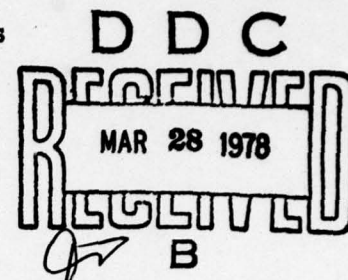Asymptotic Multinormality and Remainder Terms
of Linear Rank Vectors Under Alternatives

by
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0. Summary

Asymptotic multinormality of linear rank statistics
based on independent vector valued random variables
is obtained. Under suitable assumptions, weak esti-
mates for the remainder terms for convergence to normality
are also obtained. Results on asymptotic normality are re-
lated to Hájek (1968) and Puri and Sen (1969). Re-
sults on the remainder terms are related to those of
Jurečková and Puri (1975), Bergström and Puri (1977), and
Hušková (1977).

1. Preliminaries

Let $\underline{X}_{Ni} = (X_{Ni}^{(1)}, \ldots, X_{Ni}^{(p)})'$, $i = 1, \ldots, N$ be a sequence of

independent p-variate ($p \geq 1$) random vectors having con-

tinuous cumulative distribution functions (cdf) $F_{Ni}(\underline{x})$,

$\underline{x} = (x^{(1)}, \ldots, x^{(p)})'$, $i = 1, \ldots, N$ respectively. Consider

now the random matrix $\underline{X}_N$ corresponding to $(\underline{X}_{N1}, \ldots, \underline{X}_{NN})$, i.e.,

(1.1)
$$\underline{X}_N = ((X_{Ni}^{(\nu)}))_{\substack{i=1,\ldots,N; \\ \nu=1,\ldots,p.}} ; \quad \underline{X}_{Ni} = (X_{Ni}^{(1)}, \ldots, X_{Ni}^{(p)})'$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
Asymptotic multinormality of linear rank statistics based on independent vector valued random variables is obtained. Under suitable assumptions, weak estimates for the remainder terms for convergence to normality are related to Hájek (Ann. Math. Stat. (1968), 325-346). Results on the remainder terms are related to those of Jurečkova and Puri (Ann. Prob. (1975), 526-533), and Bergström and Puri (Ann. Stat. (1977), 671-680). compared.

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and observe that each row of $\underset{\sim}{X}_N$ is composed of N independent univariate random variables. Let $R_{Ni}^{(\nu)}$ be the rank of $X_{Ni}^{(\nu)}$ among $(X_{N1}^{(\nu)}, \ldots, X_{NN}^{(\nu)})$ for each $\nu = 1, \ldots, p$. Then corresponding to the observation matrix $\underset{\sim}{X}_N$, we have a rank collection matrix $\underset{\sim}{R}_N$, where

$$(1.2) \qquad \underset{\sim}{R}_N = ((R_{Ni}^{(\nu)})) \quad \begin{matrix} i=1,\ldots,N; \\ \nu=1,\ldots,p. \end{matrix}$$

Consider now p sets of scores $(a_{Ni}^{(\nu)}, \ 1 \leq i \leq N), \ 1 \leq \nu \leq p$ generated by known functions $\varphi_\nu: \ (0,1) \to R$ in either of the following ways:

$$(1.3) \qquad a_{Ni}^{(\nu)} = \varphi_\nu \ (i/(N+1)), \ i=1,\ldots,N; \ \nu=1,\ldots,p$$

$$(1.4) \qquad a_{Ni}^{(\nu)} = E\varphi_\nu(U_N^{(i)}), \ i=1,\ldots,N; \ \nu=1,\ldots,p$$

where $U_N^{(i)}$ is the ith order statistic in a random sample of size N from the uniform distribution over $(0,1)$. Consider now the simple linear rank vector $\underset{\sim}{S}_N$ corresponding to $\underset{\sim}{X}_N$ (or $\underset{\sim}{R}_N$) where

$$(1.5) \qquad \underset{\sim}{S}_N = (S_N^{(1)}, \ldots, S_N^{(p)})', \ S_N^{(\nu)} = \sum_{i=1}^{N} C_{Ni}^{(\nu)} \ a_N^{(\nu)} \ (R_{Ni}^{(\nu)}), \ 1 \leq \nu \leq p$$

and where $(C_{Ni}^{(\nu)}, \ 1 \leq i \leq N), \ 1 \leq \nu \leq p$ are p-sets of known (regression) constants.

In the next section we establish the asymptotic normality of $\underset{\sim}{S}_N$ by following the methods of Chernoff-Savage (1958) and Puri-Sen (1971). For a different approach, see Hájek (1968) and Puri-Sen (1969). In Section 3 we obtain the estimates for the order of normal approximation for $\underset{\sim}{S}_N$. This constitutes the generalizations of the results of Jurečková-Puri (1975), Bergstrom-Puri (1977), Puri-Rajaram (1977), and Hušková (1977b) where the problems are treated in the univariate set-ups. The multivariate extensions in the generality of our paper do not appear to exist in the literature so far. (For a rather special multivariate case, the reader is referred to Hušková (1977a)).

2. <u>Asymptotic Normality of</u> $\underset{\sim}{S}_N$. We now establish the asymptotic normality of $\underset{\sim}{S}_N$ defined in (1.5). We make the following assumptions:

(2.1) $\qquad |\varphi_\nu(t)| \leq K\{ \quad (1-t)\}^{\delta -\frac{1}{2}} , \quad |\varphi'_\nu(t)| \leq K\{t(1-t)\}^{\delta -3/2} \quad ; \quad 0<t<1$

(2.2) $\qquad \dfrac{\underset{1 \leq i \leq N}{\max} |C_{Ni}^{(\nu)}|}{\mathscr{g}_\nu} = 0(N^{-\frac{1}{2}}), \quad 1 \leq \nu \leq p$

where

(2.3) $\qquad \mathscr{g}_\nu^2 = \mathrm{Var} \, S_N^{(\nu)}, \quad 1 \leq \nu \leq p$

For convenience we shall take $\underset{\sim}{S}_N = \left(\dfrac{S_N^{(1)}}{\mathscr{g}_1}, \dots, \dfrac{S_N^{(p)}}{\mathscr{g}_p} \right)'$, where the $\mathscr{g}_\nu$'s are given by (2.3).

