AD-A051 480 FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OHIO F/G 12/1
SYSTEMS OF HYPERBOLIC DIFFERENTIAL EQUATIONS OF THE FIRST ORDER--ETC(U)
JUL 77 W HAACK, G HELLWIG

UNCLASSIFIED FTD-ID(RS)T-0926-77 NL

1 OF 1
AD
A051480

END
DATE
FILMED
5-78
DDC

FTD-ID(RS)T-0926-77

FOREIGN TECHNOLOGY DIVISION

SYSTEMS OF HYPERBOLIC DIFFERENTIAL EQUATIONS
OF THE FIRST ORDER. II

by

Wolfgnag Haack and Guenter Hellwig

FTD -ID(RS)T-0926-77

EDITED TRANSLATION

FTD-ID(RS)T-0926-77 12 July 1977

MICROFICHE NR: FTD-77-C-001291

SYSTEMS OF HYPERBOLIC DIFFERENTIAL EQUATIONS OF THE FIRST ORDER. II

By: Wolfgang Haack and Guenter Hellwig

English pages: 35

Source: Mathematische Zeitschrift, Berlin-Charlottenburg, Vol. 53, Nr. 4, 1950, pp. 340-356

Country of origin: W. Germany
Translated by: Gale M. Weisenbarger
Requester: AFWL/IN
Approved for public release; distribution unlimited.

THIS TRANSLATION IS A RENDITION OF THE ORIGINAL FOREIGN TEXT WITHOUT ANY ANALYTICAL OR EDITORIAL COMMENT. STATEMENTS OR THEORIES ADVOCATED OR IMPLIED ARE THOSE OF THE SOURCE AND DO NOT NECESSARILY REFLECT THE POSITION OR OPINION OF THE FOREIGN TECHNOLOGY DIVISION.	PREPARED BY: TRANSLATION DIVISION FOREIGN TECHNOLOGY DIVISION WP-AFB, OHIO.

FTD-ID (RS) T-0926-77 - statement A per Mr. Hughes, FTD Stinfo Officer- 20 Mar 78

0926,gw

SYSTEMS OF HYPERBOLIC DIFFERENTIAL EQUATIONS OF THE FIRST ORDER. II.

Wolfgang Haack and Guenter Hellwig

Berlin-Charlottenburg

INTRODUCTION

The following studies are a direct continuation of our first report on this topic which appeared recently in this journal. In Section I of our first report we proceeded from a hyperbolic system of linear differential equations:

$$(1)\quad \begin{aligned} a^k u_k + b^k v_k + c u + d v + e = 0, \\ \bar{a}^k u_k + \bar{b}^k v_k + \bar{c} u + \bar{d} v + \bar{e} = 0. \end{aligned}$$

(summed over k = 1, 2)

Through invariant derivatives in the characteristic directions

(2) $$\dot{x}^i = a^i; \quad \dot{x}^i = \bar{a}^i$$

and appropriate linear combinations

(3) $$U = u + \varrho v; \quad V = u + \sigma v$$

the system (1) was reduced to a normal form. In sections II and III existence theorems were proved for the functions U, V and u, v.

Below we have continued the numbering of the sections and formulas and in section IV we shall deal with several integral theorems. Two Pfaffian forms ω_1, ω_2 (I-13) belong to the vector system (2). Now if U, V, respectively, u, v are solutions of (1) and W, Z are arbitrary functions, then through the extrinsic derivative of the Pfaffian form $WV\omega_1 + ZU\omega_2$ one arrives at integral relations for the functions U, V. The requirement that certain terms in the integrals vanish leads to the adjoint system of differential equations for W and Z.

In a manner similar to that in the Riemann method which is known from the theory of a partial differential equation of the second order, one can solve the Cauchy initial value problem of system (1) using quadratures if the characteristic initial value problem is solved for the adjoint system. The result is the integral representation of the sought functions (IV. 28, 31). Section IV concludes with the proof that the classical Riemann method is contained in these integral representations as a special case.

In section V the considerations will be transformed for quasilinear systems. Systems also result in the characteristic form which was investigated by R. Courant et al. (see the first report, footnote [2]). We show that here also a normal form is always present and indicate when, through transposition of the dependent and independent variables a "linearization" is possible and outline briefly a difference method for the approximate solution of various initial value and boundary value problems. In conclusion as an example we shall deal with the the plane and dynamically balanced stationary flow of an ideal compressible fluid. We calculate the normal form of the problem and arrive at a methcd of characteristics for approximate determination of the solution which also uses the epicycloid networks and which therefore may offer certain simplifications as opposed to the familiar method (cf. section V, footnote [3]).

Finally we do not want to neglect thanking Mr. L. Bieberbach for revision of the manuscripts of both parts and for numerous suggestions for improvement.

SECTION IV.

QUASI-RIEMANN METHOD

1. Integral theorem of the normal form. The construction of the characteristic theory of a system (1) using Pfaffian forms, as was done in Section I brings the advantage that one can make direct statements about the behavior of the sought functions in general terms. Along with E. Cartan we designate the "extrinsic product" of two Pfaffian forms ω_1, ω_2 with $[\omega_1, \omega_2]$ and the "extrinsic derivative" of a Pfaffian form with $[d\omega]$. If G is a range in which the coefficients of the form ω are continuously differentiable then the integral theorem is valid [1]

[FOOTNOTE: Cf., for example, B. W. Blaschke, "Introduction to differential geometry" Berlin 1950. END FOOTNOTE]

(IV. 1) $$\int_{\text{Rand von } G} \omega = \iint_G [d\omega].$$

According to Section I two Pfaffian forms ω_1, ω_2 belong to a system of hyperbolic differential equations (1). They form the basis of a form ring, which using two auxiliary functions $W(x^1, x^2)$, $Z(x^1, x^2)$ we write in the form

