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ANALYSIS OF CENTRAL PLACE THEORY.(U)
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ANALYSIS OF CENTRAL PLACE THEORY

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1. INTRODUCTION

Christaller (1933) postulated a regular hexagonal lattice as his model for the distribution of towns over a homogeneous area. This is known as Central Place Theory. This is contrasted against the hypothesis that the spatial pattern of towns in the area is random. A survey on this and related topics is given by Cliff & Ord (1975) as well as an extensive bibliography. See also Berry & Pred (1961) and Ripley (1977).

In an ideal situation Central Place Theory would predict a pattern as given in Fig. 1 for 11 central places. As shown, the region can be covered by hexagons of the same size. We will require the following definitions.

Key words: Central Places, Delaunay Triangle, Dirichlet Cell, Miles density, Random pattern, von Mises Distribution.

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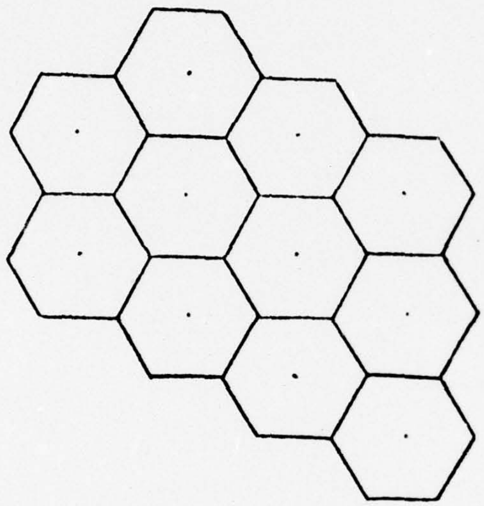

FIG. 1

Spatial pattern under Central Place Theory with 11 towns (· denotes a town)

(i) <u>Dirichlet Cells (Thiessen Polygons)</u>

Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n$ be the co-ordinates in R^2 of n central places. The ith Dirichlet cell, $\Pi(x_i)$, is defined such that for all $\underset{\sim}{x} \in \Pi(\underset{\sim}{x}_i)$,

$$||\underset{\sim}{x}-\underset{\sim}{x}_i|| < ||\underset{\sim}{x}-\underset{\sim}{x}_j||, \quad j \neq i .$$

The boundary of the cell defines the corresponding Thiessen Polygon and these cells or polygons form a non-overlapping covering of the plane.

(ii) <u>Delaunay Triangles</u>

These are formed by connecting those points whose Theissen polygons have a common side. Provided that 4 or more Dirichlet cells do not meet at any one point, the Delaunay triangulation will also form a non-overlapping covering of the plane. For 4 cells to have a common vertex, the 4 centres would have to lie at the vertices of a square or rectangle and since we are in continuous space, this probability can be neglected.

As a numerical example, we examine the data of Gaile & Burt (1977, unpublished). They consider a map of 44 Central Places in 6 counties in Iowa, namely Unwin, Ringgold, Clarke, Decatur, Lucas and Wayne Counties. Thiessen polygons were computed and the interior angles, α_1, α_2, α_3, of the 63 resulting Delaunay triangles were obtained. Of course $\alpha_1 + \alpha_2 + \alpha_3 = 180°$. Define $\alpha_{(1)}$, $\alpha_{(2)}$, $\alpha_{(3)}$ so that $\alpha_{(1)} \leq \alpha_{(2)} \leq \alpha_{(3)}$. Table I gives $\alpha_{(1)}$ and $\alpha_{(2)}$.

TABLE I

Angles $\alpha_{(1)}$ and $\alpha_{(2)}$ for the 63 Delaunay triangles formed from the 44 Central Places in Iowa. $\alpha_{(1)} \leq \alpha_{(2)} \leq \alpha_{(3)}$, $\alpha_{(1)} + \alpha_{(2)} + \alpha_{(3)} = 180^{\circ}$

$\alpha_{(1)}$	$\alpha_{(2)}$	$\alpha_{(1)}$	$\alpha_{(2)}$	$\alpha_{(1)}$	$\alpha_{(2)}$
78	82	102	108	82	100
62	136	54	90	52	58
96	106	38	120	68	76
56	78	38	128	72	110
88	100	94	110	88	108
104	104	52	86	24	100
66	112	52	74	72	118
88	134	88	92	34	146
42	132	86	134	52	114
40	82	62	134	58	78
36	144	38	122	110	118
42	80	36	108	78	132
62	124	84	106	62	98
62	140	70	136	34	120
78	80	110	124	44	110
82	118	90	108	48	96
106	122	92	120	80	120
92	112	66	98	96	112
86	114	40	158	62	118
100	122	44	86	112	116
60	112	92	114	96	120

2. RANDOMNESS HYPOTHESIS AND MILES' DENSITY

2.1 Miles' Result

When Central Place Theory does not hold, the Thiessen polygons become irregular and the Delaunay triangles follow a more random pattern. The one procedure is then to generate the points in the plane by the homogeneous Poisson process. This process has been studied in Miles (1970) with the following results.

Let X be a 'nice' domain in R^2, e.g. the interior of a circle. Let $q > 0$ be a real number and let $X(q)$ be the circle of radius q. Consider a Poisson process $\underset{\sim}{P}$ in R^2 of constant parameter ρ, and for those points lying in $X(q)$ consider the associated Delaunay triangulation. A triangle with vertices $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, $\underset{\sim}{x}_3$ can be re-parameterised in terms of $\underset{\sim}{x}_1$, R, θ_1, θ_2, θ_3, where R is the circumradius and θ_i is the angle formed between the horizontal through the centre of the circumcircle, and the line joining $\underset{\sim}{x}_i$ to the centre.