We then have the following theorem.

<u>Theorem 2.1</u> <u>Let the scores</u> $a_{Ni}^{(\nu)}$, $1 \leq \nu \leq p$ <u>be defined by</u> (1.3). <u>Then, under the assumptions</u> (2.1) <u>and</u> (2.2), <u>for every vector</u> $\underset{\sim}{\lambda}$ <u>in</u> R^p, $\underset{\sim}{\lambda}'(\underset{\sim}{S}_N - \underset{\sim}{u}_N)/(\underset{\sim}{\lambda}'\underset{\sim}{V}_N\underset{\sim}{\lambda})^{\frac{1}{2}}$ <u>has asymptotically</u> (as $N \to \infty$) <u>the</u> $n(0,1)$ <u>distribution where</u> u_N <u>and</u> ν_N <u>are defined by</u>

$$(2.4) \qquad \underset{\sim}{\mu}_N = (\mu_N^{(1)}, \ldots, \mu_N^{(p)})', \quad \mu_N^{(\nu)} = \sum_{i=1}^{N} C_{Ni}^{(\nu)} \int_{-\infty}^{\infty} \varphi_\nu(H^{(\nu)}(x) dF_{Ni}^{(\nu)}(x), 1 \leq \nu \leq p$$

$$(2.5) \qquad H^{(\nu)}(x) = N^{-1} \sum_{i=1}^{N} F_{Ni}^{(\nu)}(x); \quad F_{Ni}^{(\nu)}(x) = P(X_{Ni}^{(\nu)} \leq x), \quad 1 \leq \nu \leq p$$

$$(2.6) \qquad \underset{\sim}{V}_N = \left(\left(\frac{s_{\mu\nu}}{s_\mu s_\nu} \right) \right), \quad 1 \leq \mu, \nu \leq p$$

and

$$(2.7) \qquad s_{\mu\nu} = \text{Cov}(S_N^{(\mu)}, S_N^{(\nu)}); \quad 1 \leq \mu, \nu \leq p.$$

<u>Furthermore, the theorem remains true, if s_μ in (2.7)</u>

<u>are given by</u>

$$(2.8) \qquad s_{\mu\nu} = \sum_{i=1}^{N} \text{Cov}(A_{Ni}^{(\mu)}(X_{Ni}^{(\mu)}), A_{Ni}^{(\nu)}(X_{Ni}^{(\nu)})),$$

<u>where</u>

$$(2.9) \qquad A_{Ni}^{(\nu)}(x) = \frac{1}{N} \sum_{j=1}^{N} (C_{Nj}^{(\nu)} - C_{Ni}^{(\nu)}) \int_{-\infty}^{\infty} \{I_{[x \leq y]} - F_{Ni}^{(\nu)}(y)\} \varphi_\nu'(H_{(j)}^{(\nu)}(j))$$

$$dF_{Nj}^{(\nu)}(y)$$

<u>and</u>

$$I(x \leq y) = 1 \text{ } \underline{if} \text{ } x \leq y, \text{ } \underline{and} \text{ } 0 \text{ } \underline{otherwise}$$

Proof:

Let $(\alpha_1, \alpha_2, \ldots, \alpha_p)$ be a set of fixed but arbitrary constants.

By the Cramer-Wold criterion, it suffices to show that

$$(2.10) \qquad \sum_{\nu=1}^{p} \alpha_\nu \frac{S_N^{(\nu)}}{s_\gamma} = U_N, \text{ say is asymptotically normal.}$$

We introduce the following representation for $S_N^{(\nu)}$.

Let

$$(2.11) \qquad H_N^{(\nu)}(x) = N^{-1} \sum_{i-1}^{N} I(X_{Ni}^{(\nu)} \leq x),$$

$H^{(\nu)}(x)$ is already defined. (See (2.4) above.)

$$(2.12) \quad C_N^{(\nu)}(x) = \sum_{i=1}^{N} C_{Ni}^{(\nu)} \, I \, (X_{Ni}^{(\nu)} \leq (x))$$

$$(2.13) \quad C^{(\nu)}(x) = \sum_{i=1}^{N} C_{Ni}^{(\nu)} \, F_{Ni}^{(\nu)}(x) = E[C_N^{(\nu)}(x)]$$

We shall adhere to the convention of denoting stochastic variables $(H_N^{(\nu)}, C_N^{(\nu)})$ with subscripts and non-random functions $(H^{(\nu)}, C^{(\nu)})$ without subscripts, although depending upon N.

Then the following inequalities are immediate:

$$(2.14) \quad |C_N^{(\nu)}(x)| \leq N \max_{1 \leq i \leq N} |C_{Ni}^{(\nu)}| \, H_N^{(\nu)}(x)$$

$$(2.15) \quad |C^{(\nu)}(x)| \leq N \max_{1 \leq i \leq n} |C_{Ni}^{(\nu)}| \, H^{(\nu)}(x), \quad 1 \leq \nu \, p$$

We shall use the representation.

$$(2.16) \quad S_N^{(\nu)} = \int_{-\infty}^{\infty} \varphi_\nu \left(\frac{N}{N+1} H_N^{(\nu)}(x) \right) dC_N^{(\nu)}(x) = \mu_N^{(\nu)} + \beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)} + \sum_{j=1}^{3} D_{jN}^{(\nu)}$$

where $\mu_N^{(\nu)}$ is given by (2.3).

