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A Note on Optimal Replacement Policies for Some Shock Models.

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A Note on Optimal Replacement Policies for Some Shock Models.

Emad El-Neweihi and Peter Purdue
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Abstract:

Optimal replacement policies for systems subject to randomly occurring shocks are obtained. The models discussed include cumulative and maximum damage models with emphasis on systems with random thresholds.

1. Introduction:

The problem of the optimal replacement of an item which fails by degradation caused by cumulative damage has attracted the attention of a number of authors. A good summary of this literature is given in Taylor (1975) and the references contained therein. Taylor discusses a cumulative damage model where shocks of a random size arrive according to a Poisson Process. The item under attack can fail only at a shock arrival time and the probability of surviving a shock which brings the total damage level to x is $h(x)$. It is more costly to replace a failed item than one which is still working and the problem is to find an optimal replacement strategy which balances these costs and results in a minimum long-run average cost per unit time. The solution is given by Taylor and, in the special case where $h(x) = 1$ for $0 \leq x \leq K$ and 0 otherwise (the fixed threshold case), Nakagawa (1976) independently determined another form of the solution using a simpler approach but allowing the arrival process to be renewal. Feldman (1976) extended the model to the case of a Markov Renewal shock process. Feldman and Nakagawa approach the problem in the same spirit. Our goal is to extend the existing results to the case of a random threshold; the method of attack is in the same vein as that of Feldman and Nakagawa.

2. Maximum Shock Threshold Model.

We consider a device which is subject to failure. Shocks occur to the device according to a renewal process $\{X_i;\ i\geq 1\}$ with inter-renewal distribution F. The i-th shock causes an amount of damage W_i where the random variables $\{W_i;\ i\geq 1\}$ are independent with common distribution function G. The device fails whenever an arriving shock exceeds the level K. When the device fails it is immediately replaced by a similar device. However we have open to us the possibility of replacing the device before it fails. A cost C_1 is incurred whenever a failed item is replaced whereas a smaller cost C_2 is incurred if a device is replaced before failure. The policy sought is one which minimizes the long-run expected cost per unit time over an infinite horizon. The class of policies considered, what Feldman calls control limit policies, are those which replace the device whenever a shock of magnitude greater than some fixed level k occurs ($k \leq K$). If a cycle denotes the time span between two consecutive replacements then, as is well know, the long-run average cost per unit time is

$$= E[\text{cost/cycle}]/E[\text{length of the cycle}]$$

(a) The fixed threshold case.

In this section the threshold level K is some fixed, non-negative number. By "policy k" we mean the policy which replaces a working device whenever a shock of level greater than k occurs. Let C(k) denote the total expected cost per cycle using policy k. It is not very hard to show that

$$C(k) = \lambda\{C_1[1-G(K)] + C_2[G(K)-G(k)]\} \text{ where } \frac{1}{\lambda} = E(X_i).$$

Then as k increases, C(k) decreases and the optimal policy is to run the device to breakdown each time. This is clearly the intuitive solution to the problem.

(b) Random Threshold Model.

In this section we assume that the threshold K is a random variable with distribution function H. All of the other model assumptions are as before (w.l.o.g. we can assume $\lambda = 1$).

Letting $M(x) = \sum_{j=1}^{\infty} [G(x)]^j$ we have,

$$(1) \quad C(k) = \frac{C_1 H(k) + C_1[1+M(k)] \int_k^\infty [1-G(x)dH(x) + C_2[1+M(k)] \int_k^\infty [G(x)-G(k)dH(x)}{1 + \int_0^\infty [1-H(x)]dM(x)}$$

To derive this we simply condition with respect to K.

Now let,

$$L(k) = H(k)\{\int_0^k [\frac{1+M(x)}{H(k)}]dH(x) \int_k^\infty [\frac{\bar{G}(x)dH(x)}{\bar{H}(x)}] - 1\}$$

where $\bar{H}(x) = 1-H(x)$, $\bar{G}(x) = 1-G(x)$.

Lemma (2.1). $L(k) \leq 0$ for all $k \geq 0$ and

$L(0) = 0$.

Proof: $L(0) = 0$ is obvious. We have,

$$\int_0^k [1+M(x)]dH(x) \int_k^\infty [1-G(x)]dH(x)$$

$$= \lim_{b\to\infty} \int_0^k [1+M(x)]dH(x) \int_k^b \bar{G}(x)dH(x)$$

$$= \lim_{t\to\infty} [1+m(\xi_1)]H(k)\ \bar{G}(\xi_2)[H(b)-H(k)] \text{ where } 0\leq\xi_1\leq k,\ k\leq\xi_2\leq b.$$

$$= \lim_{t\to\infty} \frac{H(k)}{1-G(\xi_1)} \cdot [1-G(\xi_2)][H(b)-H(k)] \leq H(k)\ \bar{H}(k), \quad b \geq k.$$

the result follows at once.

We will now assume that all our distribution functions are absolutely continuous.

Theorem (2.2)

For the maximum shock model under the assumption of a random threshold, the optimal policy is to replace only on failure.

Proof:

Differentiating (1) and setting the derivitive equal to zero we get,

$$L(k) = \frac{C_2}{C_1 - C_2} .$$

But by assumption $C_1 > C_2$ and by the above lemma $L(k) \leq 0$. Hence $C'(k) < 0$ for all $k \geq 0$ and the optimal policy is to run the device to breakdown.