(IV.2) $$\omega = WV\omega_1 + ZU\omega_2.$$

In this case U, V are to be considered as solutions of the normal form (I.9)

(IV.3) $$\begin{aligned} U_u &= AU + BV + C, \\ V_{\bar{u}} &= \bar{A}U + \bar{B}V + \bar{C} \end{aligned}$$

The application of the integral theorem (IV.1) to the form ω (IV.2) yields

(IV.4) $$\oint (WV\omega_1 + ZU\omega_2) = \iint [d(WV\omega_1 + ZU\omega_2)].$$

For transformation of the right side we use the fundamental formulas known for extrinsic derivatives [1]:

If S is a position function, then

(IV.5) $$[d(S\omega)] = [(dS)\cdot\omega] + S[d\omega].$$

In addition, referred to the invariant derivatives,

(IV.6) $$dS = S_u\omega_1 + S_{\bar{u}}\omega_2.$$

Then the double integral of (IV.4) becomes

(IV.7) $$\iint [d(WV\omega_1 + ZU\omega_2)] = \iint \{[d(WV)\cdot\omega_1] + [d(ZU)\cdot\omega_2] + WV[d\omega_1] + ZU[d\omega_2]\}.$$

Since the extrinsic products are alternating it follows that

(IV.8) $$\iint [d(WV\omega_1 + ZU\omega_2)] = \iint \{(WV)_{\bar{u}} - (ZU)_u\}[\omega_2\omega_1] + \iint \{WV[d\omega_1] + ZU[d\omega_2]\}.$$

The extrinsic derivatives $[d\omega_1]$, $[d\omega_2]$ are given by the equations

(IV.9) $$[d\omega_1] = S[\omega_2\omega_1]; \quad [d\omega_2] = -R[\omega_2\omega_1]$$

with $$S = \frac{1}{D}\left(\frac{\partial \alpha_1}{\partial x^2} - \frac{\partial \alpha_2}{\partial x^1}\right); \quad R = \frac{1}{D}\left(\frac{\partial \bar{\alpha}_2}{\partial x^1} - \frac{\partial \bar{\alpha}_1}{\partial x^2}\right)$$

and
$$D = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \bar\alpha_1 & \bar\alpha_2 \end{vmatrix} \neq 0.$$

If we proceed with this in (IV.8) then the integral theorem becomes

$$\oint (WV\omega_1 + ZU\omega_2) = \iint \{(WV)_{\bar z} - (ZU)_z + WVS - ZUR\} [\omega_1 \omega_2]. \qquad \text{(IV.10)}$$

If we now note that U, V are considered as solutions of the normal form (IV.3) then we can replace U_z and $V_{\bar z}$ in (IV.10) according to (IV.3) and obtain

$$\begin{aligned} \oint (WV\omega_1 + ZU\omega_2) = & \iint V\{W_{\bar z} + W(\bar B + S) - ZB\}[\omega_1\omega_2] \\ & + \iint U\{-Z_z - Z(A + R) + W\bar A\}[\omega_1\omega_2] \\ & + \iint \{W\bar C - ZC\}[\omega_1\omega_2]. \end{aligned} \qquad \text{(IV.11)}$$

This is a general integral theorem which must be satisfied by every solution system of (IV.3) with any functions W, Z.

2. Generalization of the Riemann method. It is easy, using the integral theorem (IV.11) to reduce the Cauchy initial value problem of (IV.3) for a curve K to a "characteristic initial value problem" (Section III).

For the arbitrary functions W, Z we make the demand

(IV.12)
$$W_{\bar{\alpha}} + W(\bar{B} + S) - ZB = 0$$
$$Z_{\alpha} + Z(A + R) - W\bar{A} = 0.$$

That is a system of partial differential equations of the hyperbolic type for the functions W and Z in the normal form with the same characteristics $\alpha, \bar{\alpha}$. We call it the adjoint system to (IV.3). The adjoint system is simpler than the original (IV.3) in that it is always homogeneous: it is solvable, however, only through quadratures according to the definition of Section I when the initial system (IV.3) is solvable through quadratures. If in (IV.11) W and Z are known adjoint functions, i.e., solutions of (IV.12) then only known functions appear under the remaining double integral. It will now come to transforming the boundary integral through the choice of an appropriate range so that the sought functions U, V appear before the integral. In the case of the Cauchy initial value problem U, V are given along a curve K, the function values U(P), (VP) are sought in any point P outside of K.

We consider a region G whose boundary is formed by the initial curve K and the characteristics through point P (Fig. 1). For this region G the prerequisites of the integral theorem are satisfied. As can be seen from Fig. 1, one can split the boundary integral up into subintegrals

(IV. 13) $$\oint = \int_{P_1 P_2} + \int_{P_2 P} + \int_{P P_1}.$$

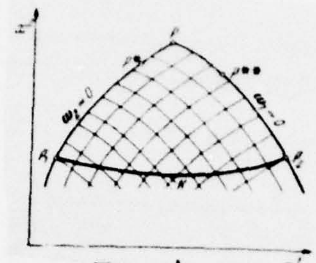

Fig 1

The first integral is to be taken along the initial curve K, on which U, V are known. For transformation of the integral over the characteristics we consider the equation (IV. 3) and write it in the form

(IV. 14) $$BV = U_{\xi} - AU - C; \quad \bar{A}U = V_{\eta} - \bar{B}V - \bar{C}.$$

In order not to have to exclude the zero places of the given functions B and $\bar{A}$ in the future, we want to write the adjoint P_1P function W on the boundary P_1P and the adjoint function Z on the boundary P_2P in the form [2]

(IV 15) $$W = \mathfrak{W} B \text{ auf } PP_1; \quad Z = \mathfrak{Z} \bar{A} \text{ auf } PP_2.$$

