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AXIALLY SYMMETRIC INCOMPRESSIBLE SHAPED CHARGE JETS

William P. Walters

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USA ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
USA BALLISTIC RESEARCH LABORATORY
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Basic equations governing the behavior of an axially symmetric, incompressible jet without body forces are presented. The jet is modeled both as an isotropic, Newtonian fluid and as a visco-plastic material.

The governing equations are normalized, standard nondimensional groups are defined and an order of magnitude analysis is performed for both the Newtonian fluid and visco-plastic representation of the stress tensor. The two models for the stress tensor are compared and shown to be equivalent for small values of the

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yield stress in pure stress.

For incompressible flow, the energy and state equations decouple from the continuity and momentum equations providing a simplified set of governing equations for both models of the stress tensor. For the Newtonian fluid assumption, an analogy to unsteady boundary layer theory is made.

TABLE OF CONTENTS

I. INTRODUCTION

A set of partial differential equations is formulated to describe the shaped charge jet behavior. Somewhat similar equations are given in the various hydrocodes but these codes employ various assumptions regarding the modeling of the stress tensor and use mixed Eulerian-Lagrangian systems. However, hydrocode analyses include compressibility effects using semi-empirical equations of state relating the density, pressure, and internal energy.

In this report, incompressible flow is assumed at the onset, since one-dimensional hydrodynamic analyses indicate that compressibility effects are not dominant during the jet development or jet penetration process (see for example Allison-Vitali[1], DiPersio-Simon-Merendino[2], Eichelberger[3], and Chou-Carleone-Tanzio-Cicarelli[4]). The major influence of compressibility is in the liner collapse process as modeled by Chou-Carleone-Karpp[5,6] especially in regard to a jet - no jet criterion. However, once it has been established that a given liner geometry and liner material will form a coherent jet, the collapse process can be satisfactorily modeled as an incompressible flow as done by Pugh-Eichelberger-Rostoker[7].

In this report, the one-dimensional hydrodynamic equations are extended to an axisymmetric hydrodynamic theory using a Newtonian fluid model for the stress tensor. Also, the material strength effects are

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[1] *F. E. Allison and R. Vitali, "A New Method of Computing Penetration Variables for Shaped-Charge Jets," BRL Report No. 1184, January 1963.* (AD #400485)

[2] *R. DiPersio, J. Simon, A. Merendino, "Penetration of Shaped-Charge Jets Into Metallic Targets," BRL Report No. 1296, September 1965.* (AD #476717)

[3] *Eichelberger, R. J., "Re-examination of the Theories of Jet Formation and Target Penetration by Lined Cavity Charges," Carnegie Institute of Technology, CEL Report No. 1, June 1954.*

[4] *P. C. Chou, J. Carleone, C. A. Tanzio, and R. D. Cicarelli, "Shaped Charge Jet Breakup Studies Using Radiograph Measurement and Surface Instability Calculations," BRL CR 337, prepared by Dyna East Corporation, April 1977.* (AD #A040444)

[5] *P.C. Chou, J. Carleone, and R. Karpp, "The Effect of Compressibility on the Formation of Shaped Charge Jets," Proceedings of the First International Symposium on Ballistics, Orlando, Fla., Nov. 13-15, 1974, sponsored by the ADPA.*

[6] *P. C. Chou, J. Carleone, and R. Karpp, "Criteria for Jet Formation from Impinging Shells and Plates," J. of Applied Physics, Vol. 47, No. 7, July 1976.*

[7] *E. M. Pugh, R. J. Eichelberger, and N. Rostoker, "Theory of Jet Formation by Charges with Lined Conical Cavities," J. of Applied Physics, Vol. 23, No. 5, May 1952.*

included directly for the stress tensor modeled as a visco-plastic material. In this sense, the incompressible Newtonian fluid or visco-plastic model represents an extension of the incompressible one-dimensional theories of Allison-Vitali[1], or DiPersio-Simon-Merendino[2]. Alternately, this model represents a relaxation of the hydrocodes to an incompressible, but viscous, flow representation.

The Newtonian fluid model is assumed since for low strain rates, the viscosity of some metals depends weakly on the strain rate, approximately justifying a Newtonian fluid model[8]. The Russians, notably Godunov-Deribas-Mali[8] and Mali-Pai-Skovpin[9] consider the viscous properties of the jet metal to be important.

Once the governing equations have been established, they will be non-dimensionalized and an order of magnitude analysis will be performed to further simplify the final set of equations. This simplified set of equations may admit approximate analytical solutions. However, determination of the appropriate boundary conditions pose the major difficulty in obtaining an approximate solution or an accurate numerical solution.

The same basic equations may be applied to the jet-target interaction where the system of equations apply independently to both the free jet and the target and are coupled by appropriate boundary conditions at the jet-target interface. In this case, a function of the ratio of the jet and target densities is important, but incompressible flow may still be assumed[1,2]. The viscous terms are also important in the jet-target interaction.

The remainder of this report deals with the development of the free jet. Specification of the jet parameters, in analytical form, prior to impact with the target will provide the first step in analyzing the target penetration. The jet formation will be coupled to the liner collapse process through initial conditions and boundary conditions. In turn, the free jet solution will provide initial and boundary conditions for the target penetration problem. Thus, the overall problem of the collapse, development, and penetration of the jet may be considered as three distinct regions: liner collapse; jet development and growth; and jet penetration.

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[8] *S. K. Godunov, A. A. Deribas and V. I. Mali, "Influence of Material Viscosity on the Jet Formation Process During Collision of Metal Plates," Novosibirsk. Translated from Fizika Goreniya i Vzryva, Vol. 11, No. 1, Jan - Feb 1975.*

[9] *V. I. Mali, V. V. Pai and A. I. Skovpin, "Investigation of the Breakdown of Flat Jets," Novosibirsk. Translated from Fizika Goreniya i Vzryva, Vol. 10, No. 5, Sept - Oct 1974.*

The following sections illustrate that the Newtonian fluid and visco-plastic stress tensor models are functionally identical for small values of the yield stress in pure shear (i.e., the visco-plastic model relaxes to a Newtonian fluid type model for small yield stresses). Also, ordering analyses are applicable to shaped charge jet studies and pertinent non-dimensional terms can be introduced. In addition, an incompressible formulation of the governing equations is presented.

