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A REFINED QUADRILATERAL FLAT PLATE FINITE ELEMENT BASED ON THE --ETC(U)
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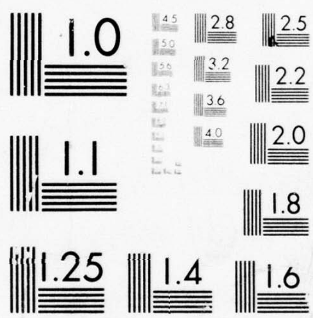

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A Refined Quadrilateral Flat Plate Finite Element Based on the Extended Field Method

by

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January, 1978

Abstract

A new quadrilaterial finite element for the problem of bending of a uniform, homogeneous, isotropic, flat plate having an arbitrary number of nodal points and allowing an arbitrary level of refinement is developed from solution functions to the governing differential equation. Interelement continuity to the appropriate level of normal derivatives is provided in least error fashion by the use of spline functions for the edge displacement quantities and minimization by the Galerkin technique. Even for general quadrilateral domains no numerical quadrature is required for developing this finite element since all required integrals are taken over straight boundary edges and can be evaluated explicitly. With this new type of element, an improvement in accuracy is expected to result when large quadrilateral regions are modeled by a single element with an appropriate number of edge nodal points, because the error of this finite element approximation is associated with the boundary region of the elements. The force vector corresponding to the transverse pressure is derived from a series solution to the inhomogeneous differential equation, so that the effect of the forcing function in the element interior can be evaluated to any degree of accuracy desired.

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1. Introduction

The vast majority of boundary value problems do not lend themselves to solution by an exact method. Analysts therefore resort to approximate methods that reduce the continuum problem to the problem of finding a finite number of parameters that characterize the solution. The most versatile approximation method to date for solving linear boundary value problems is the finite element method (FEM). This method is fully automatible and produces in general sparcely populated matrices which lend themselves to inversion by techniques that are both economic in storage and saving in computer time, thus permitting application to large systems. Although the FEM has initially been developed by structural analysts and most of its vocabulary is derived from structural mechanics, its versatility has been recognized and its application has spread to other fields of continuum mechanics.

The FEM is still a subject of vigorous research by both engineers and mathematicians alike. Efforts are going on to enlarge the field of application of the FEM, to give information about the error in the solution, and to improve upon existing finite element approximations to continuum problems. These efforts include reformulating the coefficient matrix, which in finite element nomenclature is called stiffness matrix, in such a manner as to achieve reduced computer time, refined representation of the boundary, and improved accuracy. This work is part of that effort. The point of departure for this present work is the extended field method (EXFM), a method of analysis that exhibits remarkable convergence but lacks the flexibility of application of the FEM. In this report (dissertation), the EXFM is used to formulate a new finite element stiffness matrix and the associated force vector for the problem of plate bending.

In brief, this new formulation employs solution functions in the interior of the finite element and finds some best fit along the element boundary. Inter-element compatibility in the sense of Pian (ref. 1.) is achieved by using a Galerkin-type error reduction along the boundary. Instead of deriving the force vector that corresponds to the forcing function acting over the interior of the element from work equivalence, as is customary in standard finite element formulations, the force vector is here derived from exact solutions to the inhomogeneous differential equation. The stiffness kernel is obtained from virtual work considerations that require integration of the stress resultants along the element boundary. There is no need for integration over the domain of the finite element, since all displacement functions employed in this formulation satisfy the differential equation exactly. This amounts to an equilibrium formulation. Choosing for the solution functions certain series whose recurrence relation is simply determined by an integer that multiplies the argument, this new finite element can be easily refined by increasing both the number of displacement functions and the number of nodal points along the element boundary. This process can be automated. This flexibility in the number of nodal points per element permits the analyst to use a sufficiently refined element of this type instead of an equivalent substructure consisting of a number of finite elements. Instead of progressive partioning the researcher can use the process of progressive element refinement for demonstrating convergence. It is noted, that for an equivalent refinement of the customary finite element solution the growth of degrees of freedom of the system is roughly proportional to the square of the number of edge partitions per element edge of the original element size in the case of progressive partitioning, while for element refinement, as is possible for this new formulation, the number of degrees of freedom grows proportional to the partitioning of the circumference. (Starting with a four node element, the proportionality factors are i) for progressive partitioning: $S_1 = 4$, $S_2 = 9$, $S_3 = 25$ $S_N = S_{N-1} + 16 \cdot \sum_{j=1}^{2^N} j + 8 \cdot \sum_{j=1}^{2^N - 1} j$, $N > 3$ and ii) for element refinement: $S_N = 2^{N+1}$).

2. The Plate Bending Finite Element Derived by the Direct Technique

2.1 The direct technique for the plate bending finite element

While the techniques now generally employed for the formulation of finite element stiffness matrices are based upon the method of variational calculus (Ref's 2,3,4), the approach taken here is the direct technique, or as it is termed in some literature the equilibrium technique. The general approach of the equilibrium technique is to obtain exact solutions to the governing differential equation that are expressed in terms of displacement quantities at the nodal points, equate the boundary nodal point displacement quantities of the adjacent elements and enforce equilibrium of the force quantities at the nodal points. For one-dimensional continua such as rods and beams this procedure is simple to apply, as the total boundary consists of two discrete points and the stress resultants are already discretized quantities. In the case of two dimensional problems, such as the plate bending problem, proper continuity of the primary variable across the entire element boundary cannot always be achieved and equilibrium can only be achieved in some average sense. The method described in detail later in this report allows discontinuities of the primary variable across element boundaries but minimizes these discontinuities in the Galerkin sense, and enforces equilibrium in an average sense by appeal to the priniciple of virtual work.

To develop this approach for the case of a plate bending finite element, consider the transverse displacement $w(x)$. The transverse displacement satisfies

$$D \nabla^4 w = p(x) ,$$

where $p(x)$, the forcing function, is the transverse pressure and D is the bending rigidity of the uniform plate. The transverse displacement may be expressed as the sum of a solution to the inhomogeneous equation, $w_p(x)$, satisfying come convenient boundary condition and a solution of the homogeneous equation, $w_h(x)$, with adjustable coefficients that make it possible to achieve any boundary displacement within the bounds of some required accuracy. Summing these components the transverse displacement is given by

$$W(X) = W_p(X) + W_h(X) \tag{2}$$

The solution to the homogeneous equation includes, of course, both elastic as well as rigid body displacements.

The development of the equilibrium finite element requires a finite number of displacement functions that are dependent upon undetermined parameters. Truncated series solutions are selected for the displacement functions, and they may be written in the form

$$W_h(X) = \sum_{i=1}^{N} \mathcal{H}_i(X)\, h_i \qquad \text{and} \qquad W_p(X) = \sum_{i=1}^{M} \mathcal{P}_i(X)\, p_i \quad ,$$

where the p_i are known pressure parameters that are derived from the forcing function and the h_i are the undetermined parameters. The displacement boundary conditions for the plate bending problem are the prescription of $W(\xi)$, the displacement, and of $W_{,n}(\xi)$ the rotation normal to the boundary, where ξ is a boundary coordinate. The corresponding nodal degrees of freedom at a given point ξ_i onthe boundary are

$$\begin{aligned} q_j &= W(\xi_i) \\ q_{j+1} &= W_{,x}(\xi_i) \\ q_{j+2} &= W_{,y}(\xi_i) \quad . \end{aligned}$$

Here the comma behind the variable followed by an index quantity that is a spatial variable represents partial differentiation of the dependent variable with respect to that spatial variable and n is the local normal at the boundary.

A transformation from the nodal degrees of freedom, the q_i, to the continuous parameters, the h_i, must be obtained so that the displacement functions can be expressed in terms of nodal quantities. One possible approach is to obtain the inverse transformation by simply evaluating the nodal displacement quantities in terms of the displacement functions:

$$q_1 = w(\xi_1) = \sum_{i=1}^{N} \mathcal{H}_i(\xi_1)\, h_i + \sum_{i=1}^{M} \mathcal{P}_i(\xi_1)$$
$$q_2 = w_{,x}(\xi_1) = \sum_{i=1}^{N} \mathcal{H}_{i,x}(\xi_1)\, h_i + \sum_{i=1}^{M} \mathcal{P}_{i,x}(\xi_1)$$
$$q_3 = w_{,y}(\xi_1) = \sum_{i=1}^{N} \mathcal{H}_{i,y}(\xi_1)\, h_i + \sum_{i=1}^{M} \mathcal{P}_{i,y}(\xi_1)$$
$$\vdots$$
$$q_{\hat{N}} = w_{,y}(\xi_k) = \sum_{i=1}^{N} \mathcal{H}_{i,y}(\xi_k)\, h_i + \sum_{i=1}^{M} \mathcal{P}_{i,y}(\xi_k) \quad ,$$

where $k = \hat{N}/3$.

In matrix form the above can be written as

$$\mathbb{q} = \mathbb{B}\,\mathbb{h} + \mathbb{C}\,\mathbb{p} \quad . \tag{3}$$

For the purpose of obtaining the h_i in terms of q_i it is required that $N = \hat{N}$ so that $\mathbb{B}$ is a square matrix. This transformation enforces continuity of the displacement quantities at the nodal points but says nothing about inter-element continuity of the displacement quantities along the element boundary away from the nodal points. For this work a more complex approach is taken that improves upon the match of the displacement quantities of adjacent elements along their common edges. The edge displacement quantities $w(\xi)$ and $w_{,n}(\xi)$ are expressed directly in terms of the nodal degrees of freedom

$$w(\xi) \approx \hat{w}(\xi) = \sum_{i=1}^{\hat{N}} S_i(\xi)\, q_i$$
$$w_{,n}(\xi) \approx \hat{r}(\xi) = \sum_{i=1}^{\hat{N}} R_i(\xi)\, q_i$$

in such a manner that the nodal displacement quantities on a common edge of two adjacent elements determine the same displacement and rotation of this edge for both elements. The discrepancy between this exterior representation of the displacements and rotation and the representation of these quantities by the solution functions for the element interior is then minimized.

Thus, interelement compatibility in the sense of Pian (ref.1) is achieved. Perhaps the most useful edge displacement functions $\hat{W}(\xi)$ and $\hat{r}(\xi)$ are cubic splines for the transverse displacements of the edge and linear splines for the edge rotation. Such an element would be interelement compatible with some of the finite elements currently in use, such as the QUAD4 element of NASTRAN or the NASTRAN beam element, which have this displacement pattern. These boundary displacement functions would also allow assemblies of finite elements where a corner nodal point of one element coincides with a mid-edge nodal point of another without violating the interelement compatibility requirement. The transformation from the nodal degrees of freedom, the q_i , to the distributed parameters, the h_j , is then obtained by requiring that the discrepancies of the displacement quantities between the exterior and the interior representation be orthogonal to some set of independent test functions. With the $\{\phi_j(\xi)\}$ being such a set of test functions and the discrepancies given by

$$\begin{aligned} \mathit{error}\,[W(\xi)] &= W(\xi) - \hat{W}(\xi) \\ \mathit{error}\,[W_{,n}(\xi)] &= W_{,n}(\xi) - \hat{r}(\xi) \end{aligned} ,$$

the transformation obtained from the orthogonality requirement

$$\oint \mathit{error}(\xi)\, \phi_j(\xi)\, d\xi = 0 \quad ,$$

is given by the matrix equation

$$\mathbb{A} q = \mathbb{B} h + \mathbb{C} p \quad . \tag{4}$$

The upper index limits must be chosen so that the coefficient matrix $\mathbb{B}$

of the distributed parameters will be square. The inversion properties of the coefficient matrix $\mathbb{B}$ depend strongly upon the choice of the set of test functions. In particular none of the test functions may be orthogonal to the set of component functions of the error. Similarly none of the component functions of the error may be orthogonal to the set of test functions. If $\{\phi_j(\xi)\}$ is a set of test functions satisfying these criteria, then the inverse of the coefficient matrix $\mathbb{B}$ exists. To minimize the error that results from numerical inversion it is furthermore desirable that the $\{\phi_j\}$ be an orthogonal set rather than merely being linearly independent. The choice of this set of orthogonal test functions is discussed further in sections 2.3 and 2.4a.

