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by

M. Abdel-Hameed
University of North Carolina at Charlotte

and

Allan R. Sampson
Abbott Laboratories
and
Florida State University

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Department of Statistics
Florida State University
Tallahassee, Florida 32306

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M. Abdel-Hameed[1]
University of North Carolina at Charlotte
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Allan R. Sampson[2]
Abbott Laboratories
and
Florida State University

ABSTRACT

It is shown that the bivariate density of the absolute normal distribution is totally positive of order 2. Necessary and sufficient conditions are given for the trivariate density of the absolute normal distribution to be totally positive of order 2 in pairs of arguments. These results are then used to show that certain generalized bivariate and trivariate t, χ^2 and F random variables are associated.

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1. Introduction. Motivated by needs in simultaneous inference, numerous authors have established inequalities for joint probabilities in terms of marginal probabilities. Typically in these inequalities the underlying random variables are jointly normal and most of the proofs are of an analytic nature. In this paper we obtain stronger dependence results in the bivariate and trivariate cases by using certain notions of multivariate dependence.

Suppose $(X_1,\ldots,X_p)' \sim N_p(\underset{\sim}{0}, \underset{\sim}{\Sigma})$, where $N_p(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ denotes the law of a p-variate normal random vector with mean $\underset{\sim}{0}$ and nonsingular covariance matrix $\underset{\sim}{\Sigma} = \{\rho_{ij}\sigma_i\sigma_j\}$. For $i = 1,\ldots, n$, let $\underset{\sim}{z}_i \equiv (z_{1i},\ldots,z_{pi})' \sim N_p(\underset{\sim}{0},\underset{\sim}{\psi}_i)$, where $\underset{\sim}{z}_1,\ldots,\underset{\sim}{z}_n$ are independent random variables. Further, for $i = 1,\ldots, p$, let $T_1^i,\ldots, T_{q_i}^i$ be independently and identically distributed according to $N(0, \sigma^2_{T_i})$. Now assume $(X_1,\ldots, X_p)'$, $\{\underset{\sim}{z}_i\}$, $\{T_k^1\}_{k=1}^{q_1},\ldots,\{T_k^p\}_{k=1}^{q_p}$ are mutually independent sets of random variables. Define

$$(1.1) \qquad S_k^2 = \sum_{\ell=1}^{n} z_{k\ell}^2/(\psi_\ell)_{kk}, k = 1,\ldots,p,$$

and

$$(1.2) \qquad s_k^{*2} = \sum_{\ell=1}^{q_k} (T_\ell^k)^2/\sigma^2_{T_k}, \; k = 1,\ldots,p.$$

DEFINITION. (Lehmann [1966]). The random variables $U_1,\ldots,U_\alpha$ are positively quadrant dependent (PQD) if $P[\cap(U_i \leq u_i)] \geq \Pi P[U_i \leq u_i]$, for all real numbers $u_1,\ldots,u_\alpha$.

In the case p = 2, i.e., the bivariate case, Khatri [1967] showed that $|X_1|$, $|X_2|$ are PQD and that S_1^2, S_2^2 are PQD. Šidák [1967, 1971] proved that $|X_1|/S_1$, $|X_2|/S_2$ are PQD. Halperin [1967] obtained the slightly stronger result that $|X_1|/(S_1^2 + S_1^{*2})^{\frac{1}{2}}$, $|X_2|/(S_2^2 + S_2^{*2})^{\frac{1}{2}}$ are PQD. Dunn 1958 had previously obtained similar results.

For the $p = 3$ case, similar results hold. Khatri showed that $|X_1|, |X_2|, |X_3|$ are PQD if $\underset{\sim}{\Sigma}$ is of the form $\{\beta_i\beta_j\}$, $i \neq j$. Khatri also showed under this condition that S_1^2, S_2^2, S_3^2 are PQD. Sidák [1971] proved that $|X_1|/S_1,\ldots,|X_3|/S_3$ are PQD if the correlation between X_i and X_j is of the form $\lambda_i\lambda_j\rho_{ij}$ $(i, j = 1,\ldots,3;\ i \neq j)$, $|\lambda_i| \leq 1$ $(i = 1,2,3)$, $\{\rho_{ij}\}$ is any fixed correlation matrix; and if the correlation between $z_{\ell i}$ and z_{ki} is of the form $\tau_{\ell i}\tau_{ki}$ $(\ell, k = 1,2,3;\ \ell \neq k,$ $i = 1,\ldots,n)$ where $|\tau_{\ell i}| < 1$ $(\ell = 1,2,3;\ i = 1,\ldots,n)$.