II. BASIC EQUATIONS - NEWTONIAN FLUID

The Newtonian fluid model is considered first.

The basic equations governing the behavior of an axisymmetric jet are given below. These equations are: the continuity equation; Navier-Stokes equations; and the energy equation. These equations are formulated for an axisymmetric, incompressible, isotropic Newtonian fluid without body forces. An Eulerian coordinate system is employed with the origin located at the position where the free jet is assumed to begin its formation, i.e., beyond the region of influence of the liner collapse process.

The basic equations are:

1. Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,V) + \frac{\partial u}{\partial z} = 0\,, \tag{1}$$

2. Radial Momentum:

$$\rho\left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} + u\frac{\partial V}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2} + \frac{\partial^2 V}{\partial z^2}\right), \tag{2}$$

3. Axial Momentum:

$$\rho\left(\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right)\,, \tag{3}$$

4. Energy:

Neglecting radiation heat flux, allowing for internal heat generation, and where

$$\Phi = \mu\left[2\left\{\left(\frac{\partial V}{\partial r}\right)^2 + \left(\frac{V}{r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right\} + \left(\frac{\partial V}{\partial z} + \frac{\partial u}{\partial r}\right)^2\right] + \lambda\left[\frac{\partial V}{\partial r} + \frac{V}{r} + \frac{\partial u}{\partial z}\right]^2\,,$$

the energy equation becomes

$$\rho \frac{De}{Dt} = \Phi + \frac{\partial Q}{\partial t} + k \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) . \tag{4}$$

The nomenclature is standard. Basically V represents the radial velocity, u the axial velocity, and r, z are the radial and axial coordinates, respectively. The internal heat generation per unit volume is denoted by Q, μ is the dynamic viscosity of the media encompassing the jet, i.e., the air in the case of a free jet, or the target in the case of jet-target interactions (i.e., penetration). Alternately, this viscosity could be interpreted to represent the viscosity of the metallic jet resulting from the velocity gradient within the jet. This interpretation follows from the analysis of the viscous forces acting along the free surface of the metallic jet[8]. Also, $\lambda = -\frac{2}{3}\mu$ by Stokes hypothesis, ρ is the density, T is the temperature, p is the pressure, e is the internal energy and k is the thermal conductivity. The energy equation contains temperature dependent terms since energy dissipation due to heat conduction was included in the general energy equation (4).

Finally, two state equations are required to interrelate the four thermodynamic variables e, P, ρ and T, e.g.,

$$e = e(\rho, T) \tag{5}$$

and

$$P = P(\rho, T). \tag{6}$$

The system of six equations involves six unknowns namely, V, u, P, ρ, e and T. For incompressible flow, with ρ a known constant, one state equation can be eliminated. Also, if the heat conduction terms are ignored in the energy equation, one state equation may be eliminated. In fact, for incompressible flow as assumed in Equations (1) through (4), the energy equation and both state equations are decoupled from the continuity and momentum equations. In this case, the continuity equation and the two momentum equations may be solved for the three unknowns V, u, and P, for a known ρ.

III. NORMALIZED EQUATIONS - NEWTONIAN FLUID

Next, the basic equations will be normalized and an order of magnitude analysis will be performed. The characteristic dimensions will be:

L = characteristic length = length of continuous jet, prior to breakup, for application to shaped charge jets.

u_{max} = characteristic velocity = u_o = jet tip velocity for application to shaped charge jets.

t_c = characteristic time = L/u_{max}.

Now, R_o is taken to be the maximum radial dimension, for example, the jet radius (or the maximum hole radius when shaped charge jet penetration into a target is considered). The equations derived by the order of magnitude analysis are applicable to the free jet region, exclusive of the liner collapse process.

It is assumed that:

$\frac{R_o}{L} \ll 1$, or the maximum radial dimension is much less than the continuous jet length;

$\frac{r}{L} \ll 1$ or $\frac{r}{L} \equiv \bar{r}$ is assumed small, or for ordering purposes, $R \equiv \bar{r} \ll 1$ and $\bar{r} = O(R)$, i.e., R is the "ordering symbol" and is assumed to be small;

$\frac{z}{L} = \bar{z} = O(1)$;

and define

$\bar{V} = \frac{V}{u_o}$, $\bar{u} = \frac{u}{u_o}$, $P = \frac{p}{\rho\, u_o^2}$, and $\bar{t} = t/t_c$, where u_o is the jet tip (or maximum) velocity.

Using these characteristic dimensions, the continuity, momentum, and energy equations in nondimensional form become:

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\,\bar{V}) + \frac{\partial \bar{u}}{\partial \bar{z}} = 0\ , \tag{7}$$

$$\frac{\partial \bar{V}}{\partial \bar{t}} + \bar{V}\frac{\partial \bar{V}}{\partial \bar{r}} + \bar{u}\frac{\partial \bar{V}}{\partial \bar{z}} = -\frac{\partial \bar{P}}{\partial \bar{r}} + \frac{1}{Re}\left(\frac{\partial^2 \bar{V}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{V}}{\partial \bar{r}} - \frac{\bar{V}}{\bar{r}^2} + \frac{\partial^2 \bar{V}}{\partial \bar{z}^2}\right), \tag{8}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{V}\frac{\partial \bar{u}}{\partial \bar{r}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{z}} = -\frac{\partial \bar{P}}{\partial \bar{z}} + \frac{1}{Re}\left(\frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right)\ , \tag{9}$$

where $Re = \frac{\rho\, L\, u_o}{\mu}$ = Reynolds number,

$$\frac{\partial \bar{e}}{\partial \bar{t}} + \bar{V}\frac{\partial \bar{e}}{\partial \bar{r}} + \bar{u}\frac{\partial \bar{e}}{\partial \bar{z}} = \frac{\mu}{\rho L u_o}\left[2\left\{\left(\frac{\partial \bar{V}}{\partial \bar{r}}\right)^2 + \frac{\bar{V}}{\bar{r}} + \left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2\right\}\right.$$

$$\left. + \left(\frac{\partial \bar{V}}{\partial \bar{z}} + \frac{\partial \bar{u}}{\partial \bar{r}}\right)^2 - \frac{2}{3}\left(\frac{\partial \bar{V}}{\partial \bar{r}} + \frac{\bar{V}}{\bar{r}} + \frac{\partial \bar{u}}{\partial \bar{z}}\right)^2\right] + \frac{k\,\Delta T}{\rho L u_o^3}\left(\frac{\partial^2 \bar{\theta}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{\theta}}{\partial \bar{r}} + \frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2}\right) \quad (10)$$

$$+ \frac{C\,\Delta T}{u_o^2}\frac{\partial \bar{q}}{\partial \bar{t}}$$

where, $\bar{e} = e/u_o^2$,

$\bar{q} = Q/\rho = \frac{q}{C\,\Delta T}$ = internal heat generation per unit mass,

