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TRACOR 6500 TRACOR LANE, AUSTIN, TEXAS 78721

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# NORMALIZATION OF SPECTRUM ANALYZER OUTPUT DATA

## INTRODUCTION

In general, the broadband continuum noise background consists of either self noise, shipping noise, or ambient noise or combinations thereof. In any case the spectrum of this noise will not be flat, moreover the slope of the spectrum may be highly variable depending on the source of the noise and the range of frequencies being considered. This condition is undesirable from a broadband detection point of view, but from a narrowband detection standpoint it is an intolerable situation if left as is. That this is so can be seen readily by observing the spectral representation in Fig. 1.

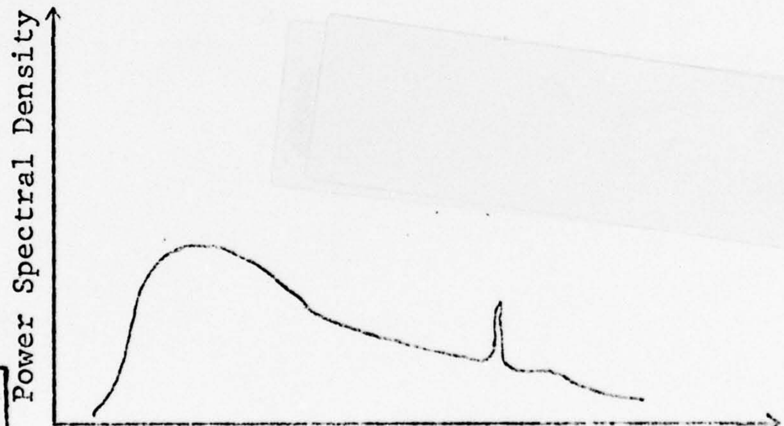


Fig. 1 REPRESENTATION OF A TYPICAL BACKGROUND SPECTRUM PLUS NARROWBAND SIGNAL





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This figure depicts a situation where a narrowband signal with a considerably large signal-to-noise ratio would, with high probability, go undetected because of a non uniform power distribution in frequency. If an OR-Gate were employed, this loss in detectability would result from capture of the OR-Gate by the frequency analyzer channels in the neighborhood of the frequency that coincides to the maximum in the power spectral density function. This situation can be rectified by normalizing the spectrum analyzer channel waveforms either before or after envelope detection has occurred. Since we are constrained to work with a hybrid spectrum analyzer which does not perform normalization prior to detection, it is necessary to normalize the data after detection. The means by which this is being accomplished is described in the remainder of this note.

PAST DETECTION NORMALIZATION

One of the primary objectives of any normalization scheme is to make the probability of exceeding the threshold independent of frequency. This may be referred to as holding the single channel false alarm probability constant. If we assume in the presence of noise alone (excluding self noise lines) that the spectrum analyzer channel outputs are Rayleigh distributed, then the single channel false alarm probability is given by

$$P'_{FA} = \int_T^\infty \frac{x}{\sigma_n^2} e^{-\frac{x^2}{2\sigma_n^2}} dx \, . \quad \text{want indep. of freq.} \qquad (1)$$

Now it is clear that this probability depends on only one parameter, namely $\sigma_n^2$. However, this parameter is the noise power in the bandwidth of the analysis filter and hence may be a function of frequency. There is a way around this, however, since the mean standard deviation, and mean square of x are related to $\sigma_n^2$ by

$$\mu_x = \sigma_n \sqrt{\frac{\pi}{2}} \tag{2}$$

$$\sigma_x = \sigma_n \sqrt{2 - \pi/2} \tag{3}$$

$$E[x^2] = 2\sigma_n^{\ 2}$$

Now suppose that we estimate $E[x^2] = P_x$ by means of the following statistic,

$$\hat{P}_x = \frac{1}{M} \sum_{i=1}^{M} x_i^{\ 2}$$

and divide each sample of x by $\hat{P}_x$, then since

$$E[\hat{P}_x] = P_x \quad ,$$

we would have a new random variable, y, given by

$$y = \frac{x}{\sqrt{\frac{1}{M} \sum^{M} x_i^{\ 2}}} \quad , \tag{4}$$

which for large M would have a Rayleigh density function with a mean square value of unity. If the mean square of y is unity, then the parameter in the distribution function of y would be 1/2 instead of $\sigma_n^{\ 2}$, thus

$$P_{FA}' = \int_T^{\infty} 2y \ e^{-y^2} \ dy \ . \tag{5}$$

This equation is independent of the parameter $\sigma_n^2$ and is thus not dependent on the noise power at the input to the detector -- hence the system is normalized.