Therefore, in the notation of Section 1,

$$\sin\alpha_1 = |\sin\tfrac{1}{2}(\theta_2 - \theta_1)|, \quad \sin\alpha_2 = |\sin\tfrac{1}{2}(\theta_3 - \theta_2)|, \quad \alpha_1, \alpha_2 \in (0,\pi).$$

Let R and (α_1, α_2) be asymptotically independent with

$$g(R) = 2(\pi\rho)^2 R^3 \exp(-\pi\rho R^2), \quad R > 0, \ \rho > 0, \tag{2.1}$$

$$f(\alpha_1, \alpha_2) = \frac{8}{3\pi} \sin\alpha_1 \sin\alpha_2 \sin(\alpha_1 + \alpha_2), \quad \alpha_1 > 0, \ \alpha_2 > 0, \ \alpha_1 + \alpha_2 < \pi, \tag{2.2}$$

and let $B \subset R^3$ be an open set in (R, α_1, α_2) space. Miles' result can be stated as

$$\frac{\text{Number of D}\Delta\text{'s in } X(q) \text{ such that } (R,\alpha_1,\alpha_2) \in B}{\text{Number of D}\Delta\text{'s in } X(q)}$$

$$\xrightarrow{a.s.} \int_B g(R) f(\alpha_1,\alpha_2) dR d\alpha_1 d\alpha_2 \tag{2.3}$$

where DΔ stands for a Delaunay triangle.

The convergence is almost sure with respect to the Poisson process $\underset{\sim}{P}$ as $q \to \infty$. In practice we have a set of data with say N_o points in a region X, and we would like to know if those points could be the realisation of a Poisson process and hence if Miles' results are applicable to the finite situation. It seems plausible that a process closer to the data would be obtained, if a priori, we condition on there being $N = N_o$ points in the region. Of course now the process is no longer Poisson and it is natural to ask if the asymptotics of Miles are valid for this conditioned process.

For $N \geq N_o$, define a conditioned Poisson process in $X(q)$ of N points. As N varies these processes are defined on different probability spaces, so it is not possible to state "almost sure' results, however, we still have convergence in distribution to (2.3).

2.2 Verification

Miles' result says nothing about the distribution of $(\alpha_{1i},\alpha_{2i})$ $i = 1,\ldots,n$. We hypothesize that the $(\alpha_{1i},\alpha_{2i})$ are independently distributed as $f(\alpha_1,\alpha_2)$ given by (2.2) for large n. To investigate this density we conducted a simulation of 44 points uniformly distributed over the rectangle with sides in the ratio 1:2, which gave rise to 64 Delaunay triangles. These numerical values correspond as closely as possible to the Iowa data. Thiessen polygons were constructed using the program of Green and Sibson (1977). We will examine this hypothesis in §2.2.1 and §2.2.2 through the density function of A, which is twice the area, and through the marginal density of the minimal angles.

2.2.1. Verification through Area

For reasons given later, we look into the behaviour of the distribution of areas of Delaunay triangles of unit radius. The areas are given by

$$\tfrac{1}{2}[\sin 2\alpha_1 + \sin 2\alpha_2 - \sin 2(\alpha_1+\alpha_2)] = \tfrac{1}{2}A,\quad \alpha_1 > 0,\ \alpha_2 > 0,\ \alpha_1+\alpha_2 < \pi \qquad (2.4)$$

The distribution function of A, $H(A)$, from $f(\alpha_1,\alpha_2)$ given by (2.2), is found to be

$$H(A) = (3\pi)^{-1}\int_{\alpha=0}^{\pi}\{s(\alpha)-\alpha\}\sin\ 2\alpha\, d\alpha$$

$$-2(3\pi)^{-1}\int_{\alpha=0}^{\pi}\sin\alpha\sin s(\alpha)\, d\alpha,$$

where

$$0 < A < \frac{3\sqrt{3}}{2} \qquad (2.5)$$

and

$$s(\alpha) = \cos^{-1}\{\max(-1, \frac{\sin 2\alpha - A}{2\sin\alpha})\}.$$

We now test the hypothesis,

$H_o : A_1,\ldots,A_n$ are independent observations from $H(A)$ given by (2.5).

There will be some form of dependence on the A's, but it has to be seen whether or not this is negligible. Table II shows a comparison of the distribution of A for the simulated Poisson process of 44 points with the simulated distribution of A under Miles density, equation (2.2).

TABLE II

The distribution of A(=2×Area) for a simulated Poisson process of 44 points giving rise to 64 Delaunay triangles (observed) and for the simulated values from the Miles distribution given by (2.5), calculated from 10,000 simulations

Range of A	Observed frequency	Expected frequency	
0.0 -0.2	1	1.11	
0.2 -0.4	1	2.25	6.50
0.4 -0.6	0	3.14	
0.6 -0.8	4	3.67	8.04
0.8 -1.0	8	4.37	
1.0 -1.2	7	5.10	
1.2 -1.4	7	5.16	
1.4 -1.6	9	5.38	
1.6 -1.8	8	6.23	
1.8 -2.0	3	6.30	
2.0 -2.2	2	6.76	
2.2 -2.4	7	7.00	
2.4 -2.6	7	7.44	

Using the χ^2 goodness of fit criterion, it is found that $\chi^2 = 14.49$. The 5% value of $\chi_9^2 = 16.92$ and hence we accept the null hypothesis.