C = specific heat at constant pressure,

$\bar{\theta} = \frac{T - T_i}{\Delta T}$,

T_i = initial jet temperature,

and $\Delta T = T_o - T_i$, where T_o is the total temperature of the jet, and ΔT may be taken to be $\frac{u_o^2}{2C}$ analogous to Schlichting[10].

The following nondimensional groups can be identified:

$\frac{\mu}{\rho L u_o} = \frac{1}{Re}$ = inverse Reynolds number,

$\frac{k\,\Delta T}{\rho L u_o^3} = \frac{1}{Re}\,\frac{1}{E}\,\frac{1}{P_r}$,

$\frac{C\,\Delta T}{u_o^2}$ = E = Eckert number, which is related to the frictional heat and the heat due to compression,

and P_r = Prandtl number = $\frac{C\mu}{k}$.

[10] *Schlichting, H., Boundary Layer Theory, 6th Edition, McGraw-Hill, 1968, Chapter XII.*

At this point, the conduction heat transfer term can be dropped from the energy equation since the shaped charge jet forms and develops in a matter of a few hundred microseconds, at most. Thus, sufficient time is not available to conduct (or radiate) heat in or out of the jet.

The final form of the energy equation becomes

$$\frac{\partial \bar{e}}{\partial \bar{t}} + \bar{V}\frac{\partial \bar{e}}{\partial \bar{r}} + \bar{u}\frac{\partial \bar{e}}{\partial \bar{z}} = \frac{1}{Re}\left[2\left\{\left(\frac{\partial \bar{V}}{\partial \bar{r}}\right)^2 + \frac{\bar{V}}{\bar{r}} + \left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2\right\} + \left(\frac{\partial \bar{V}}{\partial \bar{z}} + \frac{\partial \bar{u}}{\partial \bar{r}}\right)^2 - \frac{2}{3}\left(\frac{\partial \bar{V}}{\partial \bar{r}} + \frac{\bar{V}}{\bar{r}} + \frac{\partial \bar{u}}{\partial \bar{z}}\right)^2\right] + \frac{1}{E}\frac{\partial \bar{q}}{\partial \bar{t}} \quad (11)$$

IV. ORDER OF MAGNITUDE ANALYSIS - NEWTONIAN FLUID

The nondimensional equations are ordered as follows:

$\bar{z} = 0(1)$

$\bar{r} = 0(R)$, where R is small,

$\bar{u} = 0(1)$

$\bar{t} = 0(1)$.

Assume $0\left(\frac{\partial \bar{V}}{\partial \bar{r}}\right) \sim 0\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)$, or the radial and axial velocity gradients are of the same order of magnitude, as implied by the continuity equation. Therefore, the continuity equation is of order one. The radial momentum equation becomes

$$\underset{0(R)}{\frac{\partial \bar{V}}{\partial \bar{t}}} + \underset{0(R)}{\bar{V}\frac{\partial \bar{V}}{\partial \bar{r}}} + \underset{0(R)}{\bar{u}\frac{\partial \bar{V}}{\partial \bar{z}}} = -\frac{\partial \bar{P}}{\partial \bar{r}} + \frac{1}{Re}\left(\underset{0(\frac{1}{R})}{\frac{\partial^2 \bar{V}}{\partial \bar{r}^2}} + \underset{0(\frac{1}{R})}{\frac{1}{\bar{r}}\frac{\partial \bar{V}}{\partial \bar{r}}} - \underset{0(\frac{1}{R})}{\frac{\bar{V}}{\bar{r}^2}} + \underset{0(R)}{\frac{\partial^2 \bar{V}}{\partial \bar{z}^2}}\right), \quad (12)$$

where the order of each term is given above it. Thus, $\frac{\partial^2 \bar{V}}{\partial \bar{z}^2}$ is negligible, and for the viscous terms to be included the Reynolds number is assumed large, or $\frac{1}{Re} = 0(R^2)$. Thus,

$$\frac{\partial \bar{V}}{\partial \bar{t}} + \bar{V}\frac{\partial \bar{V}}{\partial \bar{r}} + \bar{u}\frac{\partial \bar{V}}{\partial \bar{z}} = -\frac{\partial \bar{P}}{\partial \bar{r}} + \frac{1}{Re}\left(\frac{\partial^2 \bar{V}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{V}}{\partial \bar{r}} - \frac{\bar{V}}{\bar{r}^2}\right) \tag{13}$$

where the pressure gradient has not been assigned an order of magnitude, but each remaining term in the equation is of order R.

The axial momentum equation becomes,

$$\underset{0(1)}{\frac{\partial \bar{u}}{\partial \bar{t}}} + \underset{0(1)}{\bar{V}\frac{\partial \bar{u}}{\partial \bar{r}}} + \underset{0(1)}{\bar{u}\frac{\partial \bar{u}}{\partial \bar{z}}} = -\frac{\partial \bar{P}}{\partial \bar{z}} + \frac{1}{Re}\left(\underset{0(\frac{1}{R^2})}{\frac{\partial^2 \bar{u}}{\partial \bar{r}^2}} + \underset{0(\frac{1}{R^2})}{\frac{1}{\bar{r}}\frac{\partial \bar{u}}{\partial \bar{r}}} + \underset{0(1)}{\frac{\partial^2 \bar{u}}{\partial \bar{z}^2}}\right), \tag{14}$$

where $\frac{\partial^2 \bar{u}}{\partial \bar{z}^2}$ can be neglected, and each term in the equation is then of order one, except for the pressure gradient which has not been considered. From the radial momentum equation, $\frac{\partial \bar{P}}{\partial \bar{r}}$ is assumed of order 0(R) or the pressure gradient across the jet is of order R^2. Thus, the pressure gradient across the jet is small and since the flow velocity outside the jet is nearly zero, especially if the jet-atmosphere interaction is neglected. The $\frac{\partial \bar{P}}{\partial \bar{z}}$ term is also small.