Now let us examine exactly how this normalization is being accomplished in the computer. The output of the spectrum analyzer is digitized and entered into the computer via digital tape in a form which is most conveniently represented as a matrix $\underline{x}$.

In the time interval $[t_i, t_L]$ $\underline{x}$ has a form

$$\underline{x} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1L} \\ x_{21} & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ x_{N1} & & & x_{NL} \end{bmatrix}$$

where each of the $x_{ij}$ correspond to the output of the $j^{th}$ frequency channel at the $t_i$ instant of time.

To normalize the matrix $\underline{x}$ we define an operation such that

$$y_{Lj} = x_{Lj}/\hat{P}_{Lj}$$

where

$$P_{Lj} = \sum_{i=L-S_i}^{L} \left[ \sum_{j'=j-S_j-J}^{j-S_j} x_{ij'}^2 + \sum_{j'=j+S_j}^{j+S_j+J} x_{ij'}^2 \right]$$

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$S_j$ = number of frequency samples adjacent to the $j^{th}$ sample that are skipped

J = number of samples that are used in one frequency window

$S_i$ = number of time samples that are used in the time window.

Presently, a total of 90 samples are being used to compute $\hat{P}_{ij}$ so that it is possible to get normalization of the background noise to within ± 1 dB after transient effects have died away.

The width of the normalization windows in the frequency domain is presently 5 Hz on either side of the sample being normalized. This means that only non-stationarities that are wider than this will be normalized. Similarly, in the time domain the normalizing windows are 9 seconds in duration. As a result we expect to get temporal normalization only for those waveforms that exhibit non-stationarities greater than approximately 9 seconds in duration.

The normalizer has been checked out using an array of numbers with a non-zero slope in both the time a frequency domains and it was found to produce an array of numbers all of which are constant except for those around the edges of the matrix. Presently, we are preparing to check the normalizer out using sea data.

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# ANALYSIS OF THE SIMPLE OR-GATING DATA REDUCTION TECHNIQUE

## Introduction

It is worthwhile to develop theoretical expressions for the false alarm and detection probabilities of the data reduction technique known as OR-Gating or the "maximum-of" criterion. With the expressions in hand, it will be possible to predict the performance of this processing scheme so that comparisons can be made not only with experimental results obtained using the OR-Gate, but also with the performance of other data reduction techniques. It is the purpose of this paper to develop such theoretical expressions.

## Analysis

The function of the OR-Gate is to examine N (virtually) simultaneous spectrum analyzer output samples and then to select (gate) the maximum. Symbolically, if we denote the OR-Gate output at time $t_j$ by $y(t_j) = y_j$, we have

$$y_i = \max_i \{x_{ij}\}, \quad i = 1, 2, \ldots, N. \tag{1}$$

In order to predict false alarm and detection probabilities we must compute the probability that y exceeds a threshold T when noise alone is present and when signal-plus-noise is present. To do this we proceed as follows. The distribution function of y, $F_y(Y)$, is given by

$$F_y(Y) = Pr[y < Y],$$

but

$$y = \max \{x_i\}, \; i = 1, 2, \ldots N,$$

so that

$$F_y(Y) = Pr[\max\{x_i\} < Y].$$

Now if the maximum of the set $\{x_i\}$ is less than Y, it follows that all elements of the set are less than Y, hence

$$F_y(Y) = Pr[x_1 < Y, x_2 < Y, \ldots x_N < Y].$$

If the $x_i$ are independent, which we assume here, then

$$F_y(Y) = Pr[x_1 < Y] \cdot Pr[x_2 < Y] \cdots \cdot Pr[x_N < Y],$$

but by definition the distribution function of x, $F_x(X)$ is given by

$$F_x(X) = Pr[x < X].$$

Thus,

$$F_y(Y) = \prod_{i=1}^{N} F_{x_i}(Y).$$

If all of the $x_i$ are identically distributed, which they will be in the absence of signal, and if the continuum noise is uniformly distributed (same distribution form) in frequency, and if the normalizer performs properly, then