$$(2.17) \quad \beta_{1N}^{(\nu)} = \int_{-\infty}^{\infty} \varphi_\nu \, (H^{(\nu)}(x)) \, d \, (C_N^{(\nu)}(x) - C^{(\nu)}(x))$$

$$(2.18) \quad \beta_{2N}^{(\nu)} = \int_{-\infty}^{\infty} (H_N^{(\nu)}(x) - H^{(\nu)}(x)) \, \varphi_\nu' \, (H^{(\nu)}(x)) \, dC^{(\nu)}(x)$$

$$(2.19) \quad D_{1N}^{(\nu)} = \frac{-1}{N+1} \int_{-\infty}^{\infty} H_N^{(\nu)}(x) \, \varphi_\nu' \, (H^{(\nu)}(x)) \, dC_N^{(\nu)}(x)$$

$$(2.20) \quad D_{2N}^{(\nu)} = \int_{-\infty}^{\infty} (H_N^{(\nu)}(x) - H^{(\nu)}(x)) \, \varphi_\nu \, (H^{(\nu)}(x)) \, d(C_N^{(\nu)}(x) - C^{(\nu)}(x))$$

$$(2.21) \quad D_{3N}^{(\nu)} = \int_{-\infty}^{\infty} \left\{ \varphi_\nu \left(\frac{N}{N+1} H_N^{(\nu)}(x) \right) - \varphi_\nu (H^{(\nu)}(x)) - \left(\frac{N}{N+1} H_N^{(\nu)}(x) - H^{(\nu)}(x) \right) \right.$$
$$dC_N^{(\nu)}(x)$$

Substituting (2.16) in (2.10), we have

$$(2.22) \quad U_N = \sum_{\nu=1}^{p} \frac{\alpha_\nu}{s_\nu} \left\{ \mu_N^{(\nu)} + \left(\beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)} \right) + \sum_{j=1}^{3} D_{jN}^{(\nu)} \right\}$$

The proof will be accomplished if we show the following:

(A) $\mu_N^{(\nu)}$ is finite.

(B) $\sum_{\nu=1}^{p} \alpha_\nu \frac{\left(\beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)} \right)}{s_\nu}$ is asymptotically normal.

(C) $\sum_{\nu=1}^{p} \frac{\alpha_\nu}{s_\nu} \sum_{j=1}^{3} D_{jN}^{(\nu)} \to 0$ in probability.

Proof of (A): Observe that

$$\left| \mu_N^{(\nu)} \right| = \left| \int_{-\infty}^{\infty} \varphi_\nu (H^{(\nu)}(x)) \, dC^{(\nu)}(x) \right|$$

$$\leq N \max_{1 \leq i \leq N} \left| C_{Ni}^{(\nu)} \right| \int_{-\infty}^{\infty} \varphi_\nu (H^{(\nu)}(x)) \, dH^{(\nu)}(x) < +\infty \quad \text{for each } N$$

by assumptions (2.1) and (2.15).

Proof of (B): For a fixed ν, we shall verify the Liapunov crite-

rion for the normality of $(\beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)}) s_\nu^{-1}$ and then do the same

for the sum $\sum_{\nu=1}^{p} \alpha_\nu (\beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)}) s_\nu^{-1}$, by an extension of the C_r-inequality.

In fact we do so separately for $\beta_{1N}^{(\nu)} s_\nu^{-1}$ and $\beta_{2N}^{(\nu)} s_\nu^{-1}$ and use the C_r-inequality.

Since we are considering a fixed ν, we can drop the indexing

variable ν and simplify our notation substantially.

Consider (2.18)

$$\beta_{2N} = \int_{-\infty}^{\infty} (H_N(x) - H(x)) \, \varphi'(H(x)) \, dC(x)$$

Integration by parts yields,

$$(2.23) \quad \beta_{2N} = B(x)\left\{H_N(x) - H(x)\right\}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} B(x) \, d(H_N(x) - H(x))$$

where

$$(2.24) \quad B(x) = \int_{x_0}^{x} \varphi'(H(x)) \, dC(x) \, , \quad x_0 \text{ arbitrary but } H(x_0) > 0 \ .$$

We shall later prove that the first term in (2.23) is

$O_p(\mathfrak{s})$, $\mathfrak{s}_\nu = \mathfrak{s}$.

The second term of the same can be written as

$$(2.25) \quad \int_{-\infty}^{\infty} B(x) \, d(H_N(x) - H(x)) \, N^{-1} \sum_{i=1}^{N} \left\{B(X_{Ni}) - E B(X_{Ni})\right\}$$

We shall show that

$$(2.26) \, (\mathfrak{s}N)^{-(2+\epsilon)} \sum_{i=1}^{N} E|\beta(X_{Ni}) - E \beta(X_{Ni})|^{2+\epsilon} \to 0 \, , \text{ as } N \to \infty \text{ for some } \epsilon > 0.$$

In fact, it suffices to verify,

$$(2.27) \, (\mathfrak{s}N)^{-(2+\epsilon)} \sum_{i=1}^{N} E|\beta(X_{Ni})|^{2+\epsilon} \to 0$$

which by the C_r and Jensen inequalities implies (2.26).