For the energy equation, further assume $\bar{e} = 0(1)$, and $\bar{q} = 0(1)$, or there exists no a priori reason to suppose that the internal energy or internal heat generation terms are negligible. Then the energy equation becomes:

$$\underset{0(1)}{\frac{\partial \bar{e}}{\partial \bar{t}}} + \underset{0(1)}{\bar{V}\frac{\partial \bar{e}}{\partial \bar{r}}} + \underset{0(1)}{\bar{u}\frac{\partial \bar{e}}{\partial \bar{z}}} = \frac{1}{Re}\left[2\left\{\underset{0(1)}{\left(\frac{\partial \bar{V}}{\partial \bar{r}}\right)^2} + \underset{0(1)}{\frac{\bar{V}}{\bar{r}}} + \underset{0(1)}{\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2}\right\}\right.$$

$$\left. + \left(\underset{0(R)}{\frac{\partial \bar{V}}{\partial \bar{z}}} + \underset{0(\frac{1}{R})}{\frac{\partial \bar{u}}{\partial \bar{r}}}\right)^2 - \frac{2}{3}\left(\underset{0(1)}{\frac{\partial \bar{V}}{\partial \bar{r}}} + \underset{0(1)}{\frac{\bar{V}}{\bar{r}}} + \underset{0(1)}{\frac{\partial \bar{u}}{\partial \bar{z}}}\right)^2\right] + \underset{0(1)}{\frac{1}{E}\frac{\partial \bar{q}}{\partial \bar{t}}}, \tag{15}$$

or

$$\frac{\partial \bar{e}}{\partial \bar{t}} + \bar{V}\frac{\partial \bar{e}}{\partial \bar{r}} + \bar{u}\frac{\partial \bar{e}}{\partial \bar{z}} = \frac{1}{Re}\left(\frac{\partial \bar{u}}{\partial \bar{r}}\right)^2 + \frac{1}{E}\frac{\partial \bar{q}}{\partial \bar{t}} ,$$

assuming E = 0(1), and neglecting terms of order less than one.

The final equations, returning to dimensional form, become

$$\frac{1}{r}\frac{\partial}{\partial r}(rV) + \frac{\partial u}{\partial z} = 0;\ 0(1), \tag{16}$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} + u\frac{\partial V}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{\mu}{\rho}\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2}\right);\ 0(R), \tag{17}$$

and

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right);\ 0(1). \tag{18}$$

The first and third equations are of order one. The second equation, the radial momentum equation, is of order R. The energy equation is of order one, but is not required because of the incompressible flow assumption. Since the pressure gradient terms have been previously argued to be small, the first and third equations suffice for the determination of V and u.

The final order of magnitude equations are equivalent to the Prandtl Boundary Layer equations in cylindrical coordinates and for transient flow. These equations are parabolic. The original Navier-Stokes equations are elliptic. An Eulerian formulation has been employed.

The final equations account for:

1. Two-dimensionality (axis-symmetry),

2. Transient effects, and

3. Viscous effects.

However, an ideal Newtonian incompressible fluid is assumed. This assumption is in contrast to that of Riney[11], where a compressible, but inviscid flow was assumed. Following the one-dimensional analysis of DiPersio-Simon-Merendino[2], incompressible flow is assumed, but the second order terms in the Navier-Stokes equations, i.e., the viscous terms are retained. The viscous terms are retained for two reasons: they represent dominant or high order terms in the governing equations; and these terms can be modified (by removing the Newtonian fluid assumption and Stokes hypothesis) to allow visco-plastic stress models to be considered. Also viscous effects are important in the penetration process.

Note that if steady state, inviscid, one-dimensional flow is assumed the governing momentum equations reduce to

$$\frac{1}{2}\frac{du^2}{dz} = -\frac{1}{\rho}\frac{dP}{dz}$$

or

$$\frac{u^2}{2} + \frac{P}{\rho} = \text{const},$$

which is Bernoulli's equation, which was utilized for one dimensional jet studies[1,2,3].

Also note that the so-called pressure gradient term can be modified by assuming that the stress tensor can be divided into three parts. First, the static strength of the jet is extracted from the stress tensor and designated as σ. The remainder of the stress tensor term is divided into the pressure term and the deviatoric stress term, which is then modeled as a Newtonian fluid. Thus, the axial momentum equation becomes

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial z} - \frac{\partial \sigma}{\partial z} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right),$$

or

$$\rho\frac{Du}{Dt} = -\frac{\partial (P+\sigma)}{\partial z} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right).$$

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[11] *Riney, T. D., "Numerical Evaluation of Hypervelocity Impact Phenomena," appearing in High Velocity Impact Phenomena, Chapter V, Edited by R. Kinslow, Academic Press, 1970.*

Thus, the normal stress consists of the hydrostatic pressure plus the static strength of the material. Again assuming a one-dimensional, steady steady, inviscid flow,

$$\frac{\rho}{2}\frac{du^2}{dz} + \frac{d(P+\sigma)}{dz} = 0 \text{ ,}$$

or

$$\frac{1}{2}\rho u^2 + \sigma + P = \text{constant.}$$

This equation is identical to the Bernoulli equation modified to include strength effects as given by Eichelberger[3]. Using this formulation, the governing equations can be written as

$$\frac{1}{r}\frac{\partial (rV)}{\partial r} + \frac{\partial u}{\partial z} = 0 \text{ ,} \qquad (19)$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} + u\frac{\partial V}{\partial z} = -\frac{1}{\rho}\frac{\partial P'}{\partial r} + \nu\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2}\right) \text{,} \qquad (20)$$

and

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P'}{\partial z} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \text{,} \qquad (21)$$

where $\nu = \mu/\rho$ and $P' = P + \sigma$.

When ν is assumed known, there are three equations in the three unknowns V, u and P'. The first and third equations are of dominant order.

