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A THEORETICAL TREATMENT OF CYCLIC PHENOMENA WITH PERIODIC PHASE--ETC(U)

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A THEORETICAL TREATMENT OF CYCLIC PHENOMENA WITH PERIODIC PHASE DISCONTINUITIES, PART II.
*title*

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# A Theoretical Treatment of Cyclic Phenomena with Periodic Phase Discontinuities, Part II

by

E. C. Westerfield

## Foreword

This memorandum covers only a small phase of the problem of instrumentation for signal enhancement research, and has been prepared primarily for internal distribution to aid others at NEL who may be interested in related problems. ~~Only a limited distribution outside of the laboratory is contemplated.~~ Work to August 1958 is covered.

CONTENTS



*See TM 279 for Part I.

## Introduction

Part I of this series, consisting of three sections:

1. Mathematical Background
2. Theoretical Development
3. Treatment of Single Frequency Input

was issued as TM 279.

In Part II the original numbering sequence is continued for convenience of reference. This treatment extends the results of Part I to a finite band of frequencies and considers the effect of Doppler within specified limits.

In Part III, to follow, it is planned to extend the effect of Doppler to large values requiring compression of the matching signal. It is also planned to investigate the combined effect of clipping and sampling.

## Application to Single Sideband Signals

Suppose now that the stored signal in channel b prior to speed-up is a section of duration $T = NR$, centered at $t = 0$, from the waveform

$$u(t) = \Sigma_s C_s \cos(2\pi \phi_s t + \beta_s). \qquad 4\text{-}1$$

And suppose that the signal in channel a prior to speed-up is a delayed, heterodyned, single sideband version of the same waveform, the delay $t_o$ being the time required for this section of the signal to reach and be sampled into the primary ARMS unit, $\phi$ being the heterodyne frequency prior to speed-up and $\beta$ its phase. Then for a given frequency $\phi_s$ in $u(t)$ prior to speed-up the following relations exist after speed-up for the equations in section 3.

$$\phi_a = N(\phi_s + \phi), \qquad 4\text{-}2$$

$$\phi_b = N\phi_s, \qquad 4\text{-}3$$

$$\phi_\delta = N\phi, \qquad 4\text{-}4$$

$$\beta_a = \beta_s + \beta - 2\pi\phi_s t_o, \qquad 4\text{-}5$$

$$\beta_b = \beta_s, \qquad 4\text{-}6$$

and

$$\beta_\delta = \beta - 2\pi\phi_s t_o. \qquad 4\text{-}7$$

Employing these relations the expression in (3-23) becomes

$$(1/2) \; \Sigma_r \operatorname{sinc}(\phi_s + \phi + r/R - N\phi)R \cos\left[2\pi(\phi_s + \phi + r/R)t + \beta - 2\pi\phi_s t_o\right] \qquad 4\text{-}8$$

as difference-frequency output waveform resulting from one of the input signal frequencies $\phi_s$. Ignoring the sum frequencies and the cross-product difference frequencies, the output waveform for the input $u(t)$ in equation (4-1) will be

$$(1/2) \; \Sigma_s \left\{C_s^2 \; \Sigma_r \operatorname{sinc}(\phi_s + \phi + r/R - N\phi)R \cos\left[2\pi(\phi_s + \phi + r/R)t + \beta - 2\pi\phi_s t_o\right]\right\}$$

$$= (1/2) \; \Sigma_r \left\{\Sigma_s C_s^2 \operatorname{sinc}(\phi_s + \phi + r/R - N\phi)R \cos\left[2\pi(\phi_s + \phi + r/R)t + \beta - 2\pi\phi_s t_o\right]\right\}. \qquad 4\text{-}9$$

For a rectangular input spectrum where $C_s = C$, this reduces to

$$(C^2/2) \; \Sigma_r \left\{\Sigma_s \operatorname{sinc}(\phi_s + \phi + r/R - N\phi)R \cos\left[2\pi(\phi_s + \phi + r/R)t + \beta - 2\pi\phi_s t_o\right]\right\} \qquad 4\text{-}10$$

Taking $\phi_1$ as the lowest positive frequency in the storage input signal $u(t)$, $\phi_c$ as the center frequency and $\phi_n$ as the highest frequency in the band, and restricting $\phi$ and $\phi_n$ each to be less than $1/2R$, it is seen that the spectrum of (4-10) consists of a series of component bands of width less than $1/2R$ spaced $1/R$ apart as illustrated in figure 2. The amplitude of the frequencies in these bands is determined by the sinc function. Taking $r_m$ as the nearest integer to $\left[(N-1)\phi - \phi_c\right]R$, it is seen that the component band with greatest power in (4-10) is given by

$$(C^2/2)\ \Sigma_s\ \mathrm{sinc}(\phi_s+\phi+r_m/R-N\phi)R\ \cos\left[2\pi(\phi_s+\phi+r_m/R)t+\beta-2\pi\phi_s t_o\right]\ . \qquad 4\text{-}11$$