[FOOTNOTE: [2] The next thing would be to replace the function U according to (IV.14) by $U = \frac{1}{\bar{A}}(V_{\eta} - \bar{B}V - \bar{C})$ P_2P in the characteristic parts of the boundary integral (IV.1[illegible]), respectively, (IV.13) for example in the integral $\int ZU_{dt}$ and then through partial integration with respect to V_{η} to move the function V in front of the integral. This can only be done, however, if $\bar{\lambda} \neq 0$ ist. Through introduction of the auxiliary functions $\mathfrak{W}$ and $\mathfrak{Z}$ according to (IV.15) this difficulty is eliminated. END FOOTNOTE]

The possibility of such a formulation results directly from the fact that for the adjoint functions W, Z characteristic initial values are to be sought, i.e., we can have at our disposal W and Z on the corresponding characteristics through P. With (IV.14, 15) we go into the subintegrals of (IV.13) and observe that $\omega_2 = 0$ along the characteristic P_1P and $\omega_1 = 0$ along P_2P. Then from (IV.11) it follows that

(IV.16)

$$\int_{P_1P_2} (WV\omega_1 + ZU\omega_2) + \int_{P_2P} \mathfrak{Z}(V_{\bar s} - \bar{B}V - \bar{C})\omega_2 + \int_{PP_1} \mathfrak{W}(U_s - AU - C)\omega_1 = \iint (W\bar{C} - ZC)[\omega_2\omega_1].$$

As a result of partial integration the following formulas result:

(IV.17)

$$\int_{P_2P} \mathfrak{Z}(V_{\bar s} - \bar{B}V - \bar{C})\omega_2 = \mathfrak{Z}V\Big|_{P_2}^{P} - \int_{P_2P} V(\mathfrak{Z}_{\bar s} + \bar{B}\mathfrak{Z})\omega_2 - \int_{P_2P} \mathfrak{Z}\bar{C}\omega_2$$

(IV.18)

$$\int_{PP_1} \mathfrak{W}(U_s - AU - C)\omega_1 = \mathfrak{W}U\Big|_{P}^{P_1} - \int_{PP_1} U(\mathfrak{W}_s + A\mathfrak{W})\omega_1 - \int_{PP_1} \mathfrak{W}C\omega_1.$$

Now for $\mathfrak{W}, \mathfrak{Z}$ along the corresponding characteristics through P we make the demand:

(IV.19) $$\mathfrak{Z}_\sigma + \tilde{B}\mathfrak{Z} = 0 \quad \text{auf } P_2P.$$

(IV.20) $$\mathfrak{W}_\tau + A\mathfrak{W} = 0 \quad \text{auf } PP_1.$$

Therefrom for $\mathfrak{Z}$ in a random point P** of PP_2 and for $\mathfrak{W}$ in a random point P* of PP_1 (see Fig. 1)

(IV.21) $$\mathfrak{Z}(P^{**}) = C_1 e^{-\int_{P_2}^{P^{**}} \tilde{B}\, d\sigma} \quad \text{auf } PP_2.$$

(IV.22) $$\mathfrak{W}(P^{*}) = C_2 e^{-\int_{P_1}^{P^{*}} A\, d\tau} \quad \text{auf } PP_1.$$

In this case C_1 and C_2 are integration constants which we shall later have at our disposal.

Then in accordance with (IV.15), for W, Z follows the demand:

(IV.23) $$Z(P^{**}) = C_1 \tilde{A}(P^{**}) e^{-\int_{P_2}^{P^{**}} \tilde{B}\, d\sigma} \quad \text{auf } PP_2.$$

(IV.24) $$W(P^{*}) = C_2 B(P^{*}) e^{-\int_{P_1}^{P^{*}} A\, d\tau} \quad \text{auf } PP_1.$$

These are characteristic initial values according to Section III for the solutions W, Z of the adjoint system. According to these stipulations the integral theorem (IV.16) finally acquires the form:

$$\mathfrak{W}(P)U(P) - \mathfrak{Z}(P)V(P) = \mathfrak{W}(P_1)U(P_1) - \mathfrak{Z}(P_2)V(P_2) + \int_{P_1P} \mathfrak{W}\tilde{C}\omega_1 + \int_{PP_2} \mathfrak{Z}\tilde{C}\omega_2 + \int_{P_1P_2} (WV\omega_1 + ZU\omega_2) + \iint \{W\tilde{C} - ZC\}[\omega_1\omega_2] \equiv H(W, Z). \tag{IV.25}$$

We want to summarize the result up to this point in the following manner:

If W, Z are solutions of the adjoint system (IV.12) which satisfy the characteristic initial conditions (IV.23), (IV.24) with arbitrary constants C_1, C_2 then the linear combination $\mathfrak{W}U - \mathfrak{Z}V$ in the point P can be expressed through the initial values and known values in accordance with (IV.25). Here $\mathfrak{W}, \mathfrak{Z}$ are explained by (IV.21), (IV.22). We want to call W, Z quasi-Riemann functions and their existence is assured according to Section II and III.

A further goal will be to obtain, through appropriate choice of the constants C_1, C_2, two independent linear combinations from which U, V can be calculated in point P. This can be done in various ways.

If one desires certain symmetry in the formulas, then through the demand

(IV.26) $$\text{I. } \mathfrak{W}(P) = 1; \quad \mathfrak{Z}(P) = 1$$

one can uniquely determine the constants C_1 and C_2 in (IV.21, 22) and the functions W, Z according to (IV.23, 24). A second function pair $\bar{\mathfrak{W}}, \bar{\mathfrak{Z}}$ we subject to the demand

(IV.27) $$\text{II. } \bar{\mathfrak{W}}(P) = 1; \quad \bar{\mathfrak{Z}}(P) = -1$$

and correspondingly obtain uniquely determined functions $\bar{W}, \bar{Z}$. If we regard the abbreviation H(W, Z) given in (IV.25) then it follows directly that

(IV.28a) $$U(P) = \tfrac{1}{2}\{H(W, Z) + H(\bar{W}, \bar{Z})\}.$$

(IV.28b) $$V(P) = \tfrac{1}{2}\{H(\bar{W}, \bar{Z}) - H(W, Z)\}.$$

If the characteristic initial value problem is solved for the adjoint normal form then the solution of the Cauchy initial value problem of the system (IV.3) for every initial curve K can be represented by quadratures.