$$F_y(Y)_n = F_x^{\,N}(Y)_n . \tag{2}$$

If one of the channels contains signal plus noise, then $F_y(Y)$ becomes,

$$F_y(Y)_{s+n} = F_x^{\,N-1}(Y)_n \; F_x(Y)_{x+n} . \tag{3}$$

The probability of false alarm is given by,

$$\begin{aligned} P_{FA} &= 1 - F_y(T)_n \\ &= 1 - F_x^{\,N}(T)_n , \end{aligned} \tag{4}$$

while the detection probability is given by

$$\begin{aligned} P_D &= 1 - F_y(T)_{s+n} \\ &= 1 - F_x^{\,N-1}(T)_n \; F_x(T)_{s+n} . \end{aligned} \tag{5}$$

If we define an elementary (single channel) probability of false alarm by

$$P_{FA}' = 1 - F_x(T)_n , \tag{6}$$

and similarly a single channel probability of detection by

$$P_D' = 1 - F_x(T)_{x+n} , \tag{7}$$

then by substituting Eqs. (4) and (5) we get

$$P_{FA} = 1 - (1 - P'_{FA})^N, \tag{8}$$

and

$$P_D = 1 - (1 - P'_{FA})^{N-1} (1 - P'_D). \tag{9}$$

These equations reveal that the detection and false alarm probabilities at the output of the OR-Gate can be expressed in terms of elementary probabilities at the input to the OR-Gate. Specifically, the probability of exceeding the threshold, T, at the output of the OR-Gate is given by the probability that at least one of the channels exceeds the same threshold at the input to the OR-Gate. Thus, if we can specify the density or distribution at the output of the analyzer channels, we can get the required probabilities of false alarm and detection.

When noise alone is present and when an envelope detector is used, we have as the density function of x,

$$p(x) = \frac{x}{\sigma_n^2} e^{-\frac{x^2}{2\sigma_n^2}},$$

where

$\sigma_n^2$ = noise power in each analyzer channel.

Thus,

$$P'_{FA} = \int_T^\infty \frac{x}{\sigma_n^2} e^{-\frac{x^2}{2\sigma_n^2}} dx,$$

which is

$$P'_{FA} = e^{-T^2/2\sigma_n^2} \tag{10}$$

When signal plus noise is present in the channel, we get for $P_D'$,

$$P_D' = \int_T^\infty \frac{x}{\sigma_s^2 + \sigma_n^2} e^{-\frac{x^2}{2(\sigma_s^2 + \sigma_n^2)}} dx.$$

This assumes that the signal is a narrowband Gaussian process with power $\sigma_s^{2}$*.

Letting,

$$y = \frac{x}{\sigma_n},$$

gives

$$P_D' = \int_{T/\sigma_n}^{\infty} \frac{y}{1+a} e^{-\frac{y^2}{2(1+a)}} dy \tag{11}$$

where

$$a = \frac{\sigma_s^2}{\sigma_n^2}, \text{ the signal-to-noise ratio.}$$

Equation (11) can be integrated by letting $w = \frac{y}{(1+a)}$, thus

$$P_D' = e^{-\frac{T^2}{2\sigma_n^2(1+a)}} . \tag{12}$$

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*This is a reasonable assumption when one considers the effects of the ocean channel. That is, a pure tone originating at a source will undoubtly suffer considerable fading due to multipath effects and will result in a received signal that is the superposition of many "tones" which have random amplitudes and phases with correlation time comparable to reciprocal bandwidth.

By defining a normalized threshold,

$$T' = \frac{T^2}{2\sigma_n^2} .$$

Equations (10) and (12) become

$$P'_{FA} = e^{-T'} \qquad (13)$$

$$P'_D = e^{-T'/(1+a)} . \qquad (14)$$

Using Eqs. (13) and (14) we can get the required false alarm and detection probabilities, thus

$$P_{FA} = 1 - (1 - e^{-T'})^N \qquad (15)$$

$$P_D = 1 - (1 - e^{-T'})^{N-1} (1 - e^{-T'/(1+a)}) . \qquad (16)$$

With Eqs. (15) and (16) it is possible to compute ROC curves for the simple OR-Gate as a function of the signal-to-noise ratio, a. These curves may be used for comparison with experimental results as well as for comparing different data reduction techniques.

