $\square$

The lesson of this remark is that before considering robustness, one should go back to the physical model of the system under consideration and trace out the effect of perturbations of the physical parameters of the model on the coefficients of the mathematical representation. Only in this way, the engineer will assure himself that the allowed perturbations he worries about pertain to physical reality and not to mathematical fiction.

**Remark III.2:** There is another case where perturbations are more restricted than those considered in Case III: in many models Newton's law dictates a second order pole at the origin — say in the transfer function from the external forces and the center of mass —. Clearly this second order pole is not subject to perturbations due to measurement or manufacturing errors!

**Example:** Consider the planar mechanical system described by

$$\begin{aligned} m\ddot{y}_1 - ky_2 &= u_1 \\ m\ddot{y}_2 &= u_2 \end{aligned}$$

where  $(u_1, u_2)$  is the applied force,  $(y_1, y_2)$  is the position of the particle of mass  $m$  and the physical parameters are  $m > 0$  and  $k > 0$ . Here

$$G(s) = \begin{bmatrix} m^{-1}s^{-2} & m^{-2}ks^{-4} \\ 0 & m^{-1}s^{-2} \end{bmatrix} = \begin{bmatrix} \alpha s^{-2} & \beta s^{-4} \\ 0 & \alpha s^{-2} \end{bmatrix}$$

where we put  $\alpha = m^{-1}$ ,  $\beta = m^{-2}k$ . Clearly, in this example the second order pole of  $G$  and the fourth order pole of  $G$  are not subject to perturbations when the physical parameters  $\alpha, \beta$  are perturbed. In the present case, the McMillan degree at the unstable pole ( $s=0$ ) is insensitive to small perturbations in the physical parameters. To

check this write  $G(s) = \sum_{k=0}^3 R_k s^{k-4}$ ; then  $\Delta(G;0)$  is given by the rank of the Hankel matrix [15]



$$H_0 = \begin{bmatrix} R_3 & R_2 & R_1 & R_0 \\ R_2 & R_1 & R_0 & 0 \\ R_1 & R_0 & 0 & 0 \\ R_0 & 0 & 0 & 0 \end{bmatrix}$$

where  $R_0 = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ ,  $R_2 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$  and the others are zero. It is easily checked that for all allowed physical values (namely,  $\alpha > 0$ ,  $\beta > 0$ )  $\Delta(G;0) = 4$  and also that the generic rank of  $H_0$  is 4, [17, p. 211]. Thus if this transfer function  $G$  were the forward gain of a closed-loop system with an exponentially stable feedback gain  $K(s) = \text{diag}(k_1(s), k_2(s))$  then  $p_{0+}$ , the number of required counterclockwise encirclements, would be 4 and it would be robust under any perturbation of the physical parameters  $\alpha$  and  $\beta$ .  $\square$

#### Relation between the effect of parameter perturbations in the three cases above.

Consider a given transfer function  $G(s)$ , one of its minimal state-space realization  $[A, B, C, D]$  and one of its right-coprime factorizations  $(N_r, D_r)$ . Offhand we might think that if some allowed parameter perturbation in Case I leads to the transfer function  $G+\delta G$ , then there exists some allowed parameter perturbation in Case II which leads to the same transfer function  $G+\delta G$ , and so on. In other words, we might think that the parameter perturbations in the three cases are, in some sense, equivalent. This is not the case: indeed Remark II.1 shows that for some coprime factorization  $(N_r, D_r)$  there is some arbitrarily small allowed parameter perturbation (of Case II) which will increase the degree of the characteristic polynomial, this, however, is impossible under any allowed parameter perturbation in Case I.

Similarly the example below shows that for some  $G(s)$  there is some arbitrarily small allowed parameter perturbation of Case I which leads to some transfer function  $G+\delta G$  unattainable by any allowed parameter perturbation of Case III.

Example: Consider a transfer function  $G(s)$  with the minimal state-space representation  $[A, B, C, D]$  where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, B = C = I_{5 \times 5}, D = 0_{5 \times 5}.$$

Thus,  $G(s) = (sI - A)^{-1}$

The (1,2) position minor of  $(sI - A)$  is given by

$$\det \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 1 & 0 & 0 & s \end{bmatrix} = 0, \text{ for all } s.$$

Therefore the numerator polynomial of the (2,1) element of  $G(s)$  has a degree of  $-\infty$ . Let  $\delta A$  be a  $5 \times 5$  matrix which has only one nonzero element  $\alpha$

at (3,1) position. Clearly  $[\delta A, 0, 0, 0]$  is an allowed parameter perturbation in Case I. The (1,2) position minor of  $(sI - A - \delta A)$  is given by

$$\det \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1-\alpha & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 1 & 0 & 0 & s \end{bmatrix} = -\alpha s^2. \text{ Also}$$

$$\det[sI - (A + \delta A)] = s^3(s-1)^2.$$

Therefore the numerator polynomial of (2,1) element of  $G+\delta G$  has degree zero. (Note the cancellation!) Due to the increase in degree of the numerator polynomial of (2,1) element,  $G+\delta G$  is unattainable from  $G$  by any allowed parameter perturbation of Case III.

#### VI. Conclusion

In this paper we studied the robustness of the exponential stability of continuous-time, linear, time-invariant, lumped, multi-input multi-output feedback systems. We presented robustness results for three types of system representations with corresponding sets of allowed parameter perturbations.

The robustness results above were obtained assuming that the two subsystems  $G_1$  and  $G_2$  have the same number of inputs and outputs, all the results can be extended, after simple modifications, to the case where the number of inputs and outputs of each subsystems are different.

Also since all the arguments used are purely algebraic and are based on simple properties of rational functions and polynomials, all the results above apply equally well to the discrete-time case except that in Robustness Result III.1,  $C$  should be interpreted as the unit circle with inward indentation to avoid the poles of  $G_1$  and  $G_2$ , lying on the unit circle, and  $C_1$  should be interpreted as the "outside" of  $C$  (more precisely the unbounded connected component of  $\mathbb{C} - C$ ).

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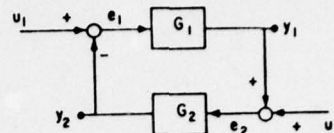


Fig. 1. Multi-input multi-output feedback system under consideration.

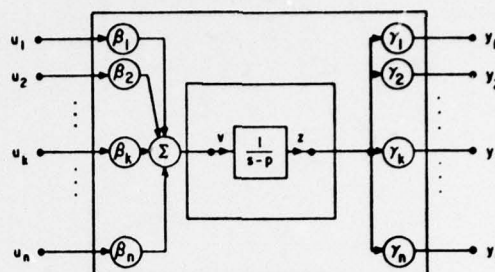


Fig. 2. Example of a system whose transfer function residue matrix has rank one for all physical values of the parameters  $\beta_i$  and  $\gamma_i$ .



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