Shorter integral representations can be obtained in the following manner: We choose the integration constants in (IV.21, 22) so that

(IV.29) $$\text{I. } \mathfrak{W}(P) = 1; \quad \mathfrak{Z} = 0 \quad \text{auf } PP_1,$$

(IV.30) $$\text{II. } \bar{\mathfrak{Z}}(P) = -1; \quad \bar{\mathfrak{W}} = 0 \quad \text{auf } PP_2.$$

Thereby W, Z, $\bar{W}$, $\bar{Z}$ are uniquely determined. From (IV.25) for the sought functions U(P), V(P) it follows immediately that:

(IV.31a) $$U(P) = \mathfrak{W}(P_1)U(P_1) + \int_{P_1P} \mathfrak{W}C\omega_1 + \int_{P_1P_2} (WV\omega_1 + ZU\omega_2) + \iint \{W\bar{C} - ZC\} [\omega_1\omega_2].$$

(IV.31b) $$V(P) = -\bar{\mathfrak{Z}}(P_2)V(P_2) + \int_{PP_2} \bar{\mathfrak{Z}}\bar{C}\omega_2 + \int_{P_1P_2} (\bar{W}V\omega_1 + \bar{Z}U\omega_2) + \iint \{\bar{W}\bar{C} - \bar{Z}C\} [\omega_1\omega_2].$$

From the integral representation (IV.31) it follows particularly graphically that U, V are continuously dependent on the initial values. Finally let us note that integral representations for the functions u, v of the Cauchy initial value problem of the general system (1) are given directly by the integral theorems (IV.28), (IV.31).

3. Relations to the "Riemann method." If a general hyperbolic differential equation of the second order is given in the form (I.17), in which we designate the independent variables with x, y

(IV.32) $$z_{xy} + a z_x + b z_y + c z + f = 0,$$

then through the introduction of both functions

(IV.33) $$u = z; \quad v = z_x$$

we can write these as a system. According to (I.19) in addition there results the normal form

(IV.34) $$\begin{aligned} U_x &= -bU + V, \\ V_y &= (b_y + ab - c)U - aV - f \end{aligned}$$

with

$$u = U; \quad bu + v = V;$$

since according to (I.18) and (I.1) $U_{\xi} = U_x; V_{\eta} = V_y$.

Two Pfaffian forms (I.13) belong to (IV.34)

(IV.35) $$\omega_1 = dx; \quad \omega_2 = dy.$$

From the integral theorem (IV.25) we immediately obtain infinitely many integral representations for a Cauchy initial value problem from (IV.32). It is now of interest whether, and in the given case, through which "marked" demands these integral representations directly yield the familiar "Riemann solution formula."

We shall show: If for the characteristic initial value problem of the "adjoint system" (IV.12), in addition to requirements (IV.19, 20) we make the demand (IV.29) for the determination of the integration constants then (IV.31a) yields precisely the Riemann solution formula.

The adjoint system (IV.12) becomes

(IV.36) $$W_y - Wa - Z = 0,$$
$$Z_x - Zb - W(b_y + ab - c) = 0.$$

The requirements for the characteristic initial value problem of (IV.36) according to (IV.15) and (IV.20) since B = 1 (IV.37) $W_x - bW = 0$ on PP_1 with $W(P) = 1$

(IV.37a) $Z = 0$ on PP_2 according to (IV.29).

From (IV.36) we eliminate Z by differentiating the first equation with respect to x and introduce Z_x frcm the second equation.

Following simple transformation it follows that

(IV.38) $$W_{xy} - (Wa)_x - (Wb)_y + Wc = 0.$$

(IV.38) is precisely the "adjoint Riemann equation" which belongs to (IV.32).

According to (IV.36) the requirement (IV.37a) goes over into

(IV.39) $W_y - aW = 0$ on PP_2.

(IV.37, 39) are the "Riemann demands" which belong to (IV.38).

According to (IV.33, 34) we set

(IV.40) $$z = u = U; \quad bz + z_x = V.$$

Now we arrive directly at the Riemann solution formula if we substitute the values (IV.40) into (IV.31a). Considering the first equation in (IV.36)

(IV.41)
$$z(P) = W(P_1)z(P_1) + \int_{P_1P_2} \{W(bz + z_x)\,dx + (W_y - Wa)\,z\,dy\} - \iint f W \{dx\,dy\}.$$

This is the Riemann representation formula for the Cauchy problem of the hyperbolic differential equation (IV.32). (Cf. perhaps Goursat "Cours D'Analyse" Paris 1927, Volume 3, page 150.)

SECTION V. SYSTEMS OF QUASILINEAR DIFFERENTIAL EQUATIONS.

In the introduction it was already pointed out that in the past ten years in connection with certain problems of fluid dynamics methods of characteristics were developed for the solution of hyperbolic systems of quasilinear differential equations with two independent variables [3].

[FOOTNOTE: [3] Even today the abundant literature is accesible here only with difficuly so that we can only mention the following:

R. Sauer, "Theoretical Introduction in Gas Dynamics," Berlin 1943.

Kl. Oswatitsch, "Methods of characteristics of Hydromechanics," ZAMM 1947, Issues 7, 8, 9.

Courant-Friedrichs, "Supersonic Flow and Shock Waves," New York 1948.