Further, it is found that for the simulated Poisson process the mean and the variance are $\bar{A} = 1.53$, $s_A^2 = .3665$. It can be shown from (2.2) that

$$\mu = E(A) = \frac{\pi}{2}, \quad \sigma^2 = \mathrm{var}(A) = \frac{35}{12} - \frac{\pi^2}{4}. \tag{2.6}$$

Under the null hypothesis for large n, $\bar{A} \sim N(\mu, \sigma^2/n)$. In fact $\sigma^{-1} n^{\frac{1}{2}} |\bar{A}-\mu| = .48$ and consequently we again accept the null hypothesis.

2.2.2. Verification through Marginal

Consider the order statistics $\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)}$ such that $\alpha_{(1)} < \alpha_{(2)} < \alpha_{(3)}$ and $\alpha_{(1)} + \alpha_{(2)} + \alpha_{(3)} = \pi$. It can easily be shown that the marginal density of $\alpha_{(1)}$, the minimum angle, becomes

$$f(\alpha_{(1)}) = (2/\pi)[(\pi - 3\alpha_{(1)}) \sin 2\alpha_{(1)} + \cos 2\alpha_{(1)} - \cos 4\alpha_{(1)}],$$

$$0 < \alpha_{(1)} < \frac{\pi}{3}. \tag{2.7}$$

We now test the hypothesis,

$H_o : \alpha_{(1)1}, \ldots, \alpha_{(1)n}$ are independent observations from (2.7).

Using the χ^2 goodness of fit criterion, $\chi^2 = 8.18$. The 5% point for $\chi^2_7 = 14.07$ and so again we accept the null hypothesis.

TABLE III (page 10)

The distribution of the minimum angle, $\alpha_{(1)}$, for a simulated Poisson process of 44 points giving rise to 64 Delaunay triangles (observed) and for the expected values for Mile's density given by (2.7)

Range of $\alpha_{(1)}$	Observed frequency	Expected frequency	
$0^o - 5^o$	0	.97	
$5^o - 10^o$	1	2.88	8.51
$10^o - 15^o$	3	4.66	
$15^o - 20^o$	8	6.22	
$20^o - 25^o$	13	7.43	
$25^o - 30^o$	9	8.19	
$30^o - 35^o$	9	8.40	
$35^o - 40^o$	6	8.02	
$40^o - 45^o$	7	7.03	
$45^o - 50^o$	4	5.49	
$50^o - 55^o$	4	3.49	10.18
$55^o - 60^o$	0	1.20	

For the simulated Poisson process $\bar{\alpha}_{(1)} = .53$. From (2.7), it can be shown that

$$\nu = E(\alpha_{(1)}) = \frac{27}{16\pi}\ ,\ \tau^2 = \mathrm{var}(\alpha_{(1)}) = \frac{\sqrt{3}}{\pi}\cdot\frac{27}{32} - \frac{1}{8} - \left(\frac{27}{16\pi}\right)^2 . \tag{2.8}$$

Under H_o, $\bar{\alpha}_{(1)} \sim N(\nu, \tau^2/n)$ and so we have $\tau^{-1} n^{\frac{1}{2}} |\bar{\alpha}_{(1)} - \nu| = .22$.

Hence we again accept the null hypothesis.

Boots (1974) has suggested using the marginal distribution of a random angle from a random Delaunay triangle for verifying whether the pattern is random. He uses Miles marginal density

$$f(\alpha) = 4\{(\pi-\alpha)\cos\alpha + \sin\alpha]\ \frac{\sin\alpha}{3\pi}\ ,\ 0 < \alpha < \pi\ , \tag{2.9}$$

and suggests randomly selecting a sub-sample of triangles and then choosing one angle at random from within each triangle. This procedure would of course throw away a substantial part of the available information.

To reach a firmer conclusion concerning the null hypothesis extensive simulations are necessary, however on the basis of the one simulation conducted, we have no grounds to reject H_o .

3. VON MISES ALTERNATIVE

For any triangle there is a unique circle, the circumcircle, which can be drawn to pass through all three vertices of the triangle. Using this fact, we can view the vertices of a particular Delaunay triangle to be observations on the circumdisc. We now make the following assumptions:

(1) Central Place Theory predicts equilateral Delaunay triangles. We propose to investigate the theory by considering the shape rather than the size of the triangles. Considering the size of the triangles to be unimportant, each triangle is scaled so that it is circumscribed by the unit circle, i.e. $R = 1$.

(11) With the angular observations thus formed, independence between triangles is assumed. We have already given some justification for this assumption in §2.

3.1. The von Mises Model

Consider $\theta_1, \theta_2, \theta_3$ independent von Mises variables with common concentration parameter κ , then θ_i is distributed as $M(\mu_i, \kappa)$ for $i=1,2,3$ and therefore the joint p.d.f. of θ_1, θ_2, θ_3 is

$$f(\theta_1,\theta_2,\theta_3) = \frac{1}{(2\pi)^3 I_o^3(\kappa)} e^{\kappa[\cos(\theta_1-\mu_1)+\cos(\theta_2-\mu_2)+\cos(\theta_3-\mu_3)]} ,$$

$$\kappa > 0 . \qquad (3.1)$$

If we take $\underset{\sim}{\mu}' = (0, 2\pi/3, 4\pi/3)$ it is obvious that as $\kappa \to \infty$, the triangles tend to become equilateral and therefore κ can be regarded as a measure of the degree of equilaterality.

3.2. von Mises Arcs

As the model stands, it cannot be directly applied to the problem as we do not know the individual marginal observations, only the angle differences. However the data in Table I can be viewed in terms of the 'arc lengths' $2\alpha_1$, $2\alpha_2$, $2\alpha_3$ on the unit circle. Therefore we investigate the distribution of these arcs which we denote by Φ_1, Φ_2, and Φ_3 such that $\Phi_1 + \Phi_2 + \Phi_3 = 2\pi$, and we can write Φ_1, Φ_2, Φ_3 in terms of $\theta_1, \theta_2, \theta_3$. There are two possibilities.