The derivation of those equations assumed that the Reynolds number is large. For hypervelocity flow through air, this is a valid assumption. Appendix A investigates this assumption for jet-target interaction, where the target material represents the penetrating media or where the jet viscosity is used in the free jet formulation.

V. BASIC EQUATIONS - VISCO-PLASTIC MATERIAL

For a visco-plastic material the basic equations are:

$$\frac{1}{r}\frac{\partial}{\partial r}(rV) + \frac{\partial u}{\partial z} = 0 \text{ ,} \qquad (22)$$

$$\rho\left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} + u\frac{\partial V}{\partial z}\right) = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,\sigma_{rr}) + \frac{\partial}{\partial z}(r\,\sigma_{zr})\right], \qquad (23)$$

$$\rho\left(\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z}\right) = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,\sigma_{rz}) + \frac{\partial(r\,\sigma_{zz})}{\partial z}\right]. \qquad (24)$$

This system has two unknowns in three equations. However, Equation (23) is not a dominant order equation, as will be shown.

Now from Cristescu[12], for a visco-plastic material,

$$\sigma_{rr} = 2\eta_* \frac{\partial V}{\partial r},$$

$$\sigma_{zz} = 2\eta_* \frac{\partial u}{\partial z},$$

$$\sigma_{rz} = \sigma_{zr} = \eta_*\left(\frac{\partial V}{\partial z} + \frac{\partial u}{\partial r}\right), \text{ and}$$

$$\eta_* = \eta\left[1 + \frac{k}{\sqrt{2}\,\eta\left[\left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial V}{\partial z} + \frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]^{1/2}}\right],$$

where for the jet material,

η is the viscosity coefficient,

η_* = variable viscosity coefficient, and

k = yield stress in pure shear.

VI. NORMALIZED AND ORDERED EQUATIONS - VISCO-PLASTIC MATERIAL

Now for $\bar{V} = V/u_o$, $\bar{u} = u/u_o$,

$\bar{r} = r/L$,

---

[12]*Cristescu, N., Dynamic Plasticity, Wiley and Sons, N.Y., 1967, Chapter X.*

$$\bar{z} = z/L,$$

$$\bar{t} = t\, u_o/L,$$

$$\bar{k} = \frac{k}{\rho u_o^2}\text{, and}$$

$$Re = \frac{\rho u_o L}{\eta},$$

the normalized continuity and momentum equations become:

Continuity:

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r}\, \bar{V}) + \frac{\partial \bar{u}}{\partial \bar{z}} = 0 \qquad : \qquad 0(1)$$

Axial Momentum:

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{V} \frac{\partial \bar{u}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{z}} = \left\{ \frac{2}{Re} \left(\frac{\partial^2 \bar{u}}{\partial \bar{z}^2} \right) \left(1 + \frac{\bar{k}\, Re}{\sqrt{2} \left[\left(\frac{\partial \bar{V}}{\partial \bar{z}} \right)^2 + \frac{1}{2} \left(\frac{\partial \bar{V}}{\partial \bar{z}} + \frac{\partial \bar{u}}{\partial \bar{r}} \right)^2 + \left(\frac{\partial \bar{u}}{\partial \bar{z}} \right)^2 \right]^{1/2}} \right) \right.$$

$$- \frac{\bar{k}}{\sqrt{2}} \frac{\partial \bar{u}}{\partial \bar{z}} \left(\frac{1}{\left(\frac{\partial \bar{V}}{\partial \bar{r}} \right)^2 + \frac{1}{2} \left(\frac{\partial \bar{V}}{\partial \bar{z}} + \frac{\partial \bar{u}}{\partial \bar{r}} \right)^2 + \left(\frac{\partial \bar{u}}{\partial \bar{z}} \right)^2} \right)^{3/2} \left(2 \frac{\partial \bar{V}}{\partial \bar{r}} \frac{\partial^2 \bar{V}}{\partial \bar{r}\, \partial \bar{z}} \right.$$

$$\left. \left. + \left(\frac{\partial \bar{V}}{\partial \bar{z}} + \frac{\partial \bar{u}}{\partial \bar{r}} \right) \left(\frac{\partial^2 \bar{V}}{\partial \bar{z}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}\, \partial \bar{r}} \right) + 2 \frac{\partial \bar{u}}{\partial \bar{z}} \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} \right) \right\}$$

$$+\left\{\frac{1}{Re}\left(\frac{\partial^2\bar{V}}{\partial\bar{r}\,\partial\bar{z}}+\frac{\partial^2\bar{u}}{\partial\bar{r}^2}\right)\left[1+\frac{\bar{k}\,Re}{\sqrt{2}\left(\left(\frac{\partial\bar{V}}{\partial\bar{r}}\right)^2+\frac{1}{2}\left(\frac{\partial\bar{V}}{\partial\bar{z}}+\frac{\partial\bar{u}}{\partial\bar{r}}\right)^2+\left(\frac{\partial\bar{u}}{\partial\bar{z}}\right)^2\right)^{1/2}}\right]\right.$$

$$-\frac{\bar{k}}{2\sqrt{2}}\left(\frac{\partial\bar{V}}{\partial\bar{z}}+\frac{\partial\bar{u}}{\partial\bar{r}}\right)\left[\frac{2\frac{\partial\bar{V}}{\partial\bar{r}}\frac{\partial^2\bar{V}}{\partial\bar{r}^2}+\left(\frac{\partial\bar{V}}{\partial\bar{z}}+\frac{\partial\bar{u}}{\partial\bar{r}}\right)\left(\frac{\partial^2\bar{V}}{\partial\bar{r}\,\partial\bar{z}}+\frac{\partial^2\bar{u}}{\partial\bar{r}^2}\right)+2\frac{\partial\bar{u}}{\partial\bar{z}}\frac{\partial^2\bar{u}}{\partial\bar{r}\,\partial\bar{z}}}{\left[\left(\frac{\partial\bar{V}}{\partial\bar{r}}\right)^2+\frac{1}{2}\left(\frac{\partial\bar{V}}{\partial\bar{z}}+\frac{\partial\bar{u}}{\partial\bar{r}}\right)^2+\left(\frac{\partial\bar{u}}{\partial\bar{z}}\right)^2\right]^{3/2}}\right]\right\}$$

$$+\left\{\frac{1}{\bar{r}\,Re}\left(\frac{\partial\bar{V}}{\partial\bar{z}}+\frac{\partial\bar{u}}{\partial\bar{r}}\right)\left[1+\frac{\bar{k}\,Re}{\sqrt{2}\left[\left(\frac{\partial\bar{V}}{\partial\bar{r}}\right)^2+\frac{1}{2}\left(\frac{\partial\bar{V}}{\partial\bar{z}}+\frac{\partial\bar{u}}{\partial\bar{r}}\right)^2+\left(\frac{\partial\bar{u}}{\partial\bar{z}}\right)^2\right]^{1/2}}\right]\right\}.$$