A. Ferri, "Elements of Aerodynamics of Supersonic Flows," New York 1949.

Abundant bibliographical information can be found in the above works. END FOOTNOTE]

In particular R. Courant, K. O. Friedrichs and Lax have recently studied the quasilinear systems in detail and given the prerequisites forthe existence of unique solutions of the Cauchy initial value problem (cf. Report I., footnote 2). Here we want to limit ourselves to pointing out the connection of the quasilinear systems with the theory of the linear systems developed in Secticns I through IV and to illustrating the advantages using a simple example of the plane and dynamically balanced flow of compressible liquids.

1. Quasilinear systems in the "characteristic form."

Let

(V-1)
$$a^k u_k + b^k v_k + c = 0, \quad \bar{a}^k u_k + \bar{b}^k v_k + \bar{c} = 0$$
(summed over k = 1, 2).

be a quasilinear system in which $a^k, \bar{a}^k \dots \bar{c}$ are functions of the independent variables x^1, x^2 and of the sought solutions u, v. If we follow the developments of Section I and assume that a solution u, v is known, then the system (V-1) is called hyperbolic with respect to this solution if the determinant (I-3) has exactly two real roots $\nu_1 : \nu_2$, μ_1, μ_2. Thereby the characteristic directions became dependent on the solutions u, v, however, equations (I.1-6) of Section I maintain their full validity, while the factors ρ_1, ρ_2, σ_1, σ_2 can now be functions of x^1, x^2, u, v. In this manner we always arrive at the following "characteristic system" in which we use the indices α, $\bar{\alpha}$ instead of points in the differential equations (I.11) of the characteristics:

(V-2a)
$$\text{I)}\quad x^1_\alpha = \alpha^1, \quad x^2_\alpha = \alpha^2. \qquad \text{II)}\quad x^1_{\bar{\alpha}} = \bar{\alpha}^1, \quad x^2_{\bar{\alpha}} = \bar{\alpha}^2.$$

The $\alpha^i, \bar{\alpha}^i$ are known functions of x^1, x^2, u, v. In addition, according to (16) there are the accompanying differential equations of the functions u, v

(V.2b)
$$\varphi_1 u_\alpha + \varphi_2 v_\alpha = -\varphi_1(\nu_1 c + \nu_2 \bar{c}),$$
$$\sigma_1 u_{\bar{\alpha}} + \sigma_2 v_{\bar{\alpha}} = -\sigma_1(\mu_1 c + \mu_2 \bar{c}).$$

The differential equations (V.2a) of the characteristic curves can also be written in the form:

(V.2c)
$$\text{I)}\ \alpha^1 x^2_\alpha - \alpha^2 x^1_\alpha = 0, \quad \text{II)}\ \bar{\alpha}^1 x^2_{\bar{\alpha}} - \bar{\alpha}^2 x^1_{\bar{\alpha}} = 0.$$

Now the equations are homogeneous and are preserved during the introduction of integrating factors. Hence the directional derivatives can be interpreted as partial derivatives. (V.2b) and (V.2c) form a system of four partial differential equations for the functions x^1, x^2, u, v of two characteristic variables. The properties and the questions of existence for such systems are studied by R. Courant et al. (cf. footnote I[2]).

2. Normal form of quasilinear systems.

Our theory of linear systems is based essentially on the normal form (see Section I). We turn to the question of whether the system of equations (V.2b) possesses a normal form in the sense of (19). One immediately recognizes the correctness of the theorem:

a) If ϱ_i, σ_i in (V.2b) are functions of x^1, x^2 alone, then one arrives at the normal form by setting:

(V.3) $$U = \varrho_1 u + \varrho_2 v; \quad V = \sigma_1 u + \sigma_2 v.$$

The normal form reads

(V.4) $$U_\alpha = C; \quad V_\beta = \bar{C}.$$

In this case C and $\bar{C}$ are known functions of x^1, x^2, U, V.

Mr. J. Nitsche pointed out to us that a corresponding theorem is also valid in the general case:

b) If ϱ_i, σ_i in (V.2b) are differentiable functions of u, v, x^i, then the system possesses a normal form.

We have to show that instead of u, v, new functions U, V of u, v, x^i can be introduced which reduce (V.2b) to the form (V.4). For this purpose we multiply the first equation (V.2b) with a function M(u, v, x^i) and the second equation with N(u, v, x^i) and define M and N so that the following equations become possible:

(V.5)
$$M\varphi_1 u_\alpha + M\varphi_2 v_\alpha = U_\alpha + \chi(u, v, x^i),$$
$$N\sigma_1 u_{\bar{\alpha}} + N\sigma_2 v_{\bar{\alpha}} = V_{\bar{\alpha}} + \bar{\chi}(u, v, x^i).$$

That is certainly the case if M and N satisfy the equations:

(V.6)
$$(M\varphi_1)_v - (M\varphi_2)_u = 0, \qquad (N\sigma_1)_v - (N\sigma_2)_u = 0.$$

In these equations the x^i are to be interpreted as constant parameters. Then

$$U = \Phi(u, v, x^i); \qquad V = \Psi(u, v, x^i)$$

such that

$$U_u = \Phi_u = M\varphi_1, \qquad U_v = \Phi_v = M\varphi_2$$
$$V_u = \Psi_u = N\sigma_1, \qquad V_v = \Psi_v = N\sigma_2$$

(V.6a)

By differentiation in the direction of the characteristics:

$$U_\alpha = M\varphi_1 u_\alpha + M\varphi_2 v_\alpha + \Phi_{x^i}\alpha^i$$
$$V_{\bar{\alpha}} = N\sigma_1 u_{\bar{\alpha}} + N\sigma_2 v_{\bar{\alpha}} + \Psi_{x^i}\bar{\alpha}^i.$$

Here the last terms are known functions of u, v, x^i if the integrating factors M, N are defined according to (V.6). The functional determinant of the functions U, V with respect to u, v is $MN(\rho_1\sigma_2-\rho_2\sigma_1)$. On account of the independence of ρ_i, σ_i, U, V are also independent with respect to u, v. Hence it follows immediately that (V.2b) can be written in the normal form (V.4).

c) For a quasilinear system in the normal form (V.2a, 4) the integral theorem (IV.11) can be accepted. For any two solutions U, V of the quasilinear system and two arbitrary functions W, Z according to (IV.11) the integral theorem is valid:

$$\begin{aligned}\oint (WV\omega_1 + ZU\omega_2) &= \iint V\{W_{\bar{z}} + WS\}[\omega_2\omega_1] \\ &+ \iint U\{-Z_z - ZR\}[\omega_2\omega_1] \\ &+ \iint (W\bar{C} - ZC)[\omega_2\omega_1]\end{aligned} \qquad \text{(V.7)}$$

whereby R, S are given by (IV.9).