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# ANALYSIS OF THE MAXIMUM LIKELIHOOD RATIO DATA REDUCTION CRITERION

## INTRODUCTION

The maximum likelihood technique for deciding which spectrum analyzer channel is most likely to contain a target spectral line is based on computing the likelihood ratio of the data at the output of each channel of the spectrum analyzer and then subjecting the set of all such likelihood ratios to an OR-Gate. That is, the output, y(t), of such a system is given by

$$y(t) = \max_i \{L_i[x_i(t)]\} , \qquad (1)$$

where

$i = 1, 2, \ldots, N$ = the spectrum analyzer channel number, and

$x_i(t)$ = *the time function at the output of the* $i^{th}$ spectrum analyzer channel.

$$L_i(x_i) = \frac{p(x_i)_{s+n}}{p(x_i)_n} , \qquad (2)$$

where

$p_{s+n}(x_i)$ = probability density function of the $i^{th}$ spectrum analyzer channel when signal plus noise is present at its output, and

$p(x_i)$ = probability density function of the $i^{th}$ spectrum analyzer channel output.

There are two interpretations that can be placed on the likelihood ratio, the first of which is that it is the likelihood ratio for single output samples. The second interpretation is that L(x) is a joint likelihood ratio obtained by operating on successive outputs of each analyzer channel. It is the purpose of this note to examine these possible implementations and the consequences of each.

SINGLE SAMPLE LIKELIHOOD RATIO

In order to establish this likelihood ratio it is necessary to first determine the form of the two required density functions, $p(x)_{s+n}$ and $p(x)_n$. The function $p(x)_n$ will be a Rayleigh density function if the spectrum analyzer output detector is an envelope detector and if the analysis filter output is Gaussian. Both of these conditions are reasonable; the first because of practical considerations and the second because of physical considerations. The Rayleigh density function is given by

$$p(x)_n = \frac{x}{\sigma_n^2} e^{-x^2/2\sigma_n^2} , \qquad (3)$$

where

$\sigma_n^2$ = the noise power level at the input to the detector, i.e., in the bandwidth of the analysis filter.

To obtain the density of the detector output in the presence of signal plus noise it is necessary to first define what is meant by signal. The most reasonable model for a narrowband signal is that of a narrowband Gaussian process. This is true because of the multipath effects that will be experienced by tone-like signals in traveling from the source to the receiver. Thus the input to the envelope detector can be modeled by the sum of two Gaussian processes, one (the signal) with power $\sigma_s^2$ and the other (the noise) with power $\sigma_n^2$. This means that the envelope detector output is described by another Rayleigh density function, i.e.,

$$p(x)_{s+n} = \frac{x}{\sigma_s^2 + \sigma_n^2} e^{-x^2/2(\sigma_s^2+\sigma_n^2)} . \qquad (4)$$

The likelihood ratio is given by the ratio of Eqs. (4) and (3), thus

$$L(x) = \frac{\sigma_n^2}{\sigma_s^2 + \sigma_n^2} e^{+(x^2/2\sigma_n^2)-[x^2/2(\sigma_s^2+\sigma_n^2)]} , \quad (5)$$

Since the logarithm is a monotonic function, we can just as easily consider the logarithm of the likelihood ratio, i.e.,

$$\ell(x) \triangleq \log_e L(x)$$

$$= C_1 x^2 + C_2 , \quad (6)$$

where

$$C_1 = \frac{a}{\sigma_n^2(1 + a)} = \frac{a}{\sigma_s^2 + \sigma_n^2} ,$$

$$C_2 = \log_e\left(\frac{1}{a + 1}\right) , \text{ and}$$

$$a = \frac{\sigma_s^2}{\sigma_n^2}$$ = the pre-detector signal-to-noise ratio, this is a so-called design signal-to-noise ratio.

Some interesting observations can be made concerning Eq. (6). First, in order to compute $\ell(x)$ we have to know or estimate both the noise power $\sigma_n^2$ and the signal-to-noise ratio, a. This means that we would have to design for some minimum value of "a" and estimate $\sigma_n^2$, the latter being a simple job if the normalization is done properly. Second, it is clear that $\ell(x)$ is a monotonic function of x and thus if the $i^{th}$ channel output, $x_i$, is the maximum of all the frequency channels, then it follows that $L_i(x_i)$ will also be the maximum over all the likelihood ratios of the channel outputs. That is to say, if the $i^{th}$ channel is chosen by a simple OR-Gating of the frequency analyzer channel outputs, then the $i^{th}$ channel will also exhibit the maximum likelihood ratio and will thus be chosen again. The next question to be considered is whether this maximum likelihood ratio technique will give the same probability of detection for equal probability of false alarm. To investigate this we proceed as follows.