(i) If $\theta_1, \theta_2, \theta_3$ are in anticlockwise order

$$\Phi_1 = (\theta_2 - \theta_1) \bmod 2\pi, \quad \Phi_2 = (\theta_3 - \theta_2) \bmod 2\pi, \quad \Phi_3 = (\theta_1 - \theta_3) \bmod 2\pi$$

or

(ii) if $\theta_1, \theta_2, \theta_3$ are in clockwise order

$$\Phi_1 = (\theta_1 - \theta_2) \bmod 2\pi, \quad \Phi_2 = (\theta_2 - \theta_3) \bmod 2\pi, \quad \Phi_3 = (\theta_3 - \theta_1) \bmod 2\pi.$$

Consider (3.1). By changing variables from $(\theta_1, \theta_2, \theta_3)$ to $((\theta_2 - \theta_1) \bmod 2\pi,\ \theta_2,\ (\theta_3 - \theta_2) \bmod 2\pi)$ and integrating out θ_2, we arrive at the joint p.d.f. (degenerate) of Φ_1, Φ_2, Φ_3. The p.d.f. of (Φ_1, Φ_2) is found to be

$$\begin{aligned} f(\Phi_1, \Phi_2) = {} & [(2\pi)^2 I_o^3(\kappa)]^{-1} [I_o\{\kappa[3 + 2\cos(\Phi_1 - \mu_2 + \mu_1) + 2\cos(\Phi_2 - \mu_3 + \mu_1) \\ & + 2\cos(\Phi_3 - \mu_1 + \mu_3)]^{\frac{1}{2}}\} \\ & + I_o\{\kappa[3 + 2\cos(\Phi_1 - \mu_1 + \mu_2) + 2\cos(\Phi_2 - \mu_2 + \mu_3) + 2\cos(\Phi_3 - \mu_3 + \mu_1]^{\frac{1}{2}}\}], \end{aligned}$$

$$\Phi_3 = 2\pi - \Phi_1 - \Phi_2, \quad 0 < \Phi_2 \leq 2\pi - \Phi_1, \quad 0 < \Phi_1 \leq 2\pi . \tag{3.2}$$

For equilateral triangles, we take $\underset{\sim}{\mu}' = (0, 2\pi/3, 4\pi/3)$ and hence the model becomes

$$f(\Phi_1, \Phi_2) = [(2\pi)^2 I_o^3(\kappa)]^{-1}[I_o\{\kappa[3+2\cos(\Phi_1 - \frac{2\pi}{3}) + 2\cos(\Phi_2 - \frac{2\pi}{3}) + 2\cos(\Phi_3 - \frac{2\pi}{3})]^{\frac{1}{2}}\}$$

$$+ I_o\{\kappa[3+2\cos(\Phi_1 + \frac{2\pi}{3}) + 2\cos(\Phi_2 + \frac{2\pi}{3}) + 2\cos(\Phi_3 + \frac{2\pi}{3})]^{\frac{1}{2}}\}] ,$$

$$\Phi_3 = 2\pi - \Phi_1 - \Phi_2, \; 0 < \Phi_2 \leq 2\pi - \Phi_1, \; 0 < \Phi_1 \leq 2\pi . \quad (3.3)$$

Particular Cases

For $\kappa = 0$, the density collapses to become

$$f(\Phi_1, \Phi_2) = 1/(2\pi^2), \; 0 < \Phi_2 \leq 2\pi - \Phi_1, \; 0 < \Phi_1 \leq 2\pi , \quad (3.4)$$

which is what one would expect from the construction. For small κ we can approximate the density by expanding the Bessel functions and ignoring $O(\kappa^4)$ and higher powers of κ which gives

$$f(\Phi_1, \Phi_2) = (2\pi^2)^{-1}\{1 - \frac{\kappa^2}{4} \sum_{j=1}^{3} \cos\Phi_j\} ,$$

$$\Phi_3 = 2\pi - \Phi_1 - \Phi_2, \; 0 < \Phi_2 \leq 2\pi - \Phi_1, \; 0 < \Phi_1 \leq 2\pi . \quad (3.5)$$

For large κ, $\Phi_i \to \frac{2\pi}{3} + \delta_i$ where δ_i is so small that we can neglect $O(\delta_i^3)$. We have therefore $\sum_{i=1}^{3} \delta_i = 0$. Defining $\bar{\delta} = \frac{1}{3} \sum_{i=1}^{3} \delta_i^2$ and neglecting $o(\kappa^{-1})$ and smaller powers of κ, (3.3) becomes

$$f(\delta_1,\delta_2) = \frac{\kappa}{2\pi\sqrt{3}}\, e^{-\frac{1}{2}\kappa\bar{\delta}}\ , \quad \frac{-2\pi}{3} < \delta_2 \leq \frac{4\pi}{3} - \delta_1\ , \quad \frac{-2\pi}{3} < \delta_1 \leq \frac{4\pi}{3}\ .$$

Hence (Φ_1,Φ_2) has a bivariate normal distribution with $\mu_1 = \mu_2 = (2\pi)/3$, $\sigma_1^2 = \sigma_2^2 = 2/\kappa$ and $\rho = -1/2$. This same result could have been found by taking the normal approximation to the von Mises density in (3.1) and using the fact that the difference of two normal variables is again normal.

