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In this paper we develop time-domain state space models for lossless layered media are developed which are described by the wave equation and boundary conditions. Our models are for non-equal one-way travel time; hence, they are more general than existing models of layered media which are usually for layers of equal one-way travel times. We develop are state space models for two cases (1) source and sensor at the surface and (2) source and sensor in the first layer.

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SYNTHETIC SEISMOGRAMS USING THE
STATE SPACE APPROACH

by

J. M. Mendel, N. E. Nahi and M. Chan
Department of Electrical Engineering
University of Southern California
Los Angeles, California 90007

ABSTRACT

In this paper we develop time-domain state space models for lossless layered media which are described by the wave equation and boundary conditions. Our models are for non-equal one-way travel times; hence, they are more general than existing models of layered media which are usually for layers of equal one-way travel times. We develop state space models for two cases: (1) source and sensor at the surface, and (2) source and sensor in the first layer.

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I. INTRODUCTION

In this paper we develop time-domain state space models for lossless layered media which are described by the wave equation and boundary conditions. We are specifically interested in models for a horizontally stratified media, nonabsorptive earth with vertically travelling plane compressional waves, and shall consider the two cases depicted in Figure 1; that is to say, we shall develop state space models for K-layer media systems in which the source and sensor are located either at the surface (the usual case, considered, for example in Refs. 1-8) or in the first layer (as is the case when the first layer is water).

In Figure 1 we characterize each layer by its one way travel time, τ_i, velocity, V_i, and normal incidence reflection coefficient, r_i (i=1, 2, ..., K). Additionally, interface-0 denotes the surface and is characterized by reflection coefficient r_0. Finally, m(t) and y(t) denote the input (e. g., seismic source signature from dynamite, airgun, etc.) to and the output (e. g., ideal seismogram as measured by a geophone or hydrophone) of the K-layer media system.

An important use of a model of a K-layer media system is to generate synthetic seismograms; i. e., to generate y(t) for a given m(t). This synthetic data can then be used either for preliminary testing and evaluation of a signal processing technique (e. g., deconvolution) or for interpretation purposes (Ref. 18). These models are also useful for developing inverse procedures by which important parameters, such as reflection coefficients and/or travel times, can be extracted from measured data.

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Our time-domain state space models, as will be seen below, are quite different from the more familiar z-transform transfer function models which have appeared in the Geophysics literature (Refs. 1-8). In the Geophysics literature, the assumption of equal one-way travel times is usually made. Layers of different travel times are built up by inserting layers whose reflection coefficients are zero. Our state space models are for non-equal one-way travel times, but can also be applied to the equal travel time case.

Why are we interested in a different class of models for what appears to be a well studied system? As is well known, there is a vast literature associated with systems which are described by time-domain state space models. Most recent results in estimation and identification theories, for example, require a state space model. These time-domain techniques have proven very beneficial outside of the geophysics field, and, we feel should also be beneficial in the geophysics field. In fact, our ultimate objective is to apply those theories to the layered media problem; but, to do so of course requires state space models.

One might argue that it should be possible to go directly from the z-transform transfer function models, already developed, to equivalent state space models. In most cases this is not practical since closed-form expressions for the transfer functions (e. g., reflection transfer function $Y(z)/M(z)$) are not available. Those transfer functions must be computed from a set of equations which are solved in a recursive manner. Additionally, those transfer functions appear to be limited by the equal one-way travel time assumption (Refs. 2-4, for example) and, they appear to only have been published for the case of source and sensor at the surface (Figure 1a).

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In this paper we develop state space models for both cases depicted in Figure 1. The case of source and sensor at the surface is treated in Section II, whereas the case of source and sensor in the first layer is treated in Section III. Our state equations turn out to be continuous-time equations with multiple time delays, and, are referred to as <u>causal functional equations</u>. There does not appear to be any literature on this class of equations; hence, we also describe some of their more important properties and two methods for their computer simulation in Section II. A connection between our state space model and transfer function models is also given in that section. In Section III we distinguish between the cases when the sensor is either above or below the source and develop models for both cases.

Some of the material which we discuss in Section II was first presented in Refs. 11, 12, and 13.

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II. A STATE EQUATION MODEL: SOURCE AND SENSOR AT THE SURFACE

A. State and State Space

The starting point for our developments is the assumption that wave motion in each layer is characterized by two signals traveling in opposite directions. This assumption is a consequence of the lossless wave equation. Symbols $u_k(t)$ and $d_k(t)$ denote the upgoing and downgoing waves in the kth layer, respectively (Figure 2). We shall refer to $u_k(t)$ and $d_k(t)$ as states.

Since the notions of states and state space may be new to many Geophysics readers, we give a brief review of them next. Our discussions paraphrase those in Refs. 9 and 10.

The state of a dynamic system at time $t=t_0$ is the amount of information at t_0 that, together with the inputs defined for all values of $t \geq t_0$, determines uniquely the behavior of the system for all $t \geq t_0$.