3. Cumulative Damage Model with a Random Threshold.

Again we assume that shocks arrive according to a renewal process with inter-renewal distribution F having mean 1. The shocks are independent random variables, W_i, which have a common distribution function G.

Let N(t) denote the number of shocks which arrive in (0,t]; the cumulative damage process, W(t), is given by,

$$W(t) = \sum_{i=1}^{N(t)} W_i, \quad t > 0.$$

The device fails only when W(t) exceeds a certain level K, which is assumed to be a random variable with distribution function H. The time to failure is

$$T_f = \inf\{t: \ W(t) > K\}.$$

Using policy k we have that the replacement time of a working device is

$$T_k = \inf\{t: \ W(t) > k\}.$$

Then the length of a cycle, T_C is given by

$$T_C = \min(T_k, T_f).$$

The cost structure is the same as in section 2 and our goal is to find a policy k which will minimize the long-run average cost per unit time.

Now, by conditioning on K, we have that, using policy k,

$$E(\text{cost/cycle}) = C_2 + (C_1-C_2)\{H(k) + \int_k^\infty [\bar{G}(\beta) + \int_0^k \bar{G}(\beta-u)dM(u)]dH(\beta)\}$$

$$\text{where } M(x) = \sum_{j=1}^{\infty} G^{*j}(x)$$

and $G^{*j}(x)$ is the j-fold convolution of G with itself. We also have,

$$E(T_C) = [1+M(k)]\bar{H}(k) + \int_0^k [1+M(\beta)]dH(\beta).$$

Finally, if C(k) denotes the expected cost per cycle,

$$C(k) = \frac{C_2 + (C_1-C_2)\{H(k) + A(k,0) + \int_0^k A(k,u)dM(u)\}}{1 + \int_0^k \bar{H}(x)dM(x)} \tag{2}$$

where,

$$A(k,u) = \int_k^\infty \bar{G}(x-u)dH(x).$$

Two extreme cases are when k=0 and ∞. Then,

$$C(0) = C_1\{1-E[G(K)]\} + C_2\, E[G(K)]$$

$$C(\infty) = \frac{C_1}{1+EM(K)}$$

To obtain the optimal policy we introduce the function L(k) defined by,

$$L(k) = \frac{A(k,k)[1+\int_0^k \bar{H}(x)dM(x)] - \bar{H}(k)[H(k)+A(k,0)+\int_0^k A(k,u)dM(u)}{\bar{H}(k)}$$

Again we will assume that all distribution functions are absolutely continuous. The next lemma describes some properties of L(k).

Lemma (3.1) L(0)=0. Under the assumption that H is Decreasing Failure Rate or that the failure rate is bounded by some finite constant,

$$\lim_{k\to\infty} L(k) = EM(K).$$

Proof: Clearly L(0)=0. We can write

$$L(k) = [1+\int_0^k \bar{H}(x)dM(x)]\,\frac{A(k,k)}{\bar{H}(k)} - [H(k)+A(k,0)+\int_0^k A(k,u)dM(u)].$$

The second term on the right goes to 1 as $k \to \infty$. Also,

$$\lim[1+\int_0^k \bar{H}(x)dM(x)] = E[1+M(K)],$$

Finally

$$\lim_{k\to\infty} \frac{A(k,k)}{\bar{H}(k)} = 1 - \lim_{k\to\infty} \int_k^\infty \frac{G(x-k)h(x)dx}{1-H(k)}$$

where $h(x)$ is the density function corresponding to H. But

$$\int_k^\infty \frac{G(x-k)h(x)dx}{1-H(k)} \leq \int_k^\infty G(x-k)[\frac{h(x)}{\bar{H}(x)}]dx$$

and, under the assumption of the lemma

$$\lim_{k\to\infty} \frac{A(k,k)}{\bar{H}(k)} = 1.$$

The result now follows.

Theorem (3.2)

Under the assumptions of Lemma 3.1 and the additional assumption that $L(k)$ is monotonically increasing then

(i) If $EM(K) > C_2/C_1-C_2$, there exists an optimum policy k^* which satisfies

$$L(k) = \frac{C_2}{C_1-C_2}$$

(ii) If $EM(K) \leq \frac{C_2}{C_1-C_2}$ then the optimal policy is to replace only after failure.

Proof: Differentiating (2) with respect to k, setting equal to 0 and using Lemma 3.1 gives the desired result. We next consider an example where the shock process is Poisson and the threshold is Exponential. Due to the lack of memory property of the exponential we would expect the optimal policy in this case to replace only on failure; this is shown to be so.

Example:

Suppose now that,

$$H(x) = \begin{cases} 0 & x < 0 \\ 1-e^{-\lambda x} & x \geq 0 \end{cases}$$

$$G(x) = \begin{cases} 0 & x < 0 \\ 1-e^{-\lambda' x} & x \geq 0 \end{cases}$$

Then we can evaluate A(k,u) explicitly

$$A(k,u) = \left(\frac{\lambda}{\lambda+\lambda'}\right) e^{\lambda'(u-k)}\, e^{-\lambda k}$$

and we then get,

$$L(k) = 0 \qquad k \geq 0 .$$

So, we see immeidately that C(k) is decreasing for all k and here the optimal policy is to replace only on failure. We note in conclusion that Nakagawa's results can be obtained by letting H be degenerate at some level K.

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