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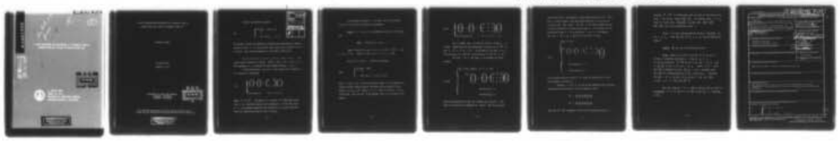

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A FINITE PROCEDURE FOR DETERMINING IF A QUADRATIC FORM IS BOUNDED BELOW ON A CLOSED POLYHEDRAL CONVEX SET

B. CURTIS EAVES

AUGUST 16, 1977

DEPARTMENT OF OPERATIONS RESEARCH
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A FINITE PROCEDURE FOR DETERMINING IF A QUADRATIC FORM IS BOUNDED BELOW ON A CLOSED POLYHEDRAL CONVEX SET

TECHNICAL REPORT

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AUGUST 16, 1977

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Consider the quadratic program

$$\text{(1)} \qquad \begin{cases} V \triangleq \inf\limits_{x}: \; x \cdot Qx + x \cdot q \\ \\ \text{s/t:} \; Ax \le a \qquad x \ge 0 \end{cases}$$

We describe a finite but inefficient procedure for determining the optimal objective value, V, of the program, and in particular, whether or not V is finite. This task was suggested to the author by David Gale.

Let Q be $n \times n$, q $n \times 1$, A $m \times n$, and a $m \times 1$. We assume that the program is feasible. Define $\mathcal{Q}(x)$ to be $x \cdot Qx + x \cdot q$. The expression $x \cdot u$ indicates the inner product between x and u.

A Kuhn-Tucker point (u,v,x,y) of the program (1) is defined to be a solution to the system

$$\text{(2)} \qquad \begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} q \\ a \end{pmatrix} + \begin{pmatrix} Q' & A^T \\ -A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \\ (u,v,x,y) \ge 0 \qquad u \cdot x = v \cdot y = 0 \end{cases}$$

where $Q' = Q + Q^T$. Of course, if (u,v,x,y) is a Kuhn-Tucker point, then x is a feasible solution to the program (1). On the other hand, if x is an optimal solution to the program (1), it can be shown that there is a Kuhn-Tucker point of form (u,v,x,y).

To determine the value of V we shall need the following result from the folklore of quadratic programming.

Lemma 1: If (u,v,x,y) is a Kuhn-Tucker point of the program then

$$\mathcal{Q}(x) = (1/2)\,(x \cdot q - y \cdot a)$$

Proof: Using (2) we have $0 = x \cdot q + x \cdot Q'x + x \cdot A^T y$, and $0 = y \cdot a - y \cdot Ax$. Hence $2x \cdot Qx + 2x \cdot q = x \cdot q - y \cdot a$ ⊠

Now for $k = 0,1,2,\ldots$ consider the programs

$$(3,k) \quad \begin{cases} V_k \triangleq \min\limits_{x}: \ \mathcal{Q}(x) \\ \\ \text{s/t:} \ Ax \leq a \quad x \geq 0 \quad ex \leq k \end{cases}$$

where $e = (1,1,\ldots 1)$. For all sufficiently large k the program has a compact nonempty feasible region, and hence, has an optimal solution. Clearly $V_k \geq V_{k+1}$ and $\lim V_k = V$ as k tends to infinity. A Kuhn-Tucker point (u,v,w,x,y,z) of the program $(3,k)$ is a solution to the system

$$(4,k)\quad \begin{cases} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} q \\ a \\ k \end{pmatrix} + \begin{pmatrix} Q' & A^T & e^T \\ -A & 0 & 0 \\ -e & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ (u,v,w,x,y,z) \geq 0 \qquad u \cdot x = v \cdot y = w \cdot z = 0 \end{cases}$$

Let $\ell = m+n+1$ and J be the set $\{1,\ell+1\} \times \{2,\ell+2\} \times \ldots \times \{\ell,\ell+\ell\}$. Observe that for any nonnegative (u,v,w,x,y,z) in $R^{2\ell}$ we have $u \cdot x = v \cdot y = w \cdot z = 0$, if and only if, for some α in J $(u,v,w,x,y,z)_\alpha = 0$, that is, $(u,v,w,x,y,z)_i = 0$ for all i in α .

