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REPRESENTING SONAR SIGNALS AS REAL AND COMPLEX FUNCTIONS, (U)
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REPRESENTING SONAR SIGNALS AS REAL AND COMPLEX FUNCTIONS

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FOREWORD

This memorandum represents a portion of work being done in support of SF 101 03 16, Task 8132 NUWC Problem Number E111 by the Sonar Signal Recognition Division.

The information contained is considered useful to those groups engaged in analysis and simulation using wideband signals.

Appendix A was developed by L. R. Weill and was previously presented with experimental verification in "Analysis of Active-Sonar Signals (U)," NUWC TP 7, Sept 1967, CONFIDENTIAL by D. G. Olson and L. R. Weill.

Only limited distribution is contemplated.

INTRODUCTION

In many circumstances an investigator wishes to use digital computers for simulating sonar receivers. Quite often the proposed implementation of beam-formers or matched-filter receivers will differ from analytically-derived filter characteristics. In order to evaluate different implementations of receivers with respect to each other and the theoretically ideal version, an investigator must use a valid representation of a sonar transmit signal in the computer simulation.

Several references ~~(1-4)~~ have described some of the differences between narrowband signal representation and wideband signal representation. However, much of the signal processing literature is directed to the radar problem which is usually a narrowband signal case. Therefore, this paper will collect the comments of several authors to demonstrate the differences of narrowband and wideband assumptions and their effects on the concepts of complex signal notation, complex modulation, and signal envelope.

REAL SIGNAL REPRESENTATION

Any transmittable radar or sonar signal belongs to a class of functions describable as

$$s(t) = a(t) \cos [2\pi f_0 t + \varphi(t)] \tag{1}$$

Whether this equation provides a particularly useful representation, however, depends on the problem. In equation (1) $a(t)$ is an amplitude modulation function, f_0 is a carrier frequency and $\varphi(t)$ is a time-varying phase modulation function. The argument of the cosine is the signal phase and the derivative of the signal phase is defined as the instantaneous frequency

$$f_i = f_0 + \frac{1}{2\pi} \frac{d\varphi(t)}{dt} \tag{2}$$

For a given real signal $s(t)$, the choice of f_0 is arbitrary since a suitable $\varphi(t)$ can be chosen to yield the given value of $s(t)$. There is, however, a physically meaningful way of defining the carrier frequency of a modulated signal.

Before proceeding, the use of the Fourier transform should be introduced where

$$s(t) = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi ft}\, df \qquad \equiv \text{signal} \tag{3}$$

$$S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft}\, dt \qquad \equiv \text{frequency spectrum} \tag{4}$$

and

$$E = \int_{-\infty}^{\infty} [s(t)]^2\, dt = \int_{-\infty}^{\infty} |S(f)|^2\, df \qquad \equiv \text{signal energy} \tag{5}$$

The frequency spectrum given by the Fourier integral is a complex function of f, and extends over all positive and negative frequencies. Since the transmitted signal $s(t)$ is real, $s(t) = s^*(t)$ and $S(-f) = S^*(f)$, where the asterisk indicates the complex conjugate. Now the carrier frequency f_0 can be defined as the first moment of one-half the energy spectrum

$$f_0 = \int_0^{\infty} f \left| \frac{S(f)}{\sqrt{\frac{E}{2}}} \right|^2 df \tag{6}$$

Since $S(f)$ extends over positive and negative frequencies, it is called the "double-sided" spectrum. The spectrum of a real signal is even in amplitude and odd in phase.

where $S(f) = |S(f)|\, e^{j\theta(f)}$ (7)