Having obtained the transformation from nodal degrees of freedom to the parameters of the solution function by multiplying equation (4) with $\mathbb{B}^{-1}$

$$\mathbb{h} = \mathbb{B}^{-1}\mathbb{A}\mathbb{q} - \mathbb{B}^{-1}\mathbb{C}\mathbb{p} \; , \tag{5}$$

it is now required to obtain the stiffness kernel. In a variational approach it would be necessary to integrate the strain energy integral over the two-dimensional domain. For the general quadrilateral element this would require numerical quadrature. In the equilibrium approach the displacement functions chosen already satisfy the differential equation, therefore there is no need to apply the Principel of the Stationary Total Potential . Instead it is required to enforce equilibrium at the boundary. An error orthogonalization scheme similar to the one used for the displacement functions, where the discrete force quantities at the nodal points are treated in the distributional sense will in general not work if the set of test functions is an orthogonal set. This becomes evident if one considers an element where all nodal points are located at the zeros of one of the test functions. The corresponding integral would yield a row of zeros for the coefficient matrix and render it singular. Using a set of merely independent test funct..ns that would circumvent this problem is deemed not satisfactory when primarily accuracy is sought.

A rational approach is to employ the Principle of Virtual Work to obtain a relation between the generalized nodal forces and the stress resultants along the boundary of the finite element. Briefly stated, write

$$\delta \mathcal{W}_n = \delta \mathcal{W}_q \, . \tag{6}$$

Here (ref.5)

$$\delta \mathcal{W}_n = \oint V_n \, \delta w \, d\xi \; - \oint M_n \, \delta w_{,n} \, d\xi \; + \sum_{k=1}^{K} (M_{ns}^{k+} - M_{ns}^{k-}) \, \delta w_k \tag{7}$$

is the virtual work done by the stress resultants and corner forces while going through the boundary edge virtual displacement and

$$\delta \mathcal{W}_q = \sum_{i=1}^{N} \delta q_i \, F_i \tag{8}$$

is the virtual work done by the generalized nodal forces while going through the nodal virtual displacements. The quantities M_n, M_{ns} and V_n in equation (7), i.e. the normal moment, the twisting moment and the Kirchhoff shear respectively at the plate edge are linear combinations of partial derivatives with respect to the local edge coordinates (ref 5) Using appropriate transformation coefficients these quantities can be expressed in terms of the reference coordinates (x,y).

Thus, from Fig. (5)

$$\begin{aligned}
w_{,n} &= \frac{dy}{ds} w_{,x} - \frac{dx}{ds} w_{,y} \\
M_n &= -D \left\{ \left[\left(\frac{dy}{ds}\right)^2 + \nu \left(\frac{dx}{ds}\right)^2 \right] w_{,xx} - 2(1-\nu)\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) w_{,xy} + \left[\left(\frac{dx}{ds}\right)^2 + \nu \left(\frac{dy}{ds}\right)^2 \right] w_{,yy} \right\} \\
V_n &= -D \left\{ \left[\left(\frac{dy}{ds}\right)^3 + (2-\nu)\left(\frac{dy}{ds}\right)\left(\frac{dx}{ds}\right)^2 \right] w_{,xxx} + \left[(1-2\nu)\left(\frac{dy}{ds}\right)^2\left(\frac{dx}{ds}\right) - (2-\nu)\left(\frac{dx}{ds}\right)^3 \right] w_{,xxy} \right. \\
&\quad \left. - \left[(1-2\nu)\left(\frac{dx}{ds}\right)^2\left(\frac{dy}{ds}\right) - (2-\nu)\left(\frac{dy}{ds}\right)^3 \right] w_{,xyy} - \left[\left(\frac{dx}{ds}\right)^3 + (2-\nu)\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right)^2 \right] w_{,yyy} \right\}
\end{aligned} \tag{9}$$

$$M_{ns} = -D(1-\nu)\left\{\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right)(W_{,xx} - W_{,yy}) + \left[\left(\frac{dy}{ds}\right)^2 - \left(\frac{dx}{ds}\right)^2\right] W_{,xy}\right\} ,$$

which when substituted into the virtual work expression give for the polygonal plate, (Fig. 1)

$$\begin{aligned}
\sum_{i=1}^{N} \delta q_i F_i = \sum_{k=1}^{K} \Big\{ & t_{1k} \cdot [W_{,xx}(\mathbb{X}_{m_k}) - W_{,yy}(\mathbb{X}_{m_k})] \, \delta W(\mathbb{X}_{m_k}) \\
& + t_{2k} \cdot W_{,xy}(\mathbb{X}_{m_k}) \, \delta W(\mathbb{X}_{m_k}) \\
& + t_{3k} \cdot \int_0^{L_k} W_{,xxx}(\xi) \, \delta W(\xi) \, d\xi \\
& + t_{4k} \cdot \int_0^{L_k} W_{,xxy}(\xi) \, \delta W(\xi) \, d\xi \\
& + t_{5k} \cdot \int_0^{L_k} W_{,xyy}(\xi) \, \delta W(\xi) \, d\xi \\
& + t_{6k} \cdot \int_0^{L_k} W_{,yyy}(\xi) \, \delta W(\xi) \, d\xi \\
& + t_{7k} \cdot \int_0^{L_k} W_{,xx}(\xi) \, \delta W_{,x}(\xi) \, d\xi \\
& + t_{8k} \cdot \int_0^{L_k} W_{,xy}(\xi) \, \delta W_{,x}(\xi) \, d\xi \\
& + t_{9k} \cdot \int_0^{L_k} W_{,yy}(\xi) \, \delta W_{,x}(\xi) \, d\xi \\
& + t_{10k} \cdot \int_0^{L_k} W_{,xx}(\xi) \, \delta W_{,y}(\xi) \, d\xi \\
& + t_{11k} \cdot \int_0^{L_k} W_{,xy}(\xi) \, \delta W_{,y}(\xi) \, d\xi \\
& + t_{12k} \cdot \int_0^{L_k} W_{,yy}(\xi) \, \delta W_{,y}(\xi) \, d\xi \Big\} ,
\end{aligned} \tag{10}$$

where

$$t_{1k} = -D(1-\nu)\left[\left(\frac{dx}{ds}\right)_k\left(\frac{dy}{ds}\right)_k - \left(\frac{dx}{ds}\right)_{k-1}\left(\frac{dy}{ds}\right)_{k-1}\right]$$

$$t_{2k} = -D(1-\nu)\left[\left(\frac{dy}{ds}\right)_k^2 - \left(\frac{dx}{ds}\right)_k^2 - \left(\frac{dy}{ds}\right)_{k-1}^2 + \left(\frac{dx}{ds}\right)_{k-1}^2\right]$$

$$t_{3k} = -D\left[\left(\frac{dy}{ds}\right)_k^3 + (2-\nu)\left(\frac{dy}{ds}\right)_k\left(\frac{dx}{ds}\right)_k^2\right]$$

$$t_{4k} = -D\left[(1-2\nu)\left(\frac{dy}{ds}\right)_k^2\left(\frac{dx}{ds}\right)_k - (2-\nu)\left(\frac{dx}{ds}\right)_k^3\right]$$

$$t_{5k} = -D\left[(1-2\nu)\left(\frac{dx}{ds}\right)_k^2\left(\frac{dy}{ds}\right)_k - (2-\nu)\left(\frac{dy}{ds}\right)_k^3\right]$$

$$t_{6k} = -D\left[\left(\frac{dx}{ds}\right)_k^3 + (2-\nu)\left(\frac{dx}{ds}\right)_k\left(\frac{dy}{ds}\right)_k^2\right]$$

$$t_{7k} = D\left[\left(\frac{dy}{ds}\right)_k^3 + \nu\left(\frac{dy}{ds}\right)_k\left(\frac{dx}{ds}\right)_k^2\right]$$

$$t_{8k} = -D\,2(1-\nu)\left(\frac{dx}{ds}\right)_k\left(\frac{dy}{ds}\right)_k^2$$

$$t_{9k} = D\left[\left(\frac{dx}{ds}\right)_k^2\left(\frac{dy}{ds}\right)_k + \nu\left(\frac{dy}{ds}\right)_k^3\right]$$

$$t_{10k} = -D\left[\left(\frac{dx}{ds}\right)_k\left(\frac{dy}{ds}\right)_k^2 + \nu\left(\frac{dx}{ds}\right)_k^3\right]$$

$$t_{11k} = D\,2(1-\nu)\left(\frac{dx}{ds}\right)_k^2\left(\frac{dy}{ds}\right)_k$$

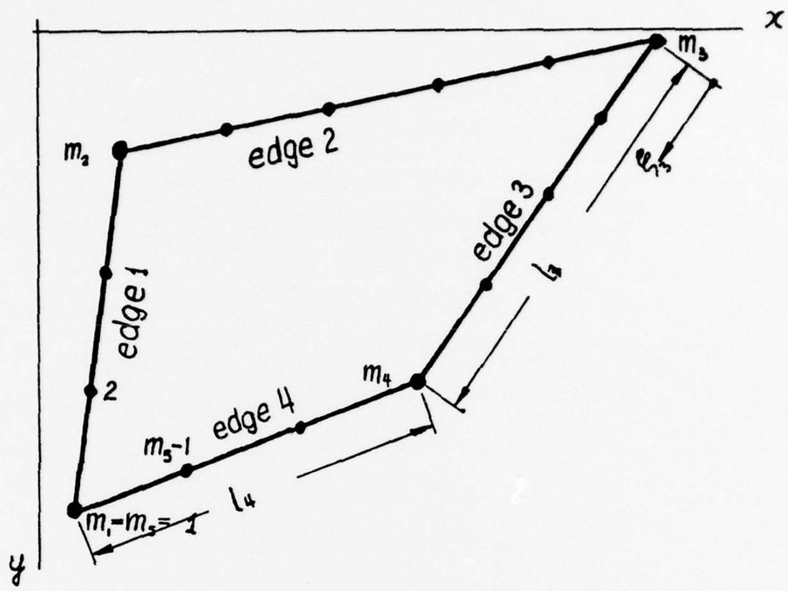
$$t_{12k} = -D\left[\left(\frac{dx}{ds}\right)_k^3 + \nu\left(\frac{dx}{ds}\right)_k\left(\frac{dy}{ds}\right)_k^2\right] .$$

Substituting for W and δW, the virtual work expression may be cast in matrix form

$$\delta \mathbb{q}^T \mathbb{F} = \delta \mathbb{h}^T \{\mathbb{G}\mathbb{h} + \mathbb{R}\mathbb{p}\} \tag{11}$$

Substituting the transformation equation, eq. (5) into the virtual work expression, eq. (11) gives

$$\delta \mathbb{q}^T \mathbb{F} = \delta \mathbb{q}^T \{\mathbb{A}^T \mathbb{B}^{-T} \mathbb{G}\, \mathbb{B}^{-1} \mathbb{A}\, \mathbb{q} - \mathbb{A}^T \mathbb{B}^{-T}(\mathbb{G}\mathbb{B}^{-1}\mathbb{C} - \mathbb{R})\mathbb{p}\} ,$$



K = 4
Quadrilateral Plate
Fig.1

which holds for arbitrary $\delta \mathbb{q}^T$, so that the equilibrium equation takes the form

$$\mathbb{F} = \mathbb{K}\mathbb{q} - \mathbb{F}_p ,$$

where

$$\mathbb{K} = \mathbb{A}^T \mathbb{B}^{-T} \mathbb{G} \mathbb{B}^{-1} \mathbb{A} \tag{12}$$

and

$$\mathbb{F}_p = \mathbb{A}^T \mathbb{B}^{-T} (\mathbb{G} \mathbb{B}^{-1} \mathbb{C} - \mathbb{R}) \mathbb{p} .$$

The required coefficient matrices in the case of a quadrilateral plate bending finite element are developed in the sequel. The components of the matrices $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are the corresponding coefficients in equations (22a) and (22b) section 2.4a and the vectors $\mathbb{h}$, $\mathbb{q}$ and $\mathbb{p}$ are defined in section 2.4b below. The components of the vector $\mathbb{F}$ correspond to those of the vector $\mathbb{q}$ in the sense that their inner product is work. As a result of substituting the series representations of the displacement function into equation (10) the matrices $\mathbb{G}$ and $\mathbb{R}$ are each linear combinations of matrices defined by equation (23) in section 2.4b below.

