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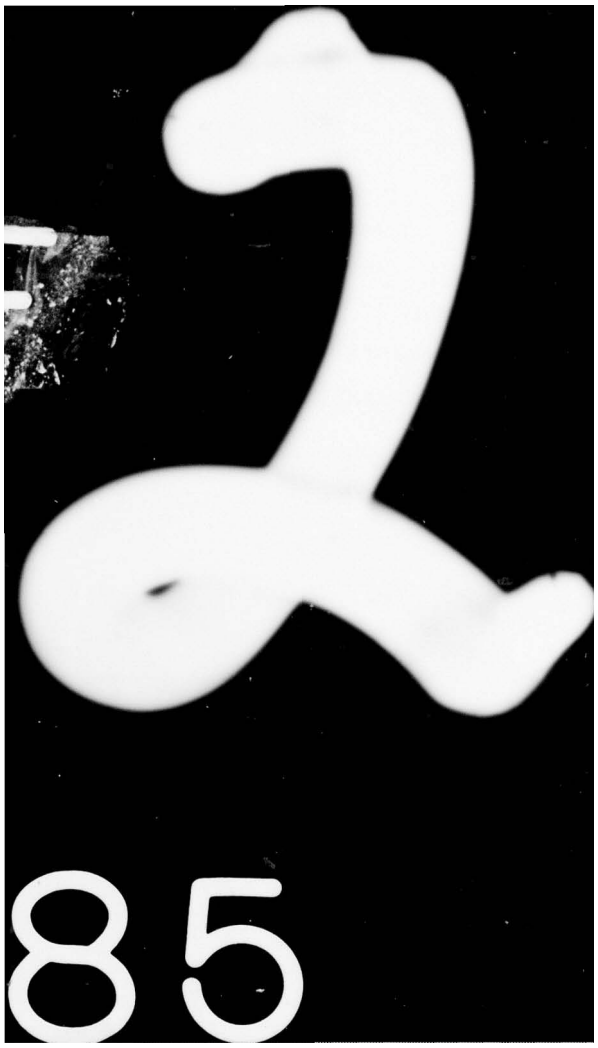
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Report CS-78-1

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FIXED CONFIGURATION REDUCED ORDER FILTERS*

Craig S. Sims
School of Electrical Engineering
Oklahoma State University
Stillwater, Oklahoma 74074

February, 1978

Final Report
TO AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

*This research was supported by the Air Force Office of Scientific Research, under Grant Number AFOSR-77-3248.

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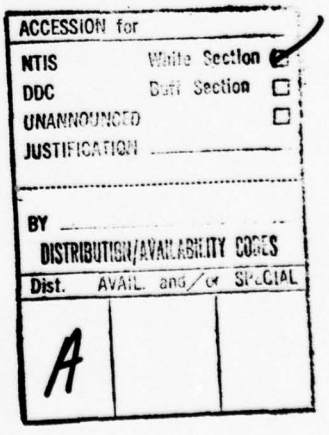
ABSTRACT

Fixed configuration filter theory provides a methodology for designing filters of reduced complexity which will provide suboptimal performance in general, and optimal performance under certain conditions. In this report we derive fixed configuration reduced order filters for continuous and discrete time systems, with and without state dependent noise.



FIXED CONFIGURATION REDUCED ORDER FILTERS

I. INTRODUCTION

Fixed configuration filter theory provides a methodology for designing filters of reduced complexity which will provide near optimal performance. Sometimes the performance is optimal, although the filter structure is simplified. The simplification is achieved by specifying the filter structure a priori, and then optimizing the free parameters of the filter. The advantage of such a filter over an optimal linear Kalman filter is that far fewer on-line calculations are necessary. This is desirable for many applications where the computational facilities are limited and/or the state vector is so large that it is not feasible to perform all the on-line calculations required for Kalman filtering. Such situations are common Air Force problems.

The fixed configuration filter discussed in this report is referred to as a reduced order filter. It has applications in several areas. One apparent area of usefulness occurs in aided navigation systems such as loran-inertial or doppler-inertial-loran systems. Another area of application is pointing and tracking problems. The criterion for applicability of reduced order filtering is to have a relatively high order system with interest in estimating only some of the state variables. This is often the case when many of the state variables occur due to a detailed model of the noise processes in the system. Here, the interest is not to estimate the noise state variables, but only their effect on the variables of concern, such as position and velocity.

Much of the previous work in fixed configuration filtering has suffered from a rather important factor. Although the on-line computational requirements were significantly reduced, the off-line computation of the optimal filter parameters was difficult. Sometimes the situation was handled by solving a difficult nonlinear two-point boundary-value problem (TPBVP) [1]. In other cases, filter parameters were not truly optimized, but only optimized with regard to the following stage of estimation, [2]. In the work presented here we continue to investigate reduced order filtering problems with the same general context set forth in [3]. The solutions obtained are basically of two types:

1. Truly optimal solutions where the TPBVP reduced to a single-point boundary-value problem.
2. Suboptimal or partially optimized solutions where only a linear TPBVP must be solved.

Both of the above mentioned solution categories have the property that the amount of off-line computation required is predictable and reasonable. This fact makes the methodology we are suggesting feasible in that the reduced order filtering problems can be solved with a realistic amount of off-line computation, while the on-line computational savings could be enormous.

The main contributions of this research are presented in the first four papers appearing in the Appendix. The first two papers deal with continuous time problems. The first paper is entitled "Reduced Order Modeling." It has been submitted to the IEEE Transactions on Automatic Control as a short paper. The second paper is entitled "Reduced Order Filtering with State Dependent Noise." It has beeen submitted to the 1978 Joint Automatic Control Conference, and will also be reviewed for the IEEE Transactions on Automatic Control.

The third and fourth papers are discrete time versions of the reduced order filtering problems. These are draft copies, not presently in the final form for submission to appropriate journals. The titles are "Linear Discrete Reduced Order Filtering," and "Discrete Reduced Order Filtering with State Dependent Noise." The discrete problems are distinctly different from the continuous problems, and represent nontrivial extensions. Perhaps this fact is one of the more important findings of the research. The fifth paper in Appendix A is not pertinent to this research, but was performed during the duration of this grant.

The details of the research are to be found in the papers in the Appendix. A discussion of the papers and an overview of the research findings are presented in the following sections of this report. The work presented herein is a continuation of an investigation which began in the summer of 1976 while the principal investigator participated in the USAF/ASEE sponsored summer faculty research program at the Frank J. Seiler Research Laboratory.

II. DISCUSSION

a) Reduced Order Modeling

The paper "Reduced Order Modeling" presents a method for modeling a linear stochastic system of high order, using a reduced order model. The problem is formulated as follows. Given a linear stochastic model

$$\dot{x}(t) = A(t)\, x(t) + w(t) \tag{1}$$

with output

$$y(t) = C(t)\, x(t) \tag{2}$$

where $w(t)$ is zero mean white noise, find a reduced order model

$$\dot{\hat{y}}(t) = F(t)\,\hat{y}(t) + K(t)\,w(t) \tag{3}$$

which approximates (1) and (2). The error criterion is quadratic

$$J = E\{\int_{t_o}^{t_f} e^T(t)\,R(t)\,e(t)\,dt + e^T(t_f)\,S\,e(t_f)\} \tag{4}$$

where e is the equation error

$$e \triangleq y - \hat{y} \tag{5}$$

This problem was motivated by the work of Obinata and Inooka [4], but uses much the same mathematics as that in [3]. It is really a model matching problem, but may be regarded as a peculiar kind of reduced order filtering problem where one only measures the input noise, and there are no noisy observations of the state vector.

The optimal results for this paper are characterized by a singular arc which exists when the matrices $F(t)$ and $K(t)$, and the vector of initial conditions, $\hat{y}(t_o)$ are selected appropriately. The correct choice of the matrices is extremely simple when it can be done, but generally the algebra will not have a solution. In this case a suboptimal approach is suggested where $F(t)$ is selected a priori and only $K(t)$ and $\hat{y}(t_o)$ are optimized. The solution is obtained as a single-point boundary-value problem.

The value of $\hat{y}(t_o)$ in this paper was not constrained a priori so that $\hat{y}(t)$ would be an unbiased estimate of $y(t)$ for $t \geq t_o$. It was obtained via application of the generalized boundary condition [5] from the calculus of variations. When the optimal choice for $F(t)$ could be obtained, this approach led to the same result as an unbiased constraint. We obtained the very important result that when $F(t)$ is not selected optimally, it is better to select $\hat{y}(t_o)$ in such a way that $\hat{y}(t)$ is a biased estimate of $y(t)$.

That is, the quadratic performance measure will be smaller if this is done. At this point it appears that the same remark will hold true in the other filtering problem which we have solved with the unbiased constraint applied prior to optimization. This may be an important property pertinent to the reduced order area, however we have not gone through the mathematics to generalize the result yet.

b) Reduced Order Filtering with State Dependent Noise

This paper represents an extension of the basic theory set forth in [3] to systems with state dependent noise as described by stochastic equations of the form

$$dx(t) = A(t)\,x(t)\,dt + dw(t) + \sum_{i=1}^{n} [x_i(t) - \mu_i(t)]\,G_i(t)\,d\nu(t) \tag{6}$$

where the disturbances are zero mean incremental wiener processes and $\mu(t)$ is the mean value of the state vector. The observation vector is also corrupted by state dependent noise. It is of the form

$$dy(t) = dv(t) + C(t)\,x(t)\,dt + \sum_{i=1}^{n} [x_i(t) - \mu_i(t)]\,M_i(t)\,d\nu \tag{7}$$

A filter is to be designed to estimate a lower order vector

$$z(t) = N(t)\,x(t) \tag{8}$$

The filter is of the form

$$d\hat{z}(t) = [F(t)\,\hat{z}(t) + g(t)]\,dt + K(t)\,dy(t) \tag{9}$$

and the estimate of z is required to be unbiased. We select $g(t)$ and $\hat{z}(t_0)$ to meet the unbiased requirement, and then optimize the choice of $F(t)$ and $K(t)$ with respect to the quadratic performance criterion

$$J = E \{ \int_{t_o}^{t_f} e^T(t)\, \bar{Q}\, e(t)\, dt + e^T(t_f)\, S\, e(t_f) \} \tag{10}$$

The solution proceeds along the same lines as that obtained in [3], with different equations resulting from the state dependent noise. There is an interesting interpretation to the result obtained when

$$\Psi_2(t) \triangleq \sum_{\substack{i=1 \\ j=1}}^{n} P_{xx_{ij}}(t)\, G_i(t)\, \Xi\, M_j^T(t) = o \tag{11}$$

where Ξ is the covariance matrix associated with ν. The solution is the same as that obtained in [3], but with R, the covariance matrix associated with v replaced by $R + \Psi_3$; and Q, the plant noise covariance matrix replaced by $Q + \Psi_1$, where

$$\Psi_1 \triangleq \sum_{i=1}^{n} P_{xx_{ij}}\, G_i\, \Xi\, G_j^T \tag{12}$$

and

$$\Psi_3 \triangleq \sum_{\substack{i=1 \\ j=1}}^{n} P_{xx_{ij}}\, M_i\, \Xi\, M_j^T \tag{13}$$

The analogy could be carried further, without requiring (11), if w and v were allowed to have non zero cross correlation. There are some interesting aspects to what we have said here. As an example, one may have meaningful problems with state dependent noise when R = o. Such problems would not be well posed without state dependent noise. It is remarkable, considering the complexity of problems involving state dependent noise, that the results obtained in [3] could be extended to this class of problems with such minor modifications.

c) Linear Discrete Reduced Order Filtering

In the proposal, it was stated that the research area would be pursued to some extent within a discrete framework. This was motivated by the fact that the discrete format would make it easy to evaluate the amount of on-line calculation required by the reduced order filters. What we found was that the character of the discrete problem was considerably different than that of the continuous time problem, and that the extension of the results was not a trivial exercise. The main reason for the difference is that matrices which occur only linearly in the continuous problem occur quadratically in the discrete problem.

The discrete problem is formulated using the dynamical model

$$x_{j+1} = A_j x_j + w_j; \; j = o, 1, \ldots \tag{14}$$

with observation model

$$y_{j+1} = C_{j+1} x_{j+1} + v_{j+1}; \; j = 1, 2, \ldots \tag{15}$$

where w_j and v_j are zero mean white noise sequences. A lower order linear transformation of the state vector

$$z_j \triangleq N_j x_j \tag{16}$$

is to be estimated by the linear filter

$$\hat{z}_{j+1} = F_j \hat{z}_j + K_j y_{j+1} + g_j \tag{17}$$

The deterministic vector sequence, g_j, and the filter initial condition, $\hat{z}_o$, are to be selected so that $\hat{z}_j$ is an unbiased estimate of z_j. The matrix sequences, F_j and K_j are then chosen to minimize a quadratic performance measure in the error

$$e_j \triangleq z_j - \hat{z}_j \tag{18}$$

The performance measure is

$$J = E \{ \sum_{j=o}^{M-1} e_j^T U_j e_j + e_M^T S e_M \} \tag{19}$$

A general TPBVP is specified, whose solution gives the required sequence of matrices, F_j and K_j. The interesting feature of the problem that differentiates it from the continuous time problem, is that the matrices F_j and K_j both appear quadratically in the Hamiltonian. The singular optimization problem is therefore not present in the discrete case. This has both good and bad aspects. The good feature is that one does not have to go through the excessively tedious mathematics of deriving conditions for a singular arc. The bad part is that it is not obvious how one should proceed to solve the TPBVP.

The way that the TPBVP simplifies is interesting, and closely related to the unbiased requirement. To satisfy the unbiased requirement, it is necessary to select

$$g_j = G_j \mu_j \tag{20}$$

where $\mu_j = E \{x_j\}$ and where

$$G_j \triangleq (N_{j+1} - K_j C_{j+1}) A_j - F_j N_j \tag{21}$$

It turns out that if one can make $G_j = o$ by selecting F_j appropriately, then the TPBVP simplifies and becomes a single-point boundary-value problem. It is not always possible to find an F_j that makes G_j equal to zero however. For this reason the problem is considered where only K_j is optimized over the entire interval, and F_j is selected prior to optimization, perhaps according to a one-stage optimization procedure [2]. The important aspect of the

result under these circumstances is that one only has to solve a linear TPBVP. Such problems can be solved using invariant imbedding techniques with the method leading to a discrete Riccati equation. Some effort was spent on a computer program using this approach during the course of this research. The author feels that the discrete problem and its solution represent a significant aspect of the research performed during the period of this grant.

d) Discrete Reduced Order Filtering with State Dependent Noise

In view of what was done in the case of continuous time systems, a natural extension of the preceding paper is to consider the case of state dependent noise. The dynamical model considered is of the form

$$x(j+1) = A(j)\, x(j) + w(j) + \sum_{i=1}^{n} \bar{x}_i(j)\, \Gamma_i(j)\, \eta(j) \tag{22}$$

where $w(j)$ and $\eta(j)$ are zero mean discrete white noise vectors. The observation model is of the form

$$y(j+1) = C(j+1)\, x(j+1) + v(j+1) + \sum_{i=1}^{n} \bar{x}_i(j)\, M_i(j)\, \eta(j) \tag{23}$$

where $v(j+1)$ is the zero mean discrete white measurement noise that is additive. In (22) and (23), the terms $\bar{x}_i(j)$ are defined as $x_i(j) - E\{x_i(j)\}$. We note that often measurements of the form

$$y(j+1) = C(j+1)\, x(j+1) + v(j+1) + \sum_{i=1}^{n} \bar{x}_i(j+1)\, M_i^*(j+1)\, \varepsilon(j+1) \tag{24}$$

can be put in the required form (23), where ε represents a disturbance influencing the observation in a multiplicative way. Hence we do not view the form of (23) as overly restrictive. The ability to treat nonadditive noise situations is a useful addition to the research findings. It enables one to

treat problems where the measurement is either present or absent in a random way, as is exemplified by the inertial example problem considered in the paper.

Just as in the continuous time problem, the results carried over with only minor modification to the case with state dependent noise, i.e. algorithms were developed for designing filters of the form

$$\hat{z}(j+1) = F(j)\,\hat{z}(j) + K(j)\,y(j+1) + g(j) \qquad (25)$$

where $\hat{z}(j)$ is a reduced order estimate for $z(j) = N(j)\,x(j)$. These algorithms differ in relatively minor ways from those developed without state dependent noise.

All of the papers discussed in this section and presented in the appendix deal with different aspects of reduced order filtering and signal processing. In all cases we have had as a goal, the idea of avoiding unrealistic off-line computation, e.g. high order matrix valued non-linear TPBVP's. It is the author's opinion that the results are practical in the sense that the off-line computation is feasible, and the on-line computational savings may be tremendously important in a system where estimates must be available in a limited time frame, and the dynamical system is of high dimension.

III. OTHER TOPICS

The most important aspects of the research, in the author's opinion, have been set forth in the publications which have been discussed. There are some other topics that we feel are worth mentioning however, and these are discussed in this section.

a) Steady State Results

Consider the steady state or stationary version of the problem considered in [3]. We assume that there is a model such as described by (1) but with

constant A matrix and stationary noise w. The observation model is

$$y(t) = C\,x(t) + v(t) \tag{26}$$

where $v(t)$ is zero mean stationary white noise. The vector to be estimated is

$$z(t) = N\,x(t) \tag{27}$$

and the filter is a reduced order filter, not time variable,

$$\dot{\hat{z}}(t) = F\,\hat{z}(t) + K\,y(t) \tag{28}$$

The optimal matrices F and K are to be found so that the performance measure

$$J = E\{e^T(t)\,U\,e(t)\} \tag{29}$$

is minimized, where $e \triangleq z - \hat{z}$.

The problem is basically a calculus problem, that can be stated as follows. Find the matrices F and K to minimize

$$J = tr\{U P_{ee}\} \tag{30}$$

where

$$P_{ee} \triangleq E\{e(t)\,e^T(t)\} \tag{31}$$

satisfies

$$o = B P_{xe} + F P_{ee} + P_{ee} F^T + P_{ex} B^T + N Q N^T + K R K^T \tag{32}$$

In (32) B is defined as

$$B \triangleq (NA - FN - KC) \tag{33}$$

and Q is the covariance matrix associated with w while R is the covariance matrix associated with v. The matrix P_{xe} is defined as

$$P_{xe} = P_{ex}^T = E\{ x e^T \} \tag{34}$$

and satisfies the requirement

$$o = A P_{xe} + P_{xx} B^T + P_{xe} F^T + Q N^T \tag{35}$$

where P_{xx} is the second moment matrix

$$P_{xx} = E\{ x x^T \} \tag{36}$$

which may be regarded as known, and satisfies the constraint

$$o = A P_{xx} + P_{xx} A^T + Q \tag{37}$$

We note that there are stability requirements which must be met before these equations are applicable.