The first term in brackets represents $\frac{\partial\sigma_{zz}}{\partial z}$; the second term in brackets represents $\frac{\partial\sigma_{rz}}{\partial r}$; and the third terms represents $\frac{\sigma_{rz}}{r}$.

Next, an order of magnitude analysis will be applied. The continuity equation is of order one. The axial momentum equation is of order one and becomes

(for $\bar{k} = 0(1)$ and $Re = 0\,(\frac{1}{R^2})$),

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{V}\frac{\partial \bar{u}}{\partial \bar{r}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{z}} = \left\{ \frac{2}{Re}\frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\left(1 + \frac{\bar{k}\ Re}{\frac{\partial \bar{u}}{\partial \bar{r}}}\right)\right.$$

$$\left. - \bar{k}\frac{\partial \bar{u}}{\partial \bar{z}}\frac{1}{\left(\frac{\partial \bar{u}}{\partial \bar{r}}\right)^2}\left[\frac{\partial^2 \bar{u}}{\partial \bar{z}\,\partial \bar{r}}\right]\right\} + \left\{\frac{1}{Re}\frac{\partial^2 \bar{u}}{\partial \bar{r}^2}\left(1 + \frac{\bar{k}\ Re}{\frac{\partial \bar{u}}{\partial \bar{r}}}\right)\right.$$

$$\left. - \bar{k}\frac{\partial \bar{u}}{\partial \bar{r}}\left(\frac{2\frac{\partial \bar{V}}{\partial \bar{r}}\frac{\partial^2 \bar{V}}{\partial \bar{r}^2} + \frac{\partial \bar{V}}{\partial \bar{z}}\frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{\partial \bar{u}}{\partial \bar{r}}\frac{\partial^2 \bar{V}}{\partial \bar{r}\,\partial \bar{z}} + \frac{\partial \bar{u}}{\partial \bar{r}}\frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + 2\frac{\partial \bar{u}}{\partial \bar{z}}\frac{\partial^2 \bar{u}}{\partial \bar{r}\,\partial \bar{z}}}{\left(\frac{\partial \bar{u}}{\partial \bar{r}}\right)^3}\right)\right\}$$

$$+ \left\{\frac{1}{Re\ \bar{r}}\frac{\partial \bar{u}}{\partial \bar{r}}\left(1 + \frac{\bar{k}\ Re}{\frac{\partial \bar{u}}{\partial \bar{r}}}\right)\right\}$$ and after the order of magnitude

analysis and algebraic grouping of the terms, the right hand side of the above equation becomes

$$= \{0\} + \overset{O(1)}{\left\{\frac{1}{Re}\frac{\partial^2 \bar{u}}{\partial \bar{r}^2}\right\}} + \left\{\overset{O(1)}{\frac{1}{\bar{r}\ Re}\frac{\partial \bar{u}}{\partial \bar{r}}} + \overset{O(1/R)}{\frac{\bar{k}}{\bar{r}}}\right\}.$$

Thus, the dominant terms of the right hand side of the axial momentum equation is governed by the $\frac{1}{r}\frac{\partial(r\,\sigma_{rz})}{\partial r}$ terms and not the $\frac{\partial \sigma_{zz}}{\partial z}$ or normal stress terms. Thus,

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{V} \frac{\partial \bar{u}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{z}} = \frac{1}{Re} \left[\frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{u}}{\partial \bar{r}} \right] + \frac{\bar{k}}{\bar{r}} ,$$

retaining terms of order one and higher, or

$$\rho \left(\frac{\partial u}{\partial t} + V \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial z} \right) = \frac{1}{r} \frac{\partial (r \sigma_{rz})}{\partial r} .$$

For $\bar{k} = O(R)$, all terms in the final axial momentum equation are of order one. If $\bar{k} = O(R^2)$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{V} \frac{\partial \bar{u}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{z}} = \frac{1}{Re} \left[\frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{1}{r} \frac{\partial \bar{u}}{\partial \bar{r}} \right]$$

or the axial momentum equation for visco-plastic flow for $\bar{k} = O(R^2)$, i.e., small, reduces to that of a Newtonian fluid, except the viscosity term in the Reynolds numbers may differ between the Newtonian fluid and the visco-plastic model unless the viscous effects in the Newtonian fluid model are assumed to result from the viscosity of the metallic jet due to the velocity gradient within the jet and the resulting shear stress within the jet. Also, the Reynolds numbers defined by the two models are identical for the penetration process when the viscosity of the target is the pertinent parameter.

The radial momentum equation is:

$$\frac{\partial \bar{V}}{\partial \bar{t}} + \bar{V} \frac{\partial \bar{V}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{V}}{\partial \bar{z}} = \frac{1}{\rho u_o^2} \left[\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} \sigma_{rr}) + \frac{\partial \sigma_{rz}}{\partial \bar{z}} \right] ,$$

and for $Re = O(1/R^2)$ and $\bar{k} = O(R)$, with

$$\frac{\sigma_{rr}}{\rho u_o^2 \bar{r}} = \frac{2}{\bar{r}\, Re} \frac{\partial \bar{V}}{\partial \bar{r}} \left[1 + \frac{\bar{k}\, Re}{\sqrt{2} \left[\left(\frac{\partial \bar{V}}{\partial \bar{r}} \right)^2 + \frac{1}{2} \left(\frac{\partial \bar{V}}{\partial \bar{z}} + \frac{\partial \bar{u}}{\partial \bar{r}} \right)^2 + \left(\frac{\partial \bar{u}}{\partial \bar{z}} \right)^2 \right]^{1/2}} \right] ,$$