2.2 The Displacement Functions for the Plate Bending Finite Element

An obvious choice for the solution function $W_p(\mathbb{x})$ for the inhomogeneous equation is provided by the Navier-solution which satisfies simply supported boundary conditions at the edges of a rectangular reference field. For the homogeneous biharmonic equation the complementary solution could be chosen to be a polynomial consisting of a complete set of linearly independent polynomial terms up to a given order that satisfy the homogeneous equation identically. Such a set of solution functions up to the 6th order is

$$
\begin{array}{llll}
1 & & & \\
x & , y & & \\
x^2 & , xy & , y^2 & \\
x^3 & , x^2y & , xy^2 & , y^3 \\
x^4-3x^2y^2 & , x^3y & , x^2y^3 & , y^4-3x^2y^2 \\
x^5-5x^3y^2 & , x^5-5xy^4 & , y^5-5x^4y & , y^5-5x^2y^3 \\
\multicolumn{4}{l}{x^6-15x^2y^4+2y^6,\ 3x^5y-5x^3y^3,\ 3xy^5-5x^3y^3,\ y^6-15x^4y^2+2x^6 \quad .}
\end{array}
$$

It is not difficult to show that for the subset of purely n^{th} order polynomial terms there exist at most four linearly independent solutions to $\nabla^4 w = 0$ and if $n \geq 3$ there exist exactly four. Considering the simpler transformation equation, eq. (3), and a polynomial series of solution functions to the homogeneous biharmonic equation, one will find it impossible to construct a truly interelement compatible finite element from this set of solution functions without resorting to subdomain techniques (Tocher, De Veubeke, ref's 6,7) Moreover, obtaining interelement compatibility in Pian's sense (ref 1) is not dependent upon polynomial representation in the interior. Furthermore, there exists no recursion formula that allows finding in an expedient manner successive terms of the series of independent polynomial solution functions to the homogeneous biharmonic equation. Therefore, such a series is not particularly well suited for the development of a family of finite elements with an arbitrary number of nodal points. Furthermore, it is generally accepted that polynomial series representations of general continuous functions in the majority of cases show poor convergence when compared with trigonometric representations of the same functions. In other words, a smaller number of parameters is needed for a given maximum error when a smooth function is represented by a trigonometric representation as opposed to a polynomial representation. Since it is required to find the inverse of the matrix $\mathbb{B}$ (eq.4), it is desirable to keep the size of this array as small as possible for a given refinement of the finite element. This speaks clearly in favor of trigonometric representation of the displacement function. Any savings in computer time in the inversion process will, however, be considerably

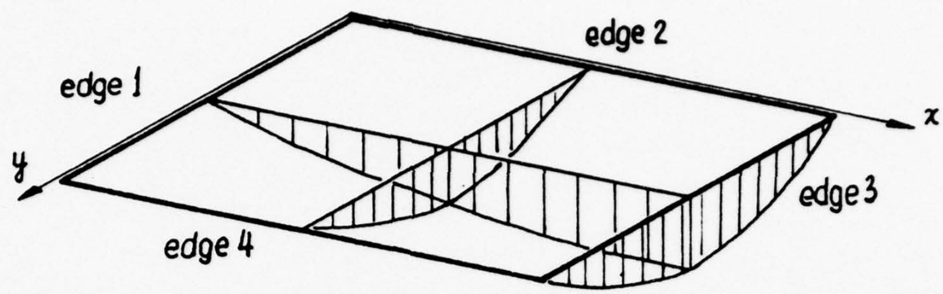


Fig. 2. 1st term of the Lévy series solution W_s

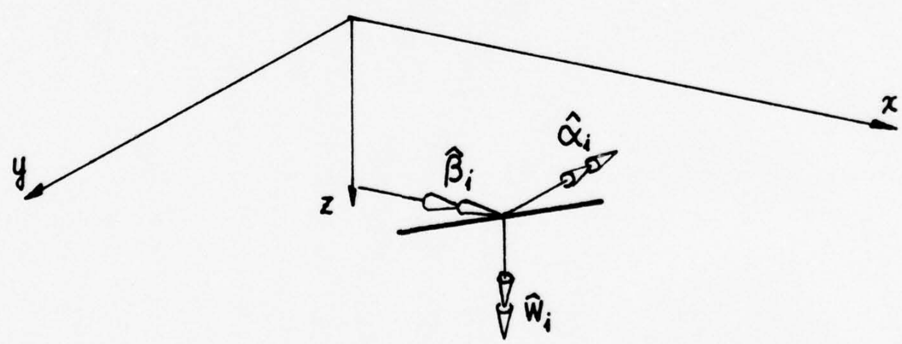


Fig. 3. Nodal point degrees of freedom

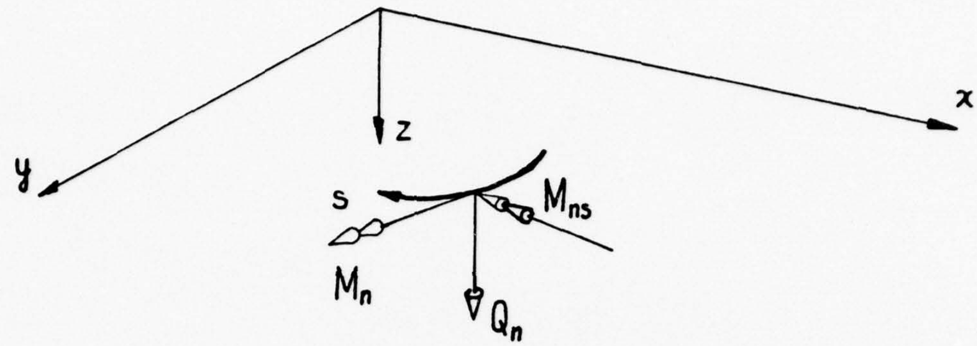


Fig. 4. Stress resultants

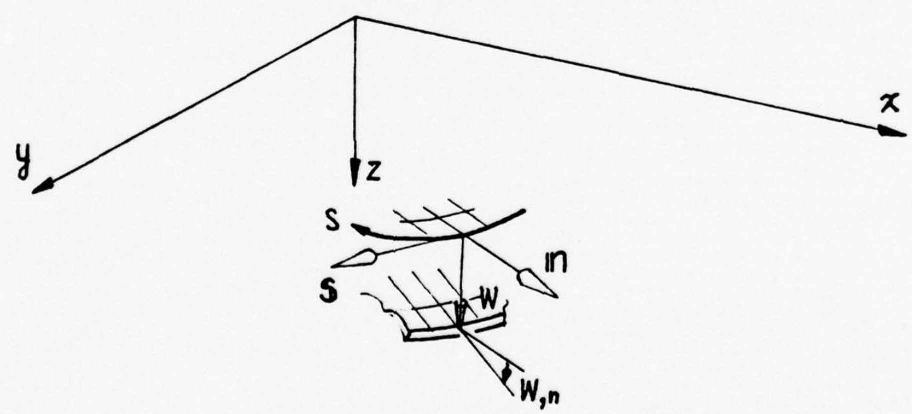


Fig. 5. Base vectors of the edge coordinates and edge displacement function

reduced since the evaluation of each trigonometric and exponential term performed on a digital computer itself involves evaluation of truncated polynomial series. Foregoing remarks indicate that the best choice of the functional representation of the transverse displacement of the plate for the equilibrium finite element with a large number of degrees of freedom is not obvious. The set of solution functions to the biharmonic equation that is chosen here are the Levy series solutions for the reference field with various boundary conditions. Donaldson et al (refs.8 thru 12) superimpose the Navier solution and Levy solutions for the dynamic plate bending equation for the purpose of finding the response to harmonic excitation and for finding the eigenvalues for the rectangular trapezoidal, and triangular plates for various boundary conditions. This method, the Extended Field Method, embeds the plate to be analyzed (the actual plate) in a larger rectangular plate (the extended plate). The Levy solutions $W_i(X), i=1,4$, chosen are those that satisfy simply supported boundary conditions on all but the i^{th} edge. The arbitrary transverse displacements and rotations of the i^{th} edge of $W_i(X)$ are determined by the free parameters of the series. The boundary conditions of the actual plate are then enforced in an approximate sense by requiring the error to be orthogonal to a set of test functions. This is precisely the technique proposed here, (eq.4), for finding the coordinate transformation for a displacement compatible finite element.

However, the Navier and Lévy series solutions to be used to produce a finite element stiffness matrix and the associated force vector are those that satisfy the static equation. The appropriate Navier solution is given in ref.5 and below. The appropriate Lévy series solutions are derived in the sequel. It may be noted here, that it is customary in finite element solutions of dynamic problems to use mass matrices that are based on a smaller number of inertia references than displacement coordinates used for developing the structural stiffness matrix. The stiffness matrix is then condensed and the solution of the dynamic problem is found from the reduced system of equations. This is justified by the fact that the solution of the dynamic problem is more sensitive to the refinement of its elastic references than to the corresponding refinement of its inertial references.

In order to produce a finite element stiffness matrix it is necessary that the displacement function used contains components that permit elastic corner displacements and rigid body displacements. This can be achieved by the use of a few polynomial terms. The completeness requirement for the plate bending element requires for the set of polynomial terms that at least all terms up to the second order are present in the displacement function. This means that in addition to all rigid body displacements, all constant strains should be included. The elastic corner displacements can be represented by the polynomial function

$$W_o(x) = C_1\left(1-\frac{x}{a}\right)\frac{y}{b} + C_2\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right) + C_3\frac{x}{a}\left(1-\frac{y}{b}\right) + C_4\frac{x}{a}\frac{y}{b} \quad . \tag{13}$$

It is observed that in addition, these functions are able to represent all rigid body displacements and constant twist. Unless the number of Lévy series terms employed is sufficiently large to approximate constant bending strain well enough, it will be necessary to include in $W_o(x)$

the functions

$$C_5 \cdot \left(\frac{x}{a}\right)^2 \quad \text{and} \quad C_6 \cdot \left(\frac{y}{b}\right)^2 .$$

The solution for the inhomogeneous biharmonic equation

$$\nabla^4 w = \frac{1}{D} p(\mathbf{x})$$

subjected to simply supported boundary conditions at the edges

$$x = 0 \quad , \quad x = a \quad , \quad y = 0 \quad , \quad y = b$$

of the reference field (the extended plate) is given by

$$w_p(\mathbf{x}) = \sum_{m=1} \sum_{n=1} p_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} ,$$

where

$$p_{mn} = \left\{ D\left(\frac{\pi}{a}\right)^4 \left[m^2 + \left(\frac{a}{b}\right)^2 n^2 \right] \right\}^{-1} \frac{4}{ab} \int_0^a \int_0^b p(\mathbf{x}) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, d\mathbf{x} . \tag{14}$$

The Lévy-series solution for the extended plate with a simply supported boundary along all edges except for example the edge $x = a$ can be found by choosing for the displacement function the Levy series

$$w_3(\mathbf{x}) = \sum_{n=1}^{N_3} X_n(x) \sin \frac{n\pi y}{b} . \tag{15}$$

The index 3 indentifies the edge of the extended field at which arbitrary displacements and rotations are prescribed (Fig. 2). This series satisfies the boundary conditions at the edges $y = 0$ and $y = b$ automatically. Substitution of this displacement function into the homogeneous differential equation and observation of the orthogonality of $\sin \frac{n\pi y}{b}$ over the interval $(0,b)$ yields the ordinary differential equation

$$X_n^{IV} - 2\left(\frac{n\pi}{b}\right)^2 X_n'' + \left(\frac{n\pi}{b}\right)^4 X_n = 0$$

whose solution is subject to the boundary conditions $X_n(0) = X_n''(0) = 0$, $X_n(a) = h_n$ and $X_n'(a) = k_n$. Substitution of the exponential function e^{rx} into the equation and cancellation of the common factor gives then the characteristic equation $r^4 - 2\left(\frac{n\pi}{b}\right)^2 r^2 + \left(\frac{n\pi}{b}\right)^4 = 0$ which has the double roots $r = \pm \frac{n\pi}{b}$. The solution of this differential equation is therefore

$$X_n(x) = \bar{A}_n e^{\lambda_n x} + \bar{B}_n e^{-\lambda_n x} + x(\bar{C}_n e^{\lambda_n x} + \bar{D}_n e^{\lambda_n x})$$

or equivalently

$$X_n(x) = A_n \cosh\lambda_n x + B_n \sinh\lambda_n x + C_n \lambda_n x \cosh\lambda_n x + D_n \lambda_n x \sinh\lambda_n x ,$$

where $\lambda_n = \frac{n\pi}{b}$. Application of the boundary conditions yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ \cosh\lambda_n a & \sinh\lambda_n a & \lambda_n a\cosh\lambda_n a & \lambda_n a\sinh\lambda_n a \\ \sinh\lambda_n a & \cosh\lambda_n a & \cosh\lambda_n a + \lambda_n a\sinh\lambda_n a & \sinh\lambda_n a + \lambda_n a\cosh\lambda_n a \end{bmatrix} \begin{Bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ h_n \\ \frac{k_n}{\lambda_n} \end{Bmatrix} .$$

Solving this matrix equation, the coefficients are

$$A_n = D_n = 0$$

$$B_n = J_n(\tfrac{a}{b}) \{ -h_n(\cosh\lambda_n a + \lambda_n a \sinh\lambda_n a) + \frac{k_n}{\lambda_n} \lambda_n a \cosh\lambda_n a \}$$

and

$$C_n = J_n(\tfrac{a}{b}) \{ h_n \cosh\lambda_n a - \frac{k_n}{\lambda_n} \sinh\lambda_n a \} ,$$

where

$$J_n(\tfrac{a}{b}) = [\lambda_n a - \sinh\lambda_n a \cosh\lambda_n a]^{-1} .$$

Substitution into the expression for W_3, eq. (15), gives

$$W_3(x) = \sum_{n=1} J_n(\tfrac{a}{b}) \sin\lambda_n y \{ h_n[-(\cosh\lambda_n a + \lambda_n a \sinh\lambda_n a)\sinh\lambda_n x + \cosh\lambda_n a \cdot \lambda_n x \cosh\lambda_n x] + \frac{k_n}{\lambda_n}[\lambda_n a \cosh\lambda_n a \sinh\lambda_n x - \sinh\lambda_n a \cdot \lambda_n x \cosh\lambda_n x] \} .$$

Similar solutions can be found for W_i , i = 1,2 and 4.