For our layered media system, depicted in Figure 1, the states consist of a finite number of variables, $u_1(t), d_1(t)$ $u_2(t), d_2(t), \ldots, u_K(t)$, and $d_K(t)$. The state of our system can then be represented by a column vector $\underline{x}$ called the state vector, whose dimension is $2K \times 1$. The components of $\underline{x}$ are called state variables; hence, $u_1(t), d_1(t), \ldots, u_K(t)$, and $d_K(t)$ are state variables. Because our input $m(t)$ is real-valued and our state vector is finite dimensional our state space, which is defined as a $2K$-dimensional space in which $u_1(t), d_1(t), \ldots, u_K(t), d_K(t)$ are coordinates, is the familiar finite-dimensional real vector space. The state at time t of our system will be defined by $2K$ equations and can then be represented by a point in $2K$-dimensional state space.

A set of state variables can be associated with a given system in many ways. In other words there exist a number of different sets of state variables for a given system for a given input set. Which of these is most relevant depends on the individual situation, that is, the nature of the problem, the nature of the input set, etc. We have chosen upgoing and downgoing signals in each layer to be state variables since these variables are most frequently used by geophysicists and also seem to be the most natural ones to use based on wave equation theory.

By a state space model we mean the set of equations that describe the unique relations between the input, output, and state. It is comprised of a state equation and an output equation. The state equation governs the behavior of the state vector, $\underline{x}$. For continuous-time dynamical systems it is usually a differential equation, whereas for discrete-time dynamical systems it is usually a difference equation. The output equation relates the output (or, outputs) to the state vector and input. An example of a continuous-time output equation is $y(t) = \underline{h}'\underline{x}(t) + m(t)$. In this equation ( )' denotes the transpose of ( ).

B. Interface Equations

The starting point for derivation of our state space model is the Figure 3 ray diagram. As in Refs. 2 and 3, we shall find it convenient to draw ray diagrams with time displacement along the horizontal axis, so that rays appear to be at non-normal incidence and so do not overlap one another. Symbols u_k and d'_k denote the upgoing and downgoing waves in the kth layer, respectively; and, we adopt the convention that waves at the top of a layer

occur at present time t. Our initial development is in terms of u_k and d'_k.

In Paragraph C we will find it more convenient to work with u_k and d_k.

From Figures 2 and 3 we see that d_k is just a time delayed version of d'_k;

i. e., $d_k(t) = d'_k(t-\tau_k)$.

As stated by Robinson (Ref. 3), "the solution of the wave equation

at each interface leads to the definition of a <u>reflection coefficient</u> r_j

associated with that interface. ... the reflection coefficient r_j, which must

satisfy $|r_j| < 1$, has these properties. A <u>downgoing wave</u> of amplitude A in

layer j, upon striking interface j, is both reflected and transmitted. The

reflected portion is an upgoing wave of amplitude $r_j A$ in layer j, so r_j

represents the <u>reflection coefficient</u>. The transmitted portion is a down-

going wave of amplitude $(1+r_j)A$ in layer j+1, so $1+r_j$ represents the

<u>transmission coefficient</u>. An <u>upgoing wave</u> of amplitude B in layer j+1 is

both reflected and transmitted when it strikes interface j. The reflected

portion is a downgoing wave of amplitude $-r_j B$ in layer j+1, and the

transmitted portion is an upgoing wave of amplitude $(1-r_j)B$. Hence $-r_j$

and $(1-r_j)$ represent, respectively, the <u>reflection coefficient</u> and trans-

mission coefficient for the upgoing wave. These properties are summarized

in Table 1 (2). "

Table 1. Reflected and Transmitted Portions

	Reflected Portion	Transmitted Portion
Downgoing wave A in layer j	Upgoing wave $r_j A$ in layer j	Downgoing wave $(1+r_j)A$ in layer j+1
Upgoing wave B in layer j+1	Downgoing wave $-r_j B$ in layer j+1	Upgoing wave $(1-r_j)B$ in layer j

Waveform $u_k(t+\tau_k)$ (Figure 3) is made up of two parts, namely the part due to the reflected portion of $d_k'(t-\tau_k)$ and the part due to the transmitted portion of $u_{k+1}(t)$. It satisfies the equation

$$u_k(t+\tau_k) = r_k d_k'(t-\tau_k) + (1-r_k)u_{k+1}(t). \tag{1}$$

In a similar manner, waveform $d_{k+1}'(t)$ satisfies the equation

$$d_{k+1}'(t) = (1+r_k)d_k'(t-\tau_k) - r_k u_{k+1}(t). \tag{2}$$

We refer to Eqs. (1) and (2) as the _interface equations_. These equations are the starting point for transfer function models, which are very popular in the Geophysics literature (Ref. 2), and, they are also our starting point.

C. _A State Space Model_

A state space model for our K layer media system is obtained directly from Eqs. (1) and (2), which are applicable at interfaces 1 through K-1 (i.e., for k=1, 2, ..., K-1), and comparable equations at the surface and K^{th} interface. At the surface (Figure 4a), we obtain

$$y(t) = r_0 m(t) + (1-r_0)u_1(t) \tag{3}$$

$$d_1'(t) = (1+r_0)m(t) - r_0 u_1(t) \tag{4}$$

and, at the Kth interface we assume that $u_{K+1}(t) = 0$ to obtain (Figure 4b)

$$u_K(t+\tau_K) = r_K d'_K(t-\tau_K) \tag{5}$$

$$d'_{K+1}(t) = (1+r_K) d'_K(t-\tau_K) \tag{6}$$

Signal $y(t)$ in Eq. (3) is the measurable system output; hence, Eq. (3) is the output equation. Signal $d'_{K+1}(t)$ is also a system output; but, since it cannot be measured, we shall ignore it in following analyses.