We now investigate the behaviour of the distribution of A and $\Phi_{(1)}$ from this model for varying κ. Since the exact form of the densities is unknown, we turned to simulations of the model. The method of simulation employed was that of the Rejection Procedure. For each value of κ, $\theta_{1i},\theta_{2i},\theta_{3i}$ were simulated independently and the corresponding values of A_i and $\Phi_{(1)i}$ were calculated for $i=1,\ldots,10{,}000$. The results are presented in Tables IV and V (p.15).

Table IV shows the density of A changing from an almost monotonic decreasing function at $\kappa=0$, to an almost monotonic increasing function at $\kappa=2.8$. Increasing κ still further, results in the density becoming heavily concentrated around equilaterality and being almost entirely contained in the upper half of the region. In Table V a similar process is occurring, but here the mode is not given by the maximum value of $\Phi_{(1)}$.

Identification with Miles' Density

The question arises whether the von Mises density is related to Miles' density for some κ. To this end, consider the area of a particular triangle which is given by

TABLE IV

10,000 simulations of A for the von Mises model for different κ

Range of A	$\kappa = 0$	$\kappa = 2.8$	$\kappa = 12$
0.0 - 0.2	2124	325	0
0.2 - 0.4	1150	318	0
0.4 - 0.6	892	348	0
0.6 - 0.8	802	372	3
0.8 - 1.0	757	464	2
1.0 - 1.2	699	547	8
1.2 - 1.4	552	612	16
1.4 - 1.6	593	689	43
1.6 - 1.8	525	890	146
1.8 - 2.0	525	1025	382
2.0 - 2.2	502	1283	961
2.2 - 2.4	471	1480	2427
2.4 - 2.6	408	1647	6012

TABLE V

10,000 simulations of $\phi_{(1)}$ for the von Mises model for different κ

Range of ϕ_1	$\kappa = 0$	$\kappa = 2.8$	$\kappa = 12$
$0^\circ - 10^\circ$	1904	426	0
$10^\circ - 20^\circ$	1442	424	0
$20^\circ - 30^\circ$	1318	515	4
$30^\circ - 40^\circ$	1126	656	13
$40^\circ - 50^\circ$	1082	813	33
$50^\circ - 60^\circ$	909	1036	123
$60^\circ - 70^\circ$	778	1153	382
$70^\circ - 80^\circ$	608	1372	806
$80^\circ - 90^\circ$	507	1357	1698
$90^\circ - 100^\circ$	346	1181	1594
$100^\circ - 110^\circ$	202	788	2931
$110^\circ - 120^\circ$	78	279	1416

$$\text{Area} = \tfrac{1}{2}A = \tfrac{1}{2}|\sin(\theta_1-\theta_2)+\sin(\theta_2-\theta_3)+\sin(\theta_3-\theta_1)|,$$

$$0 < \theta_1,\theta_2,\theta_3 \le 2\pi \qquad (3.6\ )$$

Area has a certain intuitive appeal in that for a given circumcircle, the triangle with the maximum area is an equilateral triangle. The modulus sign makes A awkward for analytical purposes, so we consider instead A^2 which has the same property of being a maximum for equilateral triangles. Hence we identify $E(A^2)$ for both densities and it is found that κ_o, the value of κ corresponding to Miles' density, should satisfy

$$\frac{35}{12} = \frac{3}{2} + \frac{3}{4}A_2^2(\kappa_o) + 3A_1^2(\kappa_o)A_2(\kappa_o) + \frac{3}{2}A_1^2(\kappa_o) \qquad (3.7)$$

where $A_r(\kappa_o) = I_r(\kappa_o)/I_o(\kappa_o)$.

The solution of (3.7) gives $\kappa_o = 2.20$.

We now investigate how close this approximation is in fact. This is done through the distribution of A and $\Phi_{(1)}$ for the Miles' density and the von Mises model. Since the exact forms of the distributions are not known, except for $\Phi_{(1)}$ under Miles' density, (2.7), most of the comparison is done through similation. As before the method employed is the Rejection Procedure. The results are shown in Figs. 2 and 3.