Let the output of each frequency analyzer channel be distributed according to Eq. (3) in the presence of noise alone and according to Eq. (4) in the presence of signal plus noise. Let the false alarm probability be given by the probability that the output of the OR-Gate exceeds a threshold T. That is,

$$P_{FA} = 1 - F_y(T) \,, \tag{7}$$

where

$$F_y(T) = P_r(y < T)$$

$$y = \max_i \, [x_i(t)] .$$

The distribution function of y is given by

$$F_y(T) = F_x^{\,N}(T) \,, \tag{8}$$

where

$F_x(T) = P_r(x < T) =$ distribution function of each of the identically distributed $x_i$'s.

We note that

$$F_x(T) = 1 - P_r(x > T)$$

$$= 1 - P'_{FA} \,, \tag{9}$$

where

$P'_{FA}$ = probability of false alarm in each spectrum analyzer channel.

Substituting Eqs. (9) and (8) into Eq. (7), we get

$$P_{FA} = 1 - (1 - P'_{FA})^N . \tag{10}$$

This equation measures the probability that at least one of the channel outputs exceeds the threshold T. What has been shown is that we can predict performance at the output of the OR-Gate by performing the analysis at the input to the OR-Gate. This approach agrees with our intuitive reasoning if we interpret the problem as one of thresholding before OR-Gating.

Substituting Eq. (3) into Eq. (10) we get

$$P_{FA} = 1 - \left(1 - \int_T^\infty \frac{x}{\sigma_n^2} e^{-x^2/2\sigma_n^2} dx\right)^N \tag{11}$$

for an explicit form of the false alarm probability.

The integral in Eq. (11) can be evaluated easily and is given by

$$\int_T^\infty \frac{x}{\sigma_n^2} e^{-x^2/2\sigma_n^2} dx = e^{-T^2/2\sigma_n^2} . \tag{12}$$

Thus Eq. (11) becomes

$$P_{FA} = 1 - \left(1 - e^{-T^2/2\sigma_n^2}\right)^N . \tag{13}$$

The detection probability, $P_D$, is given by the probability that when a signal is present in one of the channels, that the OR-Gate output will exceed the threshold T. That is,

$$P_D = 1 - (1 - P'_{FA})^{N-1}(1 - P'_D) , \tag{14}$$

where

$$P'_D = \int_T^{\infty} \frac{x}{\sigma_s^2 + \sigma_n^2} e^{-x^2/[2(\sigma_s^2+\sigma_n^2)]} dx . \tag{15}$$

But from the preceeding agreements we have

$$P'_D = e^{-T^2/[2(\sigma_s^2+\sigma_n^2)]} . \tag{16}$$

Substituting Eqs. (16) and (12) into Eq. (15) gives,

$$P_D = 1 - (1 - e^{-T^2/2\sigma_n^2})^{N-1}(1 - e^{-T^2/[2(\sigma_s^2+\sigma_n^2)]}) . \tag{17}$$

We have thus expressed the false alarm probability and detection probability for the case where the spectrum analyzer output is directly driving the OR-Gate. What must be done next is to show that when the single sample likelihood ratio is employed, the probability of detection is the same as that given by Eq. (17) when the false alarm probability is the same.

If in forming the logarithm of the likelihood ratio x is mapped according to Eq. (6), then the probability density function of the new random variable, $z = \ell(x) = C_1x^2 + C_2$, is given by

$$p(z) = p(x)\frac{dx}{dz}, \text{ with } x = \sqrt{\frac{z - C_2}{C_1}} .$$

Since

$$dz = 2C_1 x dx, \quad \frac{dx}{dz} = \frac{1}{2C_1\sqrt{\frac{z - C_2}{C_1}}} ,$$

or

$$\frac{dx}{dz} = \frac{1}{2C_1\sqrt{\frac{z - C_2}{C_1}}} ,$$

and given

$$p(x) = \frac{x}{\sigma_n^2} e^{-x^2/2\sigma_n^2} ,$$

we get for p(z),

$$p(z) = \frac{1}{2C_1\sigma_n^2} e^{-z-C_2/2C_1\sigma_n^2} .$$

The probability of false alarm, $P_{FA1}$, for this case is

$$P_{FA1} = 1 - (1 - \int_{T_1}^{\infty} \frac{1}{2C_1\sigma_n^2} e^{-z-C_2/2C_1\sigma_n^2} dz)^N ,$$

or in simplified form

$$P_{FA1} = 1 - (1 - e^{C_2/2C_1\sigma_n^2} e^{-T_1/2C_1\sigma_n^2})^N . \qquad (18)$$