It is convenient to group Eqs. (1), (2), (4), and (5) in a <u>layer ordering</u>, as follows.

$$d'_1(t) = -r_0 u_1(t) + (1+r_0) m(t)$$

$$u_1(t+\tau_1) = r_1 d'_1(t-\tau_1) + (1-r_1) u_2(t)$$

$$\left.\begin{array}{l} d'_j(t) = (1+r_{j-1}) d'_{j-1}(t-\tau_{j-1}) - r_{j-1} u_j(t) \\[2mm] u_j(t+\tau_j) = r_j d'_j(t-\tau_j) + (1-r_j) u_{j+1}(t) \end{array}\right\} \quad j=2,3,\ldots,K-1$$

$$d'_K(t) = (1+r_{K-1}) d'_{K-1}(t-\tau_{K-1}) - r_{K-1} u_K(t)$$

$$u_K(t+\tau_K) = r_K d'_K(t-\tau_K). \tag{7}$$

This system of 2K equations is not in a useful state equation format, yet, since signals in its left-hand side occur at t and delayed times, and signals on the right-hand side occur at t, $t-\tau_{j-1}$ and $t-\tau_j$. In order to put Eq. (7) into a useful state equation format, let

$$d_j(t) \triangleq d'_j(t-\tau_j) \tag{8}$$

for all $j=1,2,\ldots,K$. Observe, from Figs. 3 and 2, that the downgoing states $d_j(t)$ occur at the bottom of a layer. Equation (7) becomes

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$$d_1(t+\tau_1) = -r_0 u_1(t) + (1+r_0) m(t)$$

$$u_1(t+\tau_1) = r_1 d_1(t) + (1-r_1) u_2(t)$$

$$\left. \begin{array}{l} d_j(t+\tau_j) = (1+r_{j-1}) d_{j-1}(t) - r_{j-1} u_j(t) \\[2mm] u_j(t+\tau_j) = r_j d_j(t) + (1-r_j) u_{j+1}(t) \end{array} \right\} \quad j=2,3,\ldots,K-1$$

$$d_K(t+\tau_K) = (1+r_{K-1}) d_{K-1}(t) - r_{K-1} u_K(t)$$

$$u_K(t+\tau_K) = r_K d_K(t) \tag{9}$$

By means of transformation (8) each pair of equations in (7) now only involves two time points, $t+\tau_j$ and t. Equations (9) and (3) together represent a state space model which we refer to as the layer-ordered (L-O) model in the sequel.

In order to complete the description of our state space model we must specify <u>initial condition information</u>. The situation of interest is the one for which all states have zero values prior to application of input $m(t)$. Consider the kth layer (Figure 2). Then $d_k(t)$ is equal to zero until $t = \tau_1 + \tau_2 + \cdots + 2\tau_k$. These facts are true for all $k=1,2,\ldots,K$; i.e.,

$$d_j(t) = 0 \quad \forall\, t \in \left[0, \sum_{i=1}^{j} \tau_i \right) \tag{10a}$$

and

$$u_j(t) = 0 \quad \forall\, t \in \left[0, \sum_{i=1}^{j} \tau_i + \tau_j \right) \tag{10b}$$

where $j=1,2,\ldots,K$.

As an example consider a two layer system for which Eq. (9) becomes:

$$
\left.
\begin{aligned}
d_1(t+\tau_1) &= -r_0 u_1(t) + (1+r_0)\, m(t) \\
u_1(t+\tau_1) &= r_1 d_1(t) + (1-r_1)\, u_2(t) \\
d_2(t+\tau_2) &= (1+r_1)\, d_1(t) - r_1 u_2(t) \\
u_2(t+\tau_2) &= r_2 d_2(t)
\end{aligned}
\right\} \tag{11}
$$

When we solve this system of equations, we must remember that $d_1(t) = 0$ until $t = \tau_1$, $u_1(t) = 0$ until $t = 2\tau_1$, $d_2(t) = 0$ until $t = \tau_1 + \tau_2$, and $u_2(t) = 0$ until $t = \tau_1 + 2\tau_2$.

D. Properties of the State Space Model

State Equation (9) is a dynamical equation with multiple time delays. It is not a differential equation, nor is it a finite-difference equation. We shall refer to it as a causal functional equation. It is linear and time-invariant, and, as is the case with delay-time systems, requires initial value information over initial intervals of time. We have not been able to find any literature for causal functional equations.

Equations (9) and (3) can be expressed in more compact notation by introducing the following $2K \times 2K$ matrix operator[*]:

$$
\mathcal{Z} \triangleq \mathrm{diag}(z_1, z_1, z_2, z_2, \ldots, z_K, z_K), \tag{12}
$$

where z_i is a scalar operator used to denote a τ_i sec. time delay (i.e., $z_i f(t) = f(t-\tau_i)$). Let

$$
\underline{x}(t) = \mathrm{col}(u_1(t), d_1(t), u_2(t), d_2(t), \ldots, u_K(t), d_K(t)); \tag{13}
$$

---

[*] This idea was first suggested to us by Mr. Michael Steinberger, a former graduate student in the Electrical Engineering Department, at the University of Southern California.