An ideal filter of width $\Delta\phi = \phi_n - \phi_1$ centered at $\phi_c+\phi+r_m/R$ would select this band and exclude the others. The position of this sinc function in figure 2 depends primarily on $N\phi$, so for $N$ large compared to unity it is seen that the component bands in (4-10) are comparatively stationary. In effect the variation of $\phi$ simply slides the sinc function along thereby producing a variation in the relative amplitudes of the component bands without appreciably altering their position. To adjust the band in (4-11) to peak power it is only necessary to adjust $\phi$ to make

$$N\phi = \phi_c+\phi+r_m/R \qquad 4\text{-}12$$

that is

$$\phi = (\phi_c+r_m/R)/(N-1). \qquad 4\text{-}13$$

This centers the sinc function on the band as illustrated in figure 3, and equation (4-11) reduces to

$$(C^2/2)\ \Sigma_s\ \mathrm{sinc}(\phi_s-\phi_c)R\ \cos\left[2\pi(\phi_s+\phi+r_m/R)t+\beta-2\pi\phi_s t_o\right]\ . \qquad 4\text{-}14$$

Returning to equation (4-9) it is apparent that without the restriction on the amplitudes this component band would have the waveform

$$(1/2)\ \Sigma_s C_s^2\ \mathrm{sinc}(\phi_s-\phi_c)R\ \cos\left[2\pi(\phi_s+\phi+r_m/R)t+\beta-2\pi\phi_s t_o\right]\ . \qquad 4\text{-}15$$

Since

$$\left|\phi_s-\phi_c\right| < 1/4R \qquad 4\text{-}16$$

$\mathrm{sinc}(\phi_s-\phi_c)R$ will not differ greatly from unity and (4-15) becomes approximately

$$(1/2)\ \Sigma_s C_s^2 \ \cos\left[2\pi(\phi_s+\phi+r_m/R)t+\beta-2\pi\phi_s t_o\right] \ . \qquad 4\text{-}17$$

The signal in the storage unit on channel b is recycled indefinitely while waiting for the equivalent signal to reach and be sampled into the primary unit on channel a, hence $t-t_o$ is effectively the relative delay of the signal in channel a. Representing this relative delay by $\tau$, expression (4-17) becomes

$$(1/2)\ \Sigma_s C_s^2 \cos\ \left[2\pi(\phi_s+\phi+r_m/R)\tau+\beta_o\right] \qquad 4\text{-}18$$

where

$$\beta_o = \beta+2\pi(\phi+r_m/R)t_o \qquad 4\text{-}19$$

and

$$\tau = t-t_o \qquad 4\text{-}20$$

The expression in (4-18) begins to look a little more like an autocorrelation function. Following the method of Faran and Hills[5] the summation in this expression

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[5] James J. Faran, Jr. and Robert Hills, Jr., "Correlators for Signal Reception," Harvard Univ., Acoustics Research Laboratory, TM No. 27, 15 Sept 1952, p. 8.

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may be replaced by an integral

$$(1/2)\int C^2(f)\ \cos\ \left[2\pi(f+\phi+r_m/R)\tau+\beta_o\right]\ df \qquad 4\text{-}21$$

which for a rectangular spectrum reduces to

$$(C^2/2)\int_{\phi_1}^{\phi_n} \cos\ \left[2\pi(f+\phi+r_m/R)\tau+\beta_o\right]\ df$$

$$= E\ \mathrm{sinc}\ (\tau\Delta\phi)\cos\ \left[2\pi(\phi_c+\phi+r_m/R)\tau+\beta_o\right] \qquad 4\text{-}22$$

where

$$\Delta\phi = \phi_n-\phi_1 \qquad 4\text{-}23$$

is the frequency spread in the input $u(t)$ and

$$E = C^2\Delta\phi/2. \qquad 4\text{-}24$$

## Effect of Doppler

For non-zero doppler the situation is somewhat different to that portrayed in section 4, since now the difference frequency $\phi_\delta$ is no longer simply $N\phi$. As before it is assumed that the signal in the storage ARMS or equivalent unit is a sped-up replica of the random waveform

$$u(t) = \Sigma_s C_s \cos(2\pi\phi_s t + \beta_s). \qquad \text{5-1}$$

The signal in the primary ARMS unit at time $t = t_o$ will now however be taken as a dopplered version of that treated in section 4. It is assumed that the signal has been heterodyned up with a frequency $\phi_g$ prior to transmission of the upper side band. The doppler occurs during transmission and the resulting signal is heterodyned back down by a frequency $\phi_h$ prior to sampling and speed-up. $\phi$ now will be taken as $\phi_g - \phi_h$ so that for zero doppler it will have the same significance as in section 4. It is seen that for a given frequency $\phi_s$ in $u(t)$ the following relations will now exist for the case of a fractional doppler D:

$$\phi_a = N(\phi_s + \phi + D\phi_s + D\phi_g), \qquad \text{5-2}$$

$$\phi_b = N\phi_s, \qquad \text{5-3}$$

$$\phi_\delta = N(\phi + D\phi_s + D\phi_g), \qquad \text{5-4}$$

$$\beta_a = \beta_s + \beta - 2\pi(\phi_s + D\phi_s)t_o, \qquad \text{5-5}$$

$$\beta_b = \beta_s, \qquad \text{5-6}$$

and

$$\beta_\delta = \beta - 2\pi(\phi_s + D\phi_s)t_o. \qquad \text{5-7}$$

Corresponding to (4-9) but dropping the factor 1/2, the correlator output waveform will now be

$$\Sigma_r \left\{ \Sigma_s C_s^2 \operatorname{sinc}\left[\phi_s + r/R - (N-1)(\phi + D\phi_s + D\phi_g)\right] R \cos\left[2\pi(\phi_s + \phi + D\phi_s + D\phi_g + r/R)t + \beta - 2\pi(\phi_s + D\phi_s)t_o\right]\right\}. \qquad \text{5-8}$$

Taking $\tau = t - t_o$ as in section 4, this reduces to

$$\Sigma_r \left\{ \Sigma_s C_s^2 \operatorname{sinc}\left[\phi_s + r/R - (N-1)(\phi + D\phi_s + D\phi_g)\right] R \cos\left[2\pi(\phi_s + \phi + D\phi_s + D\phi_g + r/R)\tau + \beta_r\right]\right\}. \qquad \text{5-9}$$

where

$$\beta_r = \beta + 2\pi(\phi + r/R + D\phi_g)t_o \tag{5-10}$$

Equation (4-13) may be written

$$(N-1)\phi = \phi_c + r_m/R. \tag{5-11}$$

Taking $D_j$ as the fractional doppler satisfying the relation

$$D_j(N-1)(\phi_c + \phi_g) = j/R \tag{5-12}$$

where $j$ is an integer, and writing

$$r_j = r_m + j, \tag{5-13}$$

the center frequency of the $r_j$ band in (5-9) may be written

$$\theta_{r_j} = \phi_c + \phi + D_j(\phi_c + \phi_g) + r_j/R = \phi_c + \phi + j/(N-1)R + r_j/R, \tag{5-14}$$

and, since from (5-11) and (5-12) the corresponding sinc argument is

$$\phi_c + r_j/R - (N-1)(\phi + D_j\phi_c + D_j\phi_g) = 0, \tag{5-15}$$

it is seen that the sinc amplitude for this center frequency is unity.

For simplicity in the following discussion it will be assumed that

$$g = 2(\phi_c + \phi_g)R \tag{5-16}$$

is an odd integer. From (5-12) it follows that

$$D_j = 2j/(N-1)g \tag{5-17}$$

and

$$D_g = 2/(N-1) \tag{5-18}$$

Making use of relations (5-11,12,14, and 17) the waveform of the rth band in (5-9) may now be written

$$\begin{aligned} &\Sigma_s C_s^2 \ \text{sinc}\left[\phi_s - \phi_c + (r-r_j)/R - (N-1)(D_j\phi_s - D_j\phi_c)\right] R \\ &\qquad \cos\left\{2\pi\left[\phi_s - \phi_c + (r-r_j)/R + (D_j\phi_s - D_j\phi_c) + \theta_{r_j}\right]\tau + \beta_r\right\} \\ = \ &\Sigma_s C_s^2 \ \text{sinc}\left[\phi_s - \phi_c + (r-r_j)/R - (2j/g)(\phi_s - \phi_c)\right] R \\ &\qquad \cos\left\{2\pi\left[\phi_s - \phi_c + (r-r_j)/R + (2j/g)(\phi_s - \phi_c)/(N-1) + \theta_{r_j}\right]\tau + \beta_r\right\} \end{aligned} \tag{5-19}$$

for the fractional doppler $D_j$. From (5-19) it is apparent that the peak power occurs in the $r_j$ band

$$\sum_s C_s^2 \ \text{sinc}\left[\phi_s - \phi_c - 2(j/g)(\phi_s - \phi_c)\right] R \ \cos\left\{ 2\pi\left[\phi_s - \phi_c + 2(j/g)(\phi_s - \phi_c)/(N-1) + \theta_{r_j}\right]\tau + \beta_{r_j}\right\} . \qquad \text{5-20}$$

For values of $j$ in the range $0 \leq j \leq g$ the argument of the sinc function for a given $\phi_s$ does not exceed $\left|(\phi_s - \phi_c)R\right|$ in absolute value. From (4-16) therefore the sinc function will not differ greatly from unity over the $r_j$ band, and, analogous to (4-17), equation (5-20) becomes approximately

$$\sum_s C_s^2 \ \cos\left\{ 2\pi\left[\phi_s - \phi_c + 2\left(\frac{j}{g}\right)(\phi_s - \phi_c)/(N-1) + \theta_{r_j}\right]\tau + \beta_{r_j}\right\} \qquad \text{5-21}$$

From (5-21) it is seen that the bandwidth of the $r_j$ band is $\left[1+2j/(N-1)g\right]\Delta\phi$, and since the center is $\theta_{r_j}$ it follows that, corresponding to (4-22), for a rectangular input spectrum (5-21) reduces to

$$E \ \text{sinc}\left\{\left[1+2j/(N-1)g\right]\tau\Delta\phi\right\}\cos(2\pi\theta_{r_j}\tau + \beta_{r_j}) \qquad \text{5-22}$$