Some results have been obtained for higher dimensions by the above authors.

Note that up to constants $(X_1/S_1,\ldots,X_p/S_1)'$ is a multivariate Student's t-random vector (considered in the bivariate case by Siddiqui [1967]); $(X_1/S_1,\ldots,X_p/S_p)'$ is a generalized multivariate Student's t-random vector (Sidák [1971]); and $(S_1^2,\ldots,S_p^2)'$ is a multivariate χ^2 random vector (Krishnamoorthy and Parthasarathy [1951], Jensen [1970]). Jogdeo [1977] defined a class of multivariate random variables called "contaminated random variables" and showed that their absolute values are associated. It is worth noting that the multivariate normal distributions discussed in Khatri and Sidak papers mentioned above can be viewed as "contaminated random variables" as shown on p. 498 of Jogdeo's paper. Pitt [1977] strengthened the results of Khatri [1967] and Sidák [1967, 1971] in the bivariate case, and proved that if $n(x_1,x_2)$ is the standard normal density on R^2 and if $A = -A$ and $B = -B$ are convex subsets of R^2, then $P[(x_1,x_2) \in A \cap B] \geq P[(x_1,x_2) \in A]\, P[(x_1,x_2) \in B]$. Dykstra and Hewett [1977] established positive dependence of the roots of a Wishart matrix.

The preceding results were derived basically independently of each other and each proof involved analytic techniques specific to that result. In this paper we obtain the following basic results: (a) the density of $|X_1|$, $|X_2|$ is totally positive of order 2 (b) a necessary and sufficient condition that $|X_1|$, $|X_2|$, $|X_3|$ be totally positive of order 2 in pairs of arguments is that $\prod_{i<j} \operatorname{sgn}(\lambda_{ij}) \le 0$, where $\underset{\sim}{\Lambda} \equiv \{\lambda_{ij}\} = \underset{\sim}{\Sigma}^{-1}$;

(c) $S_1^2 + S_1^{*2}$, $S_2^2 + S_2^{*2}$, $S_3^2 + S_3^{*2}$, are associated random variables and that $|X_1|/(S_1^2 + S_1^{*2})^{\frac{1}{2}}$, $|X_2|/(S_2^2 + S_2^{*2})^{\frac{1}{2}}$, $|X_3|/(S_3^2 + S_3^{*2})^{\frac{1}{2}}$ are associated random variables. (The same results hold for $p = 2$.)

2. Total Positivity of the Bivariate Absolute Normal. We employ the following definitions and implications.

DEFINITION 2.1. (Karlin [1968]). A function $f: R^2 \to [0,\infty)$ is totally positive of order 2 (TP_2) if the second order determinant $\det\{f(u_i, v_j)\}$ is nonnegative for each choice $u_1 < u_2$, $v_1 < v_2$.

DEFINITION 2.2. (Esary, Proschan and Walkup [1967]). The random variables $U_1,\ldots,U_\alpha$ are associated if $\operatorname{Cov}[f(U_1,\ldots,U_\alpha), g(U_1,\ldots,U_\alpha)] \ge 0$ for all nondecreasing functions f, g.

DEFINITION 2.3. (Barlow and Proschan [1975]). Let α be an integer exceeding 2. A function $f: R^\alpha \to [0,\infty)$ is said to be totally positive of order 2 in pairs (TP_2 in pairs) if for any pair of arguments u_a, u_b, $f(u_1,\ldots,u_a,\ldots,u_b,\ldots,u_\alpha)$, viewed as a function of u_a, u_b with remaining arguments fixed, is TP_2.