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then, Eqs. (9) and (3) can be written, as

$$\mathcal{Z}^{-1} \underline{x}(t) = A\underline{x}(t) + \underline{b}m(t) \tag{14}$$

$$y(t) = \underline{c}'\underline{x}(t) + r_0 m(t) \tag{15}$$

where the explicit structures of A, $\underline{b}$ and $\underline{c}$ can be deduced directly from the former equations. Because we do not need this information here, we do not give explicit A, $\underline{b}$, and $\underline{c}$ structures here for Eqs. (14) and (15). [see Ref. 12 for such structures for different orderings of the states in $\underline{x}(t)$].

From Eqs. (14) and (15) we see that

$$\underline{x}(t) = (\mathcal{Z}^{-1} - A)^{-1} \underline{b}m(t) \tag{16}$$

and

$$y(t) = [\underline{c}'(\mathcal{Z}^{-1} - A)^{-1} \underline{b} + r_0]m(t) \tag{17}$$

These equations provide us (conceptually, at least) with the solution of the state equation and with the output as a function of the input.

The reflection transfer function of the K layer media system is obtained from the Laplace transform of Eq. (17), and is

$$\frac{Y(s)}{M(s)} = \underline{c}'(\hat{\mathcal{Z}}^{-1} - A)^{-1} \underline{b} + r_0 \tag{18}$$

where $\hat{\mathcal{Z}}$ is obtained from $\mathcal{Z}$ by setting $z_i = e^{-s\tau_i}$.

Equation (18) suggests a straightforward way to compute y(t) for an arbitrary m(t). First compute the system's impulse response, H(s), where [recall that Y(s) = H(s) when M(s) = 1],

$$H(s) = \underline{c}'(\hat{\mathcal{Z}}^{-1} - A)^{-1} \underline{b} + r_0 \tag{19}$$

Then, convolve h(t) with m(t) to obtain y(t). Observe that the right-hand side of (19) is an infinite series each of whose terms looks like $\alpha e^{-s\beta}$, and, that

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$\mathcal{L}^{-1}\{\alpha e^{-s\beta}\} = \alpha\delta(t-\beta)$; hence, h(t) is an infinite sequence of impulse functions.

This fact makes the convolution of h(t) and m(t) quite easy, since

$$m(t) * [\alpha\delta(t-\beta)] = \alpha m(t-\beta) \tag{20}$$

Finally, we consider the special case when $\tau_1 = \tau_2 = \cdots = \tau_K \triangleq \tau$, in which case $\mathcal{Z} = zI$, where z denotes the τ sec. time delay and I is the $2K \times 2K$ identity matrix. In this case Eq. (14) can be written as

$$\underline{x}(t+\tau) = A\underline{x}(t) + \underline{b}m(t) \tag{21}$$

Let us choose $t = k\tau$ and assume that m(t) only has values at $t = k\tau$, in which case Eq. (21) reduces to the following vector finite-difference equation:

$$\underline{x}[(k+1)\tau] = A\underline{x}(k\tau) + \underline{b}m(k\tau) \tag{22}$$

In this special case all of the usual techniques (Ref. 10, for example) associated with such equations can be used to solve for $\underline{x}(k)$, after which we can compute $y(k\tau)$ from the following sampled version of Eq. (15):

$$y(k\tau) = \underline{c}'\underline{x}(k\tau) + r_0 m(k\tau) \tag{23}$$

We do not choose to follow this uniform travel time/sampled data path, because these assumptions seem too restrictive.

E. Computational Solutions

In this paragraph we briefly describe two computational methods for practical computer solution of our state space model in Eqs. (9), (10), and (3). More detailed descriptions of both methods are given in Ref. 13.

Our first method is a ray tracing technique in which we define mapping rules to track how a state propagates at an interface. The rules are obtained by observing, in Eq. (9), what happens to downgoing or upgoing states and are illustrated for downgoing states in Figure 5. The complete set of mapping rules are:

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$$d_j(t) \begin{cases} r_0 \, m(t) & \rightarrow & y(t) & j=0 \\ r_j \, d_j(t) & \rightarrow & u_j(t+\tau_j) & j=1,\ldots,K \\ (1+r_j) \, d_j(t) & \rightarrow & d_{j+1}(t+\tau_{j+1}) & j=0,\ldots,K-1 \end{cases} \qquad (28)$$

$$u_j(t) \begin{cases} (1-r_0) \, u_1(t) & \rightarrow & y(t) & j=1 \\ (1-r_{j-1}) \, u_j(t) & \rightarrow & u_{j-1}(t+\tau_{j-1}) & j=2,\ldots,K \\ -r_{j-1} \, u_j(t) & \rightarrow & d_j(t+\tau_j) & j=1,\ldots,K \end{cases} \qquad (29)$$

In order to use Eqs. (28) and (29) we create a state reference table. Each element of the states in this table, called an event, is characterized by its time of occurrence and amplitude. Our ray tracing procedure is based on the observation that, at an interface, an incoming signal branches into reflected and transmitted signals. Every time such a branching occurs we have two event points along the time axis which are stored in the state reference table. We search along the time axis for a time at which an event has occurred. At that time point we map all d_j and u_j states which change (there are only two such states) via Eqs. (28) and (29). As each event branches into two new events, the table grows geometrically. We proceed along the time axis looking up values of d_j and u_j at event points, until we have covered the domain of interest. To restrict the growth of the table, we use two tolerance parameters, AMIN and δT, and collapse events that occur at the same time point before we store them back into the table. Parameter AMIN controls the amplitude of the computed event. If that amplitude is less than AMIN, it is set to zero. Parameter δT controls the time separation of two events below which they are considered to occur at the same time.