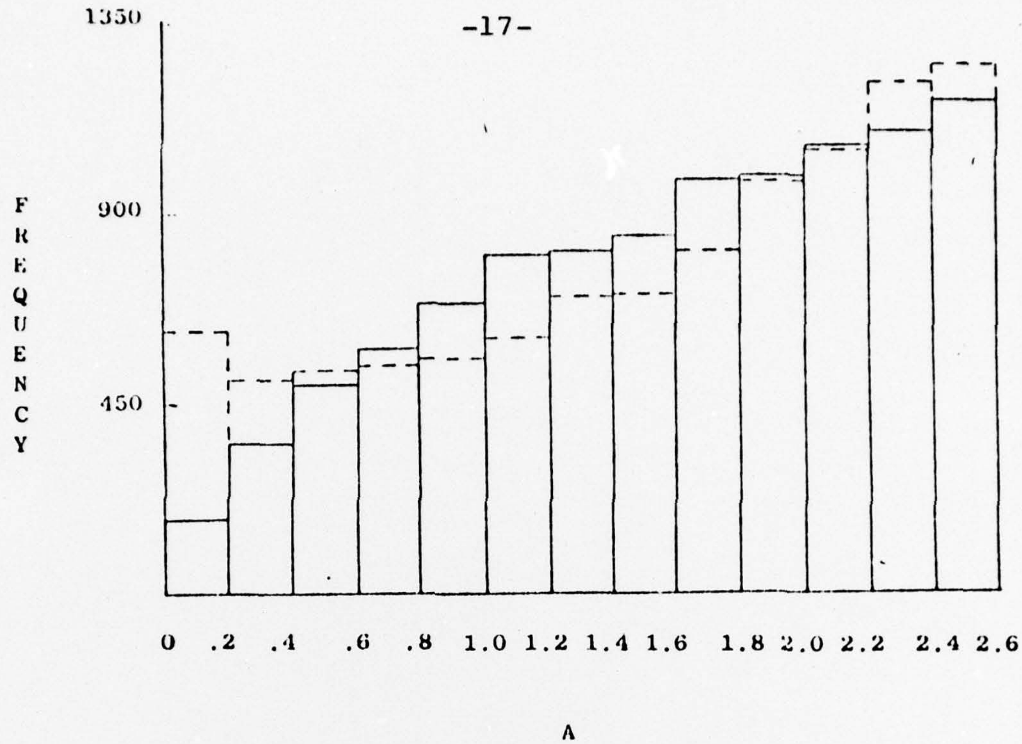

FIG. 2

Comparison between Miles' density (–) and von Mises (---) $\kappa=\kappa_o$ for A, given by 10,000 simulations

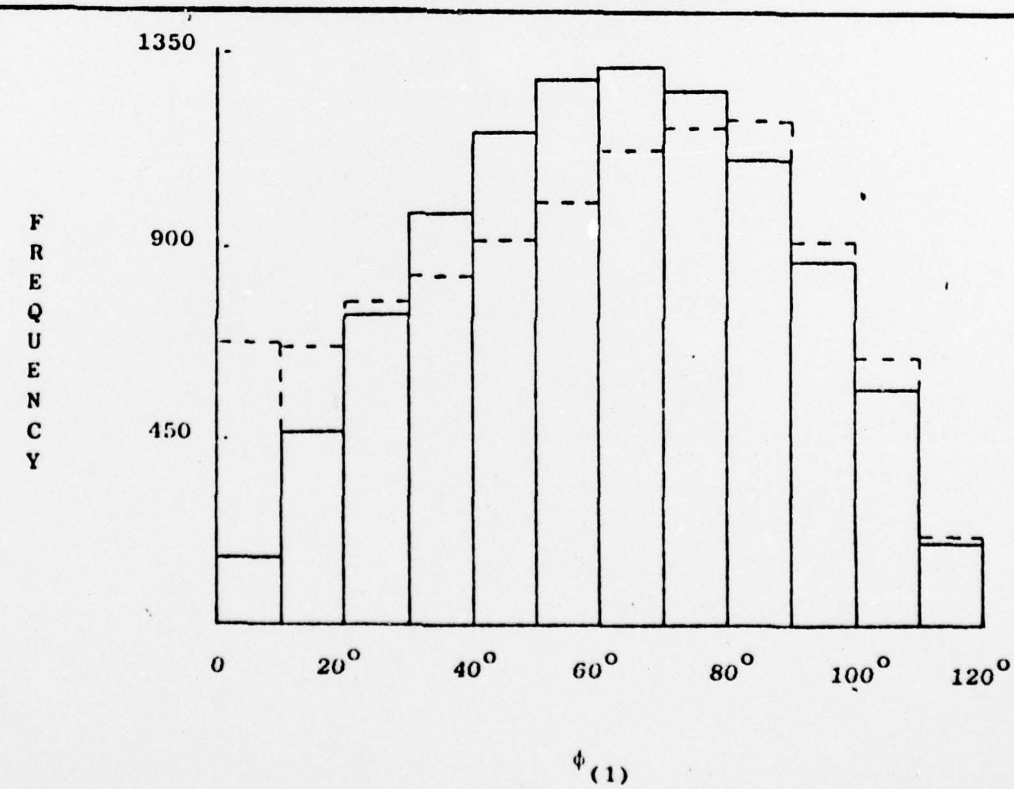

FIG. 3

Comparison of Miles' density (–) and von Mises (---) for $\kappa=\kappa_o$ for $\phi_{(1)}$. von Mises frequency is from 10,000 simulations, whilst Miles' density is the expected frequency for 10,000 observations

As the Figures show, the two distributions agree fairly well for both A and $\Phi_{(1)}$, except in the lowest region where the von Mises model predicts a much greater number of small triangles. This particular point is discussed in the next section.

3.4. Appraisal of the von Mises Model

Under the usual circumstances, the test for randomness with the von Mises density is $H_o : \kappa = 0 \vee H_1 : \kappa > 0$, however this is not applicable here. If the asymptotic approximations of Miles are valid for n as small as 44, which the simulation described in Table II would appear to confirm, the test becomes of the form $H_o : \kappa = \kappa_o \vee H_1 : \kappa > \kappa_o$ where $\kappa_o = 2.20$.

It has to be strongly emphasized that this approach is not described through a spatial process and in that sense, it has to be admitted that the procedure is 'ad hoc'. For large κ the model has certain good features (e.g. sensible m.l.e.'s) and it is certainly a viable working alternative, especially, once having rejected the hypothesis of randomness. For $\kappa = \kappa_o$ it can be thought of as similar to Miles, but for small κ, $\kappa < \kappa_o$, the density appears to be meaningless in the spatial content and so can be said to be non-spatial. This occurs due to the relatively large probability of colinearity under the von Mises model.

Under the Miles density, equation (2.2), $f(\alpha_1, \alpha_2)$ disappears when the angles are colinear, whereas since $I_o(0) = 1$, (3.3) is still equal to a positive constant, and so is bounded away from zero. As κ increases this constant tends to zero, but for $\kappa<\kappa_o$ it is quite significant.

In an actual realisation of a spatial process for a moderate number of points, for the three points of a Delaunay triangle to be almost colinear, the circumdisc of the triangle would be very large. The alternative definition of a Delaunay triangle is that 3 points form a Delaunay triangle iff their circumdisc is empty, and so given 3 points practically colinear, by the very definition of a Delaunay triangle, it is most unlikely that the 3 points would form such a triangle.