For each α in J and large k we consider the linear program

$$(5,\alpha,k)\quad \begin{cases} V_k^\alpha \triangleq \min\text{: } (1/2)(x \cdot q - y \cdot a - kz) \\ \text{s/t: } \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} q \\ a \\ k \end{pmatrix} + \begin{pmatrix} Q' & A^T & e^T \\ -A & 0 & 0 \\ -e & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ (u,v,w,x,y,z) \geq 0 \\ (u,v,w,x,y,z)_\alpha = 0 \end{cases}$$

where the minimization is over the variables (u,v,w,x,y,z) . Note that if (u,v,w,x,y,z) is feasible for $(5,\alpha,k)$, then (u,v,w,x,y,z)

solves (4,k) and is, consequently, a Kuhn-Tucker point of (3,k). Therefore, in view of Lemma 1, for any optimal solutions (u,v,w,x,y,z) to $(5,\alpha,k)$ we have $V_k^\alpha = \mathcal{Q}(x)$. For each α the linear program $(5,\alpha,k)$ is either feasible for all sufficiently large k or infeasible for all sufficiently large k ; let us partition $J = J_F \cup J_I$ accordingly. Note that α is in J_F if and only if the linear program

$$
(6,\alpha)\quad \begin{cases} \text{sup:} \quad w + ez \\[2ex] \text{s/t:} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} q \\ a \end{pmatrix} + \begin{pmatrix} Q' & A^T & e^T \\ -A & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\[2ex] \qquad (u,v,w,x,y,z) \geq 0 \\[2ex] \qquad (u,v,w,x,y,z)_\alpha = 0 \end{cases}
$$

has an optimal objective value of $+\infty$ where the maximization is over the variables (u,v,w,x,y,z) .

Assuming α is in J_F we can use the simplex method treating k parametrically to generate in a finite number of steps

$$S^\alpha = (u_1^\alpha, v_1^\alpha, w_1^\alpha, x_1^\alpha, y_1^\alpha, z_1^\alpha)$$

$$T^\alpha = (u_2^\alpha, v_2^\alpha, w_2^\alpha, x_2^\alpha, y_2^\alpha, z_2^\alpha)$$

such that $S^\alpha + kT^\alpha$ optimizes $(5,\alpha,k)$ for all sufficiently large k .

Therefore $S^\alpha + kT^\alpha$ is a Kuhn-Tucker point of (3,k) for all sufficiently large k and we have $\mathcal{Q}(x_1^\alpha + kx_2^\alpha) = V_k^\alpha$. Furthermore, given α in J_F there is a fixed triple $(C_1^\alpha, C_2^\alpha, C_3^\alpha)$ such that $V_k^\alpha = \mathcal{Q}(x_1^\alpha + kx_2^\alpha) = C_1^\alpha k^2 + C_2^\alpha k + C_3^\alpha$ for all sufficiently large k .

Select β so as to lexicographically minimize $(C_1^\alpha, C_2^\alpha, C_3^\alpha)$ over all α in J_F . Then $V_k^\beta \leq V_k^\alpha$ for all α in J_F and all sufficiently large k .

<u>Lemma 2</u>: $V_k^\beta = V_k$ for all sufficiently large k .

<u>Proof</u>: Choose $\bar{k}$ so that a) for all $k \geq \bar{k}$ $(5,\alpha,k)$ is feasible or infeasible according to α being in J_F or J_I , b) $(5,\alpha,k)$ optimized by $S^\alpha + kT^\alpha$ for all $k \geq \bar{k}$ and α in J_F , and c) $V_k^\beta \leq V_k^\alpha$ for all $k \geq \bar{k}$ and α in J_F . Assume $k \geq \bar{k}$. Since $x_1^\beta + kx_2^\beta$ is feasible to (3,k), $V_k^\beta \geq V_k$. Let x optimize (3,k), then there is a Kuhn-Tucker point of form (u,v,w,x,y,z) . Therefore, for some α in J_F we have $(u,v,w,x,y,z)_\alpha = 0$ and $V_k = \mathcal{Q}(x) = 1/2(x \cdot q - y \cdot a - kz) \geq V_k^\alpha \geq V_k^\beta$ ⊠

Hence V_k^β tends to V as k tends to infinity and the result is established; $V = -\infty$ if $C_1^\beta < 0$ or if $C_1^\beta = 0$ and $C_2^\beta < 0$, otherwise, $V = C_3^\beta$.

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