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Our second method is a <u>discretization technique</u> in which we convert

Eq. (9) into standard finite-difference equations by discretization of time.

Since the one-way travel times are in general non-uniform and not a multiple of one

another, we have to choose a very small time interval, Δ, for discretization.

We choose Δ as the greatest common factor of the one-way travel times, such

that $\tau_j = k_j \Delta$ (j=1, 2, ..., K) where k_j is a positive integer. Discretization is

equivalent to dividing a layer into equal sub-layers and inserting interfaces

whose reflection coefficients are zero. We illustrate the discretization and

labeling of intermediate states for a layer where $\tau_j = 4\Delta$ in Figure 6. The

facts that the intermediate downgoing states $\delta_1(t)$, $\delta_2(t)$, and $\delta_3(t)$ equal

$d_j(t+\Delta)$, $d_j(t+2\Delta)$, and $d_j(t+3\Delta)$, respectively, and the intermediate upgoing

states $\mu_1(t)$, $\mu_2(t)$, and $\mu_3(t)$ equal $u_j(t+\Delta)$, $u_j(t+2\Delta)$, and $u_j(t+3\Delta)$, respec-

tively, are a direct consequence of the equations written at interfaces a, b,

and c. Those interface equations use the fact that the intermediate reflection

coefficients are zero.

By the discretization technique each layer, say the jth, expands to k_j

layers, and each of the k_j layers is described by two states: hence, the

dimension of the state vector for the discretized system is $2\sum_{i=1}^{K} k_i$. The

easiest way to obtain the discrete-time state equations for the expanded

system is to imagine that our earlier state equation derivation, which was

for the system in Figure 1a, is for that same system; but, now each of

the K layers is further partitioned into k_j layers, so that Figure 1a is

now a system of $2\sum_{i=1}^{K} k_i$ layers, each of uniform one-way travel time,

Δ. Of course, many of the layers have zero reflection coefficients at

respective interfaces. The discrete-time counterparts to Equations

(9) and (3), for the expanded system, are:

$$d_1(k\Delta + \Delta) = -r_0 u_1(k\Delta) + (1 + r_0)m(k\Delta)$$

$$u_1(k\Delta + \Delta) = r_1 d_1(k\Delta) + (1-r_1)u_2(k\Delta)$$

$$\left.\begin{array}{l} d_j(k\Delta + \Delta) = (1 + r_{j-1})d_{j-1}(k\Delta) - r_{j-1}u_j(k\Delta) \\ u_j(k\Delta + \Delta) = r_j d_j(k\Delta) + (1-r_j)u_{j+1}(k\Delta) \end{array}\right\} \quad j=2, 3, \ldots, L-1$$

$$d_L(k\Delta + \Delta) = (1 + r_{L-1})d_{L-1}(k\Delta) - r_{L-1}u_L(k\Delta)$$

$$u_L(k\Delta + \Delta) = r_L d_L(k\Delta) \tag{30}$$

and

$$y(k\Delta) = r_0 m(k\Delta) + (1-r_0)u_1(k\Delta) \tag{31}$$

where

$$L = 2\sum_{i=1}^{K} k_i \tag{32}$$

The non-zero reflection coefficients are r_0, r_{k_1}, $r_{k_1 + k_2}$, $\ldots$, r_L. All other reflection coefficients are zero.

Equation (30) is solved in an iterative manner, for $k=0, 1, 2, \ldots$. Usual matrix methods for obtaining this solution (Ref. 10, for example) are terribly inefficient, due to the large value L will usually have (matrix multiplications), and the many zero reflection coefficients (the transition matrix for (30) is sparse) which are present. The ray-tracing technique applied to Eqs. (30) and (31) is much more efficient. That technique is much simpler in the present discrete-time case because we do not have to search along the time axis for a time at which an event has occurred. Events occur every Δ units of time.

The discrete method is simple, easy to implement and recursive; however, due to the non-uniform nature of the delay-terms, we have to use a Δ small enough such that it is a submultiple of all the delay terms. The storage requirement is usually not excessive. However, we compute zero for a lot of points on the time axis where no events occur. Thus the CPU time may be larger than for the ray-tracing technique.

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The ray-tracing technique requires a slightly more complex program to implement it. Its major disadvantage is the large storage requirement for the state reference table. Since this technique computes only the actual event points, it is very efficient and fast for short observation times.

The actual choice of a particular method depends on the input data set.

F. Recursive Reflection Transfer Function Relationship

We conclude this section by making a connection between our state space model in Eqs. (9) and (3) and transfer function models; however, our connection will be made for systems with non-equal one-way travel time.

For a K-layer media system, states $u_j(t)$ and $d_j(t)$ have been defined at the top and at the bottom, respectively, of the jth layer (see Fig. 2). Let $R_j(s)$ denote the transfer function between u_j and d_j <u>at the jth interface</u>; i.e.,

$$R_j(s) = \frac{\mathcal{L}\{u_j(t+\tau_j)\}}{\mathcal{L}\{d_j(t)\}} = e^{s\tau_j} \frac{U_j(s)}{D_j(s)} . \tag{33}$$

When j=0, we obtain the reflection transfer function between output y(t) and source m(t); i.e.,

$$R_0(s) = e^{s\tau_0} \frac{U_0(s)}{D_0(s)} \triangleq \frac{Y(s)}{M(s)} , \tag{34}$$

since $\tau_0 = 0$.