If $j$ is small compared to $N/2$, $j/(N-1)R$ will be small compared to $1/2R$ and for $\Delta\phi \approx 1/2R$ equation (5-14) will be represented to a good approximation by

$$\theta_{r_j} \approx \phi_c + \phi + r_j/R \qquad \text{5-23}$$

and (5-22) will reduce to

$$E \ \text{sinc}\,(\tau\Delta\phi) \ \cos\left[2\pi(\phi_c + \phi + r_j/R)\tau + \beta_{r_j}\right] \qquad \text{5-24}$$

to a good approximation. If $g$ is small compared to $N/2$ (5-24) will be a good approximation for the $r_j$ band for a Doppler $D_j$ for all values of $j$ in the range $0 \leq j \leq g$. For $j$ outside of this range the sinc function in (5-20) changes too rapidly with $\phi_s$ and (5-21) cannot be taken as a good approximation to (5-20). It follows from (5-17) that for dopplers greater than $2/(N-1)$ as well as for dopplers less than zero the stored signal in channel b should be compressed or expanded to approximately match the doppler in the signal in channel a. This process is treated in a later section.

For a fractional doppler

$$D_x = 2x/(N-1)g \qquad \text{5-25}$$

where $x$ is not necessarily an integer, it is seen from equations (5-15) and (5-17) that the argument of the sinc function in (5-9) may be written in the form

$$\left[\phi_s - \phi_c + (r - r_j)/R - 2(x/g)(\phi_s - \phi_c) - (N-1)(D_x - D_j)(\phi_c + \phi_g)\right]R, \qquad 5\text{-}26$$

and from (5-16), (5-17) and (5-25) this reduces to

$$\left[\phi_s - \phi_c + (r - r_j)/R - 2(x/g)(\phi_s - \phi_c) - (x-j)/R\right] R. \qquad 5\text{-}27$$

From (4-16) it is seen that for values of $x$ satisfying the relation

$$|x-j| \leq 1/4, \; j = 0,1,\ldots.g \qquad 5\text{-}28$$

the value of the argument of this sinc function for the $r_j$ band will not exceed 1/2 and hence the sinc function will not differ too greatly from unity over the $r_j$ band. For $x$ in this range (5-21) and (5-22) will therefore apply to a good approximation. If in addition $j$ is small compared to $N/2$ equation (5-24) and (5-23) will still apply also. A bank of $2g+1$ filters of width $1/2R$ centered at the values $\phi_c + \phi + r_j/R$ will thus be adequate to resolve the Doppler values $D_x$ for the ranges of $x$ defined by equation (5-28). For values of $x$ satisfying the relation

$$1/4 < |x-j| \leq 1/2 \qquad 5\text{-}29$$

an appropriate contraction or expansion of the comparison signal may be used to recenter the sinc function and again effectively shift the $x$ values into a more favorable range equivalent to that in (5-28). The treatment of this process is also reserved for a later section 5.

The above development is based on the assumption that $\phi$ has been chosen to satisfy (5-11) in agreement with (4-13) which was set up for the case of zero doppler. This restriction may be removed by replacing equations (5-11) and (5-12) respectively by

$$(N-1)\phi = \phi_c + (r_m + \rho)/R \qquad 5\text{-}30$$

and

$$D_j(N-1)(\phi_c + \phi_g) = (j - \rho)R \qquad 5\text{-}31$$

where $\rho$ is restricted to the range

$$-\tfrac{1}{2} \leq \rho < 1/2. \qquad 5\text{-}32$$

Equations (5-13), (5-14), (5-15) and (5-16) remain unchanged but equations (5-17) and (5-18) respectively would need to be replaced by

$$D_j = 2(j-\rho)/(N-1)g \qquad 5\text{-}33$$

and

$$D_g = 2(1-\rho/g)/(N-1). \qquad 5\text{-}34$$

Similarly (5-19) and (5-20) would change to

$$\Sigma_s C_s^2 \ \text{sinc}\left[\phi_s-\phi_c+(r-r_j)/R-2(j-\rho)(\phi_s-\phi_c)/g\right]R$$
$$\cos\left\{2\pi\left[\phi_s-\phi_c+(r-r_j)/R+2(j-\rho)(\phi_s-\phi_c)/g(N-1)+\theta_{r_j}\right]\tau+\beta_r\right\} \qquad 5\text{-}35$$

and

$$\Sigma_s C_s^2 \ \text{sinc}\left[\phi_s-\phi_c-2(j-\rho)(\phi_s-\phi_c)/g\right]R$$
$$\cos\left\{2\pi\left[\phi_s-\phi_c+2(j-\rho)(\phi_s-\phi_c)/g(N-1)+\theta_{r_j}\right]\tau+\beta_{r_j}\right\} \qquad 5\text{-}36$$