Next, choose an $\epsilon > 0$ such that $(2+\epsilon)(\delta - \frac{1}{2}) > -1$ for the

δ given by the constraints on φ . Then,

$$(2.28) \; (sN)^{-(2+\epsilon)} \sum_{i=1}^{N} |\beta(X_{Ni})|^{2+\epsilon} = (sN)^{-(2+\epsilon)} \sum_{i=1}^{N} E|B(X_{Ni})|^{2+\epsilon}$$

$$\leq (sN)^{-(2+\epsilon)} \sum_{i=1}^{N} N^{2+\epsilon} \max_{1 \leq i \leq N} |C_{Ni}|^{2+\epsilon} \int_{-\infty}^{\infty} \left| \int_{x_0}^{x} \varphi'(H(y)) \, dH(y) \right|^{2+\epsilon} dF_{Ni}(x)$$

$$= \{ \max_{1 \leq i \leq N} |s^{-1}|C_{Ni}| \}^{(2+\epsilon)} \sum_{i=1}^{N} \int_{-\infty}^{\infty} \left\{ |\varphi(H(x))| + |\varphi(H(x_0))| \right\}^{2+\epsilon} dF_{Ni}(x)$$

$$= 0(N^{-\epsilon/2}) \int_{-\infty}^{\infty} \left\{ |\varphi(H(x))| + |\varphi(H(x_0))| \right\}^{2+\epsilon} dH(x) \to 0 , \text{ as } N \to \infty .$$

because $(2+\epsilon)(\delta - \frac{1}{2}) > -1$ and $\varphi(H(x_0))$ is a constant. Observe

that we have used the fact $\max_{1 \leq i \leq N} \dfrac{|C_{Ni}|^{2+\epsilon}}{s^{2+\epsilon}} = 0(N^{-1-\epsilon/2})$.

It remains to be shown that in (2.23),

$$\frac{\beta(x)}{s} \Big|_{-\infty}^{\infty} = \frac{B(x)}{s} \left\{ H_N(x) - H(x) \right\} \Big|_{-\infty}^{+\infty} = o_p(1) .$$

$$\left| \frac{\beta(x)}{s} \right| = \sqrt{N} |H_N(x) - H(x)| \frac{1}{\sqrt{N} s} \left| \int_{x_0}^{x} \varphi'(H(y)) \, dH(y) \right|$$

$$\leq \sqrt{N} |H_N(x) - H(x)| 0(1) \left| \int_{x_0}^{x} \varphi'(H(y)) dH(y) \right|$$

$$\leq K \sqrt{N} |H_N(x) - H(x)| \left\{ H(x)(1 - H(x)) \right\}^{\delta - \frac{1}{2}}$$

since $\varphi(H(x_0))$ is a constant.

By Puri-Sen (1971), given any ϵ, δ' positive, there is a constant $C(\epsilon,\delta')$ independent of N such that, with probability $> 1-\epsilon$,

$$N^{\frac{1}{2}}|H_N(x) - H(x)| \leq C(\epsilon,\delta')\left\{H(x)(1-H(x))\right\}^{\frac{1}{2}-\delta'} \quad .$$

Thus,

$$|\bar{s}^1\beta(x)| \leq K\left\{H(x)(1-H(x))\right\}^{\delta-\delta'} C(\epsilon,\delta') \to 0$$

as $x \to \pm\infty$, by choosing $0 < \delta' < \delta$.

The verification of the Liapunov condition for $\bar{s}^1\beta_{1N}$ is similar, in fact, easier and is therefore not given here. The C_r-inequality yields the Liapunov condition for $\bar{s}^1(\beta_{1N} + \beta_{2N})$

Consider next the normalized sum,

$$\sum_{\nu=1}^{p} \alpha_\nu \bar{s}^1_\nu(\beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)}) \quad .$$

We need to verify the Liapunov condition for the above expression.

Let $\sigma^2 = \text{Var}\left(\sum_{\nu=1}^{p} \alpha_\nu \bar{s}^1_\nu(\beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)})\right)$.

We shall assume that σ^2 is bounded away from zero for all N. (If not, the sum is trivially degenerate normal as $N \to +\infty$.)

Write $\beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)} = \sum_{i=1}^{N} \beta_{Ni}^{(\nu)}$

where $\beta_{Ni}^{(\nu)}$ are independent random variables (as already done earlier). We have to show

(2.29) $\quad \lim_{N \to \infty} \sigma^{-(2+\epsilon)} \sum_{i=1}^{N} E \left| \sum_{\nu=1}^{p} \alpha_\nu \frac{\beta_{Ni}^{(\nu)}}{s_\nu} \right|^{2+\epsilon} = 0$.

We have already shown that

(2.30) $\quad \lim_{N \to \infty} (s_\nu)^{-(2+\epsilon)} \sum_{i=1}^{N} E \left| \beta_{Ni}^{(\nu)} \right|^{2+\epsilon} = 0$

(2.29) follows from (2.30) upon noting that there is a constant $C(p,\epsilon)$, depending only upon p and ϵ such that

$$E \left| \sum_{\nu=1}^{p} \alpha_\nu \frac{\beta_{Ni}^{(\nu)}}{s_\nu} \right|^{2+\epsilon} \leq C(p,\epsilon) \sum_{\nu=1}^{p} |\alpha_\nu / \delta_\nu|^{2+\epsilon} E |\beta_{Ni}^{(\nu)}|^{2+\epsilon}$$

(Generalized C_r-inequality.)

This establishes (B). We note that ϵ depends only upon δ and hence the same choice of ϵ works for all ν .

Proof of (C): Recall that we have to establish

$$\sum_{\nu=1}^{p} \alpha_\nu s_\nu^{-1} \sum_{j=1}^{3} D_{jN}^{(\nu)} = o_p(1) \quad .$$

Clearly, it suffices to prove for a particular ν since we have a finite sum $(1, \ldots, p)$. Again, we shall drop the index ν .

Consider, $|s^{-1} D_{1N}| = |-(N+1)^{-1} s^{-1} | \int_{-\infty}^{\infty} H_N(x) \varphi'(H(x)) \, dC_N(x) |$

$\leq (sN)^{-1} \sum | \varphi'(H(X_{Ni})) 1 |C_{Ni}| = \frac{1}{N} \sum_{i=1}^{N} V_{Ni}$

where

$$V_{Ni} = |\varphi'(H(X_{Ni})) \; \bar{s}^{-1} C_{Ni}| \; .$$

To prove

$$\sum_{i=1}^{N} V_{Ni} \to 0 \quad \text{in probability,}$$

it suffices to show that

$$\sum_{i=1}^{N} \bar{N}^{\alpha} \, E|V_{Ni}|^{\alpha} < +\infty \; , \text{ for all } N \text{ , for some } \alpha \; , \; 0 < \alpha < 1 \; .$$

(By particular case 1° , page 241, Loéve (1963)).