We shall now develop a simple recursive relationship between $R_j(s)$ and $R_{j+1}(s)$. Consider our earlier state equations for u_j and d_{j+1}:

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$$u_j(t+\tau_j) = r_j d_j(t) + (1-r_j) u_{j+1}(t) \tag{35}$$

$$d_{j+1}(t+\tau_{j+1}) = (1+r_j) d_j(t) - r_j u_{j+1}(t) . \tag{36}$$

From the Laplace transform of Eq. (35), we find

$$R_j(s) = r_j + (1-r_j) U_{j+1}(s)/D_j(s) \tag{37}$$

Laplace transform Eq. (36) and solve for $D_j(s)$, to show that

$$D_j(s) = [r_j U_{j+1}(s) + e^{s\tau_{j+1}} D_{j+1}(s)]/(1+r_j) \tag{38}$$

Substitute Eq. (38) into Eq. (37), and re-arrange some terms in the resulting

expression to see that

$$R_j(s) = \frac{r_j + z_{j+1}^2 R_{j+1}(s)}{1 + r_j z_{j+1}^2 R_{j+1}(s)} \quad (j=K-1, K-2, \ldots, 1, 0) \tag{39}$$

which is the desired result.

Equation (39) can be used to compute the output of a K-layer media

system in a recursive manner, beginning with a one layer system (i. e. , one

layer on top of a basement layer) for which we set j=K-1. We then iterate

Eq. (39) backwards, setting j=K-2, K-3, . . . , 1, 0. In order to compute

$R_{K-1}(s)$ we need $R_K(s)$; but, $R_K(s)$ can be obtained directly from the very

last state equation in (9), $u_K(t+\tau_K) = r_K d_K(t)$, as

$$R_K(s) = r_K . \tag{40}$$

In the special, but widely studied case of equal travel times, Eqs. (39)

and (40) simplify to

$$R_j(s) = \frac{r_j + z^2 R_{j+1}(s)}{1 + r_j z^2 R_{j+1}(s)} \quad , \quad j = K-1, K-2, \ldots, 1, 0 \tag{41.}$$

$$R_K(s) = r_K$$

where $z = e^{-s\tau}$. Equation (41) (or its discrete-time counterpart, in which Laplace transfer functions are replaced by z-transform transfer functions) is a well-known result which can be derived by widely different methods (Refs. 5 and 7, for example). Additionally, these recursive relationships occur (Ref. 14) in electric kernal functions, magnetotelluric input impedance functions, and electromagnetic modified kernal functions.

That Eq. (41) generalizes to Eq. (39) for non-equal travel times is believed to be a new result.

III. A STATE EQUATION MODEL: SOURCE AND SENSOR IN FIRST LAYER

In this section we present a state space model for which source and sensor are located in the first layer. Such a model is most important for the case of a marine environment. We shall consider the two situations depicted in Figure 7.

One approach for obtaining the state and output equations is to redo our Section II. C development in which we now include the source and sensor in the first layer. The resulting state equations are somewhat different from Eq. (9) in that the source $m(t)$ will appear in the d_1, u_1, and d_2 state equations, since $m(t)$ affects interfaces 0 and 1.

A second approach, the one we shall take, is one in which we bring $m(t)$ to the surface so that we can use our Section II. C state equations directly. By this artifice, we will reduce the derivation of the state equations for source in the first layer to a problem which has already been solved.

Observe, in Figure 8, that there are two "real" rays associated with $m(t)$. Ray (1) represents $m(t)$ as it goes directly down into the rest of the system, whereas ray (2) represents $m(t)$ which reflects off the surface interface and then goes back down into the system (i.e., the ghost). The dashed ray is "imaginary"; i.e., it is needed for construction purposes only, to bring the component of $m(t)$ along ray (1) up to the surface.

Let

$$m_e(t) = \frac{1}{(1 + r_0)} \left[m(t + \tau_a) - r_0 m(t - \tau_a) \right] \tag{42}$$

At the surface (Figure 9), we now obtain the following state equation for d_1:

$$d_1(t + \tau_1) = -r_0 u_1(t) + (1 + r_0)m_e(t) \tag{43}$$

or

$$d_1(t + \tau_1) = -r_0 u_1(t) + m(t + \tau_a) - m(t - \tau_a) \tag{44}$$

With $m(t)$ brought to the surface, all other state equations are exactly the same as in Eq. (9); hence, <u>the complete set of state equations for a source in the first layer is Eq. (9) where $m(t)$ is replaced by $m_e(t)$.</u> The following initial conditions, which make use of the fact that one component $\left[m(t + \tau_a)\right]$ of the equivalent surface source, $m_e(t)$, occurs prior to time zero, should be used:

$$
\left.
\begin{aligned}
d_1(t) &= 0 \; \forall \, t \, \varepsilon \, \left[0, \tau_1 - \tau_a = \tau_c\right) \\
d_2(t) &= 0 \; \forall \, t \, \varepsilon \, \left[0, \tau_c + \tau_2\right) \\
&\cdots \\
d_K(t) &= 0 \; \forall \, t \, \varepsilon \, \left[0, \tau_c + \tau_2 + \cdots + \tau_K\right)
\end{aligned}
\right\} \tag{45a}
$$

$$
\left.
\begin{aligned}
u_1(t) &= 0 \; \forall \, t \, \varepsilon \, \left[0, 2\tau_1 - \tau_a = \tau_c + \tau_1\right) \\
u_2(t) &= 0 \; \forall \, t \, \varepsilon \, \left[0, \tau_c + 2\tau_2\right) \\
&\cdots \\
u_K(t) &= 0 \; \forall \, t \, \varepsilon \, \left[0, \tau_c + \tau_2 + \cdots + \tau_{K-1} + 2\tau_K\right)
\end{aligned}
\right\} \tag{45b}
$$