$$\frac{\partial\ \sigma_{rr}}{(\rho\ u_o^2)\partial\bar{r}} = \frac{2}{Re}\frac{\partial^2\bar{V}}{\partial\bar{r}^2}\left[1 + \frac{\bar{k}\ Re}{\sqrt{2}\left[\left(\frac{\partial\bar{V}}{\partial\bar{r}}\right)^2 + \frac{1}{2}\left(\frac{\partial\bar{V}}{\partial\bar{z}} + \frac{\partial\bar{u}}{\partial\bar{r}}\right)^2 + \left(\frac{\partial\bar{u}}{\partial\bar{z}}\right)^2\right]^{1/2}}\right]$$

$$- \frac{\bar{k}}{\sqrt{2}}\ \frac{\partial\bar{V}}{\partial\bar{r}}\ \frac{\left[2\ \frac{\partial^2\bar{V}}{\partial\bar{r}^2} + \left(\frac{\partial\bar{V}}{\partial\bar{z}} + \frac{\partial\bar{u}}{\partial\bar{r}}\right)\left(\frac{\partial^2\bar{V}}{\partial\bar{z}\partial\bar{r}} + \frac{\partial^2\bar{u}}{\partial\bar{r}^2}\right) + 2\ \frac{\partial\bar{u}}{\partial\bar{z}}\ \frac{\partial^2\bar{u}}{\partial\bar{r}\partial\bar{z}}\right]}{\left[\left(\frac{\partial\bar{V}}{\partial\bar{r}}\right)^2 + \frac{1}{2}\left(\frac{\partial\bar{V}}{\partial\bar{z}} + \frac{\partial\bar{u}}{\partial\bar{r}}\right)^2 + \left(\frac{\partial\bar{u}}{\partial\bar{z}}\right)^2\right]^{3/2}},$$

$$\frac{\partial\ \sigma_{rz}}{\partial\bar{z}}\ \frac{1}{\rho\ u_o^2} = \frac{1}{Re}\left(\frac{\partial^2\bar{V}}{\partial\bar{z}^2} + \frac{\partial^2\bar{u}}{\partial\bar{z}\partial\bar{r}}\right)\left[1 + \frac{\bar{k}\ Re}{\sqrt{2}\left[\left(\frac{\partial\bar{V}}{\partial\bar{r}}\right)^2 + \frac{1}{2}\left(\frac{\partial\bar{V}}{\partial\bar{z}} + \frac{\partial\bar{u}}{\partial\bar{r}}\right)^2 + \left(\frac{\partial\bar{u}}{\partial\bar{z}}\right)^2\right]^{1/2}}\right]$$

$$- \frac{\bar{k}}{2\ \sqrt{2}}\left(\frac{\partial\bar{V}}{\partial\bar{z}} + \frac{\partial\bar{u}}{\partial\bar{r}}\right)\frac{\left[2\ \frac{\partial\bar{V}}{\partial\bar{r}}\ \frac{\partial^2\bar{V}}{\partial\bar{z}\partial\bar{r}} + \left(\frac{\partial\bar{V}}{\partial\bar{z}} + \frac{\partial\bar{u}}{\partial\bar{r}}\right)\left(\frac{\partial^2\bar{V}}{\partial\bar{z}^2} + \frac{\partial^2\bar{u}}{\partial\bar{r}\partial\bar{z}}\right) + 2\ \frac{\partial\bar{u}}{\partial\bar{z}}\ \frac{\partial^2\bar{u}}{\partial\bar{z}^2}\right]}{\left[\left(\frac{\partial\bar{V}}{\partial\bar{r}}\right)^2 + \frac{1}{2}\left(\frac{\partial\bar{V}}{\partial\bar{z}} + \frac{\partial\bar{u}}{\partial\bar{r}}\right)^2 + \left(\frac{\partial\bar{u}}{\partial\bar{z}}\right)^2\right]^{3/2}}.$$

To order R,

$$\frac{\bar{\sigma}_{rr}}{\bar{r}} = \frac{2}{\bar{r}\ Re}\ \frac{\partial\bar{V}}{\partial\bar{r}} + \frac{2\ \bar{k}}{\bar{r}\ \frac{\partial\bar{u}}{\partial\bar{r}}}\ \frac{\partial\bar{V}}{\partial\bar{r}}$$

$$\frac{\partial\ \bar{\sigma}_{rr}}{\partial\bar{r}} = \frac{2}{Re}\ \frac{\partial^2\bar{V}}{\partial\bar{r}^2}\left[1 + \frac{\bar{k}\ Re}{\frac{\partial\bar{u}}{\partial\bar{r}}}\right] - \frac{\partial\bar{V}}{\partial\bar{r}}\ 2\ \bar{k}\ \frac{\frac{\partial^2\bar{u}}{\partial\bar{r}^2}}{\left(\frac{\partial\bar{u}}{\partial\bar{r}}\right)^2}$$

$$\frac{\partial \bar{\sigma}_{rz}}{\partial \bar{z}} = \frac{1}{Re} \frac{\partial^2 \bar{u}}{\partial \bar{z} \partial \bar{r}} \left[1 + \frac{\bar{k}\, Re}{\frac{\partial \bar{u}}{\partial \bar{r}}} \right] - \bar{k} \frac{\frac{\partial^2 \bar{u}}{\partial \bar{r} \partial \bar{z}}}{\frac{\partial \bar{u}}{\partial \bar{r}}} ,$$

and the radial momentum equation, to order R, becomes

$$\frac{\partial \bar{V}}{\partial \bar{t}} + \bar{V} \frac{\partial \bar{V}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{V}}{\partial \bar{z}} = \left\{ \frac{2}{\bar{r}\, Re} \frac{\partial \bar{V}}{\partial \bar{r}} + \frac{2\, \bar{k}}{\bar{r} \frac{\partial \bar{u}}{\partial \bar{r}}} \frac{\partial \bar{V}}{\partial \bar{r}} \right\}$$

$$+ \left\{ \frac{2}{Re} \frac{\partial^2 \bar{V}}{\partial \bar{r}^2} + \frac{2\, \bar{k}}{\frac{\partial \bar{u}}{\partial \bar{r}}} \frac{\partial^2 \bar{V}}{\partial \bar{r}^2} - 2\, \bar{k} \frac{\partial \bar{V}}{\partial \bar{r}} \frac{\partial^2 \bar{u}}{\partial \bar{r}^2} \frac{1}{\left(\frac{\partial \bar{u}}{\partial \bar{r}} \right)^2} \right\}$$

$$+ \left\{ \frac{1}{Re} \frac{\partial^2 \bar{u}}{\partial \bar{z} \partial \bar{r}} \right\} .$$