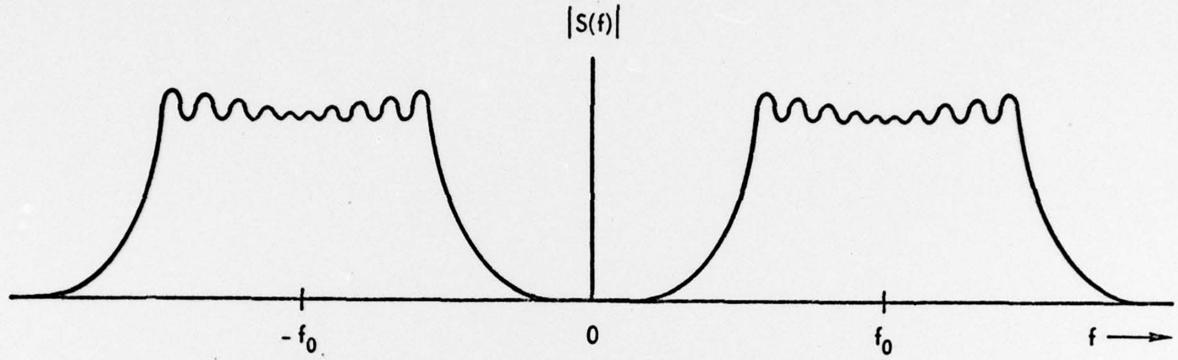
with $|S(f)| \equiv$ amplitude of the frequency spectrum (8)

and $\theta(f) \equiv$ phase of the frequency spectrum (9)

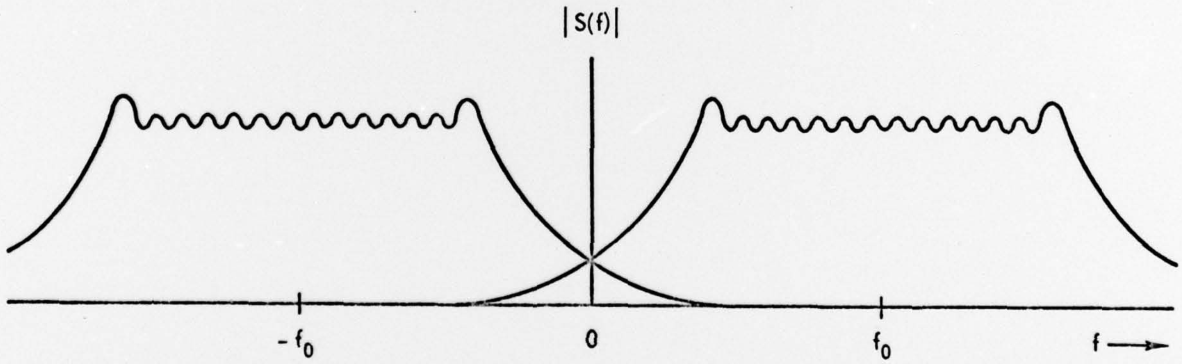
Because one-half the energy is contained in the positive spectrum, the carrier frequency was defined using $E/2$ in equation (6). A pictorial representation of a narrowband signal is given in figure 1. While the first signal amplitude spectrum in figure 1 represents a narrowband signal with no frequencies from the right-half of the amplitude spectrum spilling over into the negative frequency portion, the second amplitude spectrum represents the wide band case. All of the previous equations apply for either narrowband or wideband signals. Now the following approximations will show the utility of equation (1) for representing narrowband signals. Using equation (5) with (1) gives

$$E = \int_{-\infty}^{\infty} \{a(t) \cos [2\pi f_0 t + \varphi(t)]\}^2\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} a^2(t)\, dt + \frac{1}{2}\int_{-\infty}^{\infty} a^2(t) \cos 2[2\pi f_0 t + \varphi(t)]\, dt \tag{10}$$



Amplitude Spectra vs Frequency For a Narrowband Signal



Amplitude Spectra vs Frequency For a Wideband Signal

Figure 1.

In the case of a narrowband signal, both $a(t)$ and $\varphi(t)$ vary slowly in comparison to $\cos 2\pi f_0 t$. The second integral on the right hand side of (10) has a product term formed from the two functions as shown in figure 2 such that the integral of the product is virtually zero. Thus, as a consequence of narrow bandwidth

$$E \approx \frac{1}{2}\int_{-\infty}^{\infty} a^2(t)\,dt \tag{11}$$

This approximation greatly simplifies calculations as it permits the substitution of the amplitude modulation function for the signal.