At this instant it should be noted that $J_n(\frac{a}{b})$ is a constant for given n and ordered side ratio. Since there is no need to attach a specific meaning to the parameters h_n and k_n, convenient scaling factors may be chosen that simplify the algorithm to be developed and avoid difficulties that might arise from computing large numbers of the form $e^{2n\pi}$.

In previous work on the EXFM ,which dealt strictly with the dynamic plate equation the corresponding factors $J_n(\frac{a}{b})$ had singularities for certain side ratios $\frac{a}{b}$ and n. Therefore such simplifications were not possible. Thus let,

$$\mathcal{H}_j(X,Y;A,B) = e^{-2\lambda_j A} \sin\lambda_j Y \cdot \{\cosh\lambda_j A \cdot \lambda_j X \cosh\lambda_j X - [\cosh\lambda_j A + \lambda_j A \sinh\lambda_j A] \sinh\lambda_j X\}$$

and

$$\mathcal{K}_j(X,Y;A,B) = e^{-2\lambda_j A} \sin\lambda_j Y \cdot \{-\sinh\lambda_j A \cdot \lambda_j X \cosh\lambda_j X + \lambda_j A \cosh\lambda_j A \sinh\lambda_j X\}, \tag{16}$$

where

$$\lambda_j = j\pi/B$$

With this definition, the displacement functions $W_i(\mathbf{x})$ that correspond to simply supported boundary conditions along all edges of the extended plate, but the i^{th} edge, where arbitrary displacements are prescribed, become

$$\begin{aligned}
W_1(\mathbf{x}) &= \sum_{j=1}^{H_1} h_{1j}\,\mathcal{H}_j(a-x,y;a,b) - \sum_{j=1}^{K_1} k_{1j}\,\mathcal{K}_j(a-x,y;a,b)\\
W_2(\mathbf{x}) &= \sum_{j=1}^{H_2} h_{2j}\,\mathcal{H}_j(y,x;b,a) + \sum_{j=1}^{K_2} k_{2j}\,\mathcal{K}_j(y,x;b,a)\\
W_3(\mathbf{x}) &= \sum_{j=1}^{H_3} h_{3j}\,\mathcal{H}_j(x,y;a,b) + \sum_{j=1}^{K_3} k_{3j}\,\mathcal{K}_j(x,y;a,b)\\
W_4(\mathbf{x}) &= \sum_{j=1}^{H_4} h_{4j}\,\mathcal{H}_j(b-y,x;b,a) - \sum_{j=1}^{K_4} k_{4j}\,\mathcal{K}_j(b-y,x;b,a).
\end{aligned} \tag{17}$$

The transverse displacement of the plate can thus be expressed by

$$W(\mathbf{x}) = W_o(\mathbf{x}) + \sum_{i=1}^{4} W_i(\mathbf{x}) + W_p(\mathbf{x}). \tag{18}$$

To obtain the displacement of the plate edge in terms of the displacement parameters for the plate interior it is simply required to evaluate the displacement at the edge points. Using edgewise boundary coordinates, the transformation from boundary coordinates to reference coordinates is given by $\quad (x,y) = (x_{m_i} + (\frac{dx}{d\xi})_i \xi \; , \; y_{m_i} + (\frac{dy}{d\xi})_i \xi) \; , \; 0 \le \xi \le l_i \; , \; i = 1,4 \; ,$

where (x_{m_i}, y_{m_i}) are the coordinates of the corner nodal point i and the ratios $(\frac{dx}{d\xi})_i$ and $(\frac{dy}{d\xi})_i$ give the directions of the straight edge boundary segment i. For representation of the total boundary by one continuous parameter this transformation becomes

$$(x,y) = (x_{m_i} + (\frac{dx}{d\xi})_i (\xi - \sum_{j=0}^{i-1} l_j) \; , \; y_{m_i} + (\frac{dy}{d\xi})_i (\xi - \sum_{j=0}^{i-1} l_j) \;) \; ,$$

where $l_0 \equiv 0$.

To obtain the edge rotation, it is necessary to first take the derivative in the direction of the outside normal and then evaluate this derivative at the edge points.

The transverse displacement and rotation at the plate edge thus expressed in terms of the parameters of the interior representation, it remains to express these displacement quantities in terms of the nodal degrees of freedom. The transverse displacement and rotation of the finite element boundary have been chosen (sect. 2.1) to be splines in terms of the nodal degrees of freedom. For the edge displacements of edge j the cubic spline is

$$\hat{W}^{(j)}(\xi) = \sum_{i=m_j}^{m_{j+1}-1} f_i(\xi) \; [\, 1(\xi - \xi_i) - 1(\xi - \xi_{i+1}) \,] \; , \tag{19}$$

where

$$\begin{aligned} f_i(\xi) = & \; [\tfrac{2}{\Delta_i^3} (\xi - \xi_i)^3 - \tfrac{3}{\Delta_i^2} (\xi - \xi_i)^2 + 1 \;] \; \hat{W}_i \\ & + [\tfrac{1}{\Delta_i^2} (\xi - \xi_i)^3 - \tfrac{2}{\Delta_i} (\xi - \xi_i)^2 + (\xi - \xi_i)] \; \hat{S}_i \\ & + [-\tfrac{2}{\Delta_i^3} (\xi - \xi_i)^3 + \tfrac{3}{\Delta_i^2} (\xi - \xi_i)^2 \;] \; \hat{W}_{i+1} \\ & + [\tfrac{1}{\Delta_i^2} (\xi - \xi_i)^3 - \tfrac{1}{\Delta_i} (\xi - \xi_i)^2 \;] \; \hat{S}_{i+1} \end{aligned}$$

with $\quad \Delta_i = \xi_{i+1} - \xi_i$

and $\quad \hat{S}_i = (\frac{dx}{d\xi})_j \hat{\alpha}_i + (\frac{dy}{d\xi})_j \hat{\beta}_i$, with $\hat{\alpha}_i, \hat{\beta}_i$ and $\hat{W}_i$ defined by figure 3.

Here $1(t)$ is the Heavyside-Unit step function and ξ_i the edge coordinate at nodal point i , m_j the nodal point number corresponding to the corner node of corner j.

The linear spline for the edge rotation at edge j is given by

$$\hat{r}^{(j)}(\xi) = \sum_{i=m_j}^{m_{j+1}-1} g_i(\xi)\,[\,1(\xi-\xi_i) - 1(\xi-\xi_{i+1})\,] \,, \tag{20}$$

where

$$g_i(\xi) = [\,1 - \tfrac{1}{\Delta_i}(\xi-\xi_i)\,]\,\hat{g}_i + \tfrac{1}{\Delta_i}(\xi-\xi_i)\,\hat{g}_{i+1}$$

with

$$\hat{g}_i = \left(\frac{dy}{d\xi}\right)_j \hat{\alpha}_i - \left(\frac{dx}{d\xi}\right)_j \hat{\beta}_i \quad .$$

2.3 The Rectangular Plate Bending Finite Element

The experiences gained in references (9,10,11) can be applied to the present task of developing a rectangular finite element for plate bending via this equilibrium approach. First of all, there is little reason to have the actual rectangular finite element smaller than the rectangular extended plate. Secondly computational advantages are gained if the error of the displacement and rotation is required to be orthogonal to a set of test functions for each edge separately. That is, instead of equation (4) the boundary error reduction is given by $\int_0^{\ell_k} \mathrm{error}(\xi)\,\phi_i(\xi)\,d\xi = 0,\ k=1,4\,;\ i=1,2,\ldots$. The total number of such equations must again equal the number of parameters in the displacement functions for the plate so that the coefficients matrix associated with these parameters is square and nonsingular. Also some simplification in the equations for the force quantities are noted. It would be quite simple to derive the Force-matching conditions directly in terms of derivatives in the x and y directions, the reference coordinates, without reference to edge coordinates which require transformation coefficients. However, to maintain clarity and uniformity the structure of the development is retained and the simplification of the transformations are merely noted. The transformation coefficients t_{1k}, t_{8k} and t_{11k} vanish identically and the remaining coefficients are given in table 1 .

Transformation coefficients t_{ik} for the edge coordinates for the rectangular plate.

$i \backslash k$	1	2	3	4
2	$D(1-\nu)(-2)$	$D(1-\nu)(2)$	$D(1-\nu)(-2)$	$D(1-\nu)(2)$
3	D	0	$-D$	0
4	0	$D(2-\nu)$	0	$D(2-\nu)(-1)$
5	$D(2-\nu)(-1)$	0	$D(2-\nu)$	0
6	0	$-D$	0	D
7	$-D$	0	D	0
9	$-D\nu$	0	$D\nu$	0
10	0	$-D\nu$	0	$D\nu$
12	0	$-D$	0	D

Table 1

2.4 The Quadrilateral Plate Bending Finite Element

To obtain the transformation matrix, equation 5, from the field displacement coordinates to the nodal degrees of freedom one may require the displacement discrepancy to be orthogonal to a set of test functions for each edge separately or for the closed boundary. In either case the total number of resulting equations must equal the number of the parameters of the displacement function. The alternative procedure preferred by Donaldson and Chander, (refs. 8,9), in which the intervals over which the error is made orthogonal to the set of test functions, extend to the edges of the extended plate is certainly attractive for the case of (determined) homogeneous boundary conditions since computational advantages arise. For the purpose of developing a stiffness matrix for a finite element where the boundary conditions on the finite element remain undetermined, extension of the intervals for the orthogonalization process would add arbitrariness to the process and with it an additional source for error.

Chander also observed deterioration of convergence as the extended field becomes larger in comparison to the actual field. One should therefore choose the extended rectangular plate such that the area of the convex quadrilateral plate is as near as possible in size to the area of the extended plate.