By similar reasoning we can show that the detection probability $P_{D1}$ is given by

$$P_{D1} = 1 - (1 - P_{FA1})^{N-1}$$

$$(1 - \int_{T_1}^{\infty} \frac{1}{2C_1(\sigma_s^2 + \sigma_n^2)} e^{-z-C_2/[2C_1(\sigma_s^2+\sigma_n^2)]}) ,$$

or

$$P_{D2} = 1 - (1 - e^{+C_2/2C_1\sigma_n^2} e^{-T_1/2C_1\sigma_n^2})^{N-1}$$

$$(1 - e^{+C_2/[2C_1(\sigma_s^2+\sigma_n^2)]} e^{-T_1/[2C_1(\sigma_s^2+\sigma_n^2)]} . \qquad (19)$$

Equations (17) and (19) give the required probabilities of detection as functions of the probabilities of false alarm, so that what remains to be shown is that for equal false alarm probabilities we get equal detection probabilities.

If $P_{FA} = P_{FA1}$, this means that, using Eqs. (13) and (18)

$$T^2/2\sigma_n^2 = \frac{T_1 - C_2}{2C_1\sigma_n^2} ,$$

or that in order for our assertion to hold it must be true that

$$e^{-T^2/[2(\sigma_s^2+\sigma_n^2)]} = e^{-(T_1-C_2)/[2C_1(\sigma_s^2+\sigma_n^2)]} , \quad (20)$$

or that

$$\frac{T^2}{2(\sigma_s^2 + \sigma_n^2} = \frac{T_1 - C_2}{2C_1(\sigma_s^2 + \sigma_n^2)} , \quad (21)$$

under the condition

$$\frac{T^2}{2\sigma_n^2} = \frac{T_1 - C_2}{2C_1\sigma_n^2} ,$$

which reduces to

$$T^2 = \frac{T_1 - C_2}{C_1} . \quad (22)$$

Substituting the right hand side of Eq. (22) for $T^2$ in Eq. (21) and simplifying gives

$$\frac{T_1 - C_2}{C_1} = \frac{T_1 - C_2}{C_1} ,$$

which proves the assertion.

What we have demonstrated is the rather intuitively obvious fact that by performing a monotonic mapping on the output of the spectrum analyzer channels, the probability of detection for a given false alarm probability is not changed. Thus it may be concluded that there is nothing to be gained by performing such an operation.

JOINT LIKELIHOOD RATIO

If we make the same assumptions concerning the target signal and background noise, i.e., that they are Gaussian processes, then it is possible to compute the joint log likelihood ratio for the output of each analyzer channel. The joint likelihood ratio for M independent $i^{th}$ analyzer channel outputs is given by

$$L(\underline{x}_i) = \frac{p(x_i)_{s+n}}{p(x_i)_n}, \tag{23}$$

where

$$\underline{x}_i = [x_i(t_1), x_i(t_2), \ldots, x_i(t_M)]$$

$$p(\underline{x})_{s+n} = p(x_{i1})_{x+n}\, p(x_{i2})_{s+n} \cdots p(x_M)_{s+n}$$

$$p(\underline{x}) = p(x_{i1})_n p(s_{i2})_n \cdots p(x_{iM})_n .$$

That is, Eq. (23) can be expressed as

$$L(\underline{x}_i) = \frac{p(x_{i1})_{s+n}}{p(x_{i1})_n} \cdot \frac{p(x_{i2})_{s+n}}{p(x_{i2})_n} \cdots \frac{p(x_{iM})_{s+n}}{p(x_{iM})_n} .$$