COMPLEX SIGNAL REPRESENTATION

As a further simplification for theoretical analysis involving linear operations, the real signal is often expressed as the real part of a complex waveform $\psi(t)$

$$s(t) = \mathrm{Re}\,\{\psi(t)\} \tag{12}$$

where $\mathrm{Re}\{\ \}$ means the real part of $\{\ \}$. The analyst replaces $s(t)$ with $\psi(t)$ for these linear operations and then takes the real part of the calculation for his answer. First, rewrite equation (3) as

$$s(t) = \int_0^{\infty} S(f)\,e^{j2\pi ft}\,df + \int_{-\infty}^{0} S(f)\,e^{j2\pi ft}\,df \tag{13}$$

and using $S(-f) = S^*(f)$ for real signals

$$s(t) = \int_0^{\infty} S(f)\,e^{j2\pi ft}\,df + \int_0^{\infty} S^*(f)\,e^{-j2\pi ft}\,df \tag{14}$$

which becomes

$$s(t) = 2\,\mathrm{Re}\left\{\int_0^{\infty} S(f)\,e^{j2\pi ft}\,df\right\} \tag{15}$$

Since the negative frequencies simply mirror the positive frequencies in complex conjugate form for real signals, the negative frequencies can be omitted. Equation (16) shows that $s(t)$ can be regarded as the real part of a complex signal whose frequency spectrum is twice that of the real signal for positive frequencies and zero for negative frequencies. Therefore, using $\Psi(f)$ as the complex signal frequency spectrum

$$\begin{aligned} \Psi(f) &= 2S(f) \qquad f > 0 \\ &\ 0 \qquad\quad\ \ f < 0 \end{aligned} \tag{16}$$

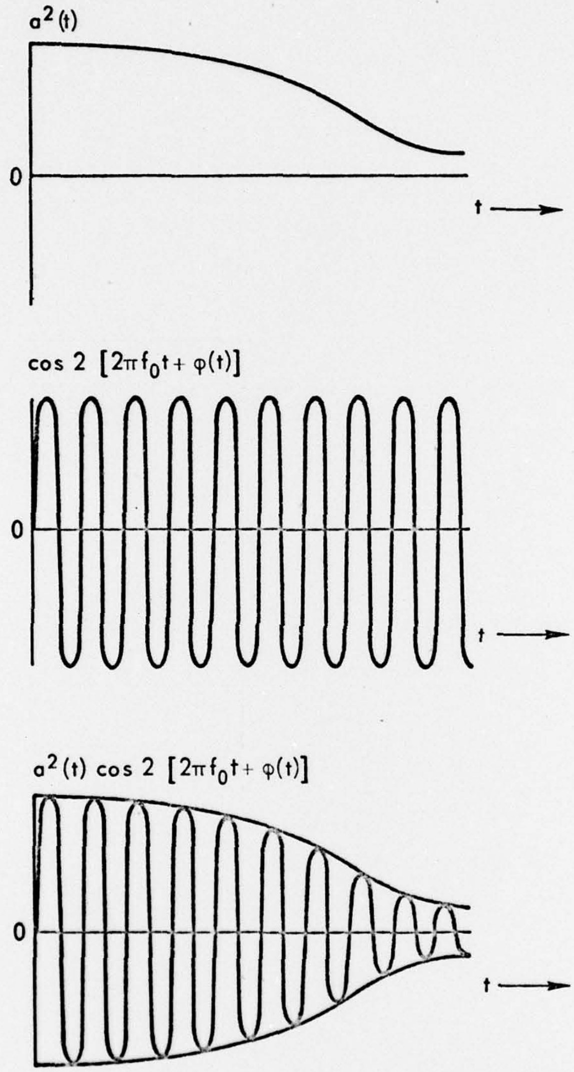


Figure 2. The integral of the product $a^2(t) \cos 2\,[2\pi f_0\, t + \varphi(t)]$ for a slowly varying $a(t)$ and $\varphi(t)$ will nearly equal zero as the contributions above and below the axis will practically cancel each other.

and $\Psi(f)$ can be called a "single-sided" spectrum.