2.4a The Coordinate Transformation for the Quadrilateral Plate Bending Finite Element

The transformation (eq. 5) that relates the nodal degrees of freedom, the q_i , to the parameters h_j of the displacement functions for the plate interior results from matrix operations on equation (4). This latter equation is obtained by demanding that the discrepancy in the external and internal representation of the same edge displacement quantitites be orthogonal to some set of test functions. Care must be taken in selecting this set of test functions so that the matrix $\mathbb{B}$ in equation (5) is square and invertable. For each of the four edges the discrepancies in the transverse displacement and in the rotation are both required to be orthogonal to an orthogonal set of functions. A convenient orthogonal set is given by $\phi_j^k(\xi) = \sin\frac{j\pi\xi}{\ell_k}$ for each edge $k = 1,4$. The resultant set of simultaneous equations is given by

$$\begin{aligned}
&\int_0^{\ell_k} [w(\xi) - \hat{w}(\xi)] \sin\frac{j\pi\xi}{\ell_k}\, d\xi = 0 \qquad j = 1, N_{1k} \quad ; \; k = 1,4 \\
&\int_0^{\ell_k} [w_{,n}(\xi) - \hat{r}(\xi)] \sin\frac{j\pi\xi}{\ell_k}\, d\xi = 0 \qquad j = 1, N_{2k} \quad ; \; k = 1,4 \; .
\end{aligned} \tag{21}$$

Substitution of the displacement functions into these equations gives the four sets for the edge displacement

$$\begin{aligned}
&C_1 \int_0^{\ell_k} [1 - \tfrac{1}{a}(x_{m_k} + (\tfrac{dx}{ds})_k \xi)][\tfrac{1}{b}(y_{m_k} + (\tfrac{dy}{ds})_k \xi)] \sin\frac{j\pi\xi}{\ell_k}\, d\xi \\
&+ C_2 \int_0^{\ell_k} [1 - \tfrac{1}{a}(x_{m_k} + (\tfrac{dx}{ds})_k \xi)][1 - \tfrac{1}{b}(y_{m_k} + (\tfrac{dy}{ds})_k \xi)] \sin\frac{j\pi\xi}{\ell_k}\, d\xi \\
&+ C_3 \int_0^{\ell_k} [\tfrac{1}{a}(x_{m_k} + (\tfrac{dx}{ds})_k \xi)][1 - \tfrac{1}{b}(y_{m_k} + (\tfrac{dy}{ds})_k \xi)] \sin\frac{j\pi\xi}{\ell_k}\, d\xi \\
&+ C_4 \int_0^{\ell_k} [\tfrac{1}{a}(x_{m_k} + (\tfrac{dx}{ds})_k \xi)][\tfrac{1}{b}(y_{m_k} + (\tfrac{dy}{ds})_k \xi)] \sin\frac{j\pi\xi}{\ell_k}\, d\xi \\
&+ C_5 \int_0^{\ell_k} (\tfrac{1}{a})^2 (x_{m_k}^2 + 2x_{m_k}(\tfrac{dx}{ds})_k \xi + (\tfrac{dx}{ds})^2 \xi^2) \sin\frac{j\pi\xi}{\ell_k}\, d\xi
\end{aligned}$$

$$
\begin{aligned}
&+ C_6 \int_0^{l_k} \left(\frac{1}{b}\right)^2 \left(y_{m_k}^2 + 2 y_{m_k} \left(\frac{dy}{ds}\right)_k \xi + \left(\frac{dy}{ds}\right)^2 \xi^2 \right) \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&+ \sum_{i=1}^{H_1} h_{1i} \int_0^{l_k} \mathcal{H}_i \left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi ,\; y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi \;;\; a, b \right) \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&- \sum_{i=1}^{K_1} k_{1i} \int_0^{l_k} \mathcal{K}_i \left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi ,\; y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi \;;\; a, b \right) \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&+ \sum_{i=1}^{H_2} h_{2i} \int_0^{l_k} \mathcal{H}_i \left(y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi ,\; x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi \;;\; b, a \right) \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&+ \sum_{i=1}^{K_2} k_{2i} \int_0^{l_k} \mathcal{K}_i \left(y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi ,\; x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi \;;\; b, a \right) \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&+ \sum_{i=1}^{H_3} h_{3i} \int_0^{l_k} \mathcal{H}_i \left(x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi ,\; y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi \;;\; a, b \right) \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&+ \sum_{i=1}^{K_3} k_{3i} \int_0^{l_k} \mathcal{K}_i \left(x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi ,\; y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi \;;\; a, b \right) \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&+ \sum_{i=1}^{H_4} h_{4i} \int_0^{l_k} \mathcal{H}_i \left(b - y_{m_k} - \left(\frac{dy}{ds}\right)_k \xi ,\; x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi \;;\; b, a \right) \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&- \sum_{i=1}^{K_4} k_{4i} \int_0^{l_k} \mathcal{K}_i \left(b - y_{m_k} - \left(\frac{dy}{ds}\right)_k \xi ,\; x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi \;;\; b, a \right) \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&+ \sum_{p=1}^{M_p} \sum_{q=1}^{N_p} p_{pq} \int_0^{l_k} \sin \frac{p\pi}{a} \left(x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi \right) \sin \frac{q\pi}{b} \left(y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi \right) \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&= \hat{W}_{m_k} \int_0^{\xi_{m_k+1}} \left[\frac{2}{\Delta_{m_k}^3} (\xi - \xi_{m_k})^3 - \frac{3}{\Delta_{m_k}^2} (\xi - \xi_{m_k})^2 + 1 \right] \sin \frac{j\pi\xi}{l_k}\, d\xi \\
&\quad + \left[\hat{\alpha}_{m_k} \left(\frac{dx}{ds}\right)_k + \hat{\beta}_{m_k} \left(\frac{dy}{ds}\right)_k \right] \int_0^{\xi_{m_k+1}} \left[\frac{1}{\Delta_{m_k}^2} (\xi - \xi_{m_k})^3 - \frac{2}{\Delta_{m_k}} (\xi - \xi_{m_k})^2 + (\xi - \xi_{m_k}) \right] \sin \frac{j\pi\xi}{l_k}\, d\xi
\end{aligned}
$$

$$+\sum_{i=m_k+1}^{m_{k+1}-1}\Big\{ \hat{W}_i \Big[\Big[\int_{\xi_{i-1}}^{\xi_i} [-\frac{2}{\Delta_{i-1}^3}(\xi-\xi_{i-1})^3+\frac{3}{\Delta_{i-1}^2}(\xi-\xi_{i-1})^2]\sin\frac{j\pi\xi}{l_k}d\xi$$

$$+\int_{\xi_i}^{\xi_{i+1}} \frac{2}{\Delta_i^3}(\xi-\xi_i)^3-\frac{3}{\Delta_i^2}(\xi-\xi_i)^2+1]\sin\frac{j\pi\xi}{l_k}d\xi$$

$$+[\hat{\alpha}_i(\frac{dx}{ds})_k+\hat{\beta}_i(\frac{dy}{ds})_k]\Big[\int_{\xi_{i-1}}^{\xi_i}[\frac{1}{\Delta_{i-1}^2}(\xi-\xi_{i-1})^3-\frac{1}{\Delta_{i-1}}(\xi-\xi_{i-1})^2]\sin\frac{j\pi\xi}{l_k}d\xi$$

$$+\int_{\xi_i}^{\xi_{i+1}}[\frac{1}{\Delta_i^2}(\xi-\xi_i)^3-\frac{2}{\Delta_i}(\xi-\xi_i)^2+(\xi-\xi_i)]\sin\frac{j\pi\xi}{l_k}d\xi\Big]\Big\}$$

$$+W_{m_{k+1}}\int_{\xi_{m_{k+1}-1}}^{l_k}[-\frac{2}{\Delta_{m_{k+1}-1}^3}(\xi-\xi_{m_{k+1}-1})^3+\frac{3}{\Delta_{m_{k+1}-1}^2}(\xi-\xi_{m_{k+1}-1})^2]\sin\frac{j\pi\xi}{l_k}d\xi$$

$$+[\hat{\alpha}_{m_{k+1}}(\frac{dx}{ds})_k+\hat{\beta}_{m_{k+1}}(\frac{dy}{ds})_k]\int_{\xi_{m_{k+1}-1}}^{l_k}[\frac{1}{\Delta_{m_{k+1}-1}^2}(\xi-\xi_{m_{k+1}-1})^3-\frac{1}{\Delta_{m_{k+1}-1}}(\xi-\xi_{m_{k+1}-1})^2]\sin\frac{j\pi\xi}{l_k}d\xi\ ,$$

$$j=1,.. \qquad ; k=1,4 \tag{22a}$$

where $\Delta_i = \xi_{i+1}-\xi_i$,

and the four sets for edge rotation

$$C_1\Big\{(\frac{dy}{ds})_k\int_0^{l_k}(-\frac{1}{a})[\frac{1}{b}(y_{m_k}+(\frac{dy}{ds})_k\xi]\sin\frac{j\pi\xi}{l_k}d\xi$$

$$-(\frac{dx}{ds})_k\int_0^{l_k}[1-\frac{1}{a}(x_{m_k}+(\frac{dx}{ds})_k\xi)](\frac{1}{b})\sin\frac{j\pi\xi}{l_k}d\xi\Big\}$$

$$+C_2\Big\{(\frac{dy}{ds})_k\int_0^{l_k}(-\frac{1}{a})[1-\frac{1}{b}(y_{m_k}+(\frac{dx}{ds})_k\xi]\sin\frac{j\pi\xi}{l_k}d\xi$$

$$-(\frac{dx}{ds})_k\int_0^{l_k}[1-\frac{1}{a}(x_{m_k}+(\frac{dx}{ds})_k\xi)](-\frac{1}{b})\sin\frac{j\pi\xi}{l_k}d\xi\Big\}$$

$$+C_3 \left\{ \left(\frac{dy}{ds}\right)_k \int_0^{l_k} \left(\frac{1}{a}\right) \left[1 - \frac{1}{b}\left(y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi\right)\right] \sin\frac{j\pi\xi}{l_k}\, d\xi \right.$$

$$\left. -\left(\frac{dx}{ds}\right)_k \int_0^{l_k} \left[\frac{1}{a}\left(x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi\right)\right]\left(-\frac{1}{b}\right) \sin\frac{j\pi\xi}{l_k}\, d\xi \right\}$$

$$+C_4 \left\{ \left(\frac{dy}{ds}\right)_k \int_0^{l_k} \left(\frac{1}{a}\right) \left[\frac{1}{b}\left(y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi\right)\right] \sin\frac{j\pi\xi}{l_k}\, d\xi \right.$$

$$\left. -\left(\frac{dx}{ds}\right)_k \int_0^{l_k} \left[\frac{1}{a}\left(x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi\right)\right]\left(\frac{1}{b}\right) \sin\frac{j\pi\xi}{l_k}\, d\xi \right\}$$

$$+C_5 \left(\frac{dy}{ds}\right)_k \int_0^{l_k} \left(\frac{2}{a^2}\right)\left(x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi\right) \sin\frac{j\pi\xi}{l_k}\, d\xi$$

$$+C_6 \left[-\left(\frac{dx}{ds}\right)_k \int_0^{l_k} \left(\frac{2}{b^2}\right)\left(y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi\right) \sin\frac{j\pi\xi}{l_k}\, d\xi\right.$$

$$+\sum_{i=1}^{H_1} h_{1i} \left\{ \left(\frac{dy}{ds}\right)_k \left[-\int_0^{l_k} \mathcal{H}_i'\left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi,\ y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi;\ a, b\right) \sin\frac{j\pi\xi}{l_k}\, d\xi\right]\right.$$

$$\left. -\left(\frac{dx}{ds}\right)_k \left[\int_0^{l_k} \mathcal{H}^{\circ}\left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi,\ y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi;\ a, b\right) \sin\frac{j\pi\xi}{l_k}\, d\xi\right]\right\}$$

$$-\sum_{i=1}^{K_1} k_{1i} \left\{ \left(\frac{dy}{ds}\right)_k \left[-\int_0^{l_k} \mathcal{K}_i'\left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi,\ y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi;\ a, b\right) \sin\frac{j\pi\xi}{l_k}\, d\xi\right]\right.$$

$$\left. -\left(\frac{dx}{ds}\right)_k \left[\int_0^{l_k} \mathcal{K}_i^{\circ}\left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi,\ y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi;\ a, b\right) \sin\frac{j\pi\xi}{l_k}\, d\xi\right]\right\}$$

$$+\sum_{i=1}^{H_2} h_{2i} \left\{ \left(\frac{dy}{ds}\right)_k \left[\int_0^{l_k} \mathcal{H}_i^{\circ}\left(y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi,\ x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi;\ b, a\right) \sin\frac{j\pi\xi}{l_k}\, d\xi\right]\right.$$

$$\left. -\left(\frac{dx}{ds}\right)_k \left[\int_0^{l_k} \mathcal{H}_i'\left(y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi,\ x_{m_k} + \left(\frac{dx}{ds}\right)_k \xi;\ b, a\right) \sin\frac{j\pi\xi}{l_k}\, d\xi\right]\right\}$$

$$+\sum_{i=1}^{K_2} k_{2i}\Big\{\left(\frac{dy}{ds}\right)_k \Big[\int_0^{l_k} \mathcal{K}_i^{\circ}\Big(y_{m_k}+\left(\frac{dy}{ds}\right)_k\xi,\ x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi;\ b,a\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big]$$