Augmenting the cost function with the constraints (32), (35), and the transpose of (35), we have

$$\begin{aligned} J^* = tr\{ U P_{ee} + [B P_{xe} + F P_{ee} + P_{ee} F^T + P_{ex} B^T + N Q N^T + K R K^T] \Lambda_{ee}^T \\ + [A P_{xe} + P_{xx} B^T + P_{xe} F^T + Q N^T] \Lambda_{xe}^T + [P_{ex} A^T + B P_{xx} + F P_{ex} \\ + N Q] \Lambda_{ex}^T \} \end{aligned} \tag{38}$$

Taking the gradients of J* with respect to P_{xe} and P_{ee} setting them equal to zero gives the equations for the Lagrange multipliers

$$o = U + F^T \Lambda_{ee} + \Lambda_{ee} F \tag{39}$$

$$o = A^T \Lambda_{xe} + \Lambda_{xe} F + B^T \Lambda_{ee} \tag{40}$$

We note that $\Lambda_{ex} = \Lambda_{xe}^T$. Setting the gradient with respect to K equal to zero gives

$$K = [P_{ex} + \Lambda_{ee}^{-1} \Lambda_{ex} P_{xx}] C^T R^{-1} \tag{41}$$

Since F appears linearly in the Hamiltonian, it is a bit troublesome. Taking the gradient of J* with respect to F and setting it equal to zero gives the requirement that

$$\Theta \triangleq (P_{ee} - N P_{xe}) \Lambda_{ee} + (P_{ex} - N P_{xx}) \Lambda_{xe} = o \tag{42}$$

It can be seen that Θ will be zero if Λ_{xe} is zero and $P_{ee} - N P_{xe} = o$. We will show that if F can be found so that B is zero, i.e.

$$NA - FN - KC = o \tag{43}$$

and the required F is nonsingular, then Θ will equal zero when K is selected optimally. When $B = o$, (40) is solved by $\Lambda_{xe} = o$. Subtracting (35) premultiplied by N from (32) gives

$$(P_{ee} - N P_{xe}) F^T + F P_{ee} - N A P_{xe} + K R K^T = o \tag{44}$$

Substituting from (43) into (44) we obtain

$$o = (P_{ee} - N P_{xe}) F^T + F(P_{ee} - N P_{xe}) + K[R K^T - C P_{xe}] \tag{45}$$

The last term is zero when K is optimal, as can be seen from (41) when $\Lambda_{xe} = o$. Hence when F is nonsingular, (45) implies that

$$P_{ee} = N P_{xe} \tag{46}$$

and consequently, Θ is zero. When a solution to (43) can be found it is of the form

$$F = [NA - KC]N^{\dagger} + \Gamma [I - NN^{\dagger}] \tag{47}$$

where Γ is arbitrary and † indicates the pseudo inverse.

We have presented the equations which must be solved for an optimal F; however, it should be noted that both F and A must have all eigenvalues with negative real parts for the solution to be applicable. The steady state value of μ, the expected value of the state must be zero therefore; and consequently, the required filter is of the form indicated by (28), with no bias removing function, g(t).

It is clear that (43) cannot always be solved, and we, therefore, look at the problem with specific values of F. If F is picked a priori, then we may solve the algebraic equations (37) and (39) for P_{xx} and Λ_{ee}, and hence these are regarded as known quantities. The remainder of the problem only involves linear algebra. We illustrate this by example. Suppose F, U, and Λ_{ee} are matrices of the form

$$\begin{aligned} F &= f\,I \\ U &= u\,I \\ \Lambda_{ee} &= \Lambda_{ee}\,I \end{aligned} \tag{48}$$

where f, u, and Λ_{ee} are scalars. Then (35) and (40), after substituting from (41) can be evaluated as

$$\left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] \left[\begin{array}{c} P_{xe} \\ \hline \Lambda_{xe} \end{array}\right] = - \left[\begin{array}{c} D_1 \\ \hline D_2 \end{array}\right] \tag{49}$$

where

$$
\begin{aligned}
G_{11} &\triangleq A + F - P_{\overline{xx}} C^T R^{-1} C \\
G_{12} &\triangleq -\Lambda_{ee}^{-1} P_{\overline{xx}} C^T R^{-1} C P_{\overline{xx}} \\
G_{21} &\triangleq \Lambda_{ee} C^T R^{-1} C \\
G_{22} &\triangleq -[A^T + F - C^T R^{-1} C P_{\overline{xx}}] \\
D_1 &\triangleq P_{\overline{xx}} [NA - FN]^T + QN^T \\
D_2 &\triangleq -[NA - FN]^T \Lambda_{ee}
\end{aligned}
\tag{50}
$$

At this point, the gain K can be evaluated by substituting the solution to (49) in (41). We note that if the solution is to be valid, then all of the eigenvalues of the matrix

$$
\left[\begin{array}{c|c} A & 0 \\ \hline B & F \end{array}\right] \tag{51}
$$

must have negative real parts. The author would like to point out that the matter of a computational search for a value of F that will lead to good performance when used in conjunction with the above equations, appears to be complicated by the fact that the matrix (51) must be checked for stability at each stage to insure that the equations are meaningful.

In summary, regarding the steady state problem, one must be aware that a solution to the reduced order problem may not be possible, even when a solution for the corresponding finite interval problem is available.

b) Computational Savings

In this section a few remarks are made about on-line computational saving, because this is the factor that motivates the work. A reduced order filter of dimension ℓ of the form indicated by (17), with an observation of dimension m requires

$$\ell\ (\ell + m) \text{ multiplies}$$

and

$$\ell\ (\ell + m - 1) \text{ adds}$$

for each on-line filter update. A Kalman filter of dimension n would require

$$n\ (n + m) \text{ multiplies}$$

$$n\ (n + m - 2) \text{ adds}$$

for each on-line filter update, assuming that all the necessary filter parameter were stored. The savings in calculations are significant when the reduced order filter is used; for example, if a filter of dimension 2 is used to estimate 2 state variables of a 10th order system, and only a scalar observation is available, then only 5% of the calculations of a Kalman filter are needed per update. Of course, this difference would be even more impressive if n were greater than 10. Performance of a reduced order filter may be very good as indicated by the example in [3] (JACC version), and the off line computation, though it may be extensive, definitely is feasible.

IV. AREAS OF FUTURE RESEARCH

A number of interesting subjects have come forth during the course of this research and have not been properly resolved at this stage. We will

mention them briefly in this section. One of the basic issues that we have not confronted is "how does one find an optimal solution when it is not possible to find an F matrix that maintains a singular arc?" We have chosen to look at suboptimal solutions in this case, rather than dealing with this central issue. Another point of interest is with regard to the control of stochastic systems. It is of interest to know how well a controller would perform if the control were required to be the output of a reduced order filter, designed according to the procedures we have presented here, as opposed to those in [6].

The reduced order modeling problem introduced a novel aspect to the work, i.e. can we do better than we have been doing with respect to minimizing a quadratic error criterion, if we drop the requirement that the estimate be unbiased? Based on the reduced order modeling paper in the appendix, it appears that the answer is yes when F is selected a priori.

We have seen that the steady state solution to the problems we have considered does not exist when A is an unstable matrix, since this would imply that P_{xx} was unbounded, and (37) would not apply of course. Another area that we must look into is the conditions for existence of the solutions of the linear two-point boundary value problems that we have derived in this research.

V. SUMMARY AND CONCLUSIONS

Significant progress has been made during the course of this research in several areas which we had sought to investigate at the onset of this project. We have looked at both continuous and discrete problems, with both additive and state dependent noise in each case. It has become clear that the discrete problems are quite different than the continuous time problems, primarily because of the quadratic occurrence of the F matrix in the discrete time

problems, while the corresponding matrix occurs only linearly in the continuous time problem, giving rise to a singular solution. It would be interesting to take the limit of the discrete solution as the sampling interval goes to zero to obtain the continuous time solution [7], to clarify the relationship between the two kinds of problems.

The author feels that the "specified F" solution obtained in this research are of considerable practical importance in the design of reduced order filters. The reason that they are important is that they maintain the TPBVP nature of the reduced order filtering problem, so that optimization looks beyond one stage ahead, however the TPBVP is linear. It is linear even in the case of having state dependent noise! The solution, when it exists, can therefore be obtained off line in a predictable number of steps. This is unlike the case of nonlinear TPBVP's which are generally solved by quasi-linearization or some such method, and which may or may not converge, and may be very sensitive to some initial guess at a solution. Since the TPBVP's we are dealing with are generally of high order, it is fortunate indeed that they are linear. A logical procedure for solving reduced order filtering problems which we are currently working on is to obtain an F matrix that is optimal with respect to the next stage, as in [2], then modify the results by solving the linear TPBVP to get the corresponding optimal gain.

As we have acknowledged in the previous section, there are still a number of unanswered questions in this area of reduced order filtering. The author believes, however, that the methods which have been investigated here are very promising.

REFERENCES

[1] Sims, C. S., and J. L. Melsa, "Specific Optimal Estimation," IEEE Transactions on Automatic Control, Vol. AC-14, pp. 183-186, April 1969.

[2] Sims, C. S., "An Algorithm for Estimating a Portion of a State Vector," IEEE Transactions on Automatic Control, Vol. AC-19, No. 4, August 1974.

[3] Sims, C. S., and R. B. Asher, "Optimal and Suboptimal Results in Full and Reduced Order Linear Filtering," Proceedings of 1977 JACC, to appear in IEEE Transactions on Automatic Control, June 1978.

[4] Obinata, G., and H. Inooka, "A Method of Reducing the Order of Multivariable Stochastic Systems," IEEE Transactions on Automatic Control, Vol. AC-22, pp. 676-677, August 1977.

[5] Schultz, D. G., and J. L. Melsa, State Functions and Linear Control Systems, McGraw-Hill, New York, 1967.

[6] Sims, C. S., and J. L. Melsa, "A Fixed Configuration Approach to the Stochastic Linear Regulator Problem," Proceedings 1970 Joint Automatic Control Conference, June 24-26, 1970, Atlanta, Georgia.

[7] Meditch, J. S., Stochastic Optimal Linear Estimation and Control, McGraw-Hill, New York, 1969.

APPENDIX
(Papers written under this grant)

Submitted to IEEE Trans. Aut. Control

REDUCED ORDER MODELING*

Craig S. Sims
School of Electrical Engineering
Oklahoma State University
Stillwater, Oklahoma 74074

ABSTRACT

This paper presents a method for modeling a linear stochastic system using a reduced order model. The parameters are selected by minimizing an integral quadratic penalty function of the equation error.

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1. INTRODUCTION

It is often convenient to model a high order system using a low order dynamical model. Several researchers have considered this type of problem [1] - [8]. In a recent paper, Obinata and Inooka [8] have treated the problem of obtaining a reduced order time invariant model which minimized a steady state quadratic error criterion. In this paper a similar viewpoint is taken, however we solve the nonstationary problem with a white noise input, and an integral quadratic performance measure. Optimal control theory is used to obtain a solution, which is conveniently found using the matrix minimum principle [9]. The problem is similar to the filtering problems posed by Lee [10], and Sims and Asher [11], in that a singular solution is obtained. When a singular solution is not possible or is considered too complex, a suboptimal solution is suggested.

2. PROBLEM STATEMENT

We consider the linear stochastic model

$$\dot{x}(t) = A(t)x(t) + w(t) \tag{1}$$

where $w(t)$ is zero mean white noise with covariance matrix

$$E\{w(t)\, w^T(\tau)\} = Q(t)\,\delta(t-\tau) \tag{2}$$

The state vector $x(t)$ is of dimension n, which is presumably too large to be desirable. The output of the system, $y(t)$, is a linear transformation of the

state vector,

$$y(t) = C(t)\, x(t) \tag{3}$$

and is of dimension m which is much smaller than n. The objective is to find a reduced order model

$$\dot{\hat{y}}(t) = F(t)\, \hat{y}(t) + K(t)\, w(t) \tag{4}$$

where $\hat{y}(t)$ is an m vector which adequately approximates $y(t)$. We assume that the initial statistics of the state vector are known

$$\begin{aligned} E\{x(t_o)\} &= \mu_o \\ \text{Var}\{x(t_o)\} &= P_o \end{aligned} \tag{5}$$

The problem is to select the matrices $F(t)$ and $K(t)$, and the initial condition $\hat{y}(t_o)$ in a way that minimizes the performance measure

$$J + E\{\int_{t_o}^{t_f} e^T(t)\, R(t)\, e(t) + e^T(t_f)\, S_e(t_f)\} \tag{6}$$

where $e(t)$, the equation error is defined as

$$e(t) \triangleq y(t) - \hat{y}(t) \tag{7}$$

Note that we do not require apriori that $\hat{y}$ be an unbiased estimate of y, although this turns out to be a property of the optimal solution. If one does not select F optimally, however, it can be advantageous to have a nonzero expectation of the error.

3. OPTIMIZATION

The first step in the optimization procedure is to transform the problem into an equivalent deterministic problem. From equations (1), (3), and (4), the error dynamics are governed by the stochastic equation

$$\dot{e} = B x + F e + (C - K) w \tag{8}$$

where B is defined as

$$B \triangleq \dot{C} + CA - FC \tag{9}$$

The second moment matrices associated with (1) and (8) are defined as

$$\begin{aligned} P_{xx}(t) &\triangleq E\{x(t)\, x^T(t)\} \\ P_{xe}(t) &\triangleq E\{x(t)\, e^T(t)\} = P_{ex}^T(t) \\ P_{ee}(t) &\triangleq E\{e(t)\, e^T(t)\} \end{aligned} \tag{10}$$

From (1) and (8), we can see that these matrices satisfy the deterministic equations

$$\begin{aligned} \dot{P}_{xx} &= A P_{xx} + P_{xx} A^T + Q \\ \dot{P}_{xe} &= A P_{xe} + P_{xx} B^T + P_{xe} F^T + Q (C - K)^T \\ \dot{P}_{ee} &= B P_{xe} + P_{ex} B^T + F P_{ee} + P_{ee} F^T + (C - K)\, Q\, (C - K)^T \end{aligned} \tag{11}$$

We make the assumption that $y(t_0)$ is to be deterministic. Then the initial conditions for (11) are

$$P_{xx}(t_o) = P_o + \mu_o \mu_o^T$$

$$P_{xe}(t_o) = (P_o + \mu_o \mu_o^T) C^T(t_o) - \mu_o \hat{y}^T(t_o)$$

$$P_{ee}(t_o) = C(t_o)(P_o + \mu_o \mu_o^T)C^T(t_o) - \hat{y}(t_o)\mu_o^T C^T(t_o) - C(t_o)\mu_o \hat{y}^T(t_o) + \hat{y}(t_o)\hat{y}^T(t_o) \quad (12)$$

The performance measure (6) may be written in terms of the matrix P_{ee} as

$$J = tr\left\{\int_{t_o}^{t_f} R(t) P_{ee}(t)\, dt + S P_{ee}(t_f)\right\} \quad (13)$$

The problem can now be stated completely within a deterministic framework. The matrices, $F(t)$ and $K(t)$, and the vector $\hat{y}(t_o)$ are to be selected to minimize (13) subject to the constraints imposed by (11) and (12). The problem is ideally suited to solution using the matrix minimum principle [9].

The Hamiltonian is formed as

$$H = tr\{R P_{ee} + \dot{P}_{ee} \Lambda_{ee}^T + \dot{P}_{xe} \Lambda_{xe}^T + \dot{P}_{ex} \Lambda_{ex}^T\} \quad (14)$$

where we have ignored the equation for P_{xx} since it is a known quantity. The costate equations are obtained from (14).

$$\dot{\Lambda}_{xe} = \frac{-\partial H}{\partial P_{xe}} = -A^T \Lambda_{xe} - \Lambda_{xe} F - B^T \Lambda_{ee} \quad (15)$$

and

$$\dot{\Lambda}_{ee} = \frac{-\partial H}{\partial P_{ee}} = -R - F^T \Lambda_{ee} - \Lambda_{ee} F \quad (16)$$

The terminal conditions for (15) and (16) are

$$\Lambda_{xe}(t_f) = o$$

$$\Lambda_{ee}(t_f) = S \quad (17)$$

It should be observed that Λ_{ex} is simply the transpose of Λ_{xe}.

The transversality condition applied at the initial time provides the correct initial condition for $\hat{y}$. The transversality condition is

$$\mathrm{tr}\,\{d P_{ee}(t_o)\,\Lambda_{ee}^{T}(t_o) + d P_{xe}(t_o)\,\Lambda_{xe}^{T}(t_o) + d P_{ex}(t_o)\,\Lambda_{ex}^{T}(t_o)\} = o \tag{18}$$

Since P_o and μ_o are fixed and the only allowable variation is in $\hat{y}(t_o)$. we have from (12),

$$\mathrm{tr}\,\{[\Lambda_{ee}(t_o)\,(\hat{v}(t_o) - C(t_o)\,\mu_o) - \Lambda_{ex}(t_o)\,\mu_o]\,d\hat{y}^{T}(t_o)$$

$$+ d\hat{y}(t_o)\,[\Lambda_{ee}(t_o)\,(\hat{y}(t_o) - C(t_o)\,\mu_o) - \Lambda_{ex}(t_o)\,\mu_o]^{T}\} = o \tag{19}$$

The above will be satisfied for arbitrary variations in $\hat{y}(t_o)$ only if the equation

$$\Lambda_{ee}(t_o)\,\hat{y}(t_o) = \Lambda_{ee}(t_o)\,C(t_o)\,\mu_o + \Lambda_{ex}(t_o)\,\mu_o \tag{20}$$

is satisfied. If $\Lambda_{ee}(t_o)$ is nonsingular then $\hat{y}(t_o)$ is given by

$$\hat{y}(t_o) = [C(t_o) + \Lambda_{ee}^{-1}(t_o)\,\Lambda_{ex}(t_o)]\,\mu_o \tag{21}$$

Next we consider optimization with respect to the gain matrix, $K(t)$. Taking the gradient of the Hamiltonian with respect to K and setting it equal to zero gives the expression

$$\Lambda_{ee}(t)\,K(t)\,Q(t) = \Lambda_{ex}(t)\,Q(t) + \Lambda_{ee}(t)\,C(t)\,Q(t) \tag{22}$$

Equation (22) will have a solution $K(t)$ if the equation,

$$\Lambda_{ee}\,\Lambda_{ee}^{\dagger}\,[\Lambda_{ex} + \Lambda_{ee}\,C]\,Q = [\Lambda_{ex} + \Lambda_{ee}\,C]\,Q \tag{23}$$

is satisfied at time t. If a solution exists, it is of the form

$$K = \Lambda_{ee}^{\dagger} \left[\Lambda_{ex} + \Lambda_{ee} C\right] Q Q^{\dagger} + \Gamma - \Lambda_{ee}^{\dagger} \Lambda_{ee} \Gamma Q Q^{\dagger} \tag{24}$$

where Γ is an arbitrary matrix. In the above expressions † is used to indicate the pseudo inverse of a matrix [12]. When Λ_{ee} is nonsingular, (22) always has a solution given by

$$K = \left[C + \Lambda_{ee}^{-1} \Lambda_{ex}\right] Q Q^{\dagger} + \Gamma \left[I - Q Q^{\dagger}\right] \tag{25}$$

In reality, Q will seldom be nonsingular, however if it is, (25) simplifies to

$$K(t) = C(t) + \Lambda_{ee}^{-1}(t) \Lambda_{ex}(t) \tag{26}$$

and a unique expression is obtained.