Now for the sensor equations, which are a bit trickier than the state equations because of the "imaginary" ray path. Consider the case where the sensor is below $m(t)$ (Figure 10a). In this case, the rays which are used to bring $m(t)$ to the surface do not pass through the sensor; hence,

$$y_b(t) = u_1(t + \tau_d) + d_1(t + \tau_b) \tag{46}$$

When the sensor is above the source (Figure 10b) both construction rays pass through the sensor, and

$$y_a(t) = u_1(t + \tau_d) + d_1(t + \tau_b) + m(t - (\tau_a - \tau_d)) - m(t + (\tau_a - \tau_d)) \qquad (47)$$

The term $m(t - (\tau_a - \tau_d))$ is due to "real" ray (2), which really passes through the sensor, but will not be part of the first two state terms in Eq. (47), since they are due to inputs at the surface. When the surface input $m(t + \tau_a)/(1 + r_0)$ travels along the "imaginary" ray on its way back down into the system, it registers the component $(1 + r_0)m(t + \tau_a - \tau_d)/(1 + r_0)$ at the sensor. Because this is an imaginary (i.e., non-existent) component, it must be removed from the sersed signal; hence, the appearance of $-m(t + (\tau_a - \tau_d))$ in Eq. (47).

Figure 11 depicts $m(t)$ and $m_e(t)$ for $r_0 = 1$ and 6 different values of τ_a, ranging from $\tau_a = 1$ msec to $\tau_a = 6$ msec. If the velocity in the first layer is 5,000 ft/sec then this range of τ_a values corresponds to source depths ranging from 50 ft to 300 ft. Observe that the most spike-like input occurs for $\tau_a = 1$ msec. As τ_a increases, $m_e(t)$ becomes larger in amplitude and more oscillatory, and eventually for $\tau_a = 5$ msec and 6 msec, distortion sets in the early portion of $m_e(t)$.

The right-hand side of the d_1 state equation in Eq. (44) suggests that our system with its equivalent surface input is non-causal; but, this is not so. We can solve Eq. (44) for $d_1(t)$, as

$$d_1(t) = -r_0 u_1(t - \tau_1) - r_0 m(t - \tau_a - \tau_1) + m(t + \tau_a - \tau_1); \qquad (48)$$

but, $\tau_1 - \tau_a = \tau_c$, so that

$$d_1(t) = -r_0 u_1(t - \tau_1) - r_0 m(t - \tau_a - \tau_1) + m(t - \tau_c) \qquad (49)$$

We observe that $m(t)$ occurs to the right of zero, which confirms the causality of our state equations.

Our subsurface source and sensor model can be simulated by either of the two methods described in Section II - E. The mapping rules for the ray tracing technique are similar to those in Eqs. (28) and (29); but, for the present case $m_e(t)$ replaces $m(t)$. In the discretization technique Δ must now be chosen as the greatest common factor of $\tau_1, \tau_2, \ldots, \tau_K$, and, τ_a, τ_b, τ_c, and τ_d. In the latter technique, we first compute the system's pulse response, then compute $u_1(t)$ and $d_1(t)$ via convolution between $m_e(t)$ and the pulse response, and finally compute $y_a(t)$ or $y_b(t)$ by delaying $u_1(t)$ and $d_1(t)$ according to Eqs. (46) and (47), respectively.

The impulse response for a three-layer system is depicted in Figure 12 for the case of a fixed sensor ($\tau_d = 0.08$ sec) and varying source locations ($\tau_a = .02, .04,$ and $.06$ sec). The system parameters are: $r_0 = 1.00$, $r_1 = 0.54$, $r_2 = 0.34$, $r_3 = 0.12$, $\tau_1 = 0.4$, $\tau_2 = 0.125$, and $\tau_3 = 0.107$ sec. As the source approaches the sensor the ghost reflections take longer to reach the sensor (see spacing of first two pulses) and many of the multiples from the two source components seem to be getting closer to one another.

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IV. CONCLUSIONS

We have developed state space models for lossless layered media
which are described by the wave equation and boundary conditions. Our
models are for non-equal one-way travel times, and are therefore more
general than traditional transfer function models, which are usually for
layers of equal one-way travel times. Our state space models represent a
new class of equations, which we call causal functional equations. These
equations are linear, time-invariant, continuous-time equations with multiple
time delays. The impulse response of our system is a sequence of unequally
spaced impulse functions.

We have developed our state space models for two cases: (1) source
and sensor at the surface and (2) source and sensor in the first layer. These
models can be used either to generate synthetic seismograms or to develop
inverse procedures for extracting reflection coefficients or reflection
coefficients and one-way travel times. Some recent work on the extraction
of reflection coefficients for source and sensor at the surface, but from
noisy data is given in Ref. 15. The extension of the Ref. 15 results to the
important case of source and sensor in the first layer is completed and will
be reported on shortly.