$$-\left(\frac{dx}{ds}\right)_k \Big[\int_0^{l_k} \mathcal{K}_i'\Big(y_{m_k}+\left(\frac{dy}{ds}\right)_k\xi,\ x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi;\ b,a\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big]\Big\}$$

$$+\sum_{i=1}^{H_3} h_{3i}\Big\{\left(\frac{dy}{ds}\right)_k \Big[\int_0^{l_k} \mathcal{H}_i'\Big(x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi,\ y_{m_k}+\left(\frac{dy}{ds}\right)_k\xi;\ a,b\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big]$$

$$-\left(\frac{dx}{ds}\right)_k \Big[\int_0^{l_k} \mathcal{H}_i^{\circ}\Big(x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi,\ y_{m_k}+\left(\frac{dy}{ds}\right)_k\xi;\ a,b\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big]\Big\}$$

$$+\sum_{i=1}^{K_3} k_{3i}\Big\{\left(\frac{dy}{ds}\right)_k \Big[\int_0^{l_k} \mathcal{K}_i'\Big(x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi,\ y_{m_k}+\left(\frac{dy}{ds}\right)_k\xi;\ a,b\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big]$$

$$-\left(\frac{dx}{ds}\right)_k \Big[\int_0^{l_k} \mathcal{K}_i^{\circ}\Big(x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi,\ y_{m_k}+\left(\frac{dy}{ds}\right)_k\xi;\ a,b\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big]\Big\}$$

$$+\sum_{i=1}^{H_4} h_{4i}\Big\{\left(\frac{dy}{ds}\right)_k \Big[\int_0^{l_k} \mathcal{H}_i^{\circ}\Big(b-y_{m_k}-\left(\frac{dy}{ds}\right)_k\xi,\ x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi;\ b,a\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big]$$

$$-\left(\frac{dx}{ds}\right)_k \Big[\int_0^{l_k} \mathcal{H}_i'\Big(b-y_{m_k}-\left(\frac{dy}{ds}\right)_k\xi,\ x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi;\ b,a\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big]\Big\}$$

$$-\sum_{i=1}^{K_4} k_{4i}\Big\{\left(\frac{dy}{ds}\right) \Big[\int_0^{l_k} \mathcal{K}_i^{\circ}\Big(b-y_{m_k}-\left(\frac{dy}{ds}\right)_k\xi,\ x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi;\ b,a\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big]$$

$$-\left(\frac{dx}{ds}\right)_k \Big[-\int_0^{l_k} \mathcal{K}_i'\Big(b-y_{m_k}-\left(\frac{dy}{ds}\right)_k\xi,\ x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi,\ b,a\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big]\Big\}$$

$$+\sum_{p=1}^{M_p}\sum_{q=1}^{N_p} p_{pq}\Big\{\left(\frac{dy}{ds}\right)_k \frac{p\pi}{a}\int_0^{l_k}\cos\frac{p\pi}{a}\Big(x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi\Big)\sin\frac{q\pi}{b}\Big(y_{m_k}+\left(\frac{dy}{ds}\right)_k\xi\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi$$

$$-\left(\frac{dx}{ds}\right)_k \frac{q\pi}{b}\int_0^{l_k}\sin\frac{p\pi}{a}\Big(x_{m_k}+\left(\frac{dx}{ds}\right)_k\xi\Big)\cos\frac{q\pi}{b}\Big(y_{m_k}+\left(\frac{dy}{ds}\right)_k\xi\Big)\sin\frac{i\pi\xi}{l_k}\,d\xi\Big\}$$

$$= [\hat{\alpha}_{m_k} (\frac{dy}{ds})_k - \hat{\beta}_{m_k} (\frac{dx}{ds})_k] \int_0^{\xi_{m_k+1}} [1 - \frac{1}{\Delta_{m_k}}(\xi - \xi_{m_k})] \sin \frac{j\pi\xi}{l_k} \, d\xi$$

$$+ \sum_{i=m_k+1}^{m_{k+1}-1} [\hat{\alpha}_i (\frac{dy}{ds})_k - \hat{\beta}_i (\frac{dx}{ds})_k] \Big\{ \int_{\xi_{i-1}}^{\xi_i} \frac{1}{\Delta_{i-1}} (\xi - \xi_{i-1}) \sin \frac{j\pi\xi}{l_k} \, d\xi$$

$$+ \int_{\xi_i}^{\xi_{i+1}} [1 - \frac{1}{\Delta_i} (\xi - \xi_i)] \sin \frac{j\pi\xi}{l_k} \, d\xi \Big\}$$

$$+ [\hat{\alpha}_{m_{k+1}} (\frac{dy}{ds})_k - \hat{\beta}_{m_{k+1}} (\frac{dx}{ds})_k] \int_{\xi_{m_{k+1}-1}}^{l_k} \frac{1}{\Delta_{m_{k+1}-1}} (\xi - \xi_{m_{k+1}-1}) \sin \frac{j\pi\xi}{l_k} \, d\xi \quad ,$$

$$j = 1, .. \qquad ; k = 1,4 \quad . \tag{22b}$$

$\mathcal{H}', \mathcal{K}'$ means differentiation of the functions $\mathcal{H}$ and $\mathcal{K}$ respectively with respect to the first argument, while $\mathcal{H}^{\circ}, \mathcal{K}^{\circ}$ means differentiation of the functions $\mathcal{H}$ and $\mathcal{K}$ respectively with respect to the second argument.

2.4b Virtual Work and the Stress Resultant-Nodal Force Relation

The transformation from the nodal degrees of freedom to the parameters of the displacement functions for the plate interior is given by the simultaneous set of equations, equations (22a) and (22b), of the previous section. It remains to obtain the elements of the stiffness kernel and the coefficient matrix of the loading parameters in the expression for virtual work. The virtual work expression (eq.10) can be written in the form

$$\delta q^T \mathbb{F} = \sum_{i=1}^{12}\sum_{k=1}^{4} t_{ik} \quad \delta h^T \int_0^{l_k} [(\mathcal{D}_i^{(1)} \mathcal{H}_p)(\mathcal{D}_i^{(2)} \mathcal{H}_q)] \; d\xi \; h$$
$$+ \sum_{i=1}^{12}\sum_{k=1}^{4} t_{ik} \quad \delta h^T \int_0^{l_k} [(\mathcal{D}_i^{(1)} \mathcal{H}_p)(\mathcal{D}_i^{(2)} \mathcal{P}_q)] \; d\xi \; q \,, \qquad (23)$$

where $\mathcal{H}_p$ are here (generically) the displacement functions, i.e. the polynomials representing rigid body displacements, elastic corner displacements and constant strains as well as the hyperbolic-trigonometric functions that represent the remaining elastic solution functions of the homogeneous equation. The $\mathcal{P}_q$ are the displacement functions corresponding to the inhomogeneous equation, the $\mathcal{D}_i^{(j)}$ represent the appropriate partial differential operators in equation (23)

$$\mathcal{D}_i^{(1)} = 1 \quad , i = 1,6 \qquad , \mathcal{D}_i^{(1)} = \frac{\partial}{\partial x} \quad , i = 7,9$$
$$\mathcal{D}_i^{(1)} = \frac{\partial}{\partial y} \quad , i = 10,12$$
$$\mathcal{D}_1^{(2)} = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \qquad , \mathcal{D}_2^{(2)} = \frac{\partial^2}{\partial x \partial y}$$
$$\mathcal{D}_3^{(2)} = \frac{\partial^3}{\partial x^3} \quad , \mathcal{D}_4^{(2)} = \frac{\partial^3}{\partial x^2 \partial y} \quad , \mathcal{D}_5^{(2)} = \frac{\partial^3}{\partial x \partial y^2} \quad , \mathcal{D}_6^{(2)} = \frac{\partial^3}{\partial y^3}$$
$$\mathcal{D}_7^{(2)} = \mathcal{D}_{10}^{(2)} = \frac{\partial^2}{\partial x^2}, \mathcal{D}_8^{(2)} = \mathcal{D}_{11}^{(2)} = \frac{\partial^2}{\partial x \partial y}, \mathcal{D}_9^{(2)} = \mathcal{D}_{12}^{(2)} = \frac{\partial^2}{\partial y^2}$$

and the vectors, $\mathbf{h}$, $\mathbf{q}$ and $\mathbf{p}$ are given by

$$\mathbf{h} = \begin{Bmatrix} c_1 \\ \vdots \\ c_6 \\ h_{11} \\ \vdots \\ h_{1H_1} \\ k_{11} \\ \vdots \\ k_{1K_1} \\ \vdots \\ k_{4K_4} \end{Bmatrix} \qquad \mathbf{q} = \begin{Bmatrix} \hat{W}_1 \\ \hat{\alpha}_1 \\ \hat{\beta}_1 \\ \vdots \\ \\ \hat{W}_{N/3} \\ \hat{\alpha}_{N/3} \\ \hat{\beta}_{N/3} \end{Bmatrix} \qquad \mathbf{p} = \begin{Bmatrix} p_{11} \\ p_{12} \\ \\ p_{1N_p} \\ p_{21} \\ \\ \\ p_{M_pN_p} \end{Bmatrix} .$$

To economize an abbreviated notation is introduced that reduces the polynomial solution functions to a single form and the hyperbolic solution functions to a single form with four different sets of arguments. This gives for the polynomial terms

$$W_0(\mathbf{x}) = \sum_{i=1}^{6} a_0^i + a_1^i x + a_2^i y + a_3^i x^2 + a_4^i xy + a_5^i y^2, \tag{24}$$

where the appropriate coefficients of the polynomial terms are collected in table 2 .

<u>The coefficients a_j^i for the polynomial solution functions</u>

$i \backslash j$	0	1	2	3	4	5
1	0	0	$\frac{1}{b}$	0	$-\frac{1}{ab}$	0
2	1	$-\frac{1}{a}$	$-\frac{1}{b}$	0	$\frac{1}{ab}$	0
3	0	$\frac{1}{a}$	0	0	$-\frac{1}{ab}$	0
4	0	0	0	0	$\frac{1}{ab}$	0
5	0	0	0	$\frac{1}{a^2}$	0	0
6	0	0	0	0	0	$\frac{1}{b^2}$

Table 2

For the hyperbolic-trigonometric functions $\mathcal{H}_j(X,Y;A,B)$ and $\mathcal{K}_j(X,Y;A,B)$ the new notation is

$$\mathcal{J}_j(X,Y;A,B;r,s) \quad ,$$

where the integer parameters r and s are to be interpreted as follows:

$$\mathcal{J} \equiv \mathcal{H} \qquad r = 1$$

and

$$\mathcal{J} \equiv \mathcal{K} \qquad r = 2$$

and the integer s matches the first subscript of the distributed parameters h_{sj} and k_{sj} with the additional meaning that if r = 2 and s = 1 or s = 4 the negative of the function $\mathcal{K}$ should be taken.