CONSTRUCTING THE ANALYTIC SIGNAL- or COMPLEX WAVEFORM

While the frequency spectrum of the desired complex signal, $\Psi(f)$, has been stated, an analyst still needs to find a suitable $\psi(t)$ corresponding to a given $s(t)$. Consider now the characteristics of the Hilbert transform defined as

$$\hat{s}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{(t-\tau)} \, d\tau \qquad (17)$$

where the principal value is intended, and $s(t)$ is a real-valued function on $-\infty < t < +\infty$. References (5-6) show several characteristics of this transform. From equation (17) the Hilbert transform is seen to be a convolution of $s(t)$ with the function $(\pi t)^{-1}$, written as $s(t) * (\pi t)^{-1}$. Remembering the property of the Fourier transform where

$$g(t) * h(t) \Longleftrightarrow G(f)\, H(f) \qquad (18)$$

or convolution in the time domain transforms to multiplication in the frequency domain, and using the Fourier transform of $(\pi t)^{-1}$ as $-j\,\mathrm{sgn}f$, the spectrum of $\hat{s}(t)$ is

$$\begin{cases} -j\,S(f) & f > 0 \\ 0 & f = 0 \\ j\,S(f) & f < 0 \end{cases} \qquad (19)$$

If we choose $\psi(t)$ as the pre-envelope discussed in reference (6) where

$$\psi(t) = s(t) + j\hat{s}(t) \qquad (20)$$

the Fourier transform

$$\begin{aligned} \Psi(f) &= S(f) + j \begin{cases} -jS(f) & f > 0 \\ 0 & f = 0 \\ jS(f) & f < 0 \end{cases} \\ &= \begin{cases} 2S(f) & f > 0 \\ S(f) & f = 0 \\ 0 & f < 0 \end{cases} \end{aligned} \qquad (21)$$

Thus, by finding $\hat{s}(t)$ from $s(t)$ and writing

$$\psi(t) = s(t) + j\,\hat{s}(t) \tag{20}$$

a complex signal can be formed from a real signal with the frequency domain behavior shown in equation (16).

Now the Hilbert transform of cos t is sin t. Reference (5) lists other transform pairs, but defines the Hilbert transform as

$$\hat{s}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{\tau - t}\, d\tau \tag{22}$$

which is the negative of equation (17). Therefore, the transforms and formulas in reference (5) must be multiplied by (-1) for equivalence with the expressions in this paper. Thus, considering a narrowband signal with a slowly varying phase modulation function $\varphi(t)$,

$$\mathcal{F}_{Hi}\{\cos[2\pi f_0 t + \varphi(t)]\} = \sin[2\pi f_0 t + \varphi(t)] \tag{23}$$

where $\mathcal{F}_{Hi}\{\ \}$ indicates the Hilbert transformation. For this narrowband case the complex waveform corresponding to s(t) becomes to a close approximation

$$\psi(t) \approx a(t)\cos[2\pi f_0 t + \varphi(t)] + j\,a(t)\sin[2\pi f_0 t + \varphi(t)] \tag{24}$$

Stating equation (24) as an equality and simplifying the notation by use of the exponential, we get

$$\begin{aligned} \psi(t) &= a(t)\, e^{j[2\pi f_0 t + \varphi(t)]} \\ &= u(t)\, e^{j2\pi f_0 t} \end{aligned} \tag{25}$$

where u(t) is a complex modulation term

$$u(t) = a(t)\, e^{j\varphi(t)} \tag{26}$$

consisting of the real amplitude modulation function and a complex phase function. The evelope of the signal is

$$|\psi(t)| = \sqrt{s^2(t) + \hat{s}^2(t)} \tag{27}$$

which gives $\quad |\psi(t)| = a(t) \qquad (28)$

for this narrowband case. While equations (20) and (27) are perfectly general and follow reference (6), the results shown in equations (25) and (28) are valid approximations only for narrowband signals.