By setting the bracketed expression in both double integrals equal to zero the adjoint equations result. If one assumes U, V to be known then the Pfaffian forms can be made integrable and the adjoint system can be reduced to the simple form

$$W_{\bar{z}} = 0; \qquad Z_z = 0.$$

Then

$$W = \text{const}, \qquad Z = \text{const}$$

are the equations for both systems of characteristics.

d) Linearizable systems. Of great interest for applications is the special case for which in (V.4)

$$C = \tilde{C} \equiv 0$$

One can transpose the role of the dependent and independent variables, i.e., one interprets U, V as an independent variable in a U, V-plane and considers [illegible] (U, V) as sought functions of the independent variables U, V. Then (V.2c) appear as a system of differential equations in characteristic form for the functions [illegible] (U, V). The curves U = const, V = const are the characteristics of the U, V-plane, which are fixed, i.e., are independent of the sought functions [illegible] (U, V). Therewith one has the same relationships as in the linear hyperbolic systems and can carry the theory over directly. We shall be satisfied with this short reference (see under (4c) but would still like to note that also in general cases such a linearization is possible, for example, if C, [illegible] and [illegible] are functions alone of U, [illegible] and V, [illegible], respectively.

3. General difference method.

From the equation system (V.2a) and (V.4) (resp. (V.2b)) one arrives directly at difference methods which approximately solve the Cauchy problem and all of the mixed initial value- and boundary value problems discussed in Section III. Let us first assume that the solutions of the systems (V.2a) and (V.4) are known and $\bar{s}$, respectively, s are the curve lengths along the characteristics. Then the equations can be writter in integral form. Approximately

$$\text{I)}\quad \begin{aligned} \Delta_\alpha x^1 &= \alpha^1 \Delta s \\ \Delta_\alpha x^2 &= \alpha^2 \Delta s, \\ \Delta_\alpha U &= C \Delta s \end{aligned} \qquad \text{II)}\quad \begin{aligned} \Delta_{\bar{\alpha}} x^1 &= \bar{\alpha}^1 \Delta \bar{s} \\ \Delta_{\bar{\alpha}} x^2 &= \bar{\alpha}^2 \Delta \bar{s} \\ \Delta_{\bar{\alpha}} V &= \bar{C} \Delta \bar{s}. \end{aligned} \tag{V.8}$$

Along the curve K let the initial values U(K), V(K) be given; along the boundary R let some function Φ (u, v), respectively, Ψ (U, V) be known (see Fig. 2). Beginning from the intersection 0 one passes the characteristic directions through a number of points (1, 2, 3) of K. They are known as quotients of the first two equations (V.8) along K. The characteristic straight lines intersect in pairs in the points 5, 6, 7. If one normalizes $\sum(\alpha^i)^2 = 1$ and $\sum(\bar{\alpha}^i)^2 = 1$, then Δs, respectively $\Delta \bar{s}$ is the spacing of the points on the α-, respectively, $\bar{\alpha}$-line. One can thus calculate $\Delta_\alpha U$ and $\Delta_{\bar{\alpha}} V$ and determine U, V in the points 5, 6, 7.

Assuming that the characteristic approaching in boundary point 4 belongs to system II of the equations (V.8) then in 4 one knows the function V and from V and the boundary condition $\psi(U, V)$ can calculate the function U (resp. u, v). In this case it is assumed that Ψ is uniquely solvable with respect to U. The method can be continued in the same manner in the range lying between K and R. With respect to the boundary curve R it is to be assumed that in the range between K and R there is one and only one characteristic passing through 0.

4) Steady flow.

As an example dynamically balanced and plane steady flow of an ideal gas will be treated in somewhat more detail. In view of the abundant literature on the subject [3] we may be brief. In cylinder coordinates r, x let $\phi(r, x)$ be the velocity potential and a be the local velocity of sound. Then the following equation is valid:

(V.9)
$$\varphi_{xx}\left(1-\frac{\varphi_x^2}{a^2}\right)+\varphi_{rr}\left(1-\frac{\varphi_r^2}{a^2}\right)-2\varphi_{xr}\frac{\varphi_x\varphi_r}{a^2}+\frac{\varphi_r}{r}=0.$$

In this case with the constants $\varkappa > 1,\ w_0 \geq w$

(V.10)
$$a^2=\frac{\varkappa-1}{2}(w_0^2-w^2),\quad w^2=\varphi_x^2+\varphi_r^2.$$

If we set

(V.11) $$\varphi_x = w \cos \vartheta, \quad \varphi_r = w \sin \vartheta,$$

then from (V.9) and the condition

(V.12) $$\frac{\partial}{\partial r}(w \cos \vartheta) = \frac{\partial}{\partial x}(w \sin \vartheta)$$

there results a quasilinear system of differential equations for the functions w, ϑ, which give the contribution and the direction of the flow velocity toward the x-axis. For w > a the system is hyperbolic.