Obtaining the matrix $F(t)$ is considerably more involved because it appears linearly in the Hamiltonian. It therefore leads to a singular type of optimization problem. The part of the Hamiltonian which depends on F can be written as

$$H^{*} = F \Theta + \Theta^{T} F^{T} \tag{27}$$

where Θ is defined as

$$\Theta \triangleq (P_{ex} - C P_{xx}) \Lambda_{xe} + (P_{ee} - C P_{xe}) \Lambda_{ee} \tag{28}$$

From (20), it is clear that $\Theta(t_o)$ is zero. If we can show that $\dot{\Theta}(t)$ is zero during the interval $[t_o, t_f]$, than a singular arc exists. Taking the time derivative of Θ gives

$$\dot{\Theta} = F \Theta - \Theta F - K Q \left[\Lambda_{xe} + (C - K)^{T} \Lambda_{ee}\right] + (C P_{xe} - P_{ee}) R \tag{29}$$

The third term in Equation (29) is always zero when K is selected optimally, as can be seen from Equation (22). Therefore, if R is zero, a singular arc

exists independent of the choice of F. Generally, however, R will be positive definite, and to insure a singular arc we must have the term $\Omega \triangleq C P_{xe} - P_{ee}$ be zero in addition to the requirement of an optimal selection of K. We can have $\Omega(t) = o$ in the interval of interest if $\Omega(t_o)$ is zero and $\dot{\Omega}(t)$ is zero in the interval $[t_o, t_f]$. The time derivative of Ω can be shown to be

$$\dot{\Omega} = \Omega F^T + F \Omega + (P_{ex} - C P_{xx}) B^T - K Q (C - K)^T \tag{30}$$

If B is zero, then from (15) and (17) it can be seen that Λ_{xe} is zero. An optimal choice of K thus insures that the last term in (30) is zero, and (30) is really a homogeneous equation in Ω when B is zero. From (20), and the fact that $\Lambda_{xe}(t_o)$ is zero, we have that $\Omega(t_o)$ is zero and consequently a singular arc will exist if K is selected optimally and B is required to be zero. This last requirement then is that

$$B = \dot{C} + CA - FC = o \tag{31}$$

A solution to (31) exists if and only if

$$(\dot{C} + CA)\, C^{\dagger} C = \dot{C} + CA \tag{32}$$

If a solution exists, it is of the form

$$F = (\dot{C} + CA)\, C^{\dagger} + \Gamma^* (I - C C^{\dagger}) \tag{33}$$

where Γ^* is an arbitrary matrix.

We have thus obtained a solution to the problem when it exists. The matrix F is to be selected according to (33), and assuming Λ_{ee} is nonsingular, from (25) we have

$$K = (C - \Gamma)\, Q Q^{\dagger} + \Gamma \tag{34}$$

where Γ is arbitrary. From (21), the appropriate initial condition is

$$\hat{y}(t_o) = C(t_o)\,\mu_o \tag{35}$$

From (3), (5), and (35), it is clear that

$$E\{e(t_o)\} = o \tag{36}$$

when the solution is optimal. Furthermore, from (8) it is clear that $B = o$ implies that $E\{e(t)\} = o$ for $t \geq t_o$. Note that although we did not require a priori that $\hat{y}$ be an unbiased estimate of y, that is the way the solution turned out.

Further remarks regarding the optimal solution are appropriate. Note that the weighting matrices R, and S, and the terminal time, t_f, do not influence the solution (except that R and S should be positive definite symmetric matrices.) More importantly from (33) and (34), we do not have to solve for any matrices P_{xe}, Λ_{xe}, etc. The two-point boundary-value problem (TPBVP) has in effect disappeared. Not even the initial variance P_o influences the solution, which has turned out to be extremely simple.

It should have occurred to the reader that (31) cannot always be satisfied. If one therefore selects F apriori, without regard to optimization, Equations (15), (16), (17), (21), and (24) can be used to specify the optimum choice of $\hat{y}(t_o)$ and K for any given F. The solution, though suboptimal in an overall sense, may be adequate. Note that in this case the weighting matrices R and S do influence the solution. It should also be observed that there is no TPBVP to be solved in this case either, although a single-point boundary-value problem must be solved off line to evaluate the optimal gain.

4. EXAMPLES

In the first example, we illustrate a case where a solution exists, and is easily obtained. The system is as described by (1) and (3) with

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} ; \; C(t) = [\cos t \quad \sin t] \tag{37}$$

and Q nonsingular. The reduced order solution obtained from (33), (34), and (35) is characterized simply and uniquely by

$$\begin{aligned} F(t) &= 0 \\ K(t) &= [\cos t \quad \sin t] \end{aligned} \tag{38}$$

and

$$\hat{y}(t_0) = [\cos t_0 \quad \sin t_0]\, \mu_0 \tag{39}$$

If Q is singular however, the result is not unique. For example if

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} \tag{40}$$

then instead of the expression (38) for K, we have

$$K(t) = [\gamma(t) \sin t] \tag{41}$$

where $\gamma(t)$ is arbitrary. This makes sense of course because (40) implies that whatever multiplies $w_1(t)$ in the solution makes no difference.

We shall now consider a problem where (31) cannot be satisfied. A first order model of the form indicated by (4) is to approximate a second order model of the form indicated by (1) and (3) with

$$A = \begin{bmatrix} \alpha_1 & o \\ o & \alpha_2 \end{bmatrix} \; ; \; C(t) = [t \quad t] \tag{42}$$

with $Q > o$ and $\alpha_1 \neq \alpha_2$. Because (31) cannot be solved, and for reasons of simplicity, F is selected to be zero. The performance measure to be minimized is

$$J = E \{ \int_o^T e^2(t)\, dt + e^2(T) \} \tag{43}$$

The optimum values for the gains are found to be

$$K_i(t) = t + (1 + T - t)^{-1} e^{-\alpha_i t} \int_t^T e^{\alpha_i \tau} u_{\alpha_i}(\tau)\, d\tau \; ; \; i = 1, 2 \tag{44}$$

where

$$u_{\alpha_i}(\tau) \triangleq (1+T) + (\alpha_i T + \alpha_i - 1)\tau - \alpha_i \tau^2 \; ; \; i = 1, 2 \tag{45}$$

We emphasize that F = o is an ad hoc decision for this second example, while a unique solution for the first case. For the second example the initial condition is, according to (21),

$$\hat{y}(o) = (1+T)^{-1} \left\{ \sum_{i=1}^{2} \mu_{oi} \int_o^T e^{\alpha_i \tau} u_{\alpha_i}(\tau)\, d\tau \right\} \tag{46}$$

SUMMARY

In this paper we have developed new methods of reduced order modeling for time variable systems with white noise inputs. The criterion used for purposes of optimization has been an integral quadratic function involving the equation error. When a singular solution to the reduced order modeling problem can be obtained, it has been demonstrated to be extremely simple. It was shown that

the optimal solution must provide an unbiased estimate. When the singular optimal solution cannot be obtained, a suboptimal easily implemented procedure has been developed for obtaining the reduced order gain matrix. Both the optimal and suboptimal solutions have been illustrated by example.

REFERENCES

[1] E. J. Davison, "A method for simplifying linear dynamic systems," IEEE Trans. Automat. Contr., vol. AC-11, pp. 93-101, Jan. 1966.

[2] M. Aoki, "Control of large scale dynamic systems by aggregation," IEEE Trans. Automat. Contr., vol. AC-13, pp. 246-253, June 1968.

[3] M. R. Chidambara, "Two simple techniques for the simplification of large dynamic systems," in Proc. 1969 Joint Automatic Control Conf., pp. 669-674.

[4] L. Meier and D. G. Luenberger, "Approximation of linear constant systems," IEEE Trans. Automat. Contr., vol AC-12, pp. 585-588, Oct. 1967.

[5] D. A. Wilson, "Optimum solution of model-reduction problem," Proc. Inst. Elec. Eng., vol. 117, pp. 1161-1165, June 1970.

[6] F. G. Galiana, "On the approximation of multiple input-multiple output constant linear systems," Int. J. Control, vol. 17, pp. 1313-1324, June 1973.

[7] G. Obinata and H. Inooka, "A method for modeling linear time-invariant systems by linear systems of low order," IEEE Trans. Automat. Contr., vol. AC-21, pp. 602-603, Aug. 1976.

[8] G. Obinata and H. Inooka, "A Method of Reducing the Order of Multivariable Stochastic Systems," IEEE Trans. Automat. Contr., vol. AC-22, pp. 676-677, Aug. 1977.

[9] M. Athans, "The Matrix Minimum Principle," Information and Control, vol. 11, pp. 592-606, 1968.

[10] J. S. Lee, "Optimal Linear Filtering and It's Relation to Singular Control," Proceedings of 1970 JACCC, June 22-26, 1970, Atlanta, Georgia.

[11] C. S. Sims and R. B. Asher, "Optimal and Suboptimal Linear Filtering," Proceedings of 1977 JACCC, June 22-24, 1977, San Francisco, California.

[12] T. L. Boullion and P. L. O'Dell, Generalized Inverse Matrices, Wiley-Interscience, New York, 1971.

REDUCED ORDER FILTERING
with
STATE DEPENDENT NOISE †

Robert B. Asher
Department of Electrical Engineering
Texas Tech University
Lubbock, Texas 79423

Craig S. Sims
School of Electrical Engineering
Oklahoma State University
Stillwater, Oklahoma 74074

For 1978 JACC

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I. INTRODUCTION

There has been considerable work dealing with the topic of filtering for problems with state dependent noise [1-3]. As well as being of theoretical interest, the topic is of some practical importance since many systems are better modeled as having multiplicative disturbances instead of additive. One example occurs in the momentum exchange method for regulating the angular procession of a rotating space craft [4]. There is a disturbance which depends on the procession rates. Another example occurs in the design of phase lock loops [2]. The phase instability of an oscillator described in rectangular coordinates appears as white, state dependent noise. If one received a signal which consisted of a large number of sinusoids of various frequencies, each having phase distortion, then one would have to build a high order filter to recover the signal using existing methods.

The design of high order filters is often problematic from the viewpoint of on-line computation. Therefore, a number of researchers have been interested in designing filters of reduced order [5-8]. It often happens that one is only interested in estimating a lower order linear transformation of a state vector, and it seems reasonable to attempt to do this with a lower order filter. Design of the filter parameters is a fixed configuration optimization problem [8-10]. In such problems, the structure is not necessarily optimal, but given the structural constraints, the parameters are selected optimally. It is interesting to note that these problems often have non-unique solutions because there are too many free parameters. This feature can be used to obtain filters which are easier to implement than well-known techniques such as Kalman filtering,

even when the fixed configuration filter is of full order [8]. In some cases, there is no performance loss associated with the alternative linear filter, [8], [11].

In this paper we seek to extend the reduced order filtering results developed in [8] to problems with state dependent noise. The problem is similar to that considered in [12], however, in [12] a discrete system model was considered, and only a single stage/optimization was performed. Here a continuous time problem is considered, and the matrix minimum principle [13] is used to obtain a solution. Because we allow a driving term in the filter to remove any a-priori bias, it turns out that the problem has singular arcs, which is not surprising considering previous works [8], [11] in the area. A very nice feature of the work is that in some cases only linear two-point boundary value problems are obtained. These can be solved either by a direct use of linear systems theory or by a Riccati equation technique. Under certain circumstances only a single-point boundary-value problem must be solved.

II. PROBLEM STATEMENT

The system of interest is assumed to be modeled by the Ito stochastic differential equation

$$dx(t) = A(t)x(t)dt + dw(t) + \sum_{i=1}^{n} \left[x_i(t)-\mu_i(t)\right] G_i(t)d\nu(t) \qquad (1)$$

where $x(t)$ is the state vector of deminsion n and $\mu(t)$ is the mean value of the state vector. The disturbances are zero mean incremental Wiener processes with covariances

$$\begin{aligned} E\{dw(t)dw^T(t)\} &= Q(t)dt \\ E\{d\nu(t)d\nu^T(t)\} &= \Xi(t)dt \end{aligned} \qquad (2)$$

It is not hard to show [14] that the mean value vector, μ satisfies

$$d\mu(t) = A(t)\mu(t)dt \tag{3}$$

The initial condition for (1) is random with known mean and variance

$$E\{x(t_o)\} = \mu_o \tag{4}$$

$$Var\{x(t_o)\} = P_o \tag{5}$$

Equation (4) is obviously the initial condition for (3).

The observation vector is also corrupted by state dependent noise.

$$dy(t) = C(t)\,x(t)\,dt + dv(t) + \sum_{i=1}^{n}\left[x_i(t) - \mu_i(t)\right] M_i(t)d\nu \tag{6}$$

In (6), $y(t)$ is the observation vector of dimension m, $dv(t)$ is the additive measurement disturbance, and $d\nu(t)$ is the multiplicative disdurbance. The vector $\nu(t)$ may be large, and some of its elements affect the dynamic model through the terms G_i, while others affect the observational model through the terms M_i. The additive disturbance, $dv(t)$ is a zero mean incremental Wiener process with covariance

$$E\{dv(t)dv^T(t)\} = R(t)dt \tag{7}$$

The terms $w(t)$, $v(t)$, $\nu(t)$ and $x(t_o)$ are uncorrelated.

Only a linear transformation of $x(t)$ is to be estimated, i.e., it is desired to estimate

$$z(t) = N(t)x(t) \tag{8}$$

where $z(t)$ is a vector of dimension $\ell \leq n$.

The estimate of $z(t)$, which we call $\hat{z}(t)$ is constrained to be obtained by the filter equation

$$d\hat{z}(t) = \left[F(t)\hat{z}(t) + g(t)\right] dt + K(t)dy \tag{9}$$

The vector $g(t)$ and the initial condition, $\hat{z}(t_o)$ are to be selected so that

$$E\{e(t)\} = 0 \quad \forall t = \left[t_o, t_f\right] \tag{10}$$

where $e(t)$ is the error vector

$$e(t) = z(t) - \hat{z}(t) \tag{11}$$

The matrices $F(t)$ and $K(t)$ are then to be selected so that a quadratic performance measure

$$J = E\left\{ \int_{t_o}^{t_f} e^T(t)\, \bar{Q} e(t) dt + e^T(t_f) S e(t_f) \right\} \tag{12}$$

is minimized. The weighting matrix S is assumed to be positive definite symmetric. The weighting matrix $\bar{Q}$ may be positive definite or zero and is critically important to the solution.

III. GENERAL SOLUTION

In order to proceed, it is convenient to develop an equation for the error. From the Ito differential rule [15], it is seen that

$$dz(t) = N(t)dx(t) + \dot{N}(t)x(t)dt \tag{13}$$

Using (6), (9), and (13) it is seen that the differential equation of the error is

$$de(t) = dz(t) - d\hat{z}(t)$$

or

$$de = \left[(NA - FN - KC + \dot{N})\, x - g \right] dt + N\, dw - K\, dv + F e dt + \left[N \sum_{i=1}^{n} \bar{x}_i G_i - K \sum_{i=1}^{n} \bar{x}_i M_i \right] d\nu \tag{14}$$

In (14) we have introduced the notation, $x = \bar{x} - \mu$. From (14) it is seen that

$$\frac{d}{dt} E\{e(t)\} = F(t) E\{e(t)\} \tag{15}$$

provided that

$$g(t) = (NA - FN - KC + \dot{N})\mu(t) \tag{16}$$

If furthermore

$$\hat{z}(t_o) = N(t_o)\mu(t_o) \tag{17}$$

it is clear that

$$E\{e(t_o)\} = 0 \tag{18}$$

From (15) and (18), one can see that (10) is satisfied so that (16) and (17) are appropriate selections. If g(t) is selected according to (16), the error differential equation can be written as

$$\begin{aligned} de = {} & (NA-FN-KC+\dot{N})\bar{x}dt + Ndw-Kdv \\ & + Fedt + \sum_{i=1}^{n} \bar{x}_i\,(NG_i-KM_i)d\nu \end{aligned} \tag{19}$$

The equation for $\bar{x}$ is

$$d\bar{x} = A\bar{x}dt + dw + \sum_{i=1}^{n} \bar{x}_i G_i d\nu \tag{20}$$

Clearly $\bar{x}$ and e are both zero mean processes.

If (19) and (20) are put in 1 equation, it is easy to see how the second moment matrix defined as

$$P(t) \triangleq \left[\begin{array}{c|c} P_{xx}(t) & P_{xe}(t) \\ \hline P_{ex}(t) & P_{ee}(t) \end{array}\right] \triangleq \left[\begin{array}{c|c} E\{\bar{x}(t)\bar{x}^T(t)\} & E\{\bar{x}(t)e^T(t)\} \\ \hline E\{e(t)\bar{x}^T(t)\} & E\{e(t)e^T(t)\} \end{array}\right] \tag{21}$$

propagates. This is useful since the performance measure (12) may be written as

$$J = tr\{\int_{t_o}^{t_f} \bar{Q}P_{ee}(t)dt + SP_{ee}(t_f)\} \tag{22}$$

If one has the appropriate constraint equation, the optimal selection of F(t) and K(t) may thus be solved with deterministic theory using the matrix minimum principle.

Equations (19) and (20) may be written as

$$\left[\begin{array}{c} d\bar{x}(t) \\ \hline de(t) \end{array}\right] = \left[\begin{array}{c|c} A & 0 \\ \hline (NA-FN-KC+\dot{N}) & F \end{array}\right] \left[\begin{array}{c} \bar{x} \\ \hline e \end{array}\right] dt + \left[\begin{array}{c} dw \\ \hline Ndw - Kdv \end{array}\right]$$

$$+ \sum_{i=1}^{n} \bar{x}_i \Gamma_i \, d\nu \tag{23}$$

where

$$\Gamma_i \triangleq \left[\begin{array}{c} G_i \\ \hline NG_i - KM_i \end{array}\right] \tag{24}$$

The second moment matrix associated with (23) satisfies [4],

$$\dot{P} = GP + PG^T + \hat{Q} + \hat{\psi} \tag{25}$$

where

$$G \triangleq \left[\begin{array}{c|c} A & 0 \\ \hline (NA-FN-KC+\dot{N}) & F \end{array}\right] \tag{26}$$

$$\hat{Q} \triangleq \left[\begin{array}{c|c} Q & QN^T \\ \hline NQ & KRK^T + NQN^T \end{array}\right] \tag{27}$$

and

$$\hat{\psi} = \sum_{\substack{i=1 \\ j=1}}^{n} P_{xx_{ij}} \Gamma_i \Xi \Gamma_j^T \tag{28}$$

Partitioning P in (25) we obtain the individual equations,

$$\dot{P}_{xx} = AP_{xx} + P_{xx}A^T + Q + \Psi_1 \tag{29}$$

$$\begin{aligned}\dot{P}_{ee} = &\left[NA-FN-KC+\dot{N}\right] P_{xe} + P_{ex}\left[NA-FN-KC+\dot{N}\right]^T \\ &+ FP_{ee} + P_{ee}F^T + NQN^T + KRK^T + K\Psi_3K^T \\ &- N\Psi_2K^T - K\Psi_2{}^TN^T + N\Psi_1N^T\end{aligned} \tag{30}$$

and

$$\begin{aligned}\dot{P}_{xe} = &AP_{xe} + P_{xx}\left(NA-FN-KC+\dot{N}\right)^T + P_{xe}F^T \\ &+ QN^T + \Psi_1N^T - \Psi_2K^T\end{aligned} \tag{31}$$

In (29), (30), and (31), the terms Ψ_1, Ψ_2, and Ψ_3 are defined as

$$\Psi_1 = \sum_{\substack{i=1\\j=1}}^{n} P_{xx_{ij}}\, G_i \Xi G_j^T \tag{32}$$

$$\Psi_2 = \sum_{\substack{i=1\\j=1}}^{n} P_{xx_{ij}}\, G_i \Xi M_j^T \tag{33}$$

$$\Psi_3 = \sum_{\substack{i=1\\j=1}}^{n} P_{xx_{ij}}\, M_i \Xi M_j^T \tag{34}$$

The term P_{ex} is simply the transpose of P_{xe}. Clearly P_{xx} can be calculated independently, and can thus be regarded as a known quantity. The problem is to select K and F so that (22) is minimized subject to the constraints imposed by (30) and (31).