While

$$\mathcal{F}_{Hi}\{\cos(2\pi f_0 t + \varphi\} = \sin(2\pi f_0 t + \varphi) \tag{29}$$

for a particular value of φ,

$$\mathcal{F}_{Hi}\{\cos[2\pi f_0 t + \varphi(t)] \neq \sin[2\pi f_0 t + \varphi(t)] \tag{30}$$

in general and the approximation of equation (23) only holds in the narrowband case. For wideband signals the Hilbert transform could be obtained by expressing

$$s(t) = a(t)\cos[2\pi f_0 t + \varphi(t)] \tag{1}$$

as a Fourier series and then transforming each term by the addition theorem [reference (5)]

$$\mathcal{F}_{Hi}\{h(t) + g(t)\} = \mathcal{F}_{Hi}\{h(t)\} + \mathcal{F}_{Hi}\{g(t)\} \tag{31}$$

In summary, the representation of any sonar or radar transmit waveform by the real signal

$$s(t) = a(t)\cos[2\pi f_0 t + \varphi(t)] \tag{1}$$

is completely general. In the case of narrowband signals - which implies a slowly varying $a(t)$ and $\varphi(t)$ - the complex signal $\psi(t)$ can be readily written as

$$\psi(t) = u(t)e^{j2\pi f_0 t} \tag{25}$$

and the evelope as

$$|\psi(t)| = a(t) \tag{28}$$

which is the amplitude modulation function. In the wideband case, finding the Hilbert transform $s(t)$ may be more difficult than simply using the real signal representation of equation (1) for analysis purposes. For computer simulation purposes which use time-series samples of the sonar signal, there is no need to consider the complex signal form anyway. Appendix A will outline a direct computational method of reconstructing the envelope defined as

$$R(t) = \sqrt{s^2(t) + \hat{s}^2(t)} \tag{32}$$

from time-series samples of the real waveform $s(t)$

USING COMPLEX NOTATION FOR WIDEBAND SIGNALS

The preceding section discussed the difficulties of finding a complex signal corresponding to a particular wideband real signal specified in the time domain. If an analyst wishes to study the sonar problem in the frequency domain, however, he can define the transmit signal as a modulated carrier. The approach is discussed in references (3) and (4).

Using the Fourier transform relations of equations (3) and (4) and the consequence of $s(t)$ being real as shown in equation (15), we can write

$$s(t) = 2\,\mathrm{Re}\left\{\int_0^{\infty} S(f)\, e^{j2\pi ft} df\right\}$$

$$= 2\,\mathrm{Re}\left\{e^{j2\pi f_0 t} \int_{-f_0}^{\infty} S(f_0 + f)\, e^{j2\pi ft} df\right\}. \tag{33}$$

If we define a function

$$V(t) = 2 \int_{-f_0}^{\infty} S(f_0 + f)\, e^{j2\pi ft} df \tag{34}$$

we can then express the real signal as

$$s(t) = 2\,\mathrm{Re}\{V(t)\, e^{j2\pi f_0 t}\} \tag{35}$$

which consists of a complex modulation function and the carrier frequency term in the exponential. The carrier frequency term f_0 is arbitrary, but the function $V(t)$ becomes unique once f_0 is chosen. Naturally, f_0 can be chosen as shown in equation (6).

To further examine the properties of $V(t)$, consider the absolute value after a change of variable

$$f_0 + f = f' \tag{36}$$

$$df = df' \tag{37}$$

such that

$$V(t) = 2 \int_0^{\infty} S(f') e^{j2\pi(f' - f_0)t} df' \tag{38}$$

$$= 2 e^{-j2\pi f_0 t} \int_0^{\infty} S(f') e^{j2\pi f' t} df' \tag{39}$$

so

$$\left| V(t) \right| = \left| 2 \int_0^{\infty} S(f') e^{j2\pi f' t} df' \right| \tag{40}$$