Take $\alpha = 2/3$:

$$\bar{N}^{-2/3} \sum_{i=1}^{N} E|V_{Ni}|^{2/3} \le K \bar{N}^{-2/3} \sum_{i=1}^{N} \left| \frac{C_{Ni}}{\bar{s}} \right|^{2/3} E|H(X_{Ni})(1 - H(X_{Ni}))|^{(\delta - 3/2)}$$

$$\le N^{1/3} \max_{1 \le i \le N} (\bar{s}^{-1}|C_{Ni}|)^{2/3} \sum_{i=1}^{N} \int_{-\infty}^{\infty} \left\{ H(x)(1 - H(x)) \right\}^{\frac{2\delta}{3} - 1} dF_{Ni}(x)$$

$$= K \cdot 0(1) \int_{-\infty}^{\infty} \left\{ H(x)(1 - H(x)) \right\}^{\frac{2\delta}{3} - 1} dH(x) < +\infty \; ;$$

where we have used the fact that

$$\max_{1 \le i \le N} |\bar{s}^{-1} C_{Ni}|^{2/3} = 0(N^{-1/3}) \; .$$

Consider next,

$$D_{2N} = \int_{-\infty}^{\infty} (H_N(x) - H(x)) \varphi'(H(x)) \; d(C_N(x) - C(x)) \; .$$

By Puri-Sen (1971), given any $\epsilon > 0$, $0 < \delta' < \frac{1}{2}$, there is a

constant $C(\epsilon, \delta')$, independent of N such that with

probability $> 1 - \epsilon$,

$$N^{\frac{1}{2}} |H_N(x) - H(x)| < C(\epsilon, \delta') \left\{ H(x)(1 - H(x)) \right\}^{\frac{1}{2}-\delta'} .$$

Thus with probability $> 1 - \epsilon$,

$$|H_N(x) - H(x)| \varphi'(H(x))| \le K C(\epsilon, \delta') N^{\frac{1}{2}} \{ H(x)(1 - H(x)) \}^{\delta^*-1}$$

taking $\delta' < \delta$ and setting $\delta^* = \delta - \delta'$.

It suffices to show that

$$\bar{g}^{-1} N^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left\{ H(x)(1 - H(x)) \right\}^{\delta^*-1} d(C_N(x) - C(x)) \to 0 \quad \text{in probability.}$$

We shall use the Liapunov criterion for degenerate convergence (page 275, Lóeve (1963)).

$$\bar{g}^{-1} N^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left\{ H(x)(1 - H(x)) \right\}^{\delta^*-1} dC_N(x)$$

$$= \bar{g}^{-1} N^{-\frac{1}{2}} \sum_{i=1}^{N} C_{Ni} \left\{ H(X_{Ni})(1 - H(X_{Ni})) \right\}^{\delta^*-1}$$

Set

$$V_{Ni} = N^{\frac{1}{2}} \bar{g}^{-1} C_{Ni} \left\{ H(X_{Ni})(1 - H(X_{Ni})) \right\}^{\delta^*-1} .$$

Then,

$$\bar{g}^{-1} N^{-\frac{1}{2}} \sum_{i=1}^{N} C_{Ni} \left\{ H(X_{Ni})(1 - H(X_{Ni})) \right\}^{\delta^*-1} = N^{-1} \sum_{i=1}^{N} V_{Ni} .$$

It remains to prove, $N^{-1} \sum_{i=1}^{N} (V_{Ni} - E V_{Ni}) \to 0$ in probability. This will be accomplished if we can show that for some $\alpha > 0$,

(2.31) $\bar{N}^{(1+\alpha)} \sum_{i=1}^{N} E \mid V_{Ni}\mid^{1+\alpha} \to 0$.

Choose $\alpha > 0$ such that, $(1+\alpha)(\delta^* - 1) > -1$; that is, $0 < \alpha < \dfrac{\delta^*}{1-\delta^*}$. Then,

$$\bar{N}^{(1+\alpha)} \sum_{i=1}^{N} E \mid V_{Ni}\mid^{1+\alpha}$$

$$\leq \bar{N}^{1(1+\alpha)} \mid \frac{\sqrt{N} \max_{1\leq i\leq N} \mid C_{Ni}\mid}{8} \mid^{1+\alpha} \sum_{i=1}^{N} E\left\{H(X_{Ni})(1-H(X_{Ni}))\right\}^{(1+\alpha)(\delta^*-1)}$$

$$= 0(1)\bar{N}^{\alpha} \int_{-\infty}^{\infty} \left\{H(x)(1-H(x))\right\}^{(1+\alpha)(\delta^*-1)} dH(x) = 0(N^{-\alpha})$$

which proves (2.31).

Next consider

$$\bar{8}^{1} D_{3N} = \bar{8}^{1} \int_{-\infty}^{\infty} \left\{\varphi\left(\frac{N H_N(x)}{N+1}\right) - \varphi(H(x)) - \left(\frac{N H_N(x)}{N+1} - H(x)\right)\varphi'(H(x))\right\}dC_N(x) .$$

We have to show that $\bar{8}^{-1} D_{3N} \to o$ in probability.

The following substitution will simplify the proof. Let

(2.32) $\quad C_{3N} = \dfrac{D_{3N}}{8\sqrt{N}}$.

Then it suffices to prove $C_{3N} = 0_p(N^{-\frac{1}{2}})$

Observe that, since $\max_{1\leq i\leq N} \mid\dfrac{C_{Ni}}{\sqrt{N}8}\mid = 0(\bar{N}^{-1})$

we have

(2.33) $|C_{3N}|$

$$\leq O(1)\int_{-\infty}^{\infty}\left|\varphi\left(\frac{N}{N+1}H_N(x)\right) - \varphi(H(x)) - \left(\frac{N\,H_N(x)}{N+1} - H(x)\right)\varphi'(H(x))\right|dH_N(x)$$

The proof that the right hand side of (2.36) is $o_p(N^{-\frac{1}{2}})$ can be found in Puri-Sen (1971), pages 401-405.