DEFINITION 2.4. (Barlow and Proschan [1975]). The random variables $U_1,\ldots,U_\alpha$ are conditionally increasing in sequence if for $i = 1,\ldots,\alpha$ $P(U_i > u_i / U_{i-1} = u_{i-1},\ldots, U_1 = u_1)$ is increasing in $u_1,\ldots, u_{i-1}$.

For $s > 0$, let

$$\gamma^{(s)}(t) = (t)^{s-1}/\Gamma(s), \quad t \geq 0$$
$$= 0 \qquad\qquad , \quad t < 0.$$

For $m > 0$, $n > 0$, define

$$\psi_{m,n}(u_1,u_2) = E[\gamma^{(m)}(U_1-u_1)\gamma^{(n)}(U_2-u_2)],$$

where the expectation in the right hand side is taken with respect to the joint distribution of U_1 and U_2.

<u>DEFINITION 2.5</u>. (Shaked [1977]). Two random variables U_1 and U_2 are said to be dependent by total positivity of order two with degree (m,n) (denoted by DTP(m,n)) if $\psi_{m,n}(u_1,u_2)$ is TP_2 in u_1 and u_2.

The following is an appropriate extension of DTP(m,n) for more than two random variables.

<u>DEFINITION 2.6</u>. The random variables $U_1,\ldots, U_\alpha$ are said to be dependent by total positivity of order two with degree (m,n) in pairs (denoted by DTP(m,n) in pairs) if for every pair of arguments u_a, u_b,

$$\psi_{m,n}(u_a,u_b) \overset{\text{def}}{=} E[\{\gamma^{(m)}(U_a - u_a)\gamma^{(n)}(U_b - u_b)\}/U_i, \ i = 1,\ldots,\alpha, \ i \neq a,b]$$

is TP_2 in u_a,u_b.

The following lemma is closely related to Theorem 4.2, p. 143, of Barlow and Proschan [1975] and Proposition 3.4 of Shaked [1977].

<u>LEMMA 2.1</u>. Let the random variables $U_1,\ldots, U_\alpha$ have joint density $f_{U_1,\ldots, U_\alpha}(u_1,\ldots, u_\alpha)$. Then the following implications hold: $f_{U_1,\ldots, U_\alpha}(u_1,\ldots, u_\alpha)$ is TP_2 in pairs $\Longleftrightarrow U_1,\ldots, U_\alpha$ are DTP$(0,0)$ in pairs $\Longrightarrow U_1,\ldots, U_\alpha$ are conditionally increasing in sequence $\Longrightarrow U_1,\ldots, U_\alpha$ are associated $\Longrightarrow U_1,\ldots, U_\alpha$ are PQD.

A more detailed examination of DTP(m,n) in pairs and its relationship to the dependence concepts given by Alam and Wallenius [1976], Esary and Proschan [1972] and Shaked [1977] are being currently examined by the authors and will appear in the future. The implications given in Lemma 2.1 are, however, sufficient for our purpose.

In order to obtain our main bivariate result, we require the following lemma.

LEMMA 2.2. Let $f(u,v) = k_1(u)\, k_2(v)\, g(uv)$ for $u \geq 0$, $v \geq 0$ and $f(u,v) = 0$, otherwise. Assume $k_1 \geq 0$, $k_2 \geq 0$, and $g \geq 0$. If g is nondecreasing and $\ln g$ is convex, then f is TP_2.

PROOF. Since $f = 0$ for $u < 0$ or $v < 0$, it suffices to consider $0 \leq u_1 < u_2$, $0 \leq v_1 < v_2$, in showing $\det\{f(u_i,v_j)\} \geq 0$. Note that $\det\{f(u_i,v_j)\} = \left[\prod_{i=1}^{2} (k_1(u_i)k_2(v_j))\right] \det\{g(u_iv_j)\}$, and thus we need only to show that $\det\{g(u_iv_j)\} \geq 0$. Define $t_1 = u_1v_1$, $t_1 + \Delta_1 = u_2v_1$, $t_2 = u_1v_2$, $t_2 + \Delta_2 = u_2v_2$, so that $0 \leq \Delta_1 < \Delta_2$. Observe that

$$
\begin{aligned}
\det\{g(u_iv_j)\} &= g(t_1)g(t_2 + \Delta_2) - g(t_1 + \Delta_1)g(t_2) \\
&\geq g(t_1)g(t_2 + \Delta_1) - g(t_1 + \Delta_1)g(t_2) \\
&\geq 0,
\end{aligned}
$$

where the first inequality follows because $g \geq 0$ and nondecreasing and the second inequality because g is logarithmically convex.