Now consider the method of computing the envelope using the Hilbert transform where

$$|\psi(t)| = |s(t) + j\hat{s}(t)| \tag{41}$$

Using the addition theorems of the Fourier and Hilbert transformations

$$|\psi(t)| = |\mathcal{F}^{-1}\{S(f)\} + j\mathcal{F}^{-1}\{-j\,\mathrm{sgn} f S(f)\}|$$

$$= \left| \int_{-\infty}^{\infty} S(f)\, e^{j2\pi ft} df + j \int_{-\infty}^{\infty} -j\,\mathrm{sgn} f S(f)\, e^{j2\pi ft} df \right|$$

$$= \left| \int_{0}^{\infty} S(f)\, e^{j2\pi ft} df + \int_{0}^{\infty} S(f)\, e^{j2\pi ft} df \right.$$

$$\left. + \int_{-\infty}^{0} S(f)\, e^{j2\pi ft} df - \int_{-\infty}^{0} S(f)\, e^{j2\pi ft} df \right|$$

$$= \left| 2 \int_{0}^{\infty} S(f)\, e^{j2\pi ft} df \right| \tag{42}$$

where $\mathcal{F}^{-1}\{-j\,\mathrm{sgn} f S(f)\} = \hat{s}(t)$, since $-j\,\mathrm{sgn} f S(f)$ is the Fourior transform of $\hat{s}(t)$ as discussed previously. Clearly, equations (40) and (42) are equivalent. Therefore, $|V(t)|$ represents the envelope of the signal and

$$|V(t)| = |s(t) + j\hat{s}(t)| \tag{43}$$

Thus, if an investigator wishes to express a wideband signal as a modulated carrier, he must first select a carrier frequency f_0 and then compute

$$V(t) \equiv 2 \int_{-f_0}^{\infty} S(f_0 + f)\, e^{j2\pi ft} df \tag{34}$$

which is related to the real signal as

$$s(t) = 2\,\mathrm{Re}\{V(t)\, e^{j2\pi f_0 t}\} \tag{35}$$

This process does not require the use of the Hilbert transform of the real signal. Only the investigator with a particular problem can select the most advantageous wideband signal representation

$$s(t) = a(t) \cos [2\pi f_0 t + \varphi(t)] \tag{1}$$

or

$$s(t) = 2\,\mathrm{Re}\{V(t)\,e^{j2\pi f_0 t}\} \tag{36}$$

In general, the relationships cannot be easily converted, although the past development shows their equivalence.

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APPENDIX

If a real analog signal is properly sampled, the signal waveform can be reconstructed. If the transmitted signal $s(t)$ is a real function defined on $-\infty < t < \infty$ and

(1) $s(t)$ has a Fourier transform $S(f)$ such that

$$\sup \{ f \mid S(f) \neq 0 \} = f_H$$

is finite

(2) the sequence of sample values

$$--- f(-n\Delta t), ---, f(-\Delta t), f(0), f(\Delta t), --- f(n\Delta t)$$

is known for all integers n

(3) $$\Delta t < \frac{1}{2 f_H}$$

then $s(t)$ can be determined exactly from the sample values alone at all t where f is continuous. Specifically, the reconstructed signal $\tilde{s}(t) = s(t)$

when
$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} s(n\Delta t) \frac{\sin \pi \left(\frac{t}{\Delta t} - n\right)}{\pi \left(\frac{t}{\Delta t} - n\right)} \quad \text{(A-1)}$$

where
$$\frac{\sin \pi \left(\frac{t}{\Delta t} - n\right)}{\pi \left(\frac{t}{\Delta t} - n\right)} = 1 \quad \text{(A-2)}$$

if
$$\frac{t}{\Delta t} - n = 0 \quad \text{(A-3)}$$

This theorem is proven in reference (7). This treats the base band signal, or a signal where the highest frequency f_H is known rather than the actual signal bandwidth $W = f_H - f_L$. The choice of sampling interval Δt when both f_H and f_L are known, and when the first two conditions leading to (A-1) are satisfied in addition to the conditions