The Hamiltonian for this problem is then

$$H = tr\left\{\bar{Q}P_{ee} + \dot{P}_{ee}\Lambda^T{}_{ee} + \dot{P}_{xe}\Lambda^T{}_{xe} + \dot{P}_{ex}\Lambda^T{}_{ex}\right\} \tag{35}$$

where Λ_{ee}, Λ_{xe}, and Λ_{ex} are Lagrange multiplier matrices associated with P_{ee}, P_{xe}, and P_{ex} respectively. The constraint equation for P_{ex} is incuded for symmetry.

The optimal solution for the gain K(t) is obtained by setting the gradient of H with respect to K equal to zero. This leads to the expression for K.

$$K = \Lambda_{ee}^{-1}\left[\Lambda_{ee}\ (P_{ex}C^T + N\Psi_2) + \Lambda_{ex}\ (P_{xx}C^T + \Psi_2)\right]\left[R + \Psi_3\right]^{-1} \tag{36}$$

where the required inverses are assumed to exist. The Lagrange multiplier matrices satisfy the equations

$$\dot{\Lambda}_{ee} = -\frac{\partial H}{\partial P_{ee}} = -\left\{\bar{Q} + \Lambda_{ee}F + F^T\Lambda_{ee}\right\} \tag{37}$$

and

$$\dot{\Lambda}_{xe} = -\frac{\partial H}{\partial P_{xe}} = -\left\{(NA - FN - KC + \dot{N})^T\Lambda_{ee} + A^T\Lambda_{xe} + \Lambda_{xe}F\right\} \tag{38}$$

The matrix Λ_{ex} is just the transpose of Λ_{xe}. The initial conditions for (29), (30), and (31) are

$$P_{xx}(t_o) = \mathrm{Var}\left\{x(t_o)\right\} = P_o \tag{39}$$

and

$$P_{xe}(t_o) = P_o\ N(t_o)^T;\ P_{ee}(t_o) = N(t_o)\ P_o\ N(t_o)^T \tag{40}$$

The terminal values for (37) and (38) are as required by the transversality condition applied at the terminal time

$$\Lambda_{ee}\ (t_f) = S \tag{41}$$

and

$$\Lambda_{xe}\ (t_f) = 0 \tag{42}$$

Notice that $\Lambda_{ee}(t)$ can be computed separately without solving the rest of the problem if F is known beforehand. However at this point, we have not yet determined how F should be selected. It will be seen that this depends in a critical way on the nature of $\bar{Q}$. We will consider two different classes of problems.

CASE I.

In this case, we assume that $\bar{Q} = 0$. The meaning of this is that the quality of the estimation algorithm is only important at the terminal time. This may make sense for rather a large class of problems. The reason that this case is of particular interest is that the selection of F does not affect the Hamiltonian, so that we are free to select its value based on other considerations.

Consider that part of the Hamiltonian which depends explicitly on F.

$$H^* = \mathrm{tr}\left\{F\Theta + \Theta^T F^T\right\} \tag{43}$$

where

$$\Theta = (P_{ee} - NP_{xe})\Lambda_{ee} + (P_{ex} - NP_{xx})\Lambda_{xe} \tag{44}$$

From (39) and (40) it is clear that $\Theta(t_0) = 0$. If it can be shown that $\Theta(t) = 0$ for all t in the interval of interest, then a singular arc exists. The Hamiltonian is independent of F. In this case, one does not need to specify F to stay on the singular arc. Differentiating Θ gives

$$\dot{\Theta} = F\Theta - \Theta F + K\left[RK^T\Lambda_{ee} + \Psi_3 K^T \Lambda_{ee} - CP_{xe}\Lambda_{ee} - CP_{xx}\Lambda_{xe} - \Psi_2{}^T\Lambda_{ee} - \Psi_2{}^T N^T \Lambda_{ee}\right] \tag{45}$$

The bracketed term in the above is zero whenever K is chosen optimally, i.e., according to (36). Hence

$$\dot{\Theta}(t) = F(t)\Theta(t) - \Theta(t)F(t) \tag{46}$$

and (46) implies that $\Theta(t) = 0$ for all $t \geq t_o$ since $\Theta(t_o) = 0$. The selection of F is thus not a performance factor. It may be selected a-priori so that $\Lambda_{ee}(t)$ can be precomputed. It may be selected so as to achieve some other objective such as reduced sensitivity, computational convenience or to minimize some alternative performance measure specifically involving F.

When one thinks about it, the singularity with respect to F is not particularly surprising. Clearly two different filters can even produce the same output at a particular time, given the same input. What is interesting, is that this fact is generally overlooked, and as the example problem will show, that an alternative filter structure can be relatively easily implemented.

CASE II.

In this case the weighting matrix, $\bar{Q}$, is a positive definite symmetric matrix. When one develops an expression for $\dot{\Theta}$, the result is

$$\dot{\Theta} = F\Theta - \Theta F + \Omega\bar{Q} \tag{47}$$

instead of (46), where

$$\Omega = NP_{xe} - P_{ee} \tag{48}$$

Thus unless Ω is zero, a singular arc does not exist.

It is easily seen that $\Omega(t)$ does not equal zero unless F is slected appropriately. From the initial conditions, $\Omega(t_o) = 0$. Taking the time derivative of Ω we get

$$\dot{\Omega} = F\Omega+\Omega F^T + (NP_{xx}-P_{ex})\,(NA-FN-KC+\dot{N})^T \\ -K\left[RK^T-\Psi_2{}^TN^T+\Psi_3K^T-CP_{xe}\right] \tag{49}$$

Examining the last equations we see that if

$$(NA-FN-KC+\dot{N}) = 0 \tag{50}$$

then

$$\dot{\Omega} = F\Omega+\Omega F^T \tag{51}$$

This follows from the fact that when (50) holds, $\Lambda_{xe}(t)$ is zero for all t in the interval. Consequently the expression for the gain becomes

$$K = \left[P_{ex}C^T + N\Psi_2\right]\left[R + \Psi_3\right]^{-1} \tag{52}$$

and (52) is sufficient to have the last term in (49) be zero. In view of (51) and the fact that $\Omega(t_o)$ is zero, it is clear that $\Omega(t)$ is zero for all $t\in\left[t_o, t_f\right]$ provided that (50) holds and that the gain is selected optimally.

When $\Omega(t)$ is zero, it may be seen that the orthogonality requirement is met in a reduced state space, i.e.

$$\Omega(t) = N(t)P_{xe}(t)-P_{ee}(t) = E\left\{\left[z(t)-e(t)\right]e^T(t)\right\} = 0 \tag{53}$$

Since $\hat{z} = z-e$, (53) may be written as

$$E\left\{\hat{z}(t)\ e^T(t)\right\} = 0 \tag{54}$$

so that what we have required for singularity is that the error and the estimate be orthogonal.

When N is the identity matrix and there is no state dependent noise, the result is the Kalman filter, with the requirement (50) that

$$F(t) = A(t)-K(t)C(t) \tag{55}$$

which of course means that the filter is of full order. When the filter is of reduced order, and N is constant, what we have is the observer constraint equation [16]

$$NA-FN-KC = 0 \tag{56}$$

In general, when $\bar{Q} > 0$, (50) is a necessary condition for a singular arc. Clearly it is not always possible to select F to satisfy (50). In such cases, the problem needs to be reformulated so that an unbounded F is not indicated. Alternatively a suboptimal solution can be accepted. We will examine this topic in the next section.

A necessary and sufficient condition that (50) have a solution F, is that

$$\left[NA-KC+\dot{N}\right] N^{\dagger}N = \left[NA-KC+\dot{N}\right] \forall t \in \{t_o, t_f\} \tag{57}$$

If (57) holds then a solution is

$$F = \left[NA-KC+\dot{N}\right] N^{\dagger} + \Gamma\left[I-NN^{\dagger}\right] \tag{58}$$

where $N^{\dagger}$ is the pseudo inverse of N and where Γ is an arbitrary $\ell x \ell$ matrix [17]. When the matrix (NN^T) is nonsingular then the solution (58) can be written as

$$F = \left[NA-KC+\dot{N}\right] N^T \left[NN^T\right]^{-1} \tag{59}$$

IV. SPECIFIC F SOLUTIONS

In the preceding section we have shown that when one is only interested in estimation at a particular time, the selection of F may be based on considerations other than optimality, so that one may pick it prior to optimization. Furthermore, when $\bar{Q} > 0$, it may not be possible to find an F which results in a singular arc. In that case one may opt to select F prior to optimization. In this section, we will see that when F is selected a priori, the two point boundary value problem which must be solved for the selection of K is linear, and hence relatively easy to solve.

Consider substituting the gain expression (36) in (31) and (38). The resulting expressions are

$$\begin{aligned}\dot{P}_{xe} = {} & AP_{xe} + P_{xx}(\dot{N}+NA-FN)^T + P_{xe}F^T + QN^T + \Psi_1 N^T \\ & -(P_{xx}C^T+\Psi_2)(R+\Psi_3)^{-1}\left[(P_{xx}C^T + \Psi_2)^T \Lambda_{xe} \Lambda_{ee}^{-1}\right. \\ & \left. + \Psi_2^T N^T + C P_{xe}\right] \end{aligned} \tag{60}$$

and

$$\begin{aligned}\dot{\Lambda}_{xe} = {} & -(NA-FN+\dot{N})^T \Lambda_{ee} - A^T \Lambda_{xe} - \Lambda_{xe}F + C^T(R+\Psi_3)^{-1} \\ & \cdot\left[(P_{xx}C^T + \Psi_2)^T \Lambda_{xe} + (C P_{xe} + \Psi_2^T N^T)\Lambda_{ee}\right] \end{aligned} \tag{61}$$

When F is known a priori, both P_{xx} and Λ_{ee} are known in the sense that they may be precomputed. The above equations are then seen to give a linear TPBVP in the matrices P_{xe} and Λ_{xe}. The solution may be obtained in a straight forward manner using linear systems theory, or alternatively by assuming that the elements of Λ_{xe} are linearly related to those of P_{xe}, and obtaining a solution involving a Riccati equation. The values obtained for P_{xe} and Λ_{xe} may then be used in the gain expression (36).

We cannot overemphasize the importance of the fact that our result is a linear TPBVP, since it is reasonable to expect to solve a linear matrix TPBVP. Often a nonlinear matrix TPBVP is so difficult to solve, that the utility of the result is questionable. We shall explain procedures for solving a linear TPBVP by looking at a particularly easy case in which Λ_{ee} is a scalar times the identity matrix. This results when both F and $\bar{Q}$ are scalars times the identity matrix. When this is true, (60) may be written as

$$\dot{P}_{xe} = L_{11} P_{xe} + L_{12} \Lambda_{xe} + D_1 \tag{62}$$

where

$$L_{11} = A + F - L^* C \tag{63}$$

$$L_{12} = -L^* (C P_{xx} + \Psi_2^T) \Lambda^{-1}_{ee} \tag{64}$$

$$D_1 = P_{xx} (NA - FN + \dot{N})^T + QN^T + \Psi_1 N^T - L^* \Psi_2^T N^T \tag{65}$$

and where

$$L^* = (P_{xx} C^T + \Psi_2) (R + \Psi_3)^{-1} \tag{66}$$

Equation (62) is of the form

$$\dot{\Lambda}_{xe} = L_{21} P_{xe} + L_{22} \Lambda_{xe} + D_2 \tag{67}$$

where

$$L_{21} = C^T (R + \Psi_3)^{-1} C \Lambda_{ee} \tag{68}$$

$$L_{22} = -A^T - F + C^T L^{*T} \tag{69}$$

and

$$D_2 = -(NA - FN + \dot{N})^T \Lambda_{ee} + C^T (R + \Psi_3)^{-1} \Psi_2^T N^T \Lambda_{ee} \tag{70}$$

Let L be the matrix

$$L = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right] \tag{71}$$

and Φ be the associated state transition matrix which can similarly be partitioned

$$\Phi = \left[\begin{array}{c|c} \Phi_{11} & \Phi_{12} \\ \hline \Phi_{21} & \Phi_{22} \end{array}\right] \tag{72}$$

Then the solution to (62) is

$$\begin{aligned} P_{xe}(t) = {} & \Phi_{11}(t, t_o) P_{xe}(t_o) + \Phi_{12}(t, t_o) \Lambda_{xe}(t_o) \\ & + \int_{t_o}^{t} \left[\Phi_{11}(t,\tau) D_1(\tau) + \Phi_{12}(t,\tau) D_2(\tau)\right] d\tau \end{aligned} \tag{73}$$

and

$$\begin{aligned} \Lambda_{xe}(t) = {} & \Phi_{21}(t,t_o) P_{xe}(t_o) + \Phi_{22}(t,t_o) \Lambda_{xe}(t_o) \\ & + \int_{t_o}^{t} \left[\Phi_{21}(t,\tau) D_1(\tau) + \Phi_{22}(t,\tau) D_2(\tau)\right] d\tau \end{aligned} \tag{74}$$

Applying (74) at time $t = t_f$ gives

$$\begin{aligned} \Lambda_{xe}(t_f) = 0 = {} & \Phi_{21}(t_f,t_o) P_{xe}(t_o) + \Phi_{22}(t_f,t_o) \Lambda_{xe}(t_o) \\ & + \int_{t_o}^{t_f} \left[\Phi_{21}(t_f,\tau) D_1(\tau) + \Phi_{22}(t_f,\tau) D_2(\tau)\right] d\tau \end{aligned} \tag{75}$$

We can solve (75) for $\Lambda_{xe}(t_o)$ and substitute the results in (73) and (74) to obtain the solution for all $t \in \left[t_o, t_f\right]$.

There is another approach which is probably preferable in most cases. We assume that Λ_{xe} is linearly related to P_{xe} by the relationship

$$\Lambda_{xe}(t) = U(t)P_{xe}(t) + B(t) \tag{76}$$

Differenatiating (76), one obtains the differential equation

$$\dot{\Lambda}_{xe} = \dot{U}P_{xe} + U\left[L_{11}P_{xe} + D_1 + L_{12}\,UP_{xe} + L_{12}\,B\right] + \dot{B} \tag{77}$$

Alternatively, from (67)

$$\dot{\Lambda}_{xe} = L_{21}P_{xe} + L_{22}\,UP_{xe} + L_{22}B + D_2 \tag{78}$$

Equating (77) and (78), we get for U

$$\dot{U} + UL_{11} + UL_{12}U = L_{21} + L_{22}U \tag{79}$$

and for B

$$UL_{12}B + \dot{B} + UD_1 = L_{22}B + D_2 \tag{80}$$

Since $\Lambda_{xe}(t_f) = 0$, the terminal conditions for U and for B are

$$U(t_f) = 0 \tag{81}$$

$$B(t_f) = 0 \tag{82}$$

The Riccati equation (79) and equation (80) can be solved backwards in time from the above terminal conditions. The optimal gain may then be expressed as

$$K = \left[P_{xe}{}^{T}C^{T} + N\Psi_2 + \Lambda_{ee}{}^{-1}\,(UP_{xe}+B)^{T}(P_{xx}C^{T}+\Psi_2)\right]\left[R+\Psi_3\right]^{-1} \tag{83}$$

and P_{xe} is evaluated as

$$\dot{P}_{xe} = L_{11}P_{xe} + L_{12}\left[UP_{xe} + B\right] + D_1 \tag{84}$$

The matrices Λ_{ee}, P_{xx}, U, and B must be evaluated off line, however, P_{xe} and K can be evaluated on line if this is desired. Most likely these would also be evaluated off line and K stored for on line calculation of $\hat{z}(t)$ using (9).

V. EXAMPLES

The first example we shall consider is of the category discussed in Case II. We assume that A(t) is zero, N(t) = C(t), and that there exists an F such that

$$FC = \dot{C} - KC \quad \forall t \in \left[t_o, t_f\right] \tag{85}$$

then if CC^T is nonsingular

$$F = \left[\dot{C}C^T - KCC^T\right](CC^T)^{-1} \tag{86}$$

The filter equation is

$$d\hat{z} = \dot{C}C^T(CC^T)^{-1}\hat{z}dt + K\left[dy - \hat{z}dt\right] \tag{87}$$

The initial condition for (87) is

$$\hat{z}(t_o) = C(t_o)\,\mu_o \tag{88}$$

The gain is of the form

$$K(t) = \left[P_{xe}{}^T C^T + C\Psi_2\right]\left[R + \Psi_3\right]^{-1} \tag{89}$$

where P_{xe} is the solution to

$$\dot{P}_{xe} = P_{xe}(CC^T)^{-1}\left[\dot{C}C^T - CC^T K^T\right] + (Q+\Psi_1)\, C^T - \Psi_2\, K^T \tag{90}$$

Alternatively since $CP_{xe} = P_{ee}$, it may be desirable to evaluate (89) as

$$K(t) = \left[P_{ee} + C\Psi_2\right]\left[R + \Psi_3\right]^{-1} \tag{91}$$

where P_{ee} is the solution to

$$\begin{aligned}\dot{P}_{ee}(t) = &\left[\dot{C}C^T(CC^T)^{-1} - K\right] P_{ee} + P_{ee}\left[\dot{C}C^T(CC^T)^{-1} - K\right]^T \\ &+ CQC^T + KRK^T + K\Psi_3 K^T - C\Psi_2 K^T \\ &- K\Psi_2{}^T C^T + C\Psi_1 C^T\end{aligned} \tag{92}$$

The reason (92) is appealing is that P_{ee} has fewer elements to calculate then P_{xe}.

The next example is concerned with the very simple problem of estimating a constant having zero mean and variance 1 prior to observations. The observation is of the form

$$dy = x\, dt + dv + Mx\, d\upsilon \tag{93}$$

where v and υ are zero mean white noise with covariance parameter 1 and M is constant. We are interested in estimating the value of x at time, T. Hence

$$J = E\left\{e^2(T)\right\} \tag{94}$$

and this is a problem of the category referred to as Case 1. For computational convenience we select F = 0. The TPBVP then is

$$\dot{P}_{xe} = -\gamma\,(P_{xe} + \Lambda_{xe}) \tag{95}$$

$$\dot{\Lambda}_{xe} = \gamma\,(P_{xe} + \Lambda_{xe}) \tag{96}$$

with $P_{xe}(0) = 1$ and $\Lambda_{xe}(T) = 0$, where

$$\gamma \triangleq (1 + M^2)^{-1} \tag{97}$$

The solution is

$$P_{xe}(t) = \frac{1 - \gamma(t-T)}{1 + \gamma T} \tag{98}$$

and

$$\Lambda_{xe}(t) = \frac{\gamma(t - T)}{1 + \gamma T} \tag{99}$$

Interestingly, because of the complimentary nature of $P_{xe}(t)$ and $\Lambda_{xe}(t)$, the gain is a constant,

$$k(t) = k = \frac{\gamma}{1 + \gamma T} \tag{100}$$

The filter is simply

$$d\hat{x}(t) = \frac{\gamma}{1 + \gamma T}\, dy(t) \tag{101}$$

and the error variance at time $t = T$ is

$$P_{ee}(T) = \frac{1}{1 + YT} = \frac{1 + M^2}{1 + M^2 + T} \tag{102}$$

The filter (101) is simpler to construct than the choice which would require $F = -K$, i.e., one of the form

$$d\hat{x}(t) = k(t)\left[dy(t) - \hat{x}(t)dt\right] \tag{103}$$

even though it is obviously a full order filter. The authors feel that the nonunique property of optimal linear filters for certain cases is a feature which one should take advantage of.