THEOREM 2.1. Let $(X_1, X_2)' \sim N_2(\underset{\sim}{0}, \underset{\sim}{\Sigma})$. Then the joint density function $f_{|X_1|, |X_2|}(x_1,x_2)$, of $|X_1|$, $|X_2|$ is TP_2.

<u>PROOF</u>. For $x_1 < 0$ or $x_2 < 0$, $f_{|X_1|,|X_2|}(x_1,x_2) = 0$, and for $x_1 \geq 0$, $x_2 \geq 0$ it is readily shown that

$$f_{|X_1|,\ |X_2|}(x_1,x_2) = k_1(x_1)k_2(x_2)g(x_1x_2),$$

where

$$k_i(s) = \exp[-s^2/\theta_i^2],\ i = 1,\ 2,$$

$$g(s) = 4c \cosh(\rho s/(\theta_1\theta_2))$$

and

$$\theta_i = (2 - 2\rho^2)^{\frac{1}{2}}\sigma_i,\ i = 1,\ 2;\ c^{-1} = 2\pi\sigma_1\sigma_2(1 - \rho^2)^{\frac{1}{2}}$$

Straightforward calculations yield that g is nondecreasing and logarithmically convex, so that Lemma 2 immediately yields that $f_{|X_1|,|X_2|}(x_1,x_2)$ is TP_2. □

<u>REMARK 2.1</u>. From Lemma 2.1, it follows that the random variables $|X_1|$, $|X_2|$ are conditionally increasing in sequence, associated, and PQD.

3. <u>Total Positivity of the Trivariate Absolute Normal</u>.

In this section we give a necessary and sufficient condition for the density function of the trivariate absolute normal variable to be TP_2 in pairs. In section 4 we use this result to show that a trivariate χ^2 and a trivariate t-distribution are associated and, hence, are PQD.

Let $(X_1,\ X_2,\ X_3)' \sim N_3(\underset{\sim}{0},\ \underset{\sim}{\Sigma})$ have a trivariate normal distribution with mean $\underset{\sim}{0}$ and covariate matrix $\underset{\sim}{\Sigma}$. Let $\underset{\sim}{\Lambda} \equiv \{\lambda_{ij}\} = \underset{\sim}{\Sigma}^{-1}$. Then the joint p.d.f., $f_{|X_1|,|X_2|,|X_3|}(x_1,x_2,x_3)$, of $|X_1|$, $|X_2|$, $|X_3|$, for $(x_1,\ x_2,\ x_3)$ in the positive octant is given by

$$(3.1)\quad f_{|X_1|,|X_2|,|X_3|}(x_1,\ x_2,\ x_3) = K_\Lambda \exp[-\tfrac{1}{2}(\lambda_{11}x_1^2 + \lambda_{22}x_2^2 + \lambda_{33}x_3^2)]g(x_1,\ x_2,\ x_3),$$

where

$$K_\Lambda = 2(\sqrt{2\pi})^{-3}|\Lambda|^{\frac{1}{2}},$$

and

$$g(x_1, x_2, x_3) = \sum_{i=0}^{1} \sum_{j=0}^{1} \exp[(-1)^i \lambda_{12} x_1 x_2 + (-1)^j \lambda_{13} x_1 x_3 + (-1)^{i+j+1} \lambda_{23} x_2 x_3].$$

The density is 0, otherwise.

Hence, to show that $f_{|X_1|,|X_2|,|X_3|}(x_1, x_2, x_3)$ is TP_2 in pairs it suffices to show that $g(x_1, x_2, x_3)$ is TP_2 in pairs. To do so we require the following two lemmas whose proofs are straightforward.