We also wish to point out that our state space models have led to

some new theoretical results which provide us with greater understanding of the very complicated internal behavior of a layered media system. The first of these results (Ref. 16) is a decomposition of a seismogram into a superposition of primaries, secondaries, tertiaries, etc. Each of the constituents (e. g., primaries, secondaries) is obtained from a state space model of dimension 2K. The second of these results (Ref. 17) uses this decomposition to quantify the reinforcement phenomena that exist within and between the constituents.

Finally, we reiterate that our ultimate objective for developing state space models is to apply time-domain estimation and identification techniques, which have proven to be very beneficial outside of the geophysics field, to the layered media problem. We are presently studying such applications.

ACKNOWLEDGEMENT

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List of Figures

Figure 1a

Figure 1b

Figure 2

Figure 3

Figure 4

$d_O(t) = m(t)$ $r_O d_O(t) \longrightarrow y(t)$ Interface – O

$(1 + r_O) d_O(t) \longrightarrow d_1(t + \tau_1)$

Layer 1   Interface 1

(a)

Interface – (J – 1)

Layer j

$d_j(t)$ $r_j d_j(t) \longrightarrow u_j(t + \tau_j)$

$(1 + r_j) d_j(t) \longrightarrow d_{j+1}(t + \tau_{j+1})$ Interface – (J)

Layer j + 1

Interface – (J + 1)

(b)

Interface – (K – 1)

Layer K

$d_K(t)$ $r_K d_K(t) \longrightarrow u_K(t + \tau_K)$ Interface – (K)

BASEMENT

(c)

Figure 5

Interface Equations

a. $\delta_2(t + \Delta) = \delta_3(t)$ & $u_j(t + \Delta) = \mu_1(t)$

b. $\delta_1(t + \Delta) = \delta_2(t)$ & $\mu_1(t + \Delta) = \mu_2(t)$

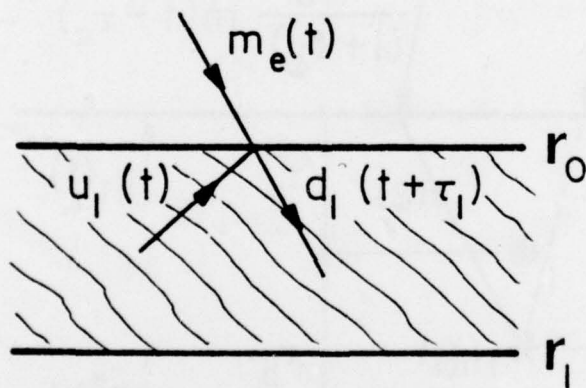c. $d_j(t + \Delta) = \delta_1(t)$ & $\mu_2(t + \Delta) = \mu_3(t)$

Figure 6

Figure 7

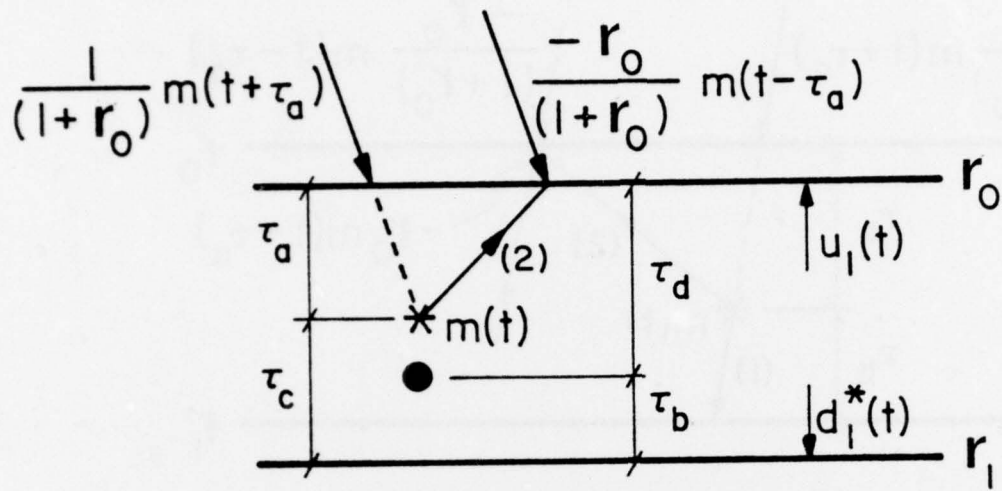$$\frac{1}{(1+r_0)} m(t+\tau_a) \qquad \frac{-r_0}{(1+r_0)} m(t-\tau_a)$$

r_0

τ_a

(2)

$-r_0 m(t-\tau_a)$

$m(t)$

τ_b

(1)

r_1

Figure 8

$m_e(t)$

r_0

$u_1(t) \qquad d_1(t+\tau_1)$

r_1

Figure 9

Figure 10

$$m(t) = -1360\, t e^{-500t} + 0.5 e^{-15.3t}\sin\left(\frac{2\pi}{0.06}t\right)$$

$\tau_a = 1$ msec

$\tau_a = 2$ msec

$\tau_a = 3$ msec

$\tau_a = 4$ msec

$\tau_a = 5$ msec

$\tau_a = 6$ msec

Figure 11

Figure 12