This proves Theorem 2.1.

For a different approach, as well as the derivation of some tests for linear hypothesis based on S_N , see Puri-Sen (1969).

Computation of V_N : It is clear that the variance contribution is from the term $\beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)}$. It is easy to check that

(2.34) $\beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)} = \sum_{i=1}^{N}\left\{A_{Ni}^{'(\nu)}\left(X_{Ni}^{(\nu)}\right) - EA_{Ni}^{'(\nu)}\left(X_{Ni}^{(\nu)}\right)\right\}$

where

$$A_{Ni}^{'(\nu)}(x) = \frac{1}{N}\sum_{j=1}^{N}(C_{Ni}^{(\nu)} - C_{Nj}^{(\nu)})\int_{x_0}^{x}\varphi_{\nu}'(H^{(\nu)}(y))\,dF_{Nj}^{(\nu)}(y) ; \quad x_0 \text{ arbitrary.}$$

Centering $A_{Ni}^{'(\nu)}(x)$ at its expectations, we note that

(2.35) $\beta_{1N}^{(\nu)} + \beta_{2N}^{(\nu)} = \sum_{i=1}^{N}A_{Ni}^{(\nu)}\left(X_{Ni}^{(\nu)}\right)$

where

$$A_{Ni}^{(\nu)}(x) = \frac{1}{N} \sum_{j=1}^{N} (C_{Ni}^{(\nu)} - C_{Nj}^{(\nu)}) \int_{-\infty}^{\infty} \left\{ I(x \le y) - F_{Ni}^{(\nu)}(y) \right\} \varphi_{\nu}'(H^{(\nu)}(y)) dF_{Nj}^{(\nu)}(y)$$

which yields the desired approximation (2.7).

We next give a corollary which extends these results to the case when the scores are generated according to (1.7)(b) .

Corollary 2.2: Let φ_{ν} be the inverse of a distribution function. Let $\underset{\sim}{S}_N^*$ be the linear rank vector defined as in (1.8) but with the scores generated according to (1.4) . Then, under the conditions of Theorem 2.1, $\underset{\sim}{S}_N^*$ is asymptotically multinormal with the same parameters μ_N and V_N .

Proof: Let

$$(2.36) \quad \underset{\sim}{S}_N^* = \left(\frac{S_N^{*(1)}}{s_1} , \ldots , \frac{S_N^{*(p)}}{s_p} \right)'$$

Clearly, it suffices to prove

$$(2.37) \quad S_N^{*(\nu)} - S_N^{(\nu)} = o_p(s_\nu) , \quad 1 \le \nu \le p$$

which then entails $\underset{\sim}{S}_N^* - \underset{\sim}{S}_N \to \underset{\sim}{0}$ in probability.

Again, since ν is fixed, $1 \le \nu \le p$, we shall drop this index. We define

$$(2.38) \quad \varphi_N(t) = \sum_{i=1}^{[Nt]} (G_N(i) - G_N(i-1)) , \quad G_N(0) = 0 , \quad 0 < t < 1 ,$$

where $[\alpha]$ is the greatest integer not exceeding α .

It is easy to check that

$$(2.39) \qquad S_N^* = \int_{-\infty}^{\infty} \varphi_N\left(\frac{N\,H_N(x)}{N+1}\right) \, d\,C_N(x) \quad.$$

The proof of Corollary 2.2 will be accomplished if we prove the following:

<u>Lemma 2.3</u>: <u>Under the hypotheses of Theorem 2.1 and Corollary 2.2, we have</u>

(i) $\lim\limits_{N \to +\infty} \varphi_N(t) = \varphi(t)$

(ii) $\left| \int_{-\infty}^{\infty} \left\{ \varphi_N\left(\frac{N}{N+1} H_N(x)\right) - \varphi\left(\frac{N}{N+1} H_N(x)\right) \right\} d\,C_N(x) \right| = o_p(\mathscr{S})$

<u>Proof of Lemma 2.3</u>: We shall only prove (ii) since the proof of (i) can be found in pages 408-409 of Puri-Sen (1971).

By inequality (2.14), we have

$$(2.40) \qquad \left| \int_{-\infty}^{\infty} \left\{ \varphi_N\left(\frac{N\,H_N(x)}{N+1}\right) - \varphi\left(\frac{N\,H_N(x)}{N+1}\right) \right\} \, d\,C_N(x) \right|$$

$$\leq \max_{1 \leq i \leq N} |C_{Ni}| \sum_{i=1}^{N} \left| \varphi_N\left(\frac{i}{N+1}\right) - \varphi\left(\frac{i}{N+1}\right) \right| \quad.$$

Hence it follows that

$$(2.41) \qquad \max_{1 \leq i \leq N} \left| \frac{C_{Ni}}{\mathscr{S}} \right| \sum_{i=1}^{N} \left| \varphi_N\left(\frac{i}{N+1}\right) - \varphi\left(\frac{i}{N+1}\right) \right|$$

$$= O(1) \, N^{-\frac{1}{2}} \sum_{i=1}^{N} \left| \varphi_N\left(\frac{i}{N+1}\right) - \varphi\left(\frac{i}{N+1}\right) \right| = A_N \, , \text{ say}$$

But, $\lim\limits_{N \to +\infty} A_N = 0$, by Puri-Sen (1971), pages 409-411.

This immediately entails $\dfrac{S_N - S_N^*}{g} \to 0$ in probability, which proves Corollary 2.2.