It would be nice if we could arrive at a von Mises type density through a non-Poisson process, but this does seem to be out of our reach at the present time, if it is indeed at all possible.

4. ESTIMATION FOR THE VON MISES-TYPE MODEL

It is reasonable to assume that the μ's are known and for which we take $\underset{\sim}{\mu}' = (0, 2\pi/3, 4\pi/3)$. Consequently we need to estimate the concentration parameter κ. Two methods of estimation are given.

4.1. Method of Moments

Defining θ_1, θ_2, θ_3 to be independent von Mises variables with $\underset{\sim}{\mu}' = (0, 2\pi/3, 4\pi/3)$ and common concentration parameter κ, it is found, as in (3.7) that

$$E(A^2) = \frac{3}{2} + \frac{3}{4}A_2^2(\kappa) + 3A_1^2(\kappa)A_2(\kappa) + \frac{3}{2}A_1^2(\kappa)\ , \qquad (4.1)$$

where $A_i(\kappa)$ is defined in §3. We take as our estimate of $E(A^2)$,

$$\hat{E}(A^2) = \frac{1}{n}\sum_{i=1}^{n} A_i^2\ , \qquad (4.2)$$

where $A_1, \ldots, A_n$ are the observed areas of the triangles. On equating (4.1) and (4.2) we can obtain the moment estimate $\bar{\kappa}$ of κ. Unfortunately the equation can only be solved numerically.

Example

From Table I we can calculate (4.2) for the Iowa data, which gives $n^{-1}\sum_{i=1}^{n} A_i^2 = 3.4073$. Substituting into (4.2) and solving numerically, it is found that $\bar{\kappa} = 2.80$.

4.2. <u>Maximum Likelihood Estimation</u>

Let $(\Phi_{1i}, \Phi_{2i}, \Phi_{3i})$, $i=1,\ldots,n$ be the observed arc lengths. From (3.8) it can be easily shown that the maximum likelihood estimate of κ, $\hat{\kappa}$, is given by

$$\frac{I_1(\hat{\kappa})}{I_o(\hat{\kappa})} = \frac{1}{3n} \sum_{i=1}^{n} \frac{u_i I_1(\hat{\kappa}u_i) + v_i I_1(\hat{\kappa}v_i)}{I_o(\hat{\kappa}u_i) + I_o(\hat{\kappa}v_i)} \tag{4.3}$$

where $u_i = [3+2\cos(\Phi_{1i}-\frac{2\pi}{3}) + 2\cos(\Phi_{2i}-\frac{2\pi}{3}) + 2\cos(\Phi_{3i}-\frac{2\pi}{3})]^{\frac{1}{2}}$

and $v_i = [3+2\cos(\Phi_{1i}+\frac{2\pi}{3}) + 2\cos(\Phi_{2i}+\frac{2\pi}{3}) + 2\cos(\Phi_{3i}+\frac{2\pi}{3})]^{\frac{1}{2}}$.

By the methods of §3.2 we can find approximations to the maximum likelihood equation for small and large values of $\hat{\kappa}$. For small $\hat{\kappa}$, by expanding the Bessel functions and neglecting $O(\hat{\kappa}^4)$, (4.3) becomes

$$\hat{\kappa}^2 = -4\sum_{i=1}^{n} c_i \Big/ \sum_{i=1}^{n} c_i^2 \tag{4.4}$$

where $c_i = \sum_{j=1}^{3} \cos \Phi_{ji}$.

For large $\hat{\kappa}$, define

$$\delta_{ji} = \Phi_{ji} - \frac{2\pi}{3}, \quad \bar{\delta} = \frac{1}{3n}\sum_{i=1}^{n}\sum_{j=1}^{3} \delta_{ji}^2, \quad j=1,2,3; \quad i=1,\ldots,n.$$

Following §3.2, (4.3) then reduces to $\hat{\kappa} = 2/\bar{\delta}$. But also we have

$$\bar{A} = \frac{1}{n}\sum_{i=1}^{n} A_i \simeq \frac{3\sqrt{3}}{4}(2 - \bar{\delta}). \quad \text{Hence,}$$

$$\hat{\kappa} \simeq (3\sqrt{3})/(3\sqrt{3} - 2\bar{A}). \tag{4.5}$$

Consequently, $\hat{\kappa}$ depends only on $\bar{A}$ for large κ.

Example

Consider $\Phi_{ji} = 2\alpha_{ji}$, $j=1,\ldots,63$; $i=1,2,3$ where the α's are given in Table I for the Iowa data. Substituting directly into (4.3) and solving numerically gives $\hat{\kappa} = 2.83$, which is in good agreement with our estimate of κ, $\tilde{\kappa} = 2.80$, given by the method of moments.

Now let us consider the large $\hat{\kappa}$ approximation. It is found that $\bar{A} = 1.7105$. Substituting this value of $\bar{A}$ into (4.5) gives the m.l.e. of κ as 2.93, which is only slightly larger than $\hat{\kappa} = 2.83$.

5. ANALYSIS OF THE IOWA DATA

We can now analyze the Iowa data given in Table I.

5.1. Models

We first enquire whether the data could be described by a von Mises type model with $\underset{\sim}{\mu}' = (0, 2\pi/3, 4\pi/3)$. This we do through the marginal distribution of A and $\Phi_{(1)}$. We note from §4.2 that the m.l.e. of κ is $\hat{\kappa} = 2.8$. Tables VI and VII give a comparison between the Iowa data and the expected frequencies for the von Mises $\kappa = 2.8$ for A and $\Phi_{(1)}$, given by 10,000 simulations.