LEMMA 3.1. Let $\underset{\sim}{\Lambda}_0$ be a fixed 3×3 positive definite matrix and define $\underset{\sim}{D}_e$ as a diagonal matrix with elements $\pm$ 1. Then the p.d.f. $f_{|X_1|, |X_2|, |X_3|}(x_1, x_2, x_3)$ given by (3.1), viewed as a function of $\underset{\sim}{\Lambda}$, if invariant on the set $\{\underset{\sim}{\Lambda}:\underset{\sim}{\Lambda} = \underset{\sim}{D}_e\underset{\sim}{\Lambda}_0\underset{\sim}{D}_e\}$.

Define $\text{sgn}(x) = 1$ if $x > 0$; $= 0$ if $x = 0$; $= -1$ if $x < 0$.

LEMMA 3.2. A necessary and sufficient condition that there exists $\underset{\sim}{D}_e$, a diagonal matrix with elements $\pm$ 1, so that the off-diagonal elements of $\underset{\sim}{D}_e\underset{\sim}{\Lambda}_0\underset{\sim}{D}_e$ are all negative (positive) is that $\prod_{i<j} \text{sgn}(\lambda_{ij}^0) = -1$ (=1), where λ_{ij}^0 is the i, jth element of $\underset{\sim}{\Lambda}_0$.

THEOREM 3.1. Let $(X_1, X_2, X_3)' \sim N_3(\underset{\sim}{0},\underset{\sim}{\Sigma})$. Then a necessary and sufficient condition that the joint density function $f_{|X_1|,|X_2|,|X_3|}(x_1, x_2, x_3)$ of $|X_1|$, $|X_2|$, $|X_3|$ be TP_2 in pairs is that $\prod_{i<j} \text{sgn}(\lambda_{ij}) \leq 0$, where $\underset{\sim}{\Lambda} \equiv \{\lambda_{ij}\} = \underset{\sim}{\Sigma}^{-1}$.

PROOF.

SUFFICIENCY. If $\prod_{i<j} \text{sgn}(\lambda_{ij}) = -1$, then by Lemmas 3.1 and 3.2, we may suppose that $-\lambda_{12} > 0$, $-\lambda_{13} > 0$, $-\lambda_{23} > 0$. Let

$$\begin{aligned} u &= -\lambda_{13} x_1 x_3, \\ v &= -\lambda_{23} x_2 x_3, \end{aligned} \tag{3.2}$$

and

$$\alpha = -\lambda_{12}/(\lambda_{13}\lambda_{23}x_3^2),$$

so that $-\lambda_{12}x_1x_2 = \alpha uv$. Without loss of generality, we only show

$f_{|X_1|,\,|X_2|,\,|X_3|}(x_1, x_2, x_3)$ is TP_2 in $x_1 > 0$, $x_2 > 0$ for $x_3 > 0$ fixed. This is equivalent to showing that for $\alpha > 0$ $h_\alpha(u,v)$ is TP_2 for $u > 0$, $v > 0$, where

$$(3.3)\quad \begin{aligned} h_\alpha(u,v) &= P_\alpha(u,v) + P_\alpha(u,-v) + P_\alpha(-u,v) + P_\alpha(-u,-v), \\ &= 2e^{\alpha uv}\cosh(u + v) + 2e^{-\alpha uv}\cosh(u - v) \end{aligned}$$

and

$$P_\alpha(u,v) = \exp[u + v + \alpha uv].$$

Let

$$(3.4)\quad \Delta_\alpha(u,v) \equiv h_\alpha(u,v)\frac{\partial^2 h_\alpha(u,v)}{\partial u\,\partial v} - \frac{\partial h_\alpha(u,v)}{\partial u}\,\frac{\partial h_\alpha(u,v)}{\partial v}.$$