A consequence of Corollary 2.2 is that in many cases of practical interest (such as the normal scores), asymptotic normality holds with the same centering sequence μ_N , whether the scores are given by (1.3) or (1.4) . Thus theorem 2.1 and corollary 2.2 serve to unify the results of Hájek (1968) and Hoeffding (1973), and also to some extent simplify the results of the latter paper.

3. <u>Remainder Terms of</u> S_N : In this section, we obtain an estimate for the remainder in the normal approximation to S_N

Recall that we can write

$$(3.1) \quad S_N - \mu_N = \sum_{i=1}^{N} A_{Ni}(X_{Ni}) + D_N$$

where

$$(3.2) \quad A_{Ni}(X_{Ni}) = \left(A_{Ni}^{(1)}(X_{Ni}^{(1)}, \ldots, A_{Ni}^{(p)}(X_{Ni}^{(p)}) \right)'$$

$$D_N = \left(D_N^{(1)}, \ldots, D_N^{(p)} \right)'$$

$$D_N^{(\nu)} = D_{1N}^{(\nu)} + D_{2N}^{(\nu)} + D_{3N}^{(\nu)} \quad , \quad 1 \le \nu \le p$$

Normalization by g_ν is of no consequence and was introduced there merely to simplify the details of the proof. We shall not

use it here.

We shall simplify the notation somewhat and also express it in a form so that we can use Corollary 17.2, page 165 of Bhattacharya-Ranga Rao (1976).

We can write,

$$(3.3) \quad \underset{\sim}{T}_{Ni} = N^{\frac{1}{2}} A_{Ni}(X_{Ni}) \ , \quad V_i = Cov(\underset{\sim}{T}_{Ni}) \ .$$

Then,

$$(3.4) \quad \underset{\sim}{S}_N = N^{-\frac{1}{2}} \sum_{i=1}^{N} \underset{\sim}{T}_{Ni} + \underset{\sim}{D}_N \ .$$

Let

$$(3.5) \quad V = N^{-1} \sum_{i=1}^{N} V_i \ .$$

Comparing (3.5) to (2.7) and (2.8), it is clear that the elements of V are $g_{\mu\nu}$.

Let, for $\underset{\sim}{x} = (x^{(1)}, \ldots, x^{(p)})'$,

$$(3.6) \quad F_N(\underline{x}) = P\left(S_N^{(1)} - \mu_N^{(1)} \le x^{(1)}, \ldots, S_N^{(p)} - \mu_N^{(p)} \le x^{(p)}\right) \ .$$

We then have the following theorem:

Theorem: Let the conditions of Theorem 2.1 be satisfied. Let in addition,

$$(3.7) \quad \underset{0 \le t \le 1}{Sup} |\varphi_\nu'(t)| = \|\varphi_\nu'\| < +\infty \ , \quad 1 \le \nu \le p \ ,$$

and V be positive definite.

Then, there exists a constant $C(p)$, independent of N, such that,

(3.8) $\quad \underset{\underset{\sim}{x} \in \mathbb{R}^p}{\text{Sup}} |F_N(\underset{\sim}{x}) - \Phi_{0,V}(\underset{\sim}{x})| \leq C(p) \lambda^{-3/2} \rho_3 N^{-\frac{1}{2}} + \Delta_N$

where

(3.9) $\quad \Phi_{0,V}(\underset{\sim}{x})$ is the distribution function (p-variate) of a Gaussian random vector with mean $\underset{\sim}{0}$ and dispersion matrix V;

(3.10) $\quad \lambda = $ smallest eigenvalue of V;

(3.11) $\rho_3 = \dfrac{1}{N} \sum\limits_{i=1}^{N} E\|T_{Ni}\|^3$, the norm being the euclidean norm;

and $\quad \Delta_N \to 0$ pointwise.

Furthermore

(3.12) $\quad N^{\frac{1}{2}} \Delta_N = 0_p\left(\sum\limits_{\nu=1}^{p} \|\varphi_\nu\| s_\nu^{-1}\left[\sum\limits_{i=1}^{N} |c_{Ni}^{(\nu)}| \right] \right)$

Remark. Since $\Delta_N \to 0$, it would be of interest to have an estimate for the fluctuations of Δ_N. But in view of the generality of the situation and the fact that we are dealing with the multivariate set up, it is not easy to secure a purely numerical estimate. (3.12) gives a probabilistic bound. The problem of obtaining a sharp bound remains open.

Proof: Writing $\underset{\sim}{T}_N = N^{-\frac{1}{2}} \sum\limits_{i=1}^{N} \underset{\sim}{T}_{Ni}$, we have

(3.13) $\quad \underset{\sim}{S}_N - \underset{\sim}{\mu}_N = \underset{\sim}{T}_N + \underset{\sim}{D}_N$.

Let G_N be the distribution function (p-variate) of $\underset{\sim}{T}_N$. Then

(3.14) $\quad F_N(\underset{\sim}{x}) = G_N(\underset{\sim}{x} - \underset{\sim}{D}_N)$.

Hence,

(3.15) $|F_N(\underset{\sim}{x}) - \Phi_{0,\nu}(\underset{\sim}{x})| \leq |G_N(\underset{\sim}{x} - \underset{\sim}{D}_N) - \Phi_{0,\nu}(\underset{\sim}{x} - \underset{\sim}{D}_N)|$

$$+ |\Phi_{0,\nu}(\underset{\sim}{x} - \underset{\sim}{D}_N) - \Phi_{0,\nu}(\underset{\sim}{x})| \ .$$

We estimate the two quantities on the right hand side separately.

First, an application of Corollary 17.2 (page 172) of Bhattacharya-Ranga Rao (1976) yields,

(3.16) $|G_N(\underset{\sim}{x} - \underset{\sim}{D}_N) - \Phi_{0,\nu}(\underset{\sim}{x} - \underset{\sim}{D}_N)| \leq C(p) \lambda^{-3/2} \rho_3 N^{-\frac{1}{2}}$

uniformly in $\underset{\sim}{x}$, wherein all the quantities have already been defined. $\|\varphi'\| < +\infty$ ensures the existence of ρ_3 .