TABLE VI

Comparison of A for the observed Iowa data and those values expected from 10,000 simulations for the von Mises model with $\kappa = 2.8$ (E_1) and for Miles' density (E_2)

Range of A	Observed frequency	E_1	E_2
0.0 - 0.6	1	6.24	6.42
0.6 - 1.0	7	5.26	7.93
1.0 - 1.4	14	7.31	10.11
1.4 - 1.8	7	9.95	11.44
1.8 - 2.2	13	14.54	12.87
2.2 - 2.6	21	19.70	14.23

TABLE VII

Comparison of $\phi_{(1)}$ values between the observed Iowa data, those values expected for the von Mises model with $\kappa = 2.8$ (E_1) given by 10,000 simulations and those values expected for Miles' density (E_2)

Range of $\phi_{(1)}$	Observed frequency	E_1	E_2
$0^\circ - 20^\circ$	0	5.35	3.80
$20^\circ - 40^\circ$	8	7.37	10.71
$40^\circ - 60^\circ$	14	11.65	15.37
$60^\circ - 80^\circ$	16	15.90	16.17
$80^\circ - 100^\circ$	18	15.99	12.32
$100^\circ - 170^\circ$	7	6.71	4.62

Using a χ^2 goodness of fit test for A and $\Phi_{(1)}$, and testing against $\chi_4^2(5\%) = 9.49$ and $\chi_4^2(1\%) = 13.28$ in both cases, it was found for A, $\chi^2 = 12.22$ and for $\Phi_{(1)}$, $\chi^2 = 6.13$. Hence we just reject the fit for A at the 5% level, but accept the fit of the model for $\Phi_{(1)}$.

The final columns in Tables VI and VII give the corresponding frequencies under Miles' model from simulation. Applying the test for Miles' density produces the mildly surprising result that we would accept the fit of this distribution to the Iowa data. For A we have $\chi^2 = 10.46$ against $\chi_5^2(5\%) = 11.07$ and for $\Phi_{(1)}$, $\chi^2 = 6.87$ against $\chi_3^2(5\%) = 7.81$. The best procedure would have been to look into the bivariate distribution of (α_1, α_2), but the sample size of $n = 63$ is anyhow too small to draw clear-cut conclusions. However, as we have seen that for $\kappa_o = 2.2$ the von Mises type model is adequate. We will study the problem more fully from this angle in the next section.

5.2. Central Place Theory

We wish to investigate whether the Delaunay triangles of the Iowa data can be regarded as having arisen as the result of a Poisson process in the plane or whether they occurred through the action of a more regular force. Under the von Mises-type model of §3, this problem can be looked upon as testing $H_o : \kappa = \kappa_o$ against $H_1 : \kappa > \kappa_o$ where $\underset{\sim}{\mu}' = (0, 2\pi/3, 4\pi/3)$. If λ is the likelihood ratio for the problem, it is found that

$$-2 \log \lambda = 6n \log (\kappa_o/\kappa_1) + 2\sum_{i=1}^{n} \log ([I_o(\kappa_1 u_i) + I_o(\kappa_1 v_i)]/$$

$$[I_o(\kappa_o u_i) + I_o(\kappa_o v_i)]).$$

where $\kappa_o = 2.20$, $\kappa_1 = 2.83$ and u_i and v_i are given by (4.3). Substituting the values from Table I gives $-2 \log \lambda = 4.08$.

$\chi_1^2(5\%) = 3.84$ and hence we reject $\kappa = \kappa_o$ at the 5% level. With the evidence before us, there seems justifiable grounds at the 5% level to suppose that the Iowa data is a result of a process that is more regular than the Poisson and hence there is some evidence that Central Place Theory holds for this particular region.

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Key words: Central Places, Delaunay triangle, Dirichlet cell, Miles' density, Random pattern, von Mises Distribution.

ABSTRACT

Central Place Theory predicts a regular spatial pattern in the plane and we observe that the Delaunay triangles will be equilateral under the theory. However, when the pattern is 'random', Miles (1970) has given the asymptotic p.d.f. of the interior angles of a random Delaunay triangle. We propose a von Mises-type model with a concentration parameter κ; the larger the value of κ, the closer we are to the Central Place

Theory. We show that the model can be approximated to the Miles' density for some value of κ. We provide the moment and maximum likelihood estimators of κ, and it is recognized that the areas of the Delaunay triangles play an important role. We construct a test of departure from the random pattern with the alternative of Central Place Theory. As a numerical example, we analyze 44 Central Places in Iowa where we find some evidence for the validity of Central Place Theory in that particular region.

RÉSUMÉ

La Théorie de la Position Centrale envisage une répartition régulière des implantations urbaines dans la plaine. Dans cette théorie on s'attend à ce que les triangles de Delaunay de la répartition soient équilatéraux. Cependant, quand la repartition est "au hasard", Miles (1970) a donné la distribution asymptotique pour les angles intérieurs d'un triangle de Delaunay pris au hasard. On nous propose un modèle de von Mises avec un paramètre de concentration "κ". La validité de la théorie de la position centrale augmente avec "κ". On démontre que l'on peut donner une approximation du modèle par la densité de Miles pour uxe valeur donnée de "κ". On trouve des évaluations de "κ" en utilisant les moments et l'équation du maximum de vraisemblance. L'importance des surfaces des triangles de Delaunay est reconnue, et une expérience entre la "priseau hasard" et la théorie de la position centrale est au cours de dévelopement. A titre d'exemple numérique, on analyse 44 positions centrales dans l'Iowa où l'on trouve des preuves de la validité de la théorie dans cette région particulière.

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* are provided