VI. REMARKS AND CONCLUSIONS

We have extended the results of [8] to problems having state dependent noise in the observation and dynamical equation. Control theoretic methods have been used to solve the problem, and optimal solutions have been shown to correspond to singular arcs. Different solutions result when there is an integral performance measure than when only estimation at the terminal time is important. In some cases, we have seen that it makes sense to select the filter matrix ahead of time and then optimize the gain. The computational algorithms associated with such prior selection are particularly convenient. There are no terribly difficult TPBVP's in this approach and that is why the authors feel that it is practical and useful, both for full order and reduced order filters. The amount of off line calculation necessary to simplify on line filtering appears to be quite realistic.

REFERENCES

[1] McLane, P. J., "Optimal Linear Filtering for Linear Systems with State Dependent Noise," International Journal of Control, Vol. 10, No. 1, pp. 41-51, 1969.

[2] Gustafson, E. G., "On Optimal Estimation and Control of Linear Systems with State Dependent Noise," Ph.D. Thesis, MIT, June, 1972.

[3] Sims, C. S., "Linear Estimation in Discrete Random Parameter Systems," SWIEEECO Record of Technical Papers, April 4-6, 1973, Houston, Texas.

[4] McLane, P. J., "Optimal Stochastic Control of Linear Systems with State and Control Dependent Disturbances," IEEE Trans. Aut. Control, Vol. AC-16, No. 6, pp. 793-798.

[5] Hutchinson, C. E., and J. A. D'Appotito, "Minimum Variance Reduced State Filter," presented at 1972 IEEE Conference on Decision and Control, New Orleans, Dec. 1972.

[6] Damiani, T. R., "A General Partial State Estimator," presented at 14th Midwest Symp. Circuit Theory, Denver, Colorado, May, 1971.

[7] Sims, C. S., "An Algorithm for Estimating a Portion of a State Vector," IEEE Trans. Aut. Control, Vol. AC-19, No. 4, August, 1974.

[8] Sims, C. S., and R. B. Asher, "Optimal and Suboptimal Linear Filtering," Proceeding of 1977 JACC, San Francisco, California.

[9] Sims, C. S., and J. L. Melsa, "A Fixed Configuration Approach to the Stochastic Linear Regulator Problem," Proceedings 1970 Joint Automatic Control Conference, June 24-26, 1970, Atlanta, Georgia.

[10] Sims, C. S., "Fixed Configuration Control of Systems with State and Control Dependent Noise," Proceedings 1971 Joint Automatic Control Conference, August 11-13, St. Louis, Missouri.

[11] Lee, J. S., "On the Relation of Singular Control to Linear Estimation," Proceedings 1970 Joint Automatic Control Conference, June 24-26, 1970, Atlanta, Georgia.

[12] Sims, C. S., "Observer-Estimators for Discrete Random Parameter Systems," Proceedings of the Eleventh Annual Allerton Conference on Circuit and Systems Theory, Monticello, Illinois, October 3-5, 1973.

[13] Athans, M., "The Matrix Minimum Principle," Information and Control, Vol. 11, pp. 592-606, 1968.

[14] Cumming, J. G., "Derivation of Moments of a Continuous Stochastic System," Int. Journal Control, Vol. 5, No. 1, pp. 85-90.

[15] Jazwinski, A. H., Stochastic Processes and Filtering Theory, Academic Press, New York, 1970.

[16] Luenberger, D. G., "Observers for Multivariable Systems," IEEE Trans. Automatic Control., Vol. AC-11, pp. 190-197, April 1966.

[17] Boullion, T. L., and P. L. O'Dell, Generalized Inverse Matrices, Wiley-Interscience, New York, 1971, pp. 4-5.

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LINEAR DISCRETE REDUCED ORDER FILTERING*

Craig S. Sims
School of Electrical Engineering
Oklahoma State University
Stillwater, Oklahoma 74074

ABSTRACT

The linear reduced order filtering problem is formulated as a matrix two-point boundary-value problem. Cases are presented in which the two-point boundary value problem simplifies, and may be solved with a reasonable amount of calculation.

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INTRODUCTION

In this paper we consider estimating only a part of a state vector, using a reduced order linear filter. The problem is motivated by the fact that sometimes one is only interested in estimating a portion of the state vector, while the entire state vector may be of large dimension. For example, the entire state vector might contain state variables associated with a detailed model of the noise generating process, and the filter designer might only be interested in estimating position and velocity. The complexity of the Kalman filter for estimating a large state vector leads one to consider less complicated reduced order filter designs for estimating the variables of interest. The reduced order filter would typically have fewer on line calculations, but a greater number of off-line calculations required for evaluating the filter parameters.

There has been considerable research in the area of reduced order filtering [1-8] because of the difficulties involved in implementing a full order filter of large dimension. This paper is a discrete version of the work presented in [8], wherein a continuous time problem was formulated as a two-point boundary-value problem (TPBVP). The discrete problem is quite different than the continuous time problem, since some of the filter variables to be optimized appear quadratically in the discrete problem and only linearly in the continuous problem. This paper also represents an extension of [5], where a discrete reduced order filtering problem was solved, but the parameters of the reduced order filter were only optimized with respect to performance at the next stage. Here parameters are optimized over an interval. A solution is

obtained using the matrix form of the minimum principle [9]. As in [8], a suboptimal approach is suggested which leads to a linear TPBVP to solve, and a method of solution is suggested.

PROBLEM STATEMENT

Consider a linear stochastic process defined by the discrete state equation

$$x_{j+1} = A_j x_j + w_j \quad ; j = 0, 1, \ldots \tag{1}$$

where x_j is the state vector at stage j, and w_j is a zero mean white noise vector. The system is observed with noisy linear measurements

$$y_j = C_j x_j + v_j \quad ; j = 1, 2, \ldots \tag{2}$$

where y_j is the observation and v_j is zero mean white noise. The covariance matrices for w_j and v_j are

$$\begin{aligned} E\{ w_j w_k^T \} &= Q_j \delta_{kj} \\ E\{ v_j v_k^T \} &= R_j \delta_{kj} \end{aligned} \tag{3}$$

The noise terms, w_j and v_j are not correlated with each other or with the initial condition, x_o. In the problem considered, a lower order linear transformation of the state vector is of interest

$$z_j = N_j x_j \tag{4}$$

The dimensions of z_j, x_j, and y_j are ℓ, n, and m respectively, and $\ell \leq n$.

Since z_j is lower dimensioned than x_j, we consider estimating it with a reduced order filter of the form

$$\hat{z}_{j+1} = F_j \hat{z}_j + K_j y_{j+1} + g_j \tag{5}$$

The objective of the problem is to select g_j and $\hat{z}_o$ to satisfy a requirement that the estimate be unbiased, i.e.,

$$E\{e_j\} = E\{z_j - \hat{z}_j\} = 0 \tag{6}$$

The matrices F_j and K_j are then subject to optimization to minimize a quadratic performance measure,

$$J = E\{\sum_{j=o}^{M-1} e_j^T U_j e_j + e_m^T S e_m\} \tag{7}$$

The solution to the unbiased requirement is easily obtained by considering the error difference equation. The optimization problem can then be solved using the matrix minimum principle.

THE UNBIASED REQUIREMENT

By considering (1), (2), (4), and (5), we find that the difference equation for the error is

$$\begin{aligned} e_{j+1} = {} & [(N_{j+1} - K_j C_{j+1}) A_j - F_j N_j] x_j + \\ & (N_{j+1} - K_j C_{j+1}) w_j + F_j e_j - K_j v_{j+1} - g_j \end{aligned} \tag{8}$$

Taking the expectation of both sides of (8) gives

$$\begin{aligned} E\{e_{j+1}\} = {} & F_j E\{e_j\} + [(N_{j+1} - K_j C_{j+1}) A_j - \\ & F_j N_j] \mu_j - g_j \end{aligned} \tag{9}$$

where $\mu_j \triangleq E\{x_j\}$. From (9) it is clear that we will have a linear homogeneous equation for the prior expectation of the error provided that g_j is

selected as

$$g_j = [(N_{j+1} - K_j C_{j+1}) A_j - F_j N_j] \mu_j \tag{10}$$

Hence $E\{e_j\} = 0 \quad \forall j \geq o$ if $E\{e_o\} = 0$, which will be true if $\hat{z}_o$ is chosen as

$$\hat{z}_o = N_o \mu_o \tag{11}$$

With g_j selected as in (10), the difference equation for the error is

$$e_{j+1} = G_j \bar{x}_j + D_j w_j + F_j e_j - K_j v_{j+1} \tag{12}$$

where we have defined

$$\begin{aligned} G_j &\triangleq [(N_{j+1} - K_j C_{j+1}) A_j - F_j N_j] \\ D_j &\triangleq (N_{j+1} - K_j C_{j+1}) \\ \bar{x}_j &\triangleq x_j - \mu_j \end{aligned} \tag{13}$$

It is clear that $\bar{x}_j$ is a zero mean process satisfying

$$\bar{x}_{j+1} = A \bar{x}_j + w_j \tag{14}$$

The remaining problem is to minimize the error criterion (7), with respect to F_j and K_j, subject to the constraints imposed by (12) and (14).

OPTIMIZATION

The performance measure (7) may be written in terms of the error variance matrix

$$J = \mathrm{tr}\{\sum_{j=o}^{M-1} U_j P_{ee}(j) + S P_{ee}(M)\} \tag{15}$$

where

$$P_{ee}(j) \triangleq E\{e_j e_j^T\} \tag{16}$$

From (12) and (14) it is seen that the error variance matrix satisfies

$$P_{ee}(j+1) = F_j P_{ee}(j) F_j^T + F_j P_{ex}(j) G_j^T + G_j P_{xe}(j) F_j^T + G_j P_{xx}(j) G_j^T + D_j Q_j D_j^T + K_j R_{j+1} K_j^T \tag{17}$$

where

$$P_{xe}(j) = P_{ex}^T(j) \triangleq E\{x_j e^T{}_j\} \tag{18}$$

propagates according to the equation

$$P_{xe}(j+1) = A_j P_{xe}(j) F_j^T + A_j P_{xx}(j) G_j^T + Q_j D_j^T \tag{19}$$

From (14), the matrix $P_{xx}(j) \triangleq E\{\bar{x}_j \bar{x}_j^T\}$ satisfies

$$P_{xx}(j+1) = A_j P_{xx}(j) A_j^T + Q_j \tag{20}$$

The initial conditions for (17), (19), and (20) are

$$\begin{aligned} P_{xx}(o) &= E\{\bar{x}_o \bar{x}_o^T\} = \mathrm{Var}\{x(o)\} = P_o \\ P_{ee}(o) &= N_o P_o N_o^T; \quad P_{xe}(o) = P_o N_o^T \end{aligned} \tag{21}$$

The problem is now completely within a deterministic frame and can be solved using the matrix minimum principle. The Hamiltonian is of the form

$$H_j = \mathrm{tr}\{U_j P_{ee}(j) + P_{ee}(j+1) \Lambda^T{}_{ee}(j+1) + P_{ex}(j+1) \Lambda_{ex}^T(j+1) + P_{xe}(j+1) \Lambda_{xe}^T(j+1)\} \tag{22}$$

where Λ_{ee} and Λ_{xe} are Lagrange multipliers, and Λ_{ex} is the transpose of

Λ_{xe}. After substituting from (17) and (19), the costate equations can be found from (22).

$$\Lambda_{ee}(j) = \frac{\partial H_j}{\partial P_{ee}(j)} = U_j + F_j^T \Lambda_{ee}(j+1) F_j \tag{23}$$

and

$$\Lambda_{xe}(j) = \frac{\partial H_j}{\partial P_{xe}(j)} = G_j^T \Lambda_{ee}(j+1) F_j + A_j^T \Lambda_{xe}(j+1) F_j \tag{24}$$

The terminal conditions for (23) and (24) are

$$\Lambda_{ee}(M) = S \tag{25}$$

and

$$\Lambda_{xe}(M) = o \tag{26}$$

Setting the gradient of the Hamiltonian with respect to $K_j = o$ gives a necessary condition for optimality,

$$\begin{aligned} \Lambda_{ee}(j+1) K_j [C_{j+1} P_{xx}(j+1) C_{j+1}^T + R_{j+1}] &= \Lambda_{ex}(j+1) P_{xx}(j+1) C_{j+1}^T \\ &+ \Lambda_{ee}(j+1) [F_j (P_{ex}(j) - N_j P_{xx}(j)) A_j^T \\ &+ N_{j+1} P_{xx}(j+1)] C_{j+1}^T \end{aligned} \tag{27}$$

Similarly, setting the gradient with respect to F_j equal to zero gives

$$\begin{aligned} &\Lambda_{ee}(j+1) F_j [P_{ee}(j) - P_{ex}(j) N_j^T - N_j P_{xe}(j) + N_j P_{xx}(j) N_j^T] \\ &= [\Lambda_{ex}(j+1) + \Lambda_{ee}(j+1) (N_{j+1} - K_j C_{j+1})] A_j [P_{xx}(j) N_j^T - P_{xe}(j)] \end{aligned} \tag{28}$$

Thus we have the TPBVP giving the necessary conditions for an optimum. In

general, such a problem is difficult to solve. There are two cases in which it is not unreasonably difficult to solve the problem. In the first case, the TPBVP simplifies into a single point boundary value problem. The Kalman filter is of this category. In the second case, F_j, is not optimized, but selected prior to optimization. The optimization of the gain K_j may then be accomplished by solving a linear TPBVP which is a routine procedure.

SIMPLIFICATION

In this section we show that there are circumstances where the TPBVP simplifies considerably. Assume that it is possible to select F_j in such a way that G_j is zero, i.e.

$$(N_{j+1} - K_j C_{j+1}) A_j - F_j N_j = 0 \tag{29}$$

Then from (24) and (26) it is clear that $\Lambda_{xe}(j)$ is zero. If Λ_{ee} is nonsingular, and if furthermore

$$\Omega(j) \triangleq P_{ee}(j) - P_{ex}(j) N_j^T = 0 \tag{30}$$

then (28) can be written as

$$F_j N_j [P_{xx}(j) N_j^T - P_{xe}(j)] = (N_{j+1} - K_j C_{j+1}) A_j [P_{xx}(j) N_j^T - P_{xe}(j)] \tag{31}$$

so it is clear that (29) implies that F_j satisfies the necessary conditions for an optimum. It really isn't necessary that Λ_{ee} be nonsingular for (29) to imply (28), but it is critical that $\Omega(j)$ be zero, so we investigate this point.

Using (17) and (19), and noting that G_j is zero, we obtain

$$\Omega(j+1) = F_j \Omega(j) F_j^T + [K_j C_{j+1} Q_j C_{j+1}^T + K_j R_{j+1} \quad (32)$$
$$-N_{j+1} Q_j C_{j+1}^T - F_j P_{ex}(j) A_j^T C_{j+1}^T] K_j^T$$

We will show that the bracketed term in (32) is zero when K_j is selected optimally. With G_j equal zero, (27) will be satisfied if

$$K_j [C_{j+1} P_{xx}(j+1) C_{j+1}^T + R_{j+1}] = [F_j P_{ex}(j) A_j^T + N_{j+1} P_{xx}(j+1)$$
$$-F_j N_j P_{xx}(j) A_j^T] C_{j+1}^T \quad (33)$$

We substitute for $F_j N_j$ from (29) in the last term of (33), obtaining

$$K_j [C_{j+1} P_{xx}(j+1) C_{j+1}^T + R_{j+1}] = [F_j P_{ex}(j) A_j^T + N_{j+1} P_{xx}(j+1)]$$
$$C_{j+1}^T - [N_{j+1} - K_j C_{j+1}] A_j P_{xx}(j) A_j^T C_{j+1}^T \quad (34)$$

Substituting from (20) for $P_{xx}(j+1)$, and simplifying the result gives

$$K_j [C_{j+1} Q_j C_{j+1}^T + R_{j+1}] = [F_j P_{ex}(j) A_j^T + N_{j+1} Q_j] C_{j+1}^T \quad (35)$$

which insures that (32) may be written as a homogeneous equation

$$\Omega(j+1) = F_j \Omega(j) F_j^T \quad (36)$$

whenever K_j is selected optimally. From the initial conditions of the problem, $\Omega(o) = 0$, so (36) implies that $\Omega(j) = 0$ for $j > o$. We therefore have established that (29) and (35) will satisfy (27) and (28), the necessary conditions for optimality. We can write (29) and (35) as

$$\left[\begin{array}{c|c} A_j^T C_{j+1}^T & N_j^T \\ \hline (C_{j+1} Q_j C_{j+1}^T + R_{j+1}) & -C_{j+1} A_j P_{xe}(j) \end{array}\right] \left[\begin{array}{c} K_j^T \\ \hline F_j^T \end{array}\right] = \left[\begin{array}{c} A_j^T N_{j+1}^T \\ \hline C_{j+1} Q_j N_{j+1}^T \end{array}\right] \quad (37)$$

or

$$L_j \left[\begin{array}{c} K_j^T \\ \hline F_j^T \end{array}\right] = B_j \tag{38}$$

where L_j and B_j are defined as the corresponding matrices in (37). A necessary and sufficient condition for (38) to have a solution is that

$$L_j \; L_j^{\dagger} B_j = B_j \tag{39}$$

where $L_j^{\dagger}$ is the pseudo inverse of L_j. If a solution exists it is of the form

$$[K_j \mid F_j] = (L_j^{\dagger} B_j)^T + \Gamma (I - L_j^{\dagger} L_j)^T \tag{40}$$

where Γ is an arbitrary matrix.

It is of interest to relate our solution to the well known Kalman filter solution when the filter is not of reduced order, but N_j=I. In this case (29) requires that F_j be selected as

$$F_j = (I - K_j C_{j+1}) A_j \tag{41}$$

which is the same as the Kalman filter result. Using (41) in (35) we have

$$\begin{aligned} &[\, C_{j+1} Q_j C_{j+1}^T + R_{j+1} + C_{j+1} A_j P_{xe}(j) A_j^T C_{j+1}^T \,] K_j^T = \\ &[\, C_{j+1} Q_j + C_{j+1} A_j P_{xe}(j) A_j^T \,] \end{aligned} \tag{42}$$

Noting that $P_{ee}(j) = P_{xe}(j)$, and defining the one stage prediction error variance matrix as

$$P_{ee}(j+1 \mid j) = A_j P_{ee}(j) A_j^T + Q_j \tag{43}$$

It is clear that (43) gives the common expression for the Kalman gain

$$K_j = P_{ee}(j+1|j)\, C_{j+1}^T [\, C_{j+1} P_{ee}(j+1|j)\, C_{j+1}^T + R_{j+1}\,]^{-1} \qquad (44)$$

Since in the reduced order case, we may not be able to solve (40), or more fundamentally, may not be able to solve the TPBVP specified in the previous section, we therefore investigate a partial solution where F_j is selected a priori, and only K_j is optimized.

SPECIFIED F_j SOLUTIONS

The idea that we suggest in this section is motivated by the fact that if we select F_j prior to optimization, the remaining TPBVP is linear, and there are a number of approaches available for solving linear TPBVP's. If a solution exists, it can be obtained in a predictable number of steps, depending on the procedure used. Thus a reasonable amount of off-line calculation can bring about a large savings in on-line calculations. The performance loss can be made small by selecting F_j in some way which relates to performance, but does not require solving a TPBVP. Such a procedure was indicated in [5] where only single stage optimization was considered. We should also point out that if one is only interested in the estimate at the terminal time, the selection of F_j does not affect the performance measure. This was proved in [8] for continuous time problems. Hence the method we propose here is appropriate for that special case, i.e. the actual optimal performance will be achieved.