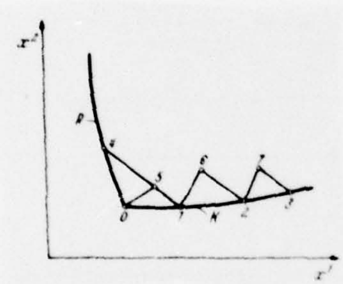

Fig 2.

Here it is recommended to not go directly to the "characteristic" derivatives, α, $\bar{\alpha}$ but to go first to another system of "invariant" derivatives.

There are

$$\beta)\quad \begin{aligned} \dot{x} &= w\cos\vartheta \\ \dot{r} &= w\sin\vartheta, \end{aligned}$$

(V.13)

$$\bar{\beta})\quad \begin{aligned} \dot{x} &= -w\,\mathrm{tg}\,\gamma\sin\vartheta \\ \dot{r} &= w\,\mathrm{tg}\,\gamma\cos\vartheta \end{aligned}$$

two vectors of which β coincides with the velocity w and $\bar{\beta}$ is perpendicular to it. In this case γ is the so-called "Mach angle" with sin γ = a/w (cf. Fig. 3). The invariant derivatives of a function S in these directions are

(V.14)
$$S_\xi = \frac{\partial S}{\partial x} w \cos\vartheta + \frac{\partial S}{\partial r} w \sin\vartheta$$
$$S_\eta = -\frac{\partial S}{\partial x} w \operatorname{tg}\gamma \sin\vartheta + \frac{\partial S}{\partial r} w \operatorname{tg}\gamma \cos\vartheta .$$

If one calculates these derivatives for the functions w and ϑ and observes (V.9) and (V.12) then the system of invariant differential equations follows after a simple intermediate calculation:

(V.15)
$$w_\xi \cdot \operatorname{ctg}\gamma - \vartheta_\eta \cdot w = \frac{w}{r} \frac{\sin\vartheta}{\operatorname{ctg}\gamma},$$
$$w_\eta \cdot \operatorname{ctg}\gamma - \vartheta_\xi \cdot w = 0.$$

a) Natural equations. The formulas (V15) allow a simple geometric interpretation. If one designates the curve length with s and the bend radius of the flow lines with ϱ_s then on account of (V.14)

(V.16a)
$$\vartheta_\xi = w \frac{d\vartheta}{ds} = \frac{1}{\varrho_s} w \quad \left(\text{wegen } \frac{d\vartheta}{ds} = \frac{1}{\varrho_s}\right).$$

If we call n and ϱ_n the curve length and the bend radius of the orthogonal trajectories of the flow lines, then according to (V.14)

(V.16b) $$\vartheta_{\bar{s}} = w \cdot \mathrm{tg}\,\gamma \cdot \frac{d\vartheta}{dn} = \frac{w \cdot \mathrm{tg}\,\gamma}{\varrho_n}$$

By introduction in (V.15) the so-called natural equations of the flow

(V.17) $$\frac{d\ln w}{ds} = \mathrm{tg}^2\gamma\left(\frac{1}{\varrho_n} + \frac{\sin\vartheta}{r}\right)$$
$$\frac{d\ln w}{dn} = \frac{1}{\varrho_s}.$$

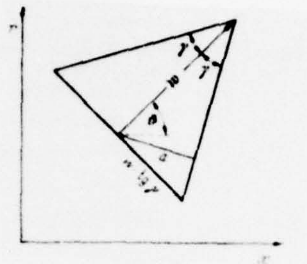

Fig. 3

They can be obtained quite effortlessly in this manner. One recognizes then that in the case of flow along a wall w only changes continuously if the curvature of the wall is continuous.

b) The normal form of the system. In order to reduce the system of equations to the characteristic form we have to add both equations (V.15) once and subtract them once and finally divide them by w. If in addition we set

$$w_\xi = w_s + w_{\bar{s}}, \quad w_\eta = w_s - w_{\bar{s}},$$

then we obtain the characteristic system

(V.18) $$\frac{\mathrm{ctg}\,\gamma}{w} w_\xi - \vartheta_\xi = \frac{w}{r}\sin\vartheta \cdot \mathrm{tg}\,\gamma,$$
$$\frac{\mathrm{ctg}\,\gamma}{w} w_\eta + \vartheta_\eta = \frac{w}{r}\sin\vartheta \cdot \mathrm{tg}\,\gamma.$$

Here the characteristic directions α, $\bar{\alpha}$, as a comparison with (V.14) shows, are given by

(V.19) $$\alpha: \; x' = \frac{w}{\cos\gamma}\cos(\gamma+\vartheta), \; y' = \frac{w}{\cos\gamma}\sin(\gamma+\vartheta), \qquad \bar{\alpha}: \; \bar{x}' = \frac{w}{\cos\gamma}\cos(\vartheta-\gamma), \; \bar{y}' = \frac{w}{\cos\gamma}\sin(\vartheta-\gamma).$$

Since in (V.18) the factor $\frac{\operatorname{ctg}\gamma}{w}$ is a function of w alone the system possesses a normal form with the functions

(V.20) $$U = \int \frac{\operatorname{ctg}\gamma}{w}\,dw - \vartheta, \quad V = \int \frac{\operatorname{ctg}\gamma}{w}\,dw + \vartheta.$$

The normal form reads:

(V.21) $$U_{\alpha} = \frac{w}{r}\sin\vartheta\,\operatorname{tg}\gamma, \quad V_{\bar{\alpha}} = \frac{w}{r}\sin\vartheta\,\operatorname{tg}\gamma,$$

where w, ϑ are to be considered as functions of U, V.

e) Let us next examine the limiting case $\frac{1}{r} = 0$.