To verify that $h_\alpha(u,v)$ is TP_2, we verify for $u > 0$, $v > 0$, $\alpha > 0$ that

$$(3.5)\quad \Delta_\alpha(u,v) \geq 0.$$

(see Karlin [1968], p. 49). Direct calculation yields that

$$\begin{aligned} \frac{\partial h_\alpha(u,v)}{\partial u} &= (1 + \alpha v)[P_\alpha(u,v) - P_\alpha(-u,v)] + \\ &\quad (1 - \alpha v)[P_\alpha(u,-v) - P_\alpha(-u,-v)], \end{aligned}$$

$$\begin{aligned} \frac{\partial h_\alpha(u,v)}{\partial v} &= (1 + \alpha u)[P_\alpha(u,v) - P_\alpha(u,-v)] + \\ &\quad (1 - \alpha u)[P_\alpha(-u,v) - P_\alpha(-u,-v)], \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 h_\alpha(u,v)}{\partial u\,\partial v} &= \alpha[P_\alpha(u,v) + P_\alpha(-u,-v) - P_\alpha(-u,v) - P_\alpha(u,-v)] \\ &\quad + (1 + \alpha v)(1 + \alpha u)P_\alpha(u,v) \\ &\quad - (1 - \alpha u)(1 + \alpha v)P_\alpha(-u,v) - (1 + \alpha u)(1 - \alpha v)P(u,-v) \\ &\quad + (1 - \alpha u)(1 - \alpha v)P_\alpha(-u,-v), \end{aligned}$$

so that after simplification, we have

$$(3.6)\quad \Delta_{\alpha}(u,v) = 2\alpha[e^{2\alpha uv}\cosh(2(u+v)) - e^{-2\alpha uv}\cosh(2(u-v))]$$
$$+ 4(2 + \alpha)\sinh(2\alpha uv) + 8\alpha v[\sinh(2u) + \alpha u \cosh(2u)]$$
$$+ 8\alpha u[\sinh(2v) + \alpha v \cosh(2v)].$$

The first term of (3.6) is nonnegative by the monotonicity of e^t and the monotonicity of $\cosh(|t|)$. The remaining 3 terms of (3.4) are nonnegative because $\cosh t \geq 0$, and $\sinh t \geq 0$ for $t \geq 0$. Thus (3.5) holds.

If $\prod_{i<j} \mathrm{sgn}(\lambda_{ij}) = 0$, then either two or more of the λ_{ij}'s equal to zero, or exactly one of the λ_{ij}'s equal to zero. The case where two or more of the λ_{ij}'s equal to zero follows from the bivariate case discussed in Section 2. If exactly one of the λ_{ij}'s equals to zero, say λ_{12}, then in Equation (3.2) divide by the other two λ_{ij}'s, so that $\alpha = 0$ and then apply a techniqe similar to the one used when $\alpha > 0$ to show that the density is TP_2 in pairs for fixed x_3. In this case, to show TP_2 in pairs for fixed x_1 or x_2 the argument would reduce to the bivariate case argument.

NECESSITY. Suppose $\prod_{i<j} \mathrm{sgn}(\lambda_{ij}) = 1$, so that by Lemmas 3.1 and 3.2 we can assume $\lambda_{12} > 0$, $\lambda_{13} > 0$, $\lambda_{23} > 0$. Define u, v, α as in the proof of the sufficiency, but note u, v, and $\alpha < 0$. We proceed to show that there exists $x_3 > 0$ so that $f_{|X_1|, |X_2|, |X_3|}(x_1, x_2, x_3)$ has negative second order determinant for certain $x_1 > 0, x_2 > 0$. To do this, we let $x_3 = [\lambda_{12}/(\lambda_{13}\lambda_{23})]^{\frac{1}{2}}$, so that $\alpha = -1$, and then show that there exists an open set so that $\Delta_{-1}(u, v)$ defined in (3.4) is negative. To find such an open set, we show that there exists $t < 0$ so that $\Delta_{-1}(t,t)$ is negative and then appeal to the continuity of $\Delta_{-1}(u,v)$. Note that

$$\Delta_{-1}(t,t) = 2e^{-2t^2}[1 - \cosh(4t)] - 16t[\sinh(2t) - t \cosh(2t)].$$

Observe that $\cosh(2t) \geq 1$ and that for suitably small negative t, $\sinh(2t) - t\cosh(2t) < 0$, so that for suitably small negative t, $\Delta_1(t, t) < 0$. Hence, we can conclude that