respectively. For values of $j$ in the range 0 to $g$ the argument of the sinc function for the $r_j$ band in (5-36) will range from $(1+2\rho/g)(\phi_s-\phi_c)R$ to $(-1+2\rho/g)(\phi_s-\phi_c)R$. For $\rho\neq 0$ this gives an unsymmetrical range for the output sinc function for the indicated range in $j$. If $j$ is restricted to the range from 0 to $g-1$ however and $\rho$ is taken as $-1/2$ the argument of the sinc function in (5-36) will range from $(1-1/g)(\phi_s-\phi_c)R$ to $(-1+1/g)(\phi_s-\phi_c)R$ and symmetry is thus restored. This case is of particular interest in later [illegible] since it results in a reduction of the number is of output filters required to cover a large range in doppler. For this case equations (5-30) and (5-33) give

$$\phi=\left[\phi_c+(r_m-1/2)/R\right]/(N-1) \qquad 5\text{-}37$$

and

$$D_j = (2j+1)/(N-1)g. \qquad 5\text{-}38$$

Lower Sideband and Double Sideband Operation

In section 4 the signal in the primary ARMS unit was assumed to be a heterodyned, single sideband version of that in the storage ARMS. A positive value for $\phi$ in section 4 corresponds to the upper sideband. The lower sideband requires special consideration.

For the lower side band $\phi$ can be replaced by $-\phi$ and $\beta$ by $-\beta$ in equations (4-2) to (4-7). From (4-13) this would lead to a negative value for $r_m$, so $r$ would need to be replaced by $-r$ in (4-8) and subsequent equations in order to reflect the principal spectral band into the positive region. With these changes (4-8) becomes

$$(1/2)\ \Sigma_r \text{ sinc } (\phi_s - \phi - r/R + N\phi)R \cos\left[2\pi(\phi_s - \phi - r/R)t - \beta - 2\pi\phi_s t_o\right]$$

$$= (1/2)\ \Sigma_r \text{ sinc } (\phi - \phi_s + r/R - N\phi)R \cos\left[2\pi(\phi - \phi_s + r/R)t + \beta + 2\pi\phi_s t_o\right] \qquad 6\text{-}1$$

from which it is seen that these changes are equivalent to replacing $\phi_s$ by $-\phi_s$ and $\beta_s$ by $-\beta_s$ in (4-8) and subsequent equations. Thus (4-13) becomes

$$\phi = (r_m/R - \phi_c)/(N-1) \qquad 6\text{-}2$$

to center the sinc function on the band. And (4-15) becomes

$$(1/2)\ \Sigma_s C_s^{\,2} \text{ sinc } (\phi_s - \phi_c)R \cos\left[2\pi(\phi - \phi_s + r_m/R)t + \beta + 2\pi\phi_s t_o\right] \qquad 6\text{-}3$$

for the resulting waveform. These same results could have been obtained equally well by replacing $\phi_s$ by $-\phi_s$ and $\beta_s$ by $-\beta_s$ in (4-2) and (4-5). This would have given

$$\phi_\sigma = N\phi \qquad 6\text{-}4$$

and

$$\beta_\sigma = \beta + 2\pi\phi_s t_o. \qquad 6\text{-}5$$

The sum-frequency waveform given in (3-25) could then have been used instead of the difference frequency waveform of (3-23) to obtain the second member of (6-1) directly. No reflection by reversing the sign of $r$ would have been necessary in this case.

Comparing (4-15) and (6-3), it is seen that the two output bands are symmetrically disposed about $\phi + r_m/R$. For double sideband operation the sinc function may therefore be centered on $\phi + r_m/R$ for peak output. This is equivalent to replacing $\phi_c$ by 0 in equations (4-12) to (4-15) and in (6-2) and (6-3). This gives

$$\phi = r_m/(N-1)R \qquad 6\text{-}6$$

for both upper and lower sidebands. The output waveform for the upper sideband is then

$$(1/2)\ \Sigma_s C_s^2 \text{ sinc } \phi_s R \cos\left[2\pi(\phi + \phi_s + r_m/R)t + \beta - 2\pi\phi_s t_o\right] \qquad 6\text{-}7$$

and for the lower sideband

$$(1/2)\ \Sigma_s C_s^2 \text{ sinc } \phi_s R \cos\left[2\pi(\phi - \phi_s + r_m/R)t + \beta + 2\pi\phi_s t_o\right]. \qquad 6\text{-}8$$

The combined output waveform for the double sideband is thus

$$(1/2)\ \Sigma_s C_s^2 \text{ sinc}\,\phi_s R \left\{\cos\left[2\pi(\phi + \phi_s + r_m/R)t + \beta - 2\pi\phi_s t_o\right] + \cos\left[2\pi(\phi - \phi_s + r_m/R)t + \beta + 2\pi\phi_s t_o\right]\right\}$$

$$= \Sigma_s C_s^2 \text{ sinc } \phi_s R \cos\left[2\pi(\phi + r_m/R)t + \beta\right] \cos\left[2\pi\phi_s(t - t_o)\right]$$

$$= \cos\left[2\pi(\phi + r_m/R)t + \beta\right] \Sigma_s C_s^2 \text{ sinc } \phi_s R \cos\left[2\pi\phi_s(t - t_o)\right]. \qquad 6\text{-}9$$