If we take the logarithm of the above equation and use the earlier $\ell(x) = \log_e L(x)$, we get

$$\ell(\underline{x}_i) = \sum_{j=1}^{M} \ell(x_{ji}) . \tag{24}$$

Recalling from Eq. (6) that

$$\ell(x) = C_1 x^2 + C_2 ,$$

we get for Eq. (24)

$$\ell(\underline{x}_i) = C_1 \sum_{j=1}^{M} x_{ji}^2 + NC_2 . \qquad (25)$$

This equation tells us that the process under consideration is square law detection and summation of M variates. The next step is to determine the probability density functions of $\ell(\underline{x}_i)$ under conditions of noise alone and signal plus noise. Define y as

$$y_i = \tfrac{1}{2} \sum_{j=1}^{M} x_{ji}^2 \; {}^{*} .$$

When signal plus noise is present we have for the density of $x_{ji}^2 = z$

$$p(z) = \frac{1}{\sigma_s^2 + \sigma_n^2} e^{-z/(\sigma_s^2+\sigma_n^2)} . \qquad (26)$$

*The factor ½ is used here for mathematical convenience and is equivalent to using a square law detector with a gain of ½.

Others[1,2] have shown that the density of $y_i$ is given by

$$p(y_i)_{s+n} = \frac{1}{(M-1)!}\left(\frac{y_i}{\sigma_s^2 + \sigma_n^2}\right)^{M-1} e^{-y_i/(\sigma_s^2+\sigma_n^2)} \quad . \qquad (27)$$

To get the density function for noise alone we let $\sigma_s^2 = 0$ and Eq. (27) becomes

$$p(y_i)_n = \frac{1}{(M-1)!}\left(\frac{y_i}{\sigma_n^2}\right)^{M-1} e^{-y_i/\sigma_n^2} \quad . \qquad (28)$$

The false alarm probability after OR-Gating is given by Eq. (10) and when Eq. (28) is substituted we get

$$P_{FA} = 1 - \left(1 - \int_{\frac{T}{\sigma_n^2}}^{\infty} \frac{1}{(M-1)!}\left(\frac{y}{\sigma_n^2}\right)^{M-1} e^{-y/\sigma_n^2} \frac{dy}{\sigma_n^2}\right)^N . \qquad (29)$$

Similarly, the detection probability is given by

$$P_D = 1 - \left(1 - \int_{\frac{T}{\sigma_n^2}}^{\infty} \frac{1}{(M-1)!}\left(\frac{y}{\sigma_n^2}\right)^{M-1} e^{-y/\sigma_n^2} \frac{dy}{\sigma_n^2}\right)^{N-1}$$
$$\cdot \left(1 - \int_{\frac{T}{\sigma_s^2+\sigma_n^2}}^{\infty} \frac{1}{(M-1)!}\left(\frac{y}{\sigma_s^2 + \sigma_n^2}\right)^{M-1} e^{-y/(\sigma_s^2+\sigma_n^2)} \frac{dy}{\sigma_s^2+\sigma_n^2}\right). \qquad (30)$$

[1] Marcum, J. I., "A Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix," RM-753, 1 July 1948, The Rand Corporation.

[2] Berkowitz, R. S., Modern Radar, John Wiley & Sons, pp. 170-185.

Equations (29) and (30) can be expressed in terms of a normalized threshold thus giving

$$P_{FA} = 1 - \left(1 - \int_{T'}^{\infty} \frac{1}{(M-1)!} w^{M-1} e^{-w} dw\right)^{N} , \quad (31)$$

and

$$P_D = 1 - \left(1 - \int_{T'}^{\infty} \frac{1}{(M-1)!} w^{M-1} e^{-w} dw\right)^{N-1}$$

$$\left(1 - \int_{\frac{T}{1+a}'}^{\infty} \frac{1}{(M-1)!} (w)^{M-1} e^{-w} dw\right). \quad (32)$$

where

$$T' = T/\sigma_n^2, \text{ and}$$

$$a = \frac{\sigma_s^2}{\sigma_n^2} , \text{ the signal-to-noise ratio.}$$

SUMMARY

We have developed two results by the analysis developed in this paper. First, it has been shown that nothing is to be gained by single-sample likelihood ratio processing. Although this is an intuitively apparent fact it never hurts to verify such assertions. The other result is that it is possible to theoretically predict the performance of the joint likelihood ratio data reduction criterion under quite realistic assumptions concerning the noise and signals. Therefore, Eqs. (31) and (32) can be evaluated numerically and receiver operating characteristic (ROC) curves can be obtained. These curves can be compared to the (ROC) curves that are obtained both theoretically and experimentally from the simple OR-Gating data reduction technique.