$f_{|X_1|, |X_2|, |X_3|}(x_1, x_2, x_3)$ is not TP_2 in pairs if $\prod_{i<j} \operatorname{sgn}(\lambda_{ij}) = 1$. □

REMARK 3.1. If $\prod_{i<j} \operatorname{sgn}(\lambda_{ij}) \leq 0$, then, using Lemma 2.1 and Theorem 3.1, we have that the random variables $|X_1|, |X_2|, |X_3|$ are conditionally increasing in sequence, and associated.

For the general multivariate normal case without absolute values, we note that Barlow and Proschan [1975, Chapter 4] proved that the multivariate normal density function is TP_2 in pairs if and only if $\lambda_{ij} \leq 0$ for $i \neq j$ where $\underset{\sim}{\Lambda} = \underset{\sim}{\Sigma}^{-1}$.

4. The Association of Bivariate and Trivariate χ^2 and t Distributions.

In this section we use the results of the previous sections to obtain the association of certain bivariate, and trivariate, χ^2, t, and F distributions.

To prove the results of this section, we make use of the following two lemmas which by themselves are quite interesting and useful. Lemma 4.1 is a special case of Theorem 4.1 of Jogdeo [1977].

LEMMA 4.1. Let $U_1, \ldots, U_\alpha$ be positive random variables. If $U_1, \ldots, U_\alpha$ are associated, then $U_\alpha^{-1}, \ldots, U_\alpha^{-1}$ are associated.

LEMMA 4.2. Suppose that the nonnegative random variables $U_1, \ldots, U_\alpha$ are independent of the nonnegative random variables $V_1, \ldots, V_\alpha$. If $U_1, \ldots, U_\alpha$ are associated and $V_1, \ldots, V_\alpha$ are associated, then $U_1V_1, \ldots, U_\alpha V_\alpha$ are associated.

THEOREM 4.1. (a) For $p = 2$, $S_1^2 + S_1^{*2}$, $S_2^2 + S_2^{*2}$ are associated random variables.

(b) For $p = 3$, if $\prod_{k<j} \operatorname{sgn}(\psi_i^{-1})_{kj} \leq 0$, $i = 1,\ldots, n$, then $S_1^2 + S_1^{*2}$, $S_2^2 + S_2^{*2}$ and $S_3^2 + S_3^{*2}$ are associated random variables, where $(\psi_i^{-1})_{kj}$ denotes k, j^{th} element of ψ_i^{-1}.

PROOF of (a): By Lemma 2.1, Theorem 2.1 and the invariance of association under nondecreasing transformations (P_4 of [4]) we have, for $i = 1,\ldots, n$, that Z_{1j}^2, Z_{2i}^2 are associated. Because $\{\underset{\sim}{Z}_i\}$, S_1^{*2}, S_2^{*2} are independent, we have that $Z_{11}^2, Z_{21}^2,\ldots,Z_{1n}^2, Z_{2n}^2, S_1^{*2}, S_2^{*2}$ are associated (P_2 of [4]). Since $S_1^2 + S_1^{*2}$, $S_2^2 + S_2^{*2}$ are nondecreasing functions of the Z_{ij}^2, S_1^{*2}, S_2^{*2}, we obtain $S_1^2 + S_1^{*2}$, $S_2^2 + S_2^{*2}$ are associated.

PROOF of (b): Using Theorem 3.1 and Lemma 2.1, we can prove (b) in a similar fashion to (a) with the obvious modifications.

REMARK 4.1. Note that Khatri's condition that $\underset{\sim}{\psi}_i$ is of the form $(\beta_i\beta_j)$ $|\beta_i| < 1$, $i = 1,2,3$ implies that $\prod_{k<j} \operatorname{sgn}(\psi_i^{-1})_{kj} \leq 0$.