Equations (V.19) to (V.21) go over into the formulas for plane flow for which: $U_{\alpha} = 0, \quad V_{\bar{\alpha}} = 0.$

The curves U = const, V = const form a fixed grid in the w, ϑ-plane and are the characteristics of the linear system (V.19) if one writes it in accordance with (V.2c) in [illegible] and [illegible] . The linearization of the differential equation of plane flow by transposition of the dependent and independent variables has been known for a long time (cf. perhaps R. Sauer [1] page 111). The curves U = const, V = const are epicycloids of the w, ϑ-plane. Epicycloid grids are available in various scales and are widely used for approximate determination of plane flow.

d) Cycloid grids can also be use to advantage for dynamically balanced flow, which to the best of our knowledge has not been mentioned in the literature up until now. We therefore want to briefly explain the method:

Along the characteristics we choose the curve lengths s and s as parameters. Then (V.19) and (V.21) can be written approximately in differences:

(V.22)
$$a)\quad \begin{aligned} \Delta x &= \cos(\gamma + \vartheta)\,\Delta s \\ \Delta r &= \sin(\gamma + \vartheta)\,\Delta s, \\ \Delta U &= \frac{\sin\gamma\sin\vartheta}{r}\,\Delta s \end{aligned} \qquad b)\quad \begin{aligned} \Delta x &= \cos(\vartheta - \gamma)\,\Delta \bar{s} \\ \Delta r &= \sin(\vartheta - \gamma)\,\Delta \bar{s} \\ \Delta V &= \frac{\sin\gamma\sin\vartheta}{r}\,\Delta \bar{s}. \end{aligned}$$

If in two points P, Q of the x, r-plane one knows the values of w, ϑ

(Fig. 4) then from the first two equations (V.22) one calculates the characteristic tangents, their point of intersection R and obtains Δs and $\Delta \bar{s}$. From them ΔU, ΔV can be determined. The values of $w(R)$, $\vartheta(R)$ can now be effortlessly read off of the cycloid grid. In order to show that, we proceed from the fact that indefinite integrals appear in (V.20). Since [illegible] vanishes for $w = a$ we can introduce the definite integrals in (V.20)

$$U = \int_a^w \frac{\ldots}{w}\, dw - \vartheta, \qquad V = \int_a^w \frac{\ldots}{w}\, dw + \vartheta.$$

Then for $w = a$

$$U(a, \vartheta) = -\vartheta, \qquad V(a, \vartheta) = +\vartheta.$$

The value ΔU appears therefore on the inner boundary circle [illegible] of the grid as $\Delta U = -\Delta\vartheta$ and correspondingly ΔV as $\Delta V = +\Delta\vartheta$. Now the following method results (Fig. 5):

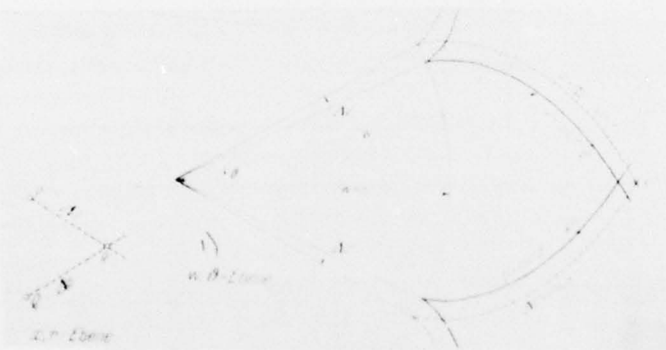

Fig4 Fig 5.

Fig. 4. ((KEY: 1) x,r-plane.))

Fig. 5. ((KEY: 1) w,ϑ-plane.))

From the given $w(P)$, $\vartheta(P)$ and $w(Q)$, $\vartheta(Q)$ by drawing the corresponding radius vectors of the w,ϑ-plane one immediately finds the points P^0, Q^0. On the U-line of the cycloid grid one approaches the boundary circle ($w=a$) through Q^0, marks off the value $\Delta U = -\Delta\vartheta$, calculated from the third equation (V.22), in radian measure and turns back on the new grid line $U + \Delta U$. If one does the same thing with P, as a section of the new grid lines one obtains [illegible] can read off $w(R)$, $\vartheta(R)$ directly. Since the transition from U to U ΔU (resp. from V to $V + \Delta V$) signifies only a stable rotation of the grid curves by the angle $(-\Delta U)$, resp., ΔV, the process can be further mechanized.

Submitted 18 July 1950.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER FTD-ID(RS)T-0926-77	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SYSTEMS OF HYPERBOLIC DIFFERENTIAL EQUATIONS OF THE FIRST ORDER. II		5. TYPE OF REPORT & PERIOD COVERED Translation
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Wolfgang Haack and Guenter Hellwig		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Foreign Technology Division Air Force Systems Command U. S. Air Force		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE 1950
		13. NUMBER OF PAGES 35
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

~~Distribution limited to U. S. Government Agencies Only; Proprietary (Copyright) Information; Other request for this document must be referred to FTD/STINFO.~~

DISTRIBUTION STATEMENT A

Approved for public release; Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

12

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DISTRIBUTION LIST

DISTRIBUTION DIRECT TO RECIPIENT

ORGANIZATION		MICROFICHE
A205	DMATC	1
A210	DMAAC	2
B344	DIA/RDS-3C	8
C043	USAMIIA	1
C509	BALLISTIC RES LABS	1
C510	AIR MOBILITY R&D LAB/FIO	1
C513	PICATINNY ARSENAL	1
C535	AVIATION SYS COMD	1
[redacted]		[redacted]
C591	FSTC	5
C619	MIA REDSTONE	1
D008	NISC	1
H300	USAICE (USAREUR)	1
P005	ERDA	1
P055	CIA/CRS/ADD/SD	1
NAVORDSTA (50L)		1
NASA/KSI		1
AFIT/LD		1
E053	AF/INAKA	1
E017	AF/ RDXTR-W	1
E404	AEDC	1
E408	AFWL	1
E410	ADTC	1
E413	ESD	2
	FTD	
	CCN	1
	ETID	3
	NIA/PHS	1
	NICD	5