Thus,

$$\mathcal{J}_j(X,Y;A,B;r,s) = e^{-2\lambda_j A}\sin\lambda_j Y(\mathcal{A}\,\lambda_j X\cosh\lambda_j X + \mathcal{B}\sinh\lambda_j X), \tag{25}$$

where $\lambda_j = j\pi/B$

and
$$\mathcal{A} = \cosh\lambda_j A \;,\; \mathcal{B} = -[\cosh\lambda_j A + \lambda_j A\sinh\lambda_j A] \quad \text{when} \quad r=1,\ s=1,4$$

$$\mathcal{A} = \sinh\lambda_j A \;,\; \mathcal{B} = -\lambda_j A\cosh\lambda_j A \quad \text{when} \quad r=2 \quad s=\{1,4\}$$

$$\mathcal{A} = -\sinh\lambda_j A \;,\; \mathcal{B} = \lambda_j A\cosh\lambda_j A \quad \text{when} \quad r=2 \quad s=\{2,3\} .$$

It is convenient to define

$$
\begin{aligned}
b_0^{ik} &= a_0^i + a_1^i x_{m_k} + a_2^i y_{m_k} + a_3^i x_{m_k}^2 + a_4^i x_{m_k} y_{m_k} + a_5^i y_{m_k}^2 \\
b_1^{ik} &= a_1^i\left(\frac{dx}{ds}\right)_k + a_2^i\left(\frac{dy}{ds}\right)_k + 2a_3^i\left(\frac{dx}{ds}\right)_k + a_4^i\left[x_{m_k}\left(\frac{dy}{ds}\right)_k + y_{m_k}\left(\frac{dx}{ds}\right)_k\right] + 2a_5^i y_{m_k}\left(\frac{dy}{ds}\right)_k \\
b_2^{ik} &= a_3^i\left(\frac{dx}{ds}\right)_k^2 + a_4^i\left(\frac{dx}{ds}\right)_k\left(\frac{dy}{ds}\right)_k + a_5^i\left(\frac{dy}{ds}\right)_k^2 \\
b_{0x}^{ik} &= a_1^i + 2a_3^i x_{m_k} + a_4^i y_{m_k}
\end{aligned}
\tag{26}
$$

$$b_{1x}^{ik} = 2a_3^i\left(\frac{dx}{ds}\right)_k + a_4^i\left(\frac{dy}{ds}\right)_k$$

$$b_{oy}^{ik} = a_2^i + a_4^i x_{m_k} + 2a_5^i y_{m_k}$$

$$b_{1y}^{ik} = a_4^i\left(\frac{dx}{ds}\right)_k + 2a_5^i\left(\frac{dy}{ds}\right)_k$$

so that

$$W_o(\mathbb{X}_{m_k}) = \sum_{i=1}^{6} b_o^{ik}\, c_i \quad ,$$

$$W_{o,xx}(\mathbb{X}_{m_k}) = W_{o,xx}(\xi) = \sum_{i=1}^{6} 2a_3^i\, c_i \quad ,$$

$$W_{o,yy}(\mathbb{X}_{m_k}) = W_{o,yy}(\xi) = \sum_{i=1}^{6} 2a_5^i\, c_i \quad ,$$

$$W_{o,xy}(\mathbb{X}_{m_k}) = W_{o,xy}(\xi) = \sum_{i=1}^{6} a_4^i\, c_i \quad ,$$

$$W_o(\xi) = \sum_{i=1}^{6}(b_o^{ik} + b_1^{ik}\xi + b_2^{ik}\xi^2)\, c_i,$$

$$W_{o,x}(\xi) = \sum_{i=1}^{6}(b_{ox}^{ik} + b_{1x}^{ik}\xi)\, c_i \quad ,$$

and $$W_{o,y}(\xi) = \sum_{i=1}^{6}(b_{oy}^{ik} + b_{1y}^{ik}\xi)\, c_i \quad .$$

Using these definitions a few typical coefficients of the matrices

$$[\mathcal{H}_{pq}]^{ik} = \left[\int_0^{l_k} (\mathcal{D}_i^{(1)}\mathcal{H}_p)(\mathcal{D}_i^{(2)}\mathcal{H}_q)\, d\xi\right]^{ik}$$

and $$[\mathcal{P}_{pq}] = \left[\int_0^{l_k} (\mathcal{D}_i^{(1)}\mathcal{H}_p)(\mathcal{D}^{(2)}\mathcal{P}_q)\, d\xi\right]^{ik}$$

are worked out and a general scheme is provided in the appendix.
Of course, the stiffness kernel and the matrix $\mathbb{R}$ in equation (11) are given by

$$\mathbb{G} = \sum_{i=1}^{12}\sum_{k=1}^{4} t_{ik}\, \mathbb{H}^{ik}$$

and $$\mathbb{R} = \sum_{i=1}^{12}\sum_{k=1}^{4} t_{ik}\, \mathbb{P}^{ik}.$$

3. Appendix-- Matrix coefficients for the Stress Resultant - Nodal Force Relation.

Components of the matrices $[\mathcal{H}_{pq}]^{ik}$ and $[\mathcal{P}_{pq}]^{ik}$:

The typical components are of the form

$$\mathcal{H}_{pq} = \int_0^{\ell_k} (\mathcal{D}_i^{(1)} \mathcal{H}_p)(\mathcal{D}_i^{(2)} \mathcal{H}_q)\, d\xi$$

$$\mathcal{P}_{pq} = \int_0^{\ell_k} (\mathcal{D}_i^{(1)} \mathcal{H}_p)(\mathcal{D}_i^{(2)} \mathcal{P}_q)\, d\xi \quad , \qquad i=1,12 \;;\; k=1,4 \,,$$

where the $\mathcal{D}_i^{(j)}$ are the differential operators defined in section 2.4b, the $\mathcal{H}_n$ and $\mathcal{P}_n$ are the solution functions for the complimentary and particular solution of the differential equation respectively. The index n in $\mathcal{H}_n$ ranges from 1 to $6+\sum_{j=1}^{4}(H_j+K_j)$. In the subranges of n : $\{1, ,6\}$, $\{7, ,6+H_1\}$, $\{7+H_1, ,6+H_1+K_1\}$, ... $\{7+\sum_{j=1}^{3}(H_j+K_j)+H_4, ,6+\sum_{j=1}^{4}(H_j+K_j)$ the function $\mathcal{H}_n$ takes different forms. The definition of $\mathcal{H}_n$ for the different subranges is given below in table A1 . The required derivatives of the polynomial functions are given in section 2.4b , those for the hyperbolic-trigonometric function $\mathcal{J}$ are

$$
\begin{aligned}
\mathcal{J}' &= \lambda_j \; e^{-2\lambda_j A} \sin\lambda_j Y \,\{\mathcal{A}_j\,[\cosh\lambda_j X+\lambda_j X\sinh\lambda_j X\,] + \mathcal{B}_j \cosh\lambda_j X\,\} \\
\mathcal{J}^{\circ} &= \lambda_j \; e^{-2\lambda_j A} \cos\lambda_j Y \,\{\mathcal{A}_j\; \lambda_j X\cosh\lambda_j X + \mathcal{B}_j \sinh\lambda_j X\,\} \\
\mathcal{J}'' &= \lambda_j^2 \; e^{-2\lambda_j A} \sin\lambda_j Y \,\{\mathcal{A}_j\,[2\sinh\lambda_j X+\lambda_j X\cosh\lambda_j X\,] + \mathcal{B}_j \sin\lambda_j X\,\} \\
\mathcal{J}'^{\circ} &= \lambda_j^2 \; e^{-2\lambda_j A} \cos\lambda_j Y \,\{\mathcal{A}_j\,[\cosh\lambda_j X+\lambda_j X\sinh\lambda_j X] + \mathcal{B}_j \cosh\lambda_j X\,\} \\
\mathcal{J}^{\circ\circ} &= -\lambda_j^2 \; e^{-2\lambda_j A} \sin\lambda_j Y \,\{\mathcal{A}_j\; \lambda_j X\cosh\lambda_j X + \mathcal{B}_j \sinh\lambda_j X\,\} \\
\mathcal{J}''' &= \lambda_j^3 \; e^{-2\lambda_j A} \sin\lambda_j Y \,\{\mathcal{A}_j\,[3\cosh\lambda_j X+\lambda_j X\sinh\lambda_j X] + \mathcal{B}_j \cosh\lambda_j X\,\} \\
\mathcal{J}''^{\circ} &= \lambda_j^3 \; e^{-2\lambda_j A} \cos\lambda_j Y \,\{\mathcal{A}_j\,[2\sinh\lambda_j X+\lambda_j X\cosh\lambda_j X] + \mathcal{B}_j \sinh\lambda_j X\,\} \\
\mathcal{J}'^{\circ\circ} &= -\lambda_j^3 \; e^{-2\lambda_j A} \sin\lambda_j Y \,\{\mathcal{A}_j\,[\cosh\lambda_j X + \lambda_j X\sin\lambda_j X\,] + \mathcal{B}_j \cosh\lambda_j X\,\} \\
\mathcal{J}^{\circ\circ\circ} &= -\lambda_j^3 \; e^{-2\lambda_j A} \cos\lambda_j Y \,\{\mathcal{A}_j\; \lambda_j X\cosh\lambda_j X + \mathcal{B}_j \sinh\lambda_j X\,\}
\end{aligned}
$$

Range of n		Function	Range of index j
1	to 6	$P^j(x) = a_0^j + a_1^j x + a_2^j y + a_3^j x^2 + a_4^j xy + a_5^j y^2$	$j = 1, .., 6$
7	to $6 + H_1$	$f_j(a-x, y; a, b; 1, 1)$	$j = 1, .., H_1$
$7 + H_1$	to $6 + H_1 + K_1$	$f_j(a-x, y; a, b; 2, 1)$	$j = 1, .., K_1$
$7 + H_1 + K_1$	to $6 + \sum_{i=1}^{2} H_i + K_1$	$f_j(y, x; b, a; 1, 2)$	$j = 1, .., H_2$
$7 + \sum_{i=1}^{2} H_i + K_1$	to $6 + \sum_{i=1}^{2}(H_i + K_i)$	$f_j(y, x; b, a; 2, 2)$	$j = 1, .., K_2$
$7 + \sum_{i=1}^{2}(H_i + K_i)$	to $6 + \sum_{i=1}^{2}(H_i + K_i) + H_3$	$f_j(x, y; a, b; 2, 3)$	$j = 1, .., H_3$
$7 + \sum_{i=1}^{2}(H_i + K_i) + H_3$	to $6 + \sum_{i=1}^{3}(H_i + K_i)$	$f_j(x, y; a, b; 2, 3)$	$j = 1, .., K_3$
$7 + \sum_{i=1}^{3}(H_i + K_i)$	to $6 + \sum_{i=1}^{3}(H_i + K_i) + H_4$	$f_j(b-y, x; b, a; 1, 4)$	$j = 1, .., H_4$
$7 + \sum_{i=1}^{3}(H_i + K_i) + H_4$	to $6 + \sum_{i=1}^{4}(H_i + K_i)$	$f_j(b-y, x; b, a; 2, 4)$	$j = 1, .., K_4$

$$f_j(X, Y; A, B; r, s) = e^{-2\lambda_j A} \sin \lambda_j Y (\mathcal{A}_j \lambda_j X \cosh \lambda_j X + \mathcal{B} \sinh \lambda_j X)$$

where $\lambda_j = j\pi/B$

and $\mathcal{A}_j = \cosh \lambda_j A$, $\mathcal{B}_j = -[\cosh \lambda_j A + \lambda_j A \sinh \lambda_j A]$, when $r = 1$, $s = 1,4$

$\mathcal{A}_j = \sinh \lambda_j A$, $\mathcal{B}_j = -\lambda_j A \cosh \lambda_j A$ $\quad r = 2$, $s = \{1,4\}$

$\mathcal{A}_j = -\sinh \lambda_j A$, $\mathcal{B}_j = \lambda_j A \cosh \lambda_j A$ $\quad r = 2$, $s = \{2,3\}$

the constants a_n^m are given in table 2 of section 2.4 b

Table A1

Solution Functions

With these definitions and those for the coefficients $b^{ik}_{m\ell}$ in section 2.4b it is a simple matter to write out the coefficients $\mathcal{H}_{pq}$ of the matrix $\mathbb{H}^{ik}$

Example 1

The elements of the matrix $\mathbb{H}^{1k}$ derive from the term

$$[\ w_{,xx}(x_{m_k}) - w_{,yy}(x_{m_k})\ ]\ \delta w(x_{m_k})$$

of equation (10 sect. 2.1).