If F_j is selected a priori, we note from (23) that Λ_{ee} may be regarded as a known quantity, as is P_{xx}, in the sense that it may be precomputed without regard for the remainder of the problem. If $\Lambda_{ee}(j)$ is nonsingular, one may solve for the gain K_j as

$$K_j = [\Lambda_{ee}^{-1}(j+1)\Lambda_{ex}(j+1)P_{xx}(j+1) + F_j\{P_{ex}(j) - N_j P_{xx}(j)\}A_j^T \quad (45)$$

$$+ N_{j+1}P_{xx}(j+1)]\ C_{j+1}^T[C_{j+1}P_{xx}(j+1)C_{j+1}^T + R_{j+1}]^{-1}$$

where we have also assumed that $[C_{j+1}P_{xx}(j+1)C_{j+1}^T + R_{j+1}]$ has an inverse. Substituting (45) in (19) and (24) we obtain the expressions

$$P_{xe}(j+1) = (I - P_{xx}(j+1)C_{j+1}^T M_{j+1}^T)A_j P_{xe}(j) F_j^T + D_1(j) \quad (46)$$

$$-P_{xx}(j+1)\ C_{j+1}^T M_{j+1}^T P_{xx}(j+1)\Lambda_{xe}(j+1)\Lambda_{ee}^{-1}(j+1)$$

$$\Lambda_{xe}(j) = A_j^T(I - C_{j+1}^T M_{j+1}^T P_{xx}(j+1))\Lambda_{xe}(j+1)F_j + D_2(j) \quad (47)$$

$$-A_j^T C_{j+1}^T M_{j+1}^T A_j P_{xe}(j) F_j^T \Lambda_{ee}(j+1) F_j$$

where

$$D_1(j) = (I - P_{xx}(j+1)C_{j+1}^T M_{j+1}^T)(P_{xx}(j+1)N_{j+1}^T - \quad (48)$$

$$A_j P_{xx}(j) N_j^T F_j^T)$$

$$D_2(j) = [(N_{j+1}A_j - F_j N_j)^T - A_j^T C_{j+1}^T M_{j+1}^T (P_{xx}(j+1) N_{j+1}^T \quad (49)$$

$$-A_j P_{xx}(j) N_j^T F_j^T)]\Lambda_{ee}(j+1) F_j$$

and

$$M_{j+1} \triangleq C_{j+1}^T (C_{j+1} P_{xx}(j+1) C_{j+1}^T + R_{j+1})^{-1} \quad (50)$$

The boundary conditions for (46) and (47) are specified by (21) and (26) respectively. Equations (46) and (47) are linear in the matrices to be computed, P_{xe} and Λ_{xe}. Therefore there are a number of ways to solve the problem. One approach is to assume a linear relationship between the elements

of P_{xe} and Λ_{xe}, i.e. after putting the elements of Λ_{xe} in a long vector Λ_{xe}^{*}, and similarly putting the elements of P_{xe} in a vector P_{xe}^{*}, we may assume that

$$\Lambda^{*}_{xe}(j) = \Xi^{*}_{j} P_{xe}^{*}(j) + \beta_{j}^{*} \tag{51}$$

Recursive equations may then be obtained for Ξ_{j}^{*} and β^{*}_{j}, and the TPBVP is effectively transformed into a single point boundary value problem.

We illustrate the above remarks by examining a special case where it is not required to first put the elements in a vector format. It is assumed that the matrices of (23) are all in a scalar form, i.e.

$$F_{j} = f_{j} I \tag{52}$$

$$\Lambda_{ee}(j) = \lambda_{ee}(j) I \tag{53}$$

which requires that S and U be similarly defined. Then (46) and (47) are of the form

$$P_{xe}(j+1) = \alpha_{11}(j) P_{xe}(j) + \alpha_{12}(j) \Lambda_{xe}(j+1) + D_{1}(j) \tag{54}$$

$$\Lambda_{xe}(j) = \alpha_{21}(j) P_{xe}(j) + \alpha_{22}(j) \Lambda_{xe}(j+1) + D_{2}(j) \tag{55}$$

where

$$\begin{aligned}
\alpha_{11}(j) &\triangleq f_{j} (I - P_{xx}(j+1) C_{j+1}^{T} M_{j+1}^{T}) A_{j} \\
\alpha_{12}(j) &\triangleq -\lambda_{ee}^{-1}(j+1) P_{xx}(j+1) C_{j+1}^{T} M_{j+1}^{T} P_{xx}(j+1) \\
\alpha_{21}(j) &\triangleq -f_{j}^{2} \lambda_{ee}(j+1) A_{j}^{T} C_{j+1}^{T} M_{j+1}^{T} A_{j} \\
\alpha_{22}(j) &\triangleq f_{j} A_{j}^{T} (I - C_{j+1}^{T} M_{j+1}^{T} P_{xx}(j+1))
\end{aligned} \tag{56}$$

A linear relationship between P_{xe} and Λ_{xe} is assumed, i.e.

$$\Lambda_{xe}(j) = \Xi P_{xe}(j) + \beta_j \tag{57}$$

Substituting (57) in (54) and (55) gives

$$P_{xe}(j+1) = \alpha_{11}(j) P_{xe}(j) + \alpha_{12}(j) \Xi_{j+1} P_{xe}(j+1) + \alpha_{12}(j)\beta_{j+1} + D_1(j) \tag{58}$$

$$\Xi_j P_{xe}(j) + \beta_j = \alpha_{21}(j) P_{xe}(j) + \alpha_{22}(j) \Xi_{j+1} P_{xe}(j+1) + \alpha_{22}(j) \beta_{j+1} + D_2(j) \tag{59}$$

Solving for $P_{xe}(j+1)$ in (58), and substituting that result in (59) gives

$$\begin{aligned}\Xi_j P_{xe}(j) + \beta_j &= \alpha_{21}(j) P_{xe}(j) + \alpha_{22}(j) \beta_{j+1} + D_2(j) \\ &+ \alpha_{22}(j) \Xi_{j+1} [I - \alpha_{12}(j) \Xi_{j+1}]^{-1} [\alpha_{11}(j) P_{xe}(j) + \alpha_{12}(j) \beta_{j+1} \\ &+ D_1(j)]\end{aligned} \tag{60}$$

If the above is to hold for arbitrary P_{xe}, then we must have

$$\Xi_j = \alpha_{21}(j) + \alpha_{22}(j) \Xi_{j+1} [I - \alpha_{12}(j) \Xi_{j+1}]^{-1} \alpha_{11}(j) \tag{61}$$

and

$$\begin{aligned}\beta_j &= \alpha_{22}(j) [I + \Xi_{j+1} (I - \alpha_{12}(j) \Xi_{j+1})^{-1} \alpha_{12}(j)]\beta_{j+1} + D_2(j) \\ &+ \alpha_{22}(j) \Xi_{j+1} (I - \alpha_{12}(j) \Xi_{j+1})^{-1} D_1(j)\end{aligned} \tag{62}$$

The terminal conditions for the above are

$$\Xi_M = 0; \quad \beta_M = 0 \tag{63}$$

Having solved backwards in time to evaluate $\{\Xi_j\}$ and $\{\beta_j\}$, one may then solve (58) forward from the initial condition specified by (21). Since Λ_{xe} is known in terms of P_{xe}, (45) may then be used to evaluate the optimum gain sequence. We note that the algorithm we have suggested will fail if any of the required matrix inverses fail to exist any where along the trajectory.

EXAMPLE PROBLEM

In this section we domonstrate the performance of a filter designed with a specified set of values for $\{F_j\}$. The example is a simplified model of a discretized inertial system as treated in [8]. The dynamics are modeled as

$$x_{j+1} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} x_j + w_j \tag{64}$$

where Δ is .02, and the observation model is

$$y_{j+1} = [1 \quad 0]\, x_{j+1} + v_{j+1} \tag{65}$$

The covariance matrix of w_j is assumed to be

$$E\{ w_j \, w_k^T \} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \delta_{jk} \tag{66}$$

and v_j is white noise with covariance parameter .1. The initial variance of the state vector is assumed to be the identity matrix. It is desired to estimate x_1 using a first order filter of the form

$$\hat{z}_{j+1} = F_j \hat{z}_j + K_{j+1} y_{j+1} + g_j \tag{67}$$

From (10), g_j must be selected as

$$g_j = [(1 - K_j - F_j)\ (1- K_j)\Delta\ ] \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}^j \mu_o \tag{68}$$

The filter parameters F_j and K_j are selected in the following way. First a single stage optimization procedure is used to obtain values for F_j and K_j as indicated in [5]. This does not require solving a TPBVP. The performance of the resulting filter is shown in Figure 1. The value of F_j is maintained, but K_j is optimized according to the method proposed in this paper, i.e. by solving a linear TPBVP to minimize the performance measure

$$J = E\ \{\sum_{j=o}^{M-1} e^2{}_j + e^2{}_M\} \tag{69}$$

The results are shown in Figure 1 where they are compared with Kalman filter results and single stage optimization results. It is seen that performance can be improved, relative to that obtained in [5], while maintaining the same amount of on line calculations. The increased number of off line calculations is within reason since only a linear TPBVP must be solved.

REMARKS AND CONCLUSIONS

In this paper we have investigated the discrete version of the reduced order filtering problem. The solution to the general problem is seen to be a nonlinear matrix TPBVP, and hence of limited usefulness. We have shown, however, that under certain conditions it is possible to find a simplified solution to the TPBVP, and that this may be of considerably less complexity than the Kalman filter. Another approach has been presented where only the gain of the reduced order filter is optimally selected, the other parameters having been selected a priori, either by means of a single stage optimization procedure as in the example problem, or by some other method. The motivation

behind this approach was that a linear TPBVP resulted, which could be solved with a predictable amount of calculation. The reduced order filters described in this paper are suited to those situations where it is important to reduce the number of on line calculations. The filter gains would ordinarily be precomputed and stored for use in filtering the data.

REFERENCES

1. R. B. Asher and R. M. Reeves, "Performance Evaluation of Suboptimal Filters," IEEE Transactions on Aerospace and Electronic Systems, May 1975, pp. 400-405.

2. R. B. Asher, K. D. Herring and J. C. Ryles, "Bias, Variance, and Estimation Error in Reduced Order Filters," Automatica, November 1976.

3. T. R. Damiani, "A General Partial State Estimator," Proceedings of the 14th Midwest Symp. Circuit Theory, Denver, Colorado, May 1971.

4. C. E. Hutchinson and J. A. D'Appolito, "Minimum Variance Reduced State Filters," Proceedings of the IEEE Conference Decision and Control, New Orleans, December 1972.

5. C. S. Sims, "An Algorithm for Estimating a Portion of a State Vector," IEEE Trans. Automatic Control, August 1974, pp. 391-393.

6. C. T. Leondes and L. M. Novak, "Optimal Minimal-Order Observers for Discrete-Time Systems - A Unified Theory," Automatica, Vol. 8, pp. 379-387.

7. M. Aoki and J. R. Huddle, "Estimation of the State Vector of a Linear Stochastic System with a Constrained Estimator," IEEE Trans. Automatic Control, Vol. AC-12, August 1967, pp. 432-433.

8. C. S. Sims and R. B. Asher," Optimal and Suboptimal Results in Full and Reduced Order Linear Filtering," Proceedings of 1977 JACC, San Francisco, California.

9. M. Athans," The Matrix Minimum Principle," Information and Control, Vol II, pp. 592-606.

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DISCRETE REDUCED ORDER FILTERING WITH STATE DEPENDENT NOISE*

Craig S. Sims
School of Electrical Engineering
Oklahoma State University
Stillwater, Oklahoma 74074

ABSTRACT

The linear reduced order filtering problem is formulated as a matrix two-point boundary-value problem for systems with state dependent noise in both the dynamic and observation models. Cases are presented in which the two-point boundary-value problem simplifies.

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INTRODUCTION

Many systems are more realistically modeled as having multiplicative noise instead of additive noise. One example occurs in the momentum exchange method for regulating the angular procession of a rotating space craft [1]. There is a disturbance which depends on the procession rates. Another example occurs in the design of phase lock loops. The phase instability of an oscillator described in rectangular coordinates appears as white, state dependent noise [2]. Multiplicative noise also appears naturally in system identification problems [3]. Because such problems are of considerable importance, there has been much research dealing with the topic of filtering and control for systems with state dependent noise [1-5].

If the order of the system is large, the design of a corresponding filter of large dimension is often problematic from the viewpoint of on-line computation. Consequently there have been many papers written in the area of reduced order filter design [5-9]. In situations where one is only interested in estimating a lower order linear transformation of the state vector, it is reasonable to attempt to do this with a filter of reduced order. The parameters of the filter may then be selected using fixed configuration optimization methods [8-13]. The structure of such a filter may then be suboptimal, but the parameters are chosen optimally, subject to the structural constraints.

In this paper we consider a discrete version of the work presented in [5], where a continuous time problem was formulated as a two-point boundary-value problem (TPBVP). The discrete problem is quite different from the continuous time problem, since some of the filter variables to be optimized

appear quadratically in the discrete problem and only linearly in the continuous time problem. The problem considered here is similar to that considered in [13], however in [13] only a single stage optimization was performed, and an observer structure was required. Here the matrix minimum principle [14] is used and optimization is performed over an interval. The observer structure is not required a-priori, as we allow a driving term in the filter to remove any a-priori bias. Cases are presented where the TPBVP simplifies considerably. In one case only a single-point boundary-value problem must be solved. In another case, a linear TPBVP is obtained which can be solved either by direct use of linear systems theory or by a Riccati equation technique.

PROBLEM STATEMENT

A stochastic process is considered which is modeled by a discrete equation with state dependent noise

$$x(j+1) = A(j)\,x(j) + w(j) + \sum_{i=1}^{n} \bar{x}_i(j)\,\Gamma_i(j)\,n(j) \tag{1}$$

where $x(j)$ is the state vector at stage j, and $w(j)$ and $n(j)$ are zero mean white noise vectors. The term $\bar{x}_i(j)$ is defined as

$$\bar{x}_i(j) \triangleq x_i(j) - \mu_i(j) = x(j) - E\{x_i(j)\} \tag{2}$$

The observation model is of the form

$$y(j+1) = C(j+1)\,x(j+1) + v(j+1) + \sum_{i=1}^{n} \bar{x}_i(j)\,M_i(j)\,n(j) \tag{3}$$

where $v(j+1)$ is zero mean additive white measurement noise. We note that measurements of the form

$$y(j+1) = C(j+1)\,x(j+1) + v(j+1) + \sum_{1=1}^{n} \bar{x}_i(j+1)\,M_i^*(j+1)\,\varepsilon(j+1) \tag{4}$$

where ε is zero mean white noise can be put in the required form (3), if η and M_i are properly defined, and if it is also true that when $\bar{x}_i(j+1)$ multiples a non-zero term in the summation of (4), then its dynamical representation (1) contains no noisy term. The covariance matrices for $w(j)$, $v(j)$, and $\eta(j)$ are

$$E\{w(j)\,w^T(k)\} = Q(j)\,\delta_{kj}$$

$$E\{v(j)\,v^T(k)\} = R(j)\,\delta_{jk} \tag{5}$$

$$E\{\eta(j)\,\eta^T(k)\} = \Xi(j)\,\delta_{jk}$$

These terms are not correlated with each other or with the initial condition x_o. The known statistics of the initial condition are

$$E\{x(o)\} = \mu_o \tag{6}$$

$$Var\{x(o)\} = P_o$$

In the problem considered, a lower order linear transformation of the state vector is considered

$$z(j) = N(j)\,x(j) \tag{7}$$

The dimensions of $z(j)$, $x(j)$, and $y(j)$ are ℓ, n, and m respectively, where $\ell \leq n$.

Since $z(j)$ is of lower dimension than $x(j)$, we consider estimating it with a reduced order filter of the form

$$\hat{z}(j+1) = F(j)\,\hat{z}(j) + K(j)\,y(j+1) + g(j) \tag{8}$$

One objective of the problem is to select $g(j)$ and $\hat{z}(o)$ to satisfy a requirement that the estimate be unbiased, i.e.

$$E\{e(j)\} = E\{z(j) - \hat{z}(j)\} = o; \forall j \geq o \tag{9}$$

We note that the expectations are prior expectations and we are not requiring that the estimate be conditionally unbiased, which is a much stronger requirement. The matrices $F(j)$ and $K(j)$ are then selected to minimize a quadratic performance measure

$$J = E\{\sum_{j=o}^{N^*-1} e^T(j)\, U(j)\, e(j) + e^T(N^*)\, S\, e(N^*)\} \tag{10}$$

The solution to the unbiased requirement is obtained by considering the error difference equation. The solution to the optimization problem is solved using the matrix minimum principle.

THE UNBIASED REQUIREMENT

The error difference equation can be shown to be

$$\begin{aligned} e(j+1) &= G(j)\, x(j) + F(j)\, e(j) + D(j)\, w(j) - K(j)\, v(j+1) - g(j) \\ &+ D(j) \sum_{i=1}^{n} \bar{x}_i(j)\, \Gamma_i(j)\, \eta(j) - K(j) \sum_{i=1}^{n} \bar{x}_i(j)\, M_i(j)\, \eta(j) \end{aligned} \tag{11}$$

where we have defined the matrices

$$G(j) = [N(j+1) - K(j)\, C(j+1)]\, A(j) - F(j)\, N(j) \tag{12}$$

and

$$D(j) = N(j+1) - K(j)\, C(j+1) \tag{13}$$

From (11) it is clear that if $g(j)$ is selected as

$$g(j) = G(j)\, \mu(j) \tag{14}$$

and if

$$\hat{z}(o) = N(o)\ \mu_o \quad , \tag{15}$$

then

$$E\{e(j)\} = o \ \forall\ j \geq o \tag{16}$$

With $g(j)$ selected according to (14), the error difference equation may be written as

$$e(j+1) = G(j)\ \bar{x}(j) + F(j)\ e(j) + D(j)\ w(j) - K(j)\ v(j+1) + \sum_{i=1}^{n} \bar{x}_i(j)\ B_i(j)\ \eta(j) \tag{17}$$

where

$$B_i(j) \triangleq D(j)\ \Gamma_i(j) - K(j)\ M_i(j) \tag{18}$$

and

$$\bar{x}(j) = x(j) - \mu(j) = x(j) - E\{x(j)\} \tag{19}$$

to be consistent with (2). It is clear that $\bar{x}(j)$ is a zero mean process satisfying

$$\bar{x}(j+1) = A(j)\ \bar{x}(j) + w(j) + \sum_{i=1}^{n} \bar{x}_i(j)\ \Gamma_i(j)\ \eta(j) \tag{20}$$

In order to solve the remaining problem, it is necessary to select $K(j)$ and $F(j)$ to minimize J as indicated by (10) subject to the constraints imposed by (17) and (20).