COROLLARY 4.1. (a) For $p = 2$, $(S_1^2 + S_1^{*2})^{-\frac{1}{2}}$, $(S_2^2 + S_2^{*2})^{-\frac{1}{2}}$ are associated random variables.

(b) For $p = 3$, if $\prod_{k<j} \operatorname{sgn}(\psi_i^{-1})_{kj} \leq 0$, $i = 1,\ldots, n$, then $(S_1^2 + S_1^{*2})^{-\frac{1}{2}}$, $(S_2^2 + S_2^{*2})^{-\frac{1}{2}}$, and $(S_3^2 + S_3^{*2})^{-\frac{1}{2}}$ are associated random variables.

PROOF: The proof follows from Theorem 4.1 and the square root analogue of Lemma 4.1. □

THEOREM 4.2. (a) For $p = 2$, the random variables $|X_1|/(S_1^2 + S_1^{*2})^{\frac{1}{2}}$ and $|X_2|/(S_2^2 + S_2^{*2})^{\frac{1}{2}}$ are associated.

(b) For $p = 3$, if $\prod_{i<j} \operatorname{sgn}(\lambda_{ij}) \le 0$,

$\prod_{k<j} \operatorname{sgn}(\psi_i^{-1})_{kj} \le 0$, $i = 1, 2, \ldots, n$, then $|X_1|/(S_1^2 + S_1^{*2})^{\frac{1}{2}}$, $|X_2|/(S_2^2 + S_2^{*2})^{\frac{1}{2}}$, $|X_3|/(S_3^2 + S_3^{*2})^{\frac{1}{2}}$ are associated random variables.

PROOF: The proof of the theorem follows from Theorem 2.1, Theorem 3.1, Lemma 4.2, and Corollary 4.1. □

Up to constants, a bivariate and a trivarite F random vector can be defined by:

$$\underset{\sim}{F}_{(2)} = (S_1^2/S_1^{*2},\ S_2^2/S_2^{*2})' ,$$

and

$$\underset{\sim}{F}_{(3)} = (S_1^2/S_1^{*2},\ S_2^2/S_2^{*2},\ S_3^2/S_3^{*2})' .$$

THEOREM 4.3. (a) S_1^2/S_1^{*2}, S_2^2/S_2^{*2} are associated random variables

(b) If $\prod_{k<j} \operatorname{sgn}(\psi_i^{-1})_{kj} \le 0$, $i = 1,\ldots, n$, then the random variables S_1^2/S_1^{*2}, S_2^2/S_2^{*2}, S_3^2/S_3^{*2} are associated.

PROOF: The proof of the theorem follows immediately from the proof of Theorem 4.1, Lemma 4.1 and Lemma 4.2.

REMARK 4.2. Theorems 4.1, 4.2, 4.3 and Corollary 4.1 remain true as long as S_1^{*2} and S_2^{*2} are any pair of positive independent random variables such that $(X_1,\ldots, X_p)'$, $\{\underset{\sim}{Z}_i\}$, S_1^{*2}, S_2^{*2} are all mutually independent sets of random variables.

We conclude this paper with the following conjecture for the TP_2 in pairs of the multivariate absolute normal, of dimension larger than 3.

CONJECTURE. Let $f_{|X_1|,\ldots,|X_p|}(x_1,\ldots, x_p)$ be the p.d.f. of the multivariate absolute normal, $p > 3$. A necessary and sufficient condition for it to be TP_2 in pairs is that there exists $\underset{\sim}{D}_e$, a diagonal matrix with elements ± 1, such that the off-diagonal elements of $\underset{\sim}{D}_e \underset{\sim}{\Sigma}^{-1} \underset{\sim}{D}_e$ are all negative.

Note that if this conjecture were true, then the corresponding result concerning the multivariate t-distribution could be proved directly in the same fashion as Theorem 4.2

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It is shown that the bivariate density of the absolute normal distribution is totally positive of order 2. Necessary and sufficient conditions are given for the trivariate density of the absolute normal distribution to be totally positive of order 2 in pairs of arguments. These results are then used to show that certain generalized bivariate and trivariate t, X^2 and F random variables are associated.

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