Thus for $p, q = 1,,6$

$$\mathcal{H}_{pq} = 2\,(a^q_3 - a^q_5)\,b^{pk}_0$$

in particular for $p = 3\ , q = 2$

$$\mathcal{H}_{3,2} = 0 \cdot b^{3k}_0 = 0$$

or for $p = 3\ , q = 5$

$$\mathcal{H}_{3,5} = \frac{2}{a^2}\left(\frac{1}{a}x_{m_k} - \frac{1}{ab}x_{m_k}y_{m_k}\right),$$

for $p = 7,\ ,6+H.\quad q = 5$

$$\mathcal{H}_{p5} = \frac{2}{a^2}\,\mathcal{S}_j(a - x_{m_k}, y_{m_k}; a, b; 1,1)$$

$$= \frac{2}{a^2}e^{-2\lambda_j a}\sin\lambda_j y_{m_k}\{\cosh\lambda_j a \cdot \lambda_j(a - x_{m_k})\cdot\cosh\lambda_j(a - x_{m_k})$$

$$-[\cosh\lambda_j a + \lambda_j a\sinh\lambda_j a]\sinh\lambda_j(a - x_{m_k})\}\quad , j = 1,,H.$$

with $\lambda_j = j\frac{\pi}{b}$

Example 2

The elements of the matrix $\mathbb{H}^{3k}$ derive from the term

$$\int_0^{\ell_k} w_{,xxx}(\xi)\ \delta w(\xi)\ d\xi$$

of equation (10 sect. 2.1)

For $p, q = 1,,6$

$$\mathcal{H}_{pq} = 0$$

For $p = 1,,6$ and $q = 7 + H_1,,6 + H_1 + K_1$

$$\mathcal{H}_{pq} = -\int_0^{l_k} \mathcal{f}_j'''\left(a - x_{m_k} - \left(\frac{d}{ds}\right)_k \xi, \; y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi; \; a, b; \; 2, 1\right)\left(b_0^{pk} + b_1^{pk}\xi + b_2^{pk}\xi^2\right) d\xi \; , \; j = 1,,K_1$$

$$= -\lambda_j^3 e^{-2\lambda_j a} \int_0^{l_k} \sin\lambda_j\left(y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi\right) \cdot \left\{ \sinh\lambda_j a \left[3\cosh\lambda_j\left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi\right) + \lambda_j\left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi\right)\right.\right.$$

$$\left.\left. \sinh\lambda_j\left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi\right) - \lambda_j a \cosh\lambda_j a \sinh\lambda_j\left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi\right)\right\} \left(b_0^{pk} + b_1^{pk}\xi + b_2^{pk}\xi^2\right) d\xi \, ,$$

where $\lambda_j = \frac{j\pi}{b}$,

specializing further to $p = 6$ then $a_m^6 = 0$ except $a_5^6 = \frac{1}{b^2}$ so that

$$b_0^{6k} + b_1^{6k}\xi + b_2^{6k}\xi^2 = \left(\frac{y_{m_k}}{b}\right)^2 + 2 y_{m_k} \frac{1}{b^2}\left(\frac{dy}{ds}\right)_k \xi + \frac{1}{b^2}\left(\frac{dy}{ds}\right)_k^2 \xi^2 .$$

Example 3

The elements of the matrix $\mathbb{H}^{12k}$ derive from the term

$$\int_0^{l_k} W_{,yy}(\xi) \; \delta W_{,y}(\xi) \; d\xi$$

of equation (10 sect. 2.1) .

For $p = 7, \; , 6 + H_1$ and $q = 7 + \sum_{i=1}^{3}(H_i + K_i) \, , , \; 6 + \sum_{i=1}^{3}(H_i + K_i) + H_4$

$$\mathcal{H}_{pq} = \lambda_m^2 \lambda_n e^{-2(\lambda_m b + \lambda_n a)} \int_0^{l_k} \mathcal{f}_m''\left(b - y_{m_k} - \left(\frac{dy}{ds}\right)_k \xi, \; x_{m_k} + \left(\frac{dx}{ds}\right)\xi; \; b, a \; ; \; 1, 4\right)$$

$$\cdot \, \mathcal{f}_n^{\circ}\left(a - x_{m_k} - \left(\frac{dx}{ds}\right)_k \xi \; ; \; y_{m_k} + \left(\frac{dy}{ds}\right)_k \xi \; ; \; a, b \; ; \; 1, 1\right) d\xi \quad ,$$

where $\lambda_m = \frac{m\pi}{a}$ and $\lambda_n = \frac{n\pi}{b}$

$$m = 1, \; , H_4 \quad \text{and} \quad n = 1, \; , H_1 \quad .$$

The coefficients of the matrices $\mathbb{P}^{ik}$ are obtained in similar manner, here the index q runs from 1 to $M_0 \times N_0$. The function $\mathcal{P}_q$ in equation (23) has the form

$$\mathcal{P}_q(x) = \sin\frac{m\pi x}{a} \, \sin\frac{n\pi y}{b} \, ,$$

where following correspondence between the index q and the index pair (m,n) is set up

q	m, n
1	1 , 1
2	2 , 1
$\vdots$	
M_o	M_o , 1
M_o+1	1 , 2
$\vdots$	
$2M_o$	M_o , 2
$2M_o+1$	1 , 3
$\vdots$	
$M_o \times N_o$	M_o , N_o

Example 4

The elements of the matrix $\mathbb{P}^{3k}$ derive as in example 2 from the term

$$\int_0^{l_k} w_{,xxx}(\xi)\delta w(\xi)\, d\xi$$

For $p = 6$ and $q = 5M_o+3$

$$\mathcal{P}_{pq} = -\left(\frac{3\pi}{a}\right)^3 \int_0^{l_k} \cos\frac{3\pi}{a}\xi \, \sin\frac{6\pi}{b}\xi \left[\left(\frac{y_{m_k}}{b}\right)^2 + 2y_{m_k}\frac{1}{b^2}\left(\frac{dy}{ds}\right)_k\xi + \frac{\xi^2}{b^2}\left(\frac{dy}{ds}\right)_k^2\xi\right] d\xi \quad .$$

4. Appendix-- Summery of the Extended Field Method of Analysis as applied to Uniform Thin Plates.

The governing differential equation for the harmonic response of thin uniform plates is

$$D\nabla^4 w - \rho\omega^2 w = p(\mathbf{x})\sin\omega t,$$

where $w(\mathbf{x},t) = W(\mathbf{x})\sin\omega t$ is the time dependent transverse displacement of the plate, D is the bending rigidity, ω is the excitation frequency and $p(\mathbf{x})$ is the spatial distribution of the transverse pressure , and ρ is the mass per unit area. The amplitude $W(\mathbf{x})$ of the transverse displacement can be expressed as the sum of the solution to the inhomogeneous biharmonic equation

$$D\nabla^4 W - \rho\omega^2 W = p(\mathbf{x})$$

for a rectangular reference field (the extended plate) with all edges simply supported (the Navier-solution) which is

$$W_5(\mathbf{x}) = \sum_{m=1}^{N_0}\sum_{n=1}^{N_0} \frac{p_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{D\left\{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2-\frac{\rho\omega^2 d}{D}\right\}}\,, \quad p_{mn}=\frac{4}{ab}\int_0^a\int_0^b p(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dy\,dx$$

and the four Levy-series solutions $W_i(\mathbf{x})$, $i=1,4$, for the same extended plate with simply supported boundary conditions on all but the i^{th} edge which undergoes arbitrary displacements that are characterized by the parameters h_{ij} and k_{ij} .For example the Levy-series solution for the 1^{st} edge is given by

$$W_1(\mathbf{x}) = \sum_j \gamma_j \{ h_{1j}[s_j\cos(h)s_ja\sinh r_j(a-x) - r_j\cosh r_ja\sin(h)s_j(a-x)]$$
$$+ k_{1j}[\sin(h)s_ja\sinh r_j(a-x) - \sinh r_ja\sin(h)s_j(a-x)]\}\sin\frac{j\pi y}{b},$$

where

$$\gamma_j = [s_j\sinh r_ja\cos(h)s_ja - r_j\cosh r_ja\sin(h)s_ja]^{-1}$$

with

$$r_j = \left[\left(\frac{j\pi}{b}\right)^2 + \omega\left(\frac{\rho}{D}\right)^{1/2} \right]^{1/2}$$

$$s_j = \left[\omega\left(\frac{\rho}{D}\right)^{1/2} - \left(\frac{j\pi}{b}\right)^2 \right]^{1/2} \qquad j < \hat{j}$$

$$s_j = \left[\left(\frac{j\pi}{b}\right)^2 - \omega\left(\frac{\rho}{D}\right)^{1/2} \right]^{1/2} \qquad j \geq \hat{j} \quad .$$

Here

$$\hat{j} = \frac{b}{\pi}\,\omega^{1/2}\left(\frac{\rho}{D}\right)^{1/4}$$

and

$$\sin(h)\, s_j x = \begin{cases} \sin\, s_j x & j < \hat{j} \\ \sinh\, s_j x & j \geq \hat{j} \end{cases}$$

$$\cos(h)\, s_j x = \begin{cases} \cos\, s_j x & j < \hat{j} \\ \cosh\, s_j x & j \geq \hat{j} \end{cases} \quad .$$

The total displacement of the plate can then be written

$$W(\mathbf{x},t) = \sin\omega t \cdot \sum_{i=1}^{5} W_i(\mathbf{x}) \quad .$$

The solution to the harmonic excitation problem of an arbitrary quadrilateral (convex region) plate that lies within the extended plate with any combination of clamped and simply supported edges and perhaps a single free edge is then obtained by approximately enforcing the desired boundary conditions at the edges of this embedded actual plate. In the case of a free corner , i.e. two adjacent free edges , the solution function must be augmented by a corner solution which , for example , for the edges 2 and 4 being free is given by

$$W_0(\mathbf{x}) = c\,\left[\, \sinh\phi(a-x)\sinh\phi(b-y) + \sin\phi(a-x)\sin\phi(b-y) \right] \; , \quad \phi = \omega^{1/2}\left(\frac{\rho}{4D}\right)^{1/4} .$$

The edge numbering corresponds to that of figure 1 of section 2.1 .

The solution to more complex structures such as beam stringer supported plate systems can be obtained by approximately enforcing the matching conditions along common boundaries (ref. 8). The set of simultaneous equations that determine the parameters h_{ij} and k_{ij} is obtained by requiring the error at the boundary edge or matching edge to be orthogonal to some set of independent test functions. The EXFM has also been successfully employed in finding the eigenfrequencies for uniform quadrilateral plates with various boundary conditions. Chander employed the method of false positioning to extract the eigenvalues ω_k which appear in a transcendental manner in the matrix equations that result from the process of error orthogonalization.

References

1. Pian, T. H. H., "Element Stiffness Matrices for Boundary Compatibility and Prescribed Boundary Stress", Proceedings of the First Conference of Matrix Methods in Structural Mechanics (AFPDL-TR-65-80) Wright Patterson Air Force Base, Dayton, Ohio, 1965.

2. Heubner, K. H., "The Finite Element Method for Engineers", John Wiley and Sons, 1975.

3. Prezemieniecki, J. S., Theory of Matrix Structural Analysis, New York, McGraw-Hill.

4. Gallagher, Finite Element Analysis Fundamentals, N. J., Prentice Hall (1974).

5. Dym, C. L. and Shames, J. H., Solid Mechanics, A Variational Approach, New York, McGraw-Hill, (1973).

6. Tocher, J. L., "Finite Element Stiffness Matrices for Analysis of Plate Bending", Proceedings of the First Conference of Matrix Methods in Structural Mechanics (AFFDL-TR-65-80) Wright Patterson Air Force Base, Dayton, Ohio, 1965.

7. DeVeubeke, B. F., Bending and Stretching of Plates - Special Models for Upper and Lower Bound, Proceedings of the First Conference of Matrix Methods in Structural Mechanics (AFFDL-TR-65-80) Wright Patterson Air Force Base, Dayton, Ohio, 1965.

8. Donaldson, B. K., "A New Approach to the Forced Vibration of Thin Plates", J. of Sound and Vibration (1973) 30 (4), 397-417.

9. Chander, S., "The Extended Field Method for the Natural Frequencies of Thin Plates and the Forced Vibration of Thin Plates with Free Corners", Ph.D. Thesis.

10. Negm, H. M., "On the Theory of Extended Fields and Its Application to the Problem of Forced Vibration of Thin Plates", Ph.D. Thesis, Univ. of Maryalnd, College Park, (1974).

11. Donaldson, B. K., "A New Approach to the Forced Vibration of Flat Skin-Stringer-Frame Structures", Journal of Sound and Vibration (1973) 30 (4) 419-435.

12. Donaldson, B. K. and Chander, S., "Numerical Results for Extended Field Method Application", J. of Sound and Vibration (1973) 30 (4), 437-444.

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Abstract

A new quadrilaterial finite element for the problem of bending of a uniform, homogeneous, isotropic, flat plate having an arbitrary number of nodal points and allowing an arbitrary level of refinement is developed from solution functions to the governing differential equation. Interelement continuity to the appropriate level of normal derivatives is provided in least error fashion by the use of spline functions for the edge displacement quantities and minimization by the Galerkin technique. Even for general quadrilateral domains no numerical quadrature is required for developing this finite element since all required integrals are taken over straight boundary edges and can be evaluated explicitly. With this new type of element, an improvement in accuracy is expected to result when large quadrilateral regions are modeled by a single element with an appropriate number of edge nodal points, because the error of this finite element approximation is associated with the boundary region of the elements. The force vector corresponding to the transverse pressure is derived from a series solution to the inhomogeneous differential equation, so that the effect of the forcing function in the element interior can be evaluated to any degree of accuracy desired.