(1) inf $\left\{ f \,\middle|\, \begin{matrix} S(f) = 0 \\ f > 0 \end{matrix} \right\} = f_L > 0$

(2) the sampling interval Δt satisfies one of the following:

$$\frac{K-1}{2f_L} < \Delta t < \frac{K}{2f_H}$$

$$\frac{K-2}{2f_L} < \Delta t < \frac{K-1}{2f_H}$$

$$\vdots$$

$$\frac{1}{2f_L} < \Delta t < \frac{1}{f_H}$$

$$0 < \Delta t < \frac{1}{2f_H}$$

for K the largest integer such that $K < \dfrac{f_L}{f_H - f_L}$

then $\tilde{s}(t) = s(t)$ where

$$\tilde{s}(t) = \Delta t \sum_{n=-\infty}^{\infty} s(n\Delta t) \left\{ 2f_H \frac{\sin 2\pi f_H (t-n\Delta t)}{2\pi f_H (t-n\Delta t)} - 2f_L \frac{\sin 2\pi f_L (t-n\Delta t)}{2\pi f_L (t-n\Delta t)} \right\} \tag{A-4}$$

as shown in reference (8).

Now consider the Hilbert transform of $\tilde{s}(t)$. The Hilbert transform of

$$\sin x \Rightarrow \frac{\cos x - 1}{x} \qquad \text{by equation (22)}$$

or

$$\Rightarrow \frac{1-\cos x}{x} \qquad \text{by equation (17)}$$

since

$$\frac{1-\cos x}{x} = \frac{2\sin^2\left(\frac{x}{2}\right)}{x} \tag{A-5}$$

we can write the Hilbert transform of the two reconstruction formulas as

$$\hat{\tilde{s}}(t) = \sum_{n=-\infty}^{\infty} s(n\Delta t) \left[\frac{2\sin^2\left[\frac{\pi}{2}\left(\frac{t}{\Delta t} - n\right)\right]}{\pi\left(\frac{t}{\Delta t} - n\right)} \right] \tag{A-6}$$

and

$$\hat{\hat{s}}(t) = \Delta t \sum_{n=-\infty}^{\infty} s(n\Delta t) \left\{ 4f_H \frac{\sin^2[\pi f_H (t-n\Delta t)]}{2\pi f_H (t-n\Delta t)} \right.$$

$$\left. - 4f_L \frac{\sin^2[\pi f_L (t-n\Delta t)]}{2\pi f_L (t-n\Delta t)} \right\} \tag{A-7}$$

Using a sampling interval (a) such that

$$t = ga,\ g = 0,\ 1,\ 2,\ ---,\ N\text{-}1$$

a simplified expression for computing the envelope

$$R(ga) = |S(ga)|$$

$$= \sqrt{s^2(ga) + \left\{ \sum_{n=0}^{n-1} s(na) \left[\frac{2\sin^2\left[\frac{\pi}{2}(g-n)\right]}{\pi(g-n)} \right] \right\}^2}$$

$$= \sqrt{s^2(ga) + \frac{4}{\pi^2} \left\{ \sum_{n=0}^{n=1} \frac{s(na)}{(g-n)} \left[\frac{(-1)^{(g-n)} - 1}{2} \right] \right\}^2} \tag{A-8}$$

$$\sin^2\left[\frac{\pi}{2}(g-n)\right] = \begin{cases} 0 & \text{for } (g-n) \text{ even} \\ 1 & \text{for } (g-n) \text{ odd} \end{cases}$$

and for the other case of known bandwidth

$$R(ga) = \sqrt{s^2(ga) + \left[\Delta t \sum_{n=0}^{n-1} s(na) \left\{ 2f_H \frac{\sin^2 \pi(\Delta t\, f_H)(g-n)}{\pi(\Delta t\, f_H)(g-n)} - 2f_L \frac{\sin^2(\Delta t\, f_L)(g-n)}{\pi(\Delta t\, f_L)(g-n)} \right\} \right]^2} \tag{A-9}$$