The first term in brackets on the right hand side represents $\frac{\bar{\sigma}_{rr}}{\bar{r}}$; the second term $\frac{\partial \bar{\sigma}_{rr}}{\partial \bar{r}}$; the third term $\frac{\partial \bar{\sigma}_{rz}}{\partial \bar{z}}$. Also

$$\frac{\partial \bar{V}}{\partial \bar{t}} + \bar{V} \frac{\partial \bar{V}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{V}}{\partial \bar{z}}$$

$$= \frac{1}{Re} \left[\frac{1}{\bar{r}} \frac{\partial \bar{V}}{\partial \bar{r}} + \frac{\partial^2 \bar{V}}{\partial \bar{r}^2} + \frac{\bar{V}}{\bar{r}^2} \right] + \frac{2\, \bar{k}}{\frac{\partial \bar{u}}{\partial \bar{r}}} \left[\frac{\partial \bar{V}}{\partial \bar{r}} \left[\frac{1}{\bar{r}} - \frac{\frac{\partial^2 \bar{u}}{\partial \bar{r}^2}}{\frac{\partial \bar{u}}{\partial \bar{r}}} \right] + \frac{\partial^2 \bar{V}}{\partial \bar{r}^2} \right]$$

using the continuity equation.

For $k = O(R^2)$,

$$\frac{\partial\bar{V}}{\partial\bar{t}} + \bar{V}\frac{\partial\bar{V}}{\partial\bar{r}} + \bar{u}\frac{\partial\bar{V}}{\partial\bar{z}} = \frac{1}{Re}\left[\frac{1}{\bar{r}}\frac{\partial\bar{V}}{\partial\bar{r}} + \frac{\bar{V}}{\bar{r}^2} + \frac{\partial^2\bar{V}}{\partial\bar{r}^2}\right] = \frac{1}{Re}\left[\frac{\partial^2\bar{V}}{\partial\bar{r}^2} - \frac{1}{\bar{r}}\frac{\partial\bar{u}}{\partial\bar{z}}\right] .$$

Thus, the continuity and momentum equations for a visco-plastic model reduce to the equations for a Newtonian fluid for small values of the normalized yield stress in pure shear, i.e., $\bar{k} = O(R^2)$, where the Reynolds number may differ between the two models, as discussed earlier.

VII. DISCUSSION OF EQUATIONS

The governing equations formulated for the Newtonian fluid model and for the visco-plastic model have been simplified by assuming incompressible flow, axisymmetric flow, and by retaining only dominant order terms. However, the final equations are still nonlinear and analytical solutions, even approximate solutions, are not readily apparent.

Nevertheless, the governing equations for the Newtonian fluid are equivalent to the transient boundary layer equations. The boundary layer equations have been treated extensively in the literature. However, the boundary and initial conditions differ, of course, between shaped charge jets and boundary layer flow. For steady state conditions, the equations can be solved by similarity transformations for generalized boundary conditions. Steady state solutions however, may not realistically describe the true jet behavior.

The visco-plastic model governing equations are similar to the Newtonian fluid model except that an additional term $\frac{\bar{k}}{\bar{r}}$ (for $\bar{k} = O(R)$) is present.

If a one dimensional, transient flow is assumed and the normal stress terms are retained, a somewhat simpler set of equations results. Chou and Carleone[4,13] are investigating the solution of the one dimensional transient, inviscid, incompressible equations using a strain hardening model for the normal (axial) stress. Their equations employ a Lagrangian coordinate system.

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[13] *Chou, P. C. and J. Carleone, "The Stability and Breakup of Shaped Charge Jets," Submitted to Third International Symposium on Ballistics, Karlsrue, Germany, March 1977.*

VIII. CONCLUSIONS

Governing equations have been developed for an Eulerian, axi-symmetric, incompressible flow. The stress tensor was modeled both as a Newtonian fluid and as a visco-plastic material.

The visco-plastic material governing equations were shown to relax to the Newtonian fluid equations when the yield stress in shear was assumed very small. However, the Reynolds numbers defined in the two models are not necessarily identical.

For both models appropriate nondimensional groups were defined and the dominant terms were determined by an order of magnitude analysis. Other models of the stress tensor can be analyzed by a similar order of magnitude analysis as long as the stress tensor can be expressed as functions of the velocities or velocity gradients, (or displacements).

The assumption of incompressible flow is extremely important. This assumption decouples the continuity and momentum equations from the energy and state equations. The energy equation given for the Newtonian fluid, incompressible flow model involves an internal heat generation term. Terms of this nature (frictional heat) are not included in the energy equations formulated for the hydrocodes. This term will include the jet temperature and may influence the balance between the kinetic energy and the internal energy.

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APPENDIX A

THE JET REYNOLDS NUMBER

The derivation of the final momentum and energy equations requires that the Reynolds number be relatively large to allow the inclusion of the viscous terms in the final order-of-magnitude equations. This assumption is now investigated.

The Reynolds number is given by

$$Re = \frac{\rho \; u_o \; L}{\mu} \quad :$$

a representative jet tip velocity may range from 3 - 9mm/μsec,

a representative jet length (measured from the virtual orgin) may range from 100 to 600mm,

a representative kinematic viscosity, μ/ρ can range from 0.25 to 5.5[8,9,14] mm^2/μsec for most jet or target metals of interest.

Thus, the Reynolds number may range from 10^2 to 10^5. Usually a Reynolds number greater than 10^3 should suffice for a large Reynolds number assumption.

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[14]*F. Harlow and W. Pracht, "Formation and Penetration of High-Speed Collapse Jets," Physics of Fluids, Vol. 9, No. 10, October 1966.*

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