THE OPTIMIZATION PROBLEM

The performance measure (10) may be written in terms of the error variance matrix as

$$J = \mathrm{tr}\{\sum_{j=o}^{N^*-1} U(j)\, P_{ee}(j) + S\, P_{ee}(N^*)\} \tag{21}$$

where

$$P_{ee}(j) \triangleq E\{e(j)\, e^T(j)\} \tag{22}$$

It is possible to set up the optimization problem completely within a deterministic framework if one can find a set of equations describing the propagation of P_{ee}. We first define the matrices

$$P_{xe}(j) = P_{ex}^T(j) \triangleq E\{\bar{x}(j)\, e^T(j)\} \tag{23}$$

and

$$P_{xx}(j) \triangleq E\{\bar{x}(j)\, \bar{x}^T(j)\} \tag{24}$$

Then from Equations (17) and (20) it can be seen that the above matrices satisfy the equations

$$\begin{aligned} P_{ee}(j+1) = {} & F(j)\, P_{ee}(j)\, F^T(j) + G(j)\, P_{xx}(j)\, G^T(j) + G(j)\, P_{xe}(j)\, F^T(j) \\ & + F(j)\, P_{ex}(j)\, G^T(j) + D(j)\, Q(j)\, D^T(j) + K(j)\, R(j+1)\, K^T(j) \\ & + D(j)\, \Psi(j)\, D^T(j) + K(j)\, \Psi_1(j)\, K^T(j) - K(j)\, \Psi_2(j)\, D^T(j) \\ & - D(j)\, \Psi_2^T(j)\, K^T(j) \end{aligned} \tag{25}$$

$$P_{xe}(j+1) = A(j) P_{xx}(j) G^T(j) + A(j) P_{xe}(j) F^T(j) + Q(j) D^T(j) + \Psi(j) D^T(j) - \Psi_2^T(j) K^T(j) \tag{26}$$

and

$$P_{xx}(j+1) = A(j) P_{xx}(j) A^T(j) + Q(j) + \Psi(j) \tag{27}$$

where we have used the definitions

$$\Psi(j) \triangleq \sum_{i,k=1}^{n} \Gamma_i(j) \Xi(j) \Gamma_k^T(j) P_{xx_{ik}}(j) \tag{28}$$

$$\Psi_1(j) \triangleq \sum_{i,k=1}^{n} M_i(j) \Xi(j) M_k^T(j) P_{xx_{ik}}(j) \tag{29}$$

and

$$\Psi_2(j) \triangleq \sum_{i,k=1}^{n} M_i(j) \Xi(j) \Gamma_k^T(j) P_{xx_{ik}}(j) \tag{30}$$

The initial conditions for (25), (26), and (27) are

$$P_{xx}(o) = P_o$$
$$P_{xe}(o) = P_o N_o^T \; ; \qquad P_{ee}(o) = N_o P_o N_o^T \tag{31}$$

The problem may be stated using only deterministic equations. It is desired to choose $K(j)$ and $F(j)$ to minimize J as indicated by (21) subject to the constraints imposed by (25), (26), and (27). Actually $P_{xx}(j)$ may be pre-computed and considered a known sequence of matrices. The remaining part of the problem is solved using the matrix minimum principle. The Hamiltonian is of the form

$$H_j = \text{tr}\,\{U(j)\,P_{ee}(j) + P_{ee}(j+1)\,\Lambda_{ee}^T(j+1) + P_{ex}(j+1)\,\Lambda_{ex}^T(j+1) + P_{xe}(j+1)\,\Lambda_{xe}^T(j+1)\} \tag{32}$$

where Λ_{ee} and Λ_{xe} are Lagrange multiplier matrices, and Λ_{ex} is the transpose of Λ_{xe}. The costate equations can be found from (32), after substituting from (25), (26), and (27).

$$\Lambda_{ee}(j) = \frac{\partial H_j}{\partial P_{ee}(j)} = U(j) + F^T(j)\,\Lambda_{ee}(j+1)\,F(j) \tag{33}$$

$$\Lambda_{xe}(j) = \frac{\partial H_j}{\partial P_{xe}(j)} = G^T(j)\,\Lambda_{ee}(j+1)\,F(j) + A^T(j)\,\Lambda_{xe}(j+1)\,F(j) \tag{34}$$

The terminal conditions for (33) and (34) are

$$\Lambda_{ee}(N^*) = S$$
$$\Lambda_{xe}(N^*) = 0 \tag{35}$$

A necessary condition for optimality of the filter gain can be found by setting the gradient of the Hamiltonian with respect to K equal to zero. The resulting expression is

$$\Lambda_{ee}(j+1)\{N(j+1)\,P_{xx}(j+1)\,C^T(j+1) + F(j)[P_{ex}(j) - N(j)\,P_{xx}(j)]\cdot A^T(j)\,C^T(j+1) + N(j+1)\,\Psi_2^T(j)\} + \Lambda_{ex}(j+1)\{P_{xx}(j+1)\,C^T(j+1) + \Psi_2^T(j)\} = \Lambda_{ee}(j+1)\,K(j)\{C(j+1)\,P_{xx}(j+1)\,C^T(j+1) + R(j+1) + \Psi_1(j) + \Psi_2(j)\,C^T(j+1) + C(j+1)\,\Psi_2^T(j)\} \tag{36}$$

Similarly, differentiating H_j with respect to $F(j)$ and setting the result equal to zero gives the equation

$$\Lambda_{ee}(j+1)\,F(j)\,[P_{ee}(j) - P_{ex}(j)\,N^T(j) - N(j)\,P_{xe}(j) + N(j)\,P_{xx}(j)$$

$$\cdot N^T(j)\,] = [\Lambda_{ex}(j+1) + \Lambda_{ee}(j+1)\,(N(j+1) - K(j)\,C(j+1))\;\; A(j)$$

$$\cdot [P_{xx}(j)\,N^T(j) - P_{xe}(j)\,] \qquad (37)$$

We have presented the TPBVP giving the necessary equations for an optimium choice of K and F, the reduced order filter design matrices. In general the TPBVP will be rather difficult to solve. There are two cases we shall consider in which the problem simplifies considerably and may be solved with a reasonable amount of effort. In the first case the TPBVP simplifies into a single point boundary value problem. In the second case, the problem is really modified so that F is selected prior to optimization and only K is optimally selected. The resulting TPBVP is therefore linear and can be solved by well known procedures.

SIMPLIFCATION

In the first case we assume that it is possible to choose $F(j)$ in such a way that

$$G(j) = N(j+1)\,A(j) - K(j)\,C(j+1)\,A(j) - F(j)\,N(j) = o \qquad (38)$$

Then from (34) and (35) we can see that $\Lambda_{xe}(j)$ is zero. If it is also true that

$$\Omega(j) \triangleq P_{ee}(j) - P_{ex}(j)\,N^T(j) \qquad (39)$$

is zero, then (37) simplifies to

$$\Lambda_{ee}(j+1)\,F(j)\,N(j)\,[P_{xx}(j)\,N^T(j) - P_{xe}(j)] = \Lambda_{ee}(j+1) \cdot [N(j+1) - K(j)\,C(j+1)]\,A(j)\,[P_{xx}(j)\,N^T(j) - P_{xe}(j)] \tag{40}$$

Thus (38) is sufficient to insure that (37) is satisfied under the indicated conditions. We must investigate the requirement that $\Omega(j)$ is zero, and see what the implications are. Using (25) and (26), a difference equation for $\Omega(j+1)$ may be derived, i.e.

$$\begin{aligned}\Omega(j+1) = {} & F(j)\,\Omega(j)\,F^T(j) + [K(j)\,C(j+1)\{Q(j) + \Psi(j)\}\,C^T(j+1) \\ & + K(j)\{R(j+1) + \Psi_1(j)\} - N(j+1)\{Q(j) + \Psi(j)\}C^T(j+1) \\ & - F(j)\,P_{ex}(j)\,A^T(j)\,C^T(j+1) - N(j+1)\,\Psi_2^T(j) \\ & + K(j)\{\Psi_2(j)\,C^T(j+1) + C(j+1)\,\Psi_2^T(j)\}]K^T(j)\end{aligned} \tag{41}$$

We will show that the bracketed term that multiplies $K^T(j)$ in (41) is zero when K is selected optimally.

If $G(j)$ is zero and $\Lambda_{ee}(j+1)$ is nonsingular, then Equation (36) becomes

$$\begin{aligned}& K(j)[C(j+1)\,P_{xx}(j+1)\,C^T(j+1) + R(j+1) + \Psi_1(j) + \Psi_2(j)\,C^T(j+1) \\ & + C(j+1)\,\Psi_2^T(j)] = [N(j+1)\,P_{xx}(j+1) + F(j)\,P_{ex}(j)\,A^T(j)]\,C^T(j+1) \\ & + N(j+1)\,\Psi_2^T(j) - [N(j+1) - K(j)\,C(j+1)]\,A(j)\,P_{xx}(j)\,A^T(j) \\ & C^T(j+1)\end{aligned} \tag{42}$$

where we have substituted for $F(j)\,N(j)$ from (38) in the last equation. If we further substitute from (27), replacing $P_{xx}(j+1)$, we obtain

$$K(j)\,[C(j+1)\,\{Q(j)+\Psi(j)\}\,C^T(j+1)+R(j+1)+\Psi_1(j)+\Psi_2(j)\,C^T(j+1)$$
$$+\,C(j+1)\,\Psi_2^T(j)] = F(j)\,P_{ex}(j)\,A^T(j)\,C^T(j+1)+N(j+1)[\{\Psi(j) \tag{43}$$
$$+\,Q(j)\}\,C^T(j+1)+\Psi_2^T(j)]$$

Equation (43) is sufficient to show that (41) may be written as

$$\Omega(j+1) = F(j)\,\Omega(j)\,F^T(j) \tag{44}$$

whenever $K(j)$ is selected optimally. Since $\Omega(o)$ is zero, (44) implies that $\Omega(j) = o\ \forall\ j \geq o$. We have therefore developed alternative equations for selecting $K(j)$ and $F(j)$, i.e. (38) and (43). Furthermore these equations do not involve the Lagrange multipliers so that we no longer have a TPBVP. If we define

$$\gamma_1(j) \triangleq C(j+1)\,\{Q(j)+\Psi(j)\}\,C^T(j+1)+R(j+1)+\Psi_1(j)+\Psi_2(j)\,C^T(j+1)$$
$$+\,C(j+1)\,\Psi_2^T(j) \tag{45}$$

and

$$\gamma_2(j) \triangleq [\Psi_2(j)+C(j+1)\,\{\Psi(j)+Q(j)\}]\,N^T(j+1) \tag{46}$$

then (38) and (43) can be written as

$$\left[\begin{array}{c|c} A^T(j)\,C^T(j+1) & N^T(j) \\ \hline \gamma_1(j) & -C(j+1)\,A(j)\,P_{xe}(j) \end{array}\right]\left[\begin{array}{c} K^T(j) \\ \hline F^T(j) \end{array}\right]=\left[\begin{array}{c} A^T(j)\,N^T(j+1) \\ \hline \gamma_2(j) \end{array}\right] \tag{47}$$

or equivalently

$$L(j)\left[\begin{array}{c} K^T(j) \\ \hline F^T(j) \end{array}\right] = \nu(j) \tag{48}$$

where $L(j)$ and $\nu(j)$ are defined as the corresponding matrices in (48). A necessary and sufficient condition for (48) to have a solution is that

$$L(j)\, L^{\dagger}(j)\, \nu(j) = \nu(j) \tag{49}$$

where $L^{\dagger}(j)$ is the pseudo inverse of $L^{\dagger}(j)$. If a solution exists it is of the form

$$[K(j)\,|\,F(j)] = [L^{\dagger}(j)\, \nu(j)]^{T} + \Gamma^{*}\, [I - L^{\dagger}(j)\, L(j)]^{T} \tag{50}$$

where Γ^{*} is an arbitrary matrix.

Since it is clearly possible that we might not be able to solve (48), and more generally, we might not be able to solve the TPBVP specified in the previous section, it seems appropriate to investigate a suboptimal approach. We therefore investigate a partial solution where $F(j)$ is selected a priori, and only $K(j)$ is optimized.

SOLUTIONS WITH SPECIFIC F(j)

Selecting $F(j)$ a priori has an important consequence, i.e. the TPBVP to be solved in optimizing $K(j)$ is linear. There are a number of approaches available for solving linear TPBVP's, so that one can solve for an optimal filter gain with a reasonable amount of off line calculation. Performance will of course be suboptimal because $F(j)$ is not optimized, however the loss in performance can be kept small by selecting $F(j)$ in some way which relates to performance but does not require solving a TPBVP. A procedure similar to that indicated in [8] can be used to select $F(j)$ where only single stage optimization was considered.

From Equation (33) it is clear that with $F(j)$ selected a priori, Λ_{ee} may be regarded as a known quantity, just as P_{xx} is known. With Λ_{ee} known and nonsingular, one may solve for $K(j)$ using Equation (36). The resulting expression is

$$K(j) = [\Lambda_{ee}^{-1}(j+1)\ \Lambda_{ex}(j+1)\ \{P_{xx}(j+1)\ C^T(j+1) + \Psi_2^T(j)\} + N(j+1)\ \Psi_2^T(j)$$

$$+ \{N(j+1)\ P_{xx}(j+1) + F(j)\ [P_{ex}(j) - N(j)\ P_{xx}(j)]\ A^T(j)\}\ C^T(j+1)]$$

$$\cdot M^*(j+1) \tag{51}$$

where

$$M^*(j+1) = [R(j+1) + \Psi_1(j) + \Psi_2(j)\ C^T(j+1) + C(j+1)\ \Psi_2^T(j)$$

$$+ C(j+1)\ P_{xx}(j+1)\ C^T(j+1)]^{-1} \tag{52}$$

If we substitute from (51) in Equations (26) and (24), we obtain the equations

$$P_{xe}(j+1) = [I - \{P_{xx}(j+1)\ C^T(j+1) + \Psi_2^T(j)\}\ M^*(j+1)\ C(j+1)]$$

$$\cdot A(j)\ P_{xe}(j)\ F^T(j) - [P_{xx}(j+1)\ C^T(j+1) + \Psi_2^T(j)]\ M^*(j+1)\ [\Psi_2(j)$$

$$+ C(j+1)\ P_{xx}(j+1)]\ \Lambda_{xe}(j+1)\ \Lambda_{ee}^{-1}(j+1) + \rho_1(j) \tag{53}$$

and

$$\Lambda_{xe}(j) = A^T(j)\ [I - C^T(j+1)\ M^*(j+1)\ \{\Psi_2(j) + C(j+1)\ P_{xx}(j+1)\}]$$

$$\cdot \Lambda_{xe}(j+1)\ F(j) - A^T(j)\ C^T(j+1)\ M^*(j+1)\ C(j+1)\ A(j)\ P_{xe}(j)\ F^T(j)$$

$$\cdot \Lambda_{ee}(j+1)\ F(j) + \rho_2(j) \tag{54}$$

where $\rho_1(j)$ and $\rho_2(j)$ are defined as

$$\rho_1(j) \triangleq P_{xx}(j+1)\ N^T(j+1) - A(j)\ P_{xx}(j)\ N^T(j)\ F^T(j) - [P_{xx}(j+1)$$

$$\cdot C^T(j+1) + \Psi_2^T(j)]\ M^*(j+1)\ [\{C(j+1)\ P_{xx}(j+1) + \Psi_2(j)\}$$

$$\cdot N^T(j+1) - C(j+1)\ A(j)\ P_{xx}(j)\ N^T(j)\ F^T(j)] \tag{55}$$

and

$$\rho_2(j) = [N(j+1)\ A(j) - F(j)\ N(j)]^T \Lambda_{ee}(j+1)\ F(j) - A^T(j)$$
$$\cdot C^T(j+1)\ M^*(j+1)\{[C(j+1)\ P_{xx}(j+1) + \Psi_2(j)]\ N^T(j+1) - C(j+1)$$
$$\cdot A(j)\ P_{xx}(j)\ N^T(j)\ F^T(j)]\ \Lambda_{ee}(j+1)\ F(j) \tag{56}$$

The boundary conditions for (53) and (54) are given by (31) and (35) respectively. Since (53) and (54) are linear in P_{xe} and Λ_{xe}, there are many ways to solve the TPBVP. One approach is to assume a linear relationship between the elements of P_{xe} and Λ_{xe}. We may put the distinct elements of Λ_{xe} in a long vector Λ^*_{xe}, and similarly form P^*_{xe} from the distinct elements of P_{xe}. If we then assume the relationship

$$\Lambda^*_{xe}(j) = V^*(j)\ P^*_{xe}(j) + \beta^*(j) \tag{57}$$

then recursive equations may be obtained for $V^*(j)$ and $\beta^*(j)$ and the problem is transformed into a single point boundary value problem. We will illustrate the procedure with a specific example where it is not necessary to put the distinct elements of P_{xe} and Λ_{xe} in a vector form.

It is assumed that the matrices of (23) are selected to be of a scalar form, i.e.

$$F(j) = f(j)\ I$$
$$\Lambda_{ee}(j) = \lambda_{ee}(j)\ I \tag{58}$$

where f and λ_{ee} are scalars; the terminal matrix s must be of the same form. We may then reposition Λ_{ee} and F in (53) and (54) and obtain equations of the form

$$P_{xe}(j+1) = \alpha_{11}(j)\ P_{xe}(j) + \alpha_{12}(j)\ \Lambda_{xe}(j+1) + \rho_1(j) \tag{59}$$

$$\Lambda_{xe}(j) = \alpha_{21}(j)\, P_{xe}(j) + \alpha_{22}(j)\, \Lambda_{xe}(j+1) + \rho_2(j) \tag{60}$$

where

$$\alpha_{11}(j) \triangleq f(j)\,[I - \{P_{xx}(j+1)\, C^T(j+1) + \Psi_2^T(j)\} M^*(j+1)\, C(j+1)]\; A(j)$$

$$\alpha_{12}(j) \triangleq -\lambda_{ee}^{-1}(j+1)\; [P_{xx}(j+1)\, C^T(j+1) + \Psi_2^T(j)]\, M^*(j+1)\, [\Psi_2(j) + C(j+1)\; P_{xx}(j+1)\,] \tag{61}$$

$$\alpha_{21}(j) \triangleq -f^2(j)\, \lambda_{ee}(j+1)\, A^T(j)\, C^T(j+1)\, M^*(j+1)\, C(j+1)\, A(j)$$

$$\alpha_{22}(j) \triangleq f(j)\, A^T(j)\; [I - C^T(j+1)\, M^*(j+1)\, \{\Psi_2(j) + C(j+1)\, P_{xx}(j+1)\}]$$

A linear relationship between P_{xe} and Λ_{xe} is assumed, i.e.

$$\Lambda_{xe}(j) = V(j)\, P_{xe}(j) + \beta(j) \tag{62}$$

Substituting (62) in (59) and (60) gives

$$P_{xe}(j+1) = \alpha_{11}(j)\, P_{xe}(j) + \alpha_{12}(j)\, V(j+1)\, P_{xe}(j+1) + \alpha_{12}(j)\, \beta(j+1) + \rho_1(j) \tag{63}$$

$$V(j)\, P_{xe}(j) + \beta(j) = \alpha_{21}(j)\, P_{xe}(j) + \alpha_{22}(j)\, V(j+1)\, P_{xe}(j+1) + \alpha_{22}(j)\; \beta(j+1) + \rho_2(j) \tag{64}$$

Solving for $P_{xe}(j+1)$ in (63), and substituting that result in (64) gives

$$V(j)\, P_{xe}(j) + \beta(j) = \alpha_{21}(j)\, P_{xe}(j) + \alpha_{22}(j)\, \beta(j+1) + \rho_2(j) + \alpha_{22}(j)\, V(j+1)\, [I - \alpha_{12}(j)\, V(j+1)]^{-1}\; [\alpha_{11}(j)\, P_{xe}(j) + \alpha_{12}(j) \cdot \beta(j+1) + \rho_1(j)\,] \tag{65}$$

If the above is to hold for arbitrary P_{xe}, then we must have

$$V(j) = \alpha_{21}(j) + \alpha_{22}(j)\, V(j+1)\, [I - \alpha_{12}(j)\, V(j+1)]^{-1}\, \alpha_{11}(j) \quad (66)$$

and

$$\beta(j) = \alpha_{22}(j)\, [I + V(j+1)\ (I - \alpha_{12}(j)\, V(j+1))^{-1}\, \alpha_{12}(j)]\ \beta(j+1) + \rho_2(j) + \alpha_{22}(j)\, V(j+1)\, (I - \alpha_{12}(j)\, V(j+1))^{-1} \rho_1(j) \quad (67)$$

The terminal conditions for the above are

$$V(N^*) = 0;\ \beta(N^*) = 0 \quad (68)$$

We may thus compute V and β backward in time so that Λ_{xe} is known in terms of P_{xe}. The gain may then be calculated forward in time using Equation (51) and solving for P_{xe} from (59).