## Effect of Finite Duration of Sample

In section 3 the effect of sampling was investigated by use of a repeated $\delta$ function. For samples of short duration compared to the interval between samples, the $\delta$ function provides a fair approximation. In normal operation of the ARMS or DELTIC, however, the instantaneous values of the sample may be stretched out to approximate a square pulse of duration as great as 1/2 or even all of the interval between samples. In normal operation the beginning of this square pulse is the time at which the instantaneous sample is taken. That is, the position of the $\delta$ function would correspond to the beginning of the square pulse. For simplicity in the following development, however, the $\delta$ function will be taken to coincide with the middle of the square pulse. For most purposes this simplification will make no significant difference in the final result.

From (3-5) the impulse sampled output waveform of the storage ARMS may be written

$$\sum_n \left\{ \text{rect}\,(t-nR)/R \cos\left[2\pi\phi_b(t-nR)+\beta_b\right] \;\text{rep}_S\, \delta(t-nR)\right\}$$

$$= \sum_n \left\{ \text{rect}\,(t-nR)/R \cos\left[2\pi\phi_b(t-nR)+\beta_b\right] \sum_r \delta(t-rS-nR)\right\}$$

$$= \sum_n \left\{ \text{rect}\,(t-nR)/R \sum_r \left[\cos(2\pi\phi_b rS+\beta_b)\;\delta(t-rS-nR)\right]\right\} . \qquad \text{7-1}$$

To represent the stretching of these pulse samples into square pulses of duration $Q \leq S$, it is necessary to replace the $\delta$ function in the last member of (7-1) by a rect function, obtaining

$$\sum_n \left\{ \text{rect}(t-nR)/R \sum_r \left[\cos(2\pi\phi_b rS+\beta_b)\;\text{rect}(t-rS-nR)/Q\right]\right\} \qquad \text{7-2}$$

as the resulting output waveform. From (2-24), the corresponding spectrum is

$$|Q/2S| \sum_n \Big[\text{sinc}(f-\phi_b-n/S)R\;\text{sinc}(\phi_b+n/S)Q\;\exp i\beta_b$$

$$+\;\text{sinc}\,(f+\phi_b-n/S)R\;\text{sinc}(\phi_b-n/S)Q\;\exp(-i\beta_b)\Big]\;\text{rep}_{1/R}\,\delta(f). \qquad \text{7-3}$$

This may be compared with the expression in (3-6) obtained by use of the $\delta$ function. For sample pulses sufficiently short it is seen that the two expressions give essentially the same results. To obtain a corresponding expression to replace (3-11) as the spectrum of the primary ARMS output, the impulse sampled waveform may first be written as

$$\sum_n \left\{ \mathrm{rect}(t-nR)/R \cos\left[2\pi\phi_a(t-nP)+\beta_a\right] \mathrm{rep}_S\,\delta(t-nR)\right\}$$

$$= \sum_n \left\{ \mathrm{rect}(t-nR)/R \cos\left[2\pi\phi_a(t-nP)+\beta_a\right] \sum_r \delta(t-nR-rS)\right\}$$

$$= \sum_n \left\{ \mathrm{rect}(t-nR)/R \sum_r \left[\cos\left\{2\pi\phi_a(nR+rS-nP)+\beta_a\right\} \delta(t-nR-rS)\right]\right\} . \qquad 7\text{-}4$$

Since, from (3-7), R-P = S, this reduces to

$$\sum_n \left\{ \mathrm{rect}(t-nR)/R \sum_r \left[\cos\left\{2\pi\phi_a(n+r)S+\beta_a\right\} \delta(t-nR-rS)\right]\right\} . \qquad 7\text{-}5$$

As in the case of the storage unit, in order to represent the stretched pulse samples it is now necessary to replace the $\delta$ function by a rect function. Thus (7-5) is converted to

$$\sum_n \left\{ \mathrm{rect}(t-nR)/R \sum_r \left[\cos\left\{2\pi\phi_a(n+r)S+\beta_a\right\} \mathrm{rect}(t-nR-rS)/Q\right]\right\} \qquad 7\text{-}6$$

and, from (2-25), the corresponding spectrum is

$$\left|Q/2S\right| \left\{ \sum_n \left[\mathrm{sinc}(f-\phi_a+n/S)R\ \mathrm{sinc}(\phi_a-n/S)Q \exp i\beta_a\right] \mathrm{rep}_{1/R}\ \delta(f-\phi_3)\right.$$

$$\left. + \sum_n \left[\mathrm{sinc}(f+\phi_a-n/S)R\ \mathrm{sinc}\ (\phi_a-n/S)Q \exp(-i\beta_a)\right] \mathrm{rep}_{1/R}\ \delta(f+\phi_3)\right\} \qquad 7\text{-}7$$