Estimating the second term is rather more involved. However, since $\Phi_{0,\nu}$ is continuous, the convergence of the distribution functions is uniform and hence

(3.17) $\underset{\underset{\sim}{x} \in \mathbb{R}^p}{\text{Sup}} |\Phi_{0,\nu}(\underset{\sim}{x} - \underset{\sim}{D}_N) - \Phi_{0,\nu}(\underset{\sim}{x})| = \Delta_N$

converges to zero pointwise. We only need to obtain (3.12).

Note that, by the Mean Value Theorem (in $\mathbb{R}^p$) , there is a point $\underset{\sim}{\xi}$ on the Line segment joining $\underset{\sim}{x} - \underset{\sim}{D}_N$ and $\underset{\sim}{x}$ such that

(3.18) $\Phi_{0,\nu}(\underset{\sim}{x}) - \Phi_{0,\nu}(\underset{\sim}{x} - \underset{\sim}{D}_N) = \underset{\sim}{D}'_N L(\underset{\sim}{\xi})$

where $L(\underset{\sim}{\xi})$ is the differential of $\Phi_{0,\nu}$ at $\underset{\sim}{\xi}$.

Next, for the right hand side of (3.18), we have

(3.19) $\underset{\sim}{D}'_N L(\underset{\sim}{\xi}) = \sum_{\nu=1}^{p} D_N^{(\nu)} \frac{\delta}{\delta x^{(\nu)}} \Phi_{0,\nu}(\underset{\sim}{x})\Big|_{\underset{\sim}{x} = \underset{\sim}{\xi}}$

Further, for each ν, $1 \leq \nu \leq p$, it is easy to check that

$$\left| \frac{\delta \Phi_{0,\nu}}{\delta x^{(\nu)}} \right| \text{ is dominated by the } \nu^{th} \text{ marginal density .}$$

Hence, $\left| \dfrac{\delta \Phi_{0,\nu}}{\delta x^{(\nu)}} \right| \leq s_\nu^{-1} (2\pi)^{-\frac{1}{2}}$.

Consequently,

$$(3.19) \quad |\Delta_N| \leq \sum_{\nu=1}^{p} s_\nu^{-1} |D_N^{(\nu)}| (2\pi)^{-\frac{1}{2}} .$$

Thus, (3.12) will be established if we show that for each ν ,

$$(3.20) \quad \sqrt{N} \, D_N^{(\nu)} = 0_p \left(\|\varphi_\nu'\| \sum_{i=1}^{N} |C_{Ni}^{(\nu)}| \right) .$$

Again, we shall drop the index ν in the rest of the proof. Rearranging terms of D_N , we get,

$$(3.21) \quad D_N = \int_{-\infty}^{\infty} \left\{ \varphi\left(\frac{N}{N+1} H_N(x) \right) - \varphi(H(x)) \right\} d\,C_N(x)$$

$$- \int_{-\infty}^{\infty} (H_N(x) - H(x)) \varphi'(H(x)) \, d\,C(x) .$$

In order to simplify the proof, we shall drop the factor $\frac{N}{N+1}$. In view of the fact that φ' is bounded, the conclusion will not be affected.

Consider $\varphi(H_N(x)) - \varphi(H(x))$. By the Mean Value Theorem,

$$\varphi(H_N(x)) - \varphi(H(x)) = (H_N(x) - H(x)) \varphi'(\xi_N(x))$$

for some $\xi_N(x)$.

Hence,

$$(3.22) \quad \left| \sqrt{N} \int_{-\infty}^{\infty} (H_N(x) - H(x)) \varphi'(\xi_N(x)) d\, C_N(x) \right|$$

$$\leq \|\varphi'\| \int_{-\infty}^{\infty} \sqrt{N} \, |H_N(x) - H(x)| d|C_N|(x)$$

Let $\epsilon > 0$ be given. Then by Puri-Sen (1971), there is a constant $C(\epsilon)$ independent of N such that with probability $> 1 - \epsilon$

$$\sqrt{N} \, |H_N(x) - H(x)| < C(\epsilon) \quad .$$

Thus in (3.22), with probability $> 1 - \epsilon$,

$$(3.23) \quad \left| \int \sqrt{N} \, (H_N(x) - H(x)) \varphi'(\xi_N(x)) d\, C_N(x) \right| \leq C(\epsilon) \|\varphi'\| \sum_{i=1}^{N} |C_{Ni}| \quad .$$

The proof of

$$(3.24) \quad \left| \int_{-\infty}^{\infty} \sqrt{N} \, (H_N(x) - H(x)) \varphi'(H(x)) d\, C(x) \right| \leq C(\epsilon) \|\varphi'\| \sum_{i=1}^{N} |C_{Ni}|$$

in probability is identical.

(3.23) and (3.24) establish (3.20) and (3.19), which in turn entail (3.12). The proof is completed.

<u>Remark</u>: From the expressions for $g_{\mu\nu}$ and g_ν^2, it is clear that λ is extremely hard to compute. Recall that λ is the smallest eigenvalue of the dispersion matrix V. However, if the components $X_{Ni}^{(\nu)}$ of the vectors $\underset{\sim}{X}_{Ni}$ satisfy the

condition of "weak dependence" in the sense

$$(3.25) \quad s_\nu^2 - \sum_{\mu \neq \mu'} |s_{\mu\mu'}| > 0 \quad , \quad \nu = 1, \ldots, p \quad ;$$

the classical Grischgorin theorem (cf. Dahlquist-Björek, (1974)), permits us to replace λ by much simpler expression (3.25) , assuming without loss of generality, $\lambda > 0$. If one is prepared to work numerically, much better estimates for λ can be obtained (op.cit. Dahlquist-Björck).

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