In this section we consider a simplified model of a discretized inertial system as considered in [8]. The dynamics are modeled as

$$x(j+1) = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} x(j) + w(j) \quad (69)$$

where Δ is .2 and the observation model is

$$y(j+1) = [\,1 + \varepsilon(j+1),\ o\,]\ x(j+1) + v(j+1) \quad (70)$$

where $\varepsilon(j)$ represents a sequence of independent random variables such that

$$\begin{aligned} &Pr\ [\ \varepsilon(j) = 1\ ] = \tfrac{1}{2} \\ &Pr\ [\ \varepsilon(j) = -1\ ] = \tfrac{1}{2} \end{aligned} \quad (71)$$

and $\frac{1}{2}$ then is the probabilty that there will be no signal measured at a given stage. If we define $\eta(j)$ as $\varepsilon(j+1)$ then (70) may be written as

$$y(j+1) = [1,\ o]\ x(j+1) + v(j+1) + [x_1(j) + \Delta x_2(j)]\ n(j) \tag{72}$$

where it has been assumed that $w_1(j)$ is zero. Equation (72) is of the same form as (3). The initial conditions for the example have mean value zero and the initial variance matrix is the identity matrix. The variance of the measurement noise is R = .1, and the variance of the plant noise parameter, $w_2(j)$ is 1. It is desired to estimate x_1 using a first order filter of the form

$$\hat{z}(j+1) = F(j)\ \hat{z}(j) + K(j)\ y\ (j+1) \tag{73}$$

There is no driving term, g(j), since $\mu(j)$ is zero. The parameter F(j) is not selected optimally, but selected using a one stage optimization procedure as indicated in [8]. The parameter K(j) is then selected by solving a linear TPBVP as suggested in this paper. Optimization is with respect to the performance measure

$$J = E\{\sum_{j=o}^{N^*-1} e^2(j) + e^2(N^*)\} \tag{74}$$

In Figure 1, the performance of the reduced order filter, designed to account for the probability that there will be no signal present at a given stage, is compared with that of a second order Kalman filter designed using the assumption that the parameter multiplying $x_1(j+1)$ in (70) is at its mean value of unity.

SUMMARY AND CONCLUSIONS

In this paper we have investigated the discrete reduced order filtering problem for systems with state dependent noise. The general solution is shown to be a nonlinear matrix TPBVP, however the result becomes considerably less

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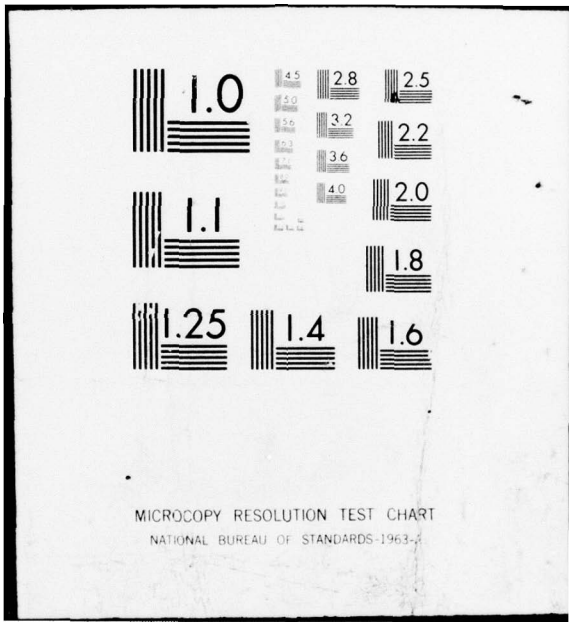
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complicated under certain circumstances, resulting in an initial value problem. Simplification also results when some of the filter parameters are selected optimally and the others are chosen a priori. In this case a linear TPBVP results, which can be solved by standard procedures. Because systems with state dependent noise occur frequently, we feel that it is important to consider the reduced order filtering problem for such systems. As indicated by the example problem, such models can represent intermittent observation data as well as a number of other situations. The reduced order filters described in this paper are suited to those situations where it is important to reduce the number of on line calculations, and the gains may be precomputed and stored for use in the filtering process.

REFERENCES

[1] McLane, P. J., "Optimal Stochastic Control of Linear Systems with State and Control Dependent Disturbances," IEEE Trans. Aut. Control, Vol. AC-16, No. 6, pp. 793-798.

[2] Gustafson, E. G., "On Optimal Estimation and Control of Linear Systems with State Dependent Noise," Ph.D. Thesis, MIT, June, 1972.

[3] Kotob, S., and H. Kaufman, "Analysis and Application of Minimum Variance Discrete Linear System Identification," IEEE Trans. Aut. Control, Vol. AC-22, No. 5, October 1977, pp. 807-815.

[4] McLane, P. J., "Optimal Linear Filtering for Linear Systems with State Dependent Noise," International Journal of Control, Vol. 10, No. 1, PP. 41-51, 1969.

[5] Asher, R. B., and C. S. Sims, "Reduced Order Filtering with State Dependent Noise," submitted to 1978 JACC, Philadelphia, Pa., October 18-20, 1978.

[6] Hutchinson, C. E., and J. A. D'Appotito, "Minimum Variance Reduced State Filter," presented at 1972 IEEE Conference on Decision and Control, New Orleans, Dec. 1972.

[7] Damiani, T. R., "A General Partial State Estimator," presented at 14th Midwest Sump. Circuit Theory, Denver, Colorado, May 1971.

[8] Sims, C. S., "An Algorithm for Estimating a Portion of a State Vector," IEEE Trans. Aut. Control, Vol. AC-19, No. 4, August, 1974.

[9] Sims, C. S., and R. B. Asher, "Optimal and Suboptimal Linear Filtering," Proceeding of 1977 JACC, San Francisco, California.

[10] Sims, C. S., and J. L. Melsa, "A Fixed Configuration Approach to the Stochastic Linear Regulator Problem," Proceedings 1970 Joint Automatic Control Conference, June 24-26, 1970, Atlanta, Georgia.

[11] Sims, C. S. "Fixed Configuration Control of Systems with State and Control Dependent Noise," Proceedings 1971 Joint Automatic Control Conference, August 11-13, St. Louis, Missouri.

[12] Lee, J. S., "On the Relation of Singular Control to Linear Estimation," Proceedings 1970 Joint Automatic Control Conference, June 24-26, 1970, Atlanta, Georgia.

[13] Sims, C. S., "Observer-Estimators for Discrete Random Parameter Systems," Proceedings of the Eleventh Annual Allerton Conference on Circuit and Systems Theory, Monticello, Illinois, October 3-5, 1973.

[14] Athans, M., "The Matrix Minimum Principle," Information and Control, Vol. II, pp. 592-606, 1968.

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OPTIMAL TRACKING OF A MARKOV JUMP PROCESS*

Craig S. Sims
School of Electrical Engineering
Oklahoma State University
Stillwater, Oklahoma 74074

Abstract

A feedback control is obtained for a linear random tracking problem with a quadratic performance measure. Both inputs to the system and the reference vector, which is to be tracked, are capable of instantaneous change.

INTRODUCTION

Tracking problems are an important category of control problems where the output of a system is required to follow a reference vector. Such problems have been extensively treated in the literature [1], [2]. In this paper, we consider this class of problems within a format which allows the reference vector to change values suddenly. Thus the system may be tracking one vector, and suddenly it is required to track another vector. For added generality, we also allow for the possibility of system inputs which can change instantly.

It is assumed that the input and reference vectors may change instantly and randomly in time, so that the problem is a stochastic control problem. The elements of these vectors are modeled as Markov jump processes with a finite number of states [3]. Sworder introduced a method for solving related stochastic control problems in [4], where the parameters of a linear system were modeled as Markov jump processes. We apply similar methodology to solve the random tracking problem in this paper.

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PROBLEM STATEMENT

Consider a system described by the linear differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) + \gamma(t) \tag{1}$$

where $\gamma(t)$ is a random input vector. The output equation is also linear

$$y(t) = Cx(t) \tag{2}$$

The control vector, $u(t)$, is to be selected to minimize a quadratic performance measure

$$J = \frac{1}{2} E\{\int_t^T \Big[[y(\tau) - \eta(\tau)]^T Q[y(\tau) - \eta(\tau)] + u^T(\tau)\, Ru(\tau)\Big] d\tau \,|\, x(t),\, r(t)\} \tag{3}$$

In (3), $\eta(\tau)$ is the random vector to be tracked, and $r(t)$ is an indicator function which indicates the values of $\eta(t)$ and $\gamma(t)$. The stochastic vectors, $\gamma(t)$ and $\eta(t)$, are Markov jump processes described using the transition probabilities

$$\begin{aligned} &Pr[\eta(t+\Delta), \gamma(t+\Delta) = \eta_i, \gamma_i \mid \eta(t), \gamma(t) = \eta_j, \gamma_j] \\ &\quad = q_{ji}\Delta + O(\Delta), \quad i \neq j \\ &\quad = 1 + q_{ii}\Delta + O(\Delta), \quad i = j \\ &\qquad \text{for } i,j = 1, 2, \ldots, s \end{aligned} \tag{4}$$

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Thus we are designing a control to track a signal which is capable of instantaneous change, and are also subject to instantaneous input disturbances. The controller at all times has perfect information, r(t), which indicates the current values of the signal to be tracked and the input disturbances. Using notation corresponding to that of Sworder [4], we denote the event

$$\{\eta(t),\ \gamma(t) = \eta_i,\ \gamma_i\} \longrightarrow r(t) \in [i] \tag{5}$$

Note that there are only a finite number of possible values which can be assumed by η and γ. From the definition (4) it is clear that

$$\sum_{i=1}^{s} q_{ji} = 0 \tag{6}$$

and we will make use of this fact in the development of the optimal controller.

THE OPTIMAL SOLUTION

To obtain the solution we follow steps similar to those used in [4]. The Hamiltonion for this problem is

$$H(x,u) = \frac{1}{2}(Cx-\eta)^T Q(Cx-\eta) + \frac{1}{2}u^T Ru + \lambda^T[Ax + Bu + \gamma] \tag{7}$$

where λ is a stochastic Lagrange Multiplier. It is necessary to minimize $E\{H[x(t), u(t)]|x(t), r(t)\}$ with respect to the contorl, u. This leads to the equation

$$u[x(t), r(t)] = -R^{-1}B^T E\{\lambda(t)|x(t), r(t)\} \tag{8}$$

It is convenient to assume a linear form for λ

$$\lambda(t) = P(t)\,x(t) + \alpha(t) \tag{9}$$

so that u(.) can be written as

$$u[x(t), r(t)] = -R^{-1}B^T\Big[E\{P(t)|r(t)\}\,x(t) + E\{\alpha(t)|r(t)\}\Big] \triangleq K(t)\,x(t) + \beta(t) \tag{10}$$

Substituting for u in (7) and differentiating with respect to x, we obtain the differential equation for λ,

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -(C^TQC + K^TRK)x + C^TQ\eta - K^TR\beta - (A+BK)^T\lambda \tag{11}$$

which has terminal condition,

$$\lambda(T) = 0 \tag{12}$$

Another expression for $\dot{\lambda}$ is found using (9), i.e.

$$\dot{\lambda}(t) = \dot{P}(t)\,x(t) + P(t)\,\dot{x}(t) + \dot{\alpha}(t) \tag{13}$$

or, substituting from (1),

$$\dot{\lambda} = (\dot{P} + PA + PBK)x + PB\beta + P\gamma + \dot{\alpha} \tag{14}$$

If (11) and (14) are equated, and we substitute for λ from (9), then two equations are obtained:

$$\dot{P} + PA + PBK = -C^TQC - K^TRK - (A+BK)^TP \tag{15}$$

$$C^TQ\eta - K^TR\beta - (A+BK)^T\alpha = PB\beta + P\gamma + \dot{\alpha} \tag{16}$$

These equations are required if (11) and (14) are to be equivalent for arbitrary x.

From (15) we calculate $E\{\dot{P}(t)|r(t)\in[j]\}$. The resulting expression is

$$E\{\dot{P}(t)|r(t)\in[j]\} = -P_j(t)A - A^TP_j(t) - C^TQC + P_j(t)\,BR^{-1}B^TP_j(t) \tag{17}$$

where we have defined

$$P_j(t) \triangleq E\{P(t)|r(t)\in[j]\} \tag{18}$$

and used the fact that

$$E\{K(t)|r(t)\in[j]\} = -R^{-1}B^TP_j(t) \tag{19}$$

The left hand side of (17) can be calculated a different way as

$$E\{\dot{P}(t)|r(t)\in[j]\} = \lim_{\Delta\to 0} \frac{E\{P(t+\Delta)|r(t)\in[j]\} - E\{P(t)|r(t)\in[j]\}}{\Delta} \tag{20}$$

Using (4), we then obtain

$$E\{\dot{P}(t)|r(t)\in[j]\} = \sum_{i=1}^{s} q_{ji}\,P_i(t) + \dot{P}_j(t) \tag{21}$$

Hence combining (17) and (21), there results

$$\dot{P}_j + \sum_{i=1}^{s} q_{ji}\,P_i = -P_jA - A^TP_j - C^TQC + P_jBR^{-1}B^TP_j \tag{22}$$

Noting that (22) gives the same expression for all $\dot{P}_j$, and using (6), we have

$$\dot{P} = \dot{P}_j = -PA - A^TP - C^TQC + PBR^{-1}B^TP \qquad j=1,2,\ldots,s \tag{23}$$

Equation (23) is fortunate, since it indicates that it is only be necessary to solve one Riccati equation and not s coupled Riccati equations.

Using equation (16), we calculate $E\{\dot{\alpha}(t)|r(t)\in[j]\}$, obtaining

$$E\{\dot{\alpha}(t)|r(t)\in[j]\} = C^TQ\eta_j - A^T\alpha_j(t) - P_j(t)\gamma_j + P_j(t)BR^{-1}B^T\alpha_j(t) \tag{24}$$

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where we have defined the vector

$$\alpha_j(t) \triangleq E\{\alpha(t)|r(t)\varepsilon[j]\} \tag{25}$$

and noted that

$$E\{\beta(t)|r(t)\varepsilon[j]\} = -R^{-1}B^T\alpha_j \tag{26}$$

Proceeding as in obtaining (21), we find another expression,

$$E\{\dot{\alpha}(t)|r(t)\varepsilon[j]\} = \sum_{i=1}^{s} q_{ji}\,\alpha_i(t) + \dot{\alpha}_j(t) \tag{27}$$

Combining (24) and (27), we then find

$$\dot{\alpha}_j = C^TQ\eta_j - A^T\alpha_j - P\,\gamma_j + P\,BR^{-1}B^T\alpha_j - \sum_{i=1}^{s} q_{ji}\,\alpha_i$$

$$j=1,2,\ldots,s \tag{28}$$

Hence there are s vector equations to be solved for the terms $\{\alpha_j\}$, and these are coupled through the parameters q_{ji}. From (12) it is clear that the terminal conditions for (23) and (28) are

$$\begin{aligned} P(T) &= 0 \\ \alpha_j(T) &= 0 \end{aligned} \tag{29}$$

The optimal control strategy, is thus found to be

$$u[x(t), r(t)] = -R^{-1}B^T[P(t)x(t) + \alpha_j(t)] \tag{30}$$

$$\text{when } r(t)\varepsilon[j]$$

where $P(t)$ and $\alpha_j(t)$ are found by solving (23) and (28) respectively.

EXAMPLE

We will consider a simple scalar example with an input which can change suddenly. The dynamical description is

$$\dot{x} = u + \gamma(t) \tag{31}$$

where $\gamma(t)$ has two probable states, $\gamma(t) = \gamma^*$ or $\gamma(t) = 0$. With γ^* a transient state and 0 an absorbing state, we have

$$\begin{aligned} &Pr[\gamma(t+\Delta) = \gamma^*|\gamma(t) = \gamma^*] = 1 - q\Delta \\ &Pr[\gamma(t+\Delta) = \gamma^*|\gamma(t) = 0\ ] = 0 \\ &Pr[\gamma(t+\Delta) = 0\ |\gamma(t) = \gamma^*] = q\Delta \\ &Pr[\gamma(t+\Delta) = 0\ |\gamma(t) = 0\ ] = 1 \end{aligned} \tag{32}$$

where $q > 0$. The problem is an infinite horizon problem,

$$J = E\{\int_t^\infty [x^2(\tau) + u^2(\tau)]d\tau|x(t), r(t)\}$$

Since we are looking for the steady state solution, corresponding to (22) we have the degenerate Riccati solutions,

$$P_1 = 1 = P = P_2 = \tfrac{1}{2}\left[\sqrt{q^2 + 4(q+1)} - q\right] \tag{33}$$

where we have assumed that if $r(t)\varepsilon[1]$, it means $\gamma(t) = 0$, and $r(t)\varepsilon[2]$ means $\gamma(t) = \gamma^*$. We have corresponding equations for α_1 and α_2

$$\dot{\alpha}_1 = \alpha_1 = 0 \tag{34}$$

$$\dot{\alpha}_2 = -\gamma^* + (1+q)\alpha_2 = 0 \tag{35}$$

which gives

$$\alpha_2 = \frac{\gamma^*}{1+q} \tag{36}$$

The optimal control is thus

$$\begin{aligned} &u[x(t), r(t)] = -x(t) - \gamma^*/(1+q) \text{ when } r(t)\varepsilon[2] \\ &u[x(t), r(t)] = -x(t) \text{ when } r(t)\varepsilon[1] \end{aligned} \tag{37}$$

Suppose that to begin with, $r(t)\varepsilon[2]$, then substitution of (37) in (31) gives

$$\dot{x} = -x + \gamma^*\left(\frac{q}{1+q}\right) \tag{38}$$

which describes the trajectory until $\gamma(t)$ switches to zero, at which time we begin to have a decaying exponential described by

$$\dot{x} = -x \tag{39}$$

The results appear to be reasonable, although they are not intuitive.

SUMMARY

In this paper we have applied the stochastic minimum principle to solve a random tracking problem where the values to be tracked were capable of changing instantly. In addition, input disturbances have been modeled as Markov jump processes which could suddenly change values. A feedback control was found which minimized a quadratic performance measure. The control law was found to be linear in the state vector and contained an additional additive term which changed whenever the input disturbance or reference vector changed.

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LIST OF REFERENCES

[1] Kwakernaak, H., and R. Sivan, Linear Optimal Control Systems, Wiley-Interscience, 1972.

[2] Sage, A. P., Optimum Systems Control, Prentice Hall, 1968.

[3] Cox, D. R., and H. D. Miller, The Theory of Stochastic Processes, Mithuen and Co. Ltd., 1965.

[4] Sworder, D. D., "Feedback Control of a Class of Linear Systems with Jump Parameters," IEEE Trans. Automatic Control, Vol. AC-14, No. 1, February, 1969.

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Fixed configuration filter theory provides a methodology for designing filters of reduced complexity which will provide suboptimal performance in general, and optimal performance under certain conditions. In this report we derive fixed configuration reduced order filters for continuous and discrete time systems, with and without state dependent noise.

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