where, from (2-26) and (3-4),

$$\phi_3 = \phi_a\, S/R = \phi_a/N. \qquad 7\text{-}8$$

It should be possible to obtain the spectrum of the correlator output by combining (7-3) and (7-7), however it seems simpler to start from the product waveform as was done in section 3. From (3-12) the impulse sampled output waveform of the correlator is given by

$$\Sigma_n \left\{ \text{rect}\,\frac{t-nR}{R}\; \cos\left[2\pi\phi_a(t-nP)+\beta_a\right] \cos\left[2\pi\phi_b(t-nR)+\beta_b\right] \text{rep}_S\,\delta(t-nR)\right\}$$

$$= \Sigma_n \left\{ \text{rect}\,\frac{t-nR}{R}\; \cos\left[2\pi\phi_a(t-nP)+\beta_a\right] \cos\left[2\pi\phi_b(t-nR)+\beta_b\right] \Sigma_r\, \delta(t-nR-rS)\right\}$$

$$= \Sigma_n \left\{ \text{rect}\,\frac{t-nR}{R}\; \Sigma_r \left[\cos\left\{2\pi\phi_a(nR+rS-nP)+\beta_a\right\} \cos(2\phi_b rS+\beta_b)\,\delta(t-nR-rS)\right]\right\}$$

$$= \Sigma_n \left\{ \text{rect}\,\frac{t-nR}{R}\; \Sigma_r \left[\cos\left\{2\pi\phi_a(n+r)S+\beta_a\right\} \cos(2\pi\phi_b rS+\beta_b)\,\delta(t-nR-rS)\right]\right\}. \tag{7-9}$$

Replacing the $\delta$ function by the corresponding rect gives

$$\Sigma_n \left\{ \text{rect}\,\frac{t-nR}{R}\; \Sigma_r \left[\cos\left\{2\pi\phi_a(n+r)S+\beta_a\right\} \cos(2\pi\phi_b rS+\beta_b)\,\text{rect}(t-nR-rS)/Q\right]\right\}$$

$$= (1/2)\ \Sigma_n \left\{ \text{rect}\,\frac{t-nR}{R}\; \Sigma_r \left[\cos\left\{2\pi\left[\phi_a nS+(\phi_a-\phi_b)rS\right]+\beta_a-\beta_b\right\} \text{rect}(t-nR-rS)/Q\right]\right\}$$

$$+ (1/2)\ \Sigma_n \left\{ \text{rect}\,\frac{t-nR}{R}\; \Sigma_r \left[\cos\left\{2\pi\left[\phi_a nS+(\phi_a+\phi_b)rS\right]+\beta_a+\beta_b\right\} \text{rect}(t-nR-rS)/Q\right]\right\}. \tag{7-10}$$

From (2-27) the corresponding spectrum is

$$\begin{aligned} |Q/4S| \Big\{ & \Sigma_n \left[\text{sinc}(f-\phi_\delta+n/S)R\ \text{sinc}(\phi_\delta-n/S)Q\ \exp i\beta_\delta\ \text{rep}_{1/R}\ \delta(f-\phi_3)\right] \\ & + \Sigma_n \left[\text{sinc}(f+\phi_\delta-n/S)R\ \text{sinc}(\phi_\delta-n/S)Q\ \exp(-i\beta_\delta)\ \text{rep}_{1/R}\ \delta(f+\phi_3)\right] \\ & + \Sigma_n \left[\text{sinc}(f-\phi_\sigma+n/S)R\ \text{sinc}(\phi_\sigma-n/S)Q\ \exp i\beta_\sigma\ \text{rep}_{1/R}\ \delta(f-\phi_3)\right] \\ & + \Sigma_n \left[\text{sinc}(f+\phi_\sigma-n/S)R\ \text{sinc}(\phi_\sigma-n/S)Q\ \exp(-i\beta_\sigma)\text{rep}_{1/R}\ \delta(f+\phi_3),\right]\Big\} \end{aligned} \tag{7-11}$$

where $\phi_3$ is still represented by (7-8),

$$\phi_\delta = \phi_a - \phi_b\,, \tag{7-12}$$

$$\phi_\sigma = \phi_a + \phi_b\,, \tag{7-13}$$

$$\beta_\delta = \beta_a - \beta_b\,, \tag{7-14}$$

and $$\beta_\sigma = \beta_a + \beta_b\,. \tag{7-15}$$

As noted following equation (2-23) the $n$ in (7-11) is unrelated to the $n$ in (7-10). As in section 3 the sum terms and higher order difference terms will usually be eliminated by filtering leaving

$$\left|Q/4S\right| \Big[\mathrm{sinc}(f-\phi_\delta)R \ \mathrm{sinc}\,\phi_\delta Q \ \exp(i\beta_\delta) \ \mathrm{rep}_{1/R} \ \delta(f-\phi_a/N)$$

$$+ \ \mathrm{sinc}(f+\phi_\delta)R \ \mathrm{sinc}(\phi_\delta Q) \ \exp(-i\beta_\delta)\mathrm{rep}_{1/R} \ \delta(f+\phi_a/N)\Big] \qquad 7\text{-}16$$

corresponding to (3-21). Corresponding to (3-22) this reduces to

$$\left|Q/2S\right| \ \mathrm{sinc}\,\phi_\delta Q \ \Sigma_r \ \Big\{\mathrm{sinc}(\phi_a/N+r/R-\phi_\delta)R \ \mathrm{spec} \ \cos\left[2\pi(\phi_a/N+r/R)t+\beta_\delta\right]\Big\}$$

$$= \left|Q/2S\right| \ \mathrm{sinc}\,\phi_\delta Q \ \mathrm{spec} \ \Sigma_r \ \Big\{\mathrm{sinc}(\phi_a/N+r/R-\phi_\delta)R \ \cos\left[2\pi(\phi_a/N+r/R)t+\beta_\delta\right]\Big\}. \qquad 7\text{-}17$$

The resulting output waveform corresponding to (3-23) becomes, neglecting the constant factor $\left|Q/2S\right|$,

$$\mathrm{sinc}\,\phi_\delta Q \ \Sigma_r \ \Big\{\mathrm{sinc}(\phi_a/N+r/R-\phi_\delta)T \ \cos\left[2\pi(\phi_a/N+r/R)t+\beta_\delta\right]\Big\} \qquad 7\text{-}18$$

For the condition

$$\phi_\delta \leq 1/4Q \qquad . \ 7\text{-}19$$

it is seen that sinc $\phi_\delta Q$ does not differ greatly from unity. Hence (7-18) is essentially identical with (3-23) for this condition.

Figure 2. Illustration of band spectra output of Deltic Correlator

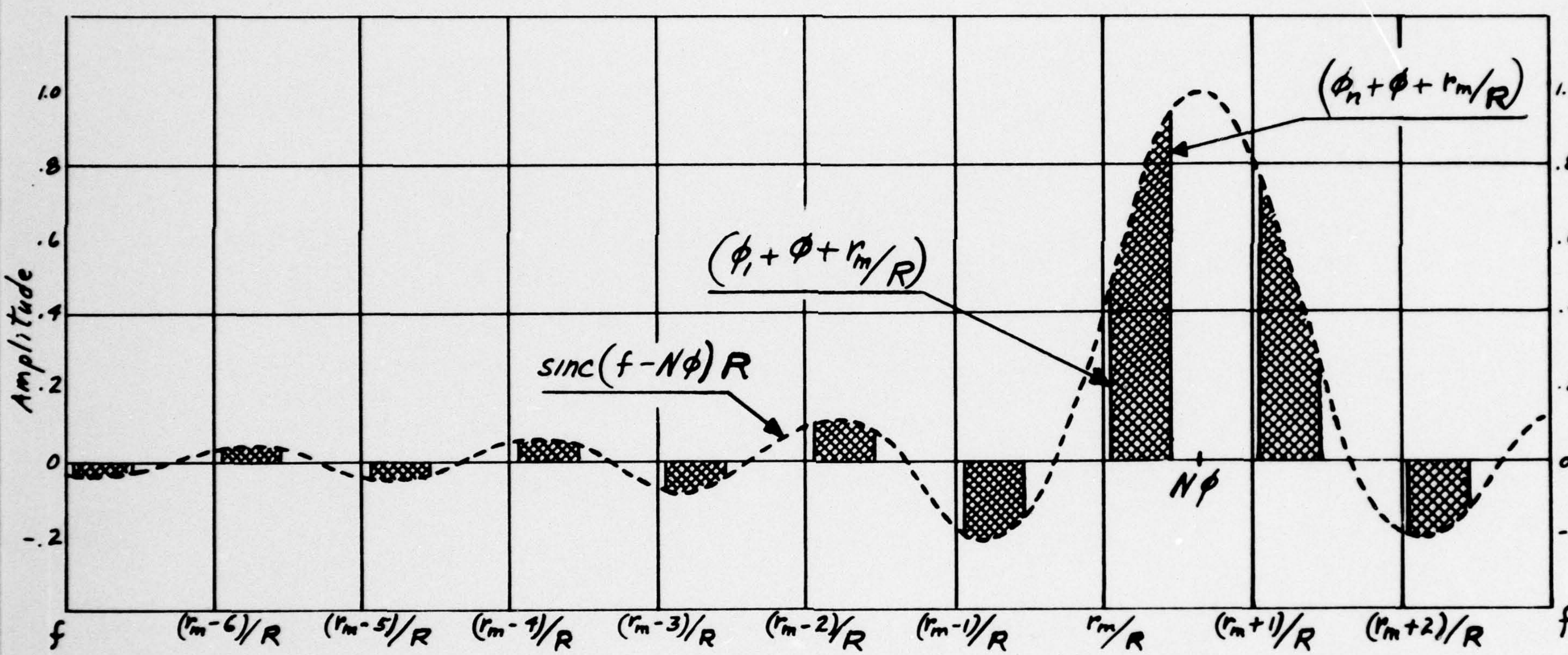

Figure 3. Illustration of band-spectra output of Deltic Correlator for $\phi = (\phi_c + r_m/R)/(N-1)$