

ASSIFICATION OF THIS PAGE (W) en Data Entered) READ INSTRUCTIONS REPORT DOCUMENTATION PAGE BEFORE COMPLETING FORM 2. GOVT ACCESSION NO. 3. RECIPIENT'S CATALOG NUMBER 5. TYPE OF REPORT & PERIOD COVERED SQUARE-ROOT ALGORITHMS FOR MODEL Interim rcp SENSITIVITY ANALYSIS . 6. PERFORMING ORG. REPORT NUMBER 8. CONTRACT OR GRANT NUMBER(\*) 7. AUTHOR(s) M Morf, E Verriest, J Dobbins AD A 0509 7.51 F44-620-74-C-0068 -Kailath Т 9. PERFORMING ORGANIZATION NAME AND ADDRESS Stanford University 61102F 1612304 Dept. of Electrical Engineering Stanford, CA. 94305 11. CONTROLLING OFFICE NAME AND ADDRESS 12. REPORT DATE Air Force Office of Scientific Research (NM) 11 [Mare Bolling AFB, Bldg. 410 NUMBER OF B Washington, D.C. 20332 6 14. MONITORING AGENCY NAME & ADDRESS(II different from Controlling Office) 15. SECURITY CLASS UNCLASSIFIED 15. DECLASSIFICATION/DOWNGRADING SCHEDULE 16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) 18. SUPPLEMENTARY NOTES Proc. 1977 Conference on Information Sciences and Systems, pp. 64-69, Johns Hopkins Univ., Maryland, March 30 - April 1, 1977. 19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Morf-Kailath Discrete Square-Root Filtering Algorithms Sensitivity Information Model Parameters Variance Ratios Orthogonal Transforms 20 BSTRACT (Continue on reverse side it necessary and identify by block number) The Morf-Kailath discrete square-root filtering algorithms are extended to incorporate sensitivity information with respect to the assumed model and statistics is available, considered to be the "actual" model. The sensitivity analysis is carried out by computing the so-called "variance ratios" that can be obtained by a second set of orthogonal transforms. These operations can be performed either in covariance or information filter form, as the actual filters. This leads to four basic variants for the propagation of the variance ratios one of them has been obtained by Bierman. DD 1 JAN 73 1473 ine UNCLASSIFIED EDITION OF I NOV 65 IS OBSOLETE 400 852 SECURITY CLASSIFICATION OF THIS PAGE (When Date En

# UNCLASSIFIED

# SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

Sec.

A. S. A. S. S. S. S.

......

20. These ideas can also be applied to other related areas such as smoothing and the dual control problems, where it is of interest to study the sensitivity of the total cost with respect to different models and weights of the cost.

ACCESSION for

UNANNOUNCED JUSTIFICATION \_

There in the maintener

NTIS

DDC

BY \_

..

White Section

Buff Section

DISTRIBUTION/AVAILABILITY CODES ......

Dist. AVAIL and/or SPECIAL

D

.: 9

SQUARE-ROOT ALCORITHMS FOR MODEL SENSITIVITY ANALYSIS

AFOSR-TR-78-0307 M. Morf, E. Verriest, J. Dobbins and T. Kailath Information Systems Laboratory Stanford University Stanford, CA 94305

#### ABSTRACT

The Morf-Kailath discrete square-root filtering algorithms are extended to incorporate sensitivity information with respect to the assumed model and statistics. If the model and statistics are incorrect, we presume that another set of model parameters and statistics is available, considered to be the "actual" model. The sensitivity analysis is carried out by computing the so-called "variance ratios" that can be obtained by a second set of orthogonal transforms. These operations can be performed either in covariance or information filter form, as the actual filters. This leads to four basic variants for the propagation of the variance ratios, one of them has been obtained by Bierman.

These ideas can also be applied to other related areas such as smoothing and the dual control problems, where it is of interest to study the sensitivity of the total cost with respect to different models and weights of the cost.

# I. Square-Root Filters Applied to Sensitivity Analysis

In this paper we consider the linear discretetime dynamic model, where the superscript "a" denotes "actual" model parameters

$$\mathbf{x}_{i+1} = \phi_{i}^{\mathbf{a}} \mathbf{x}_{i} + \Gamma_{i}^{\mathbf{a}} \mathbf{u}_{i} \quad \phi^{\mathbf{a}} : n \times n \quad \Gamma_{i}^{\mathbf{a}} : n \times p \tag{1}$$

$$y_{i} = H_{i}^{a} x_{i} + v_{i} \quad H_{i}^{a} : m \times n$$
 (2)

for simplicity, all random variables have zero mean and covariances

$$cov x_0 = P_0^a$$
(3)

$$\operatorname{cov} \begin{bmatrix} u_{i} \\ v_{i} \end{bmatrix} = \begin{bmatrix} Q_{i}^{a} & C_{i}^{a'} \\ C_{i}^{a} & R_{i}^{a} \end{bmatrix}$$
(4)

The propagation of the square root array is based on the identity in [1]

<sup>T</sup>This work was supported by the Air Force Office of Scientific Research, Air Force Systems Command, under Contract AF44-620-74-C-0068, and in part by the National Science Foundation under Contract NSF-Eng-75-18952 and the Joint Services Electronics Program under Contract N00014-75-C-0601.

$$\begin{bmatrix} R_{1}^{\epsilon} + H_{1}P_{1}H_{1}^{i} & H_{1}P_{1}\Phi_{1}^{i+c}\Gamma_{1}^{i} & C_{1} \\ \Phi_{1}P_{1}H_{1}^{i+r}\Gamma_{1}C_{1}^{i} & \Phi_{1}P_{1}\Phi_{1}^{i+r}\Gamma_{1}Q_{1}\Gamma_{1}^{i} & \Gamma_{1}Q_{1} \\ C_{1}^{i} & Q_{1}\Gamma_{1}^{i} & Q_{1} \end{bmatrix} = \\ \begin{bmatrix} R_{1}^{\epsilon} & K_{1}^{i} & C_{1} \\ K_{1} & P_{1+1}+K_{1}R_{1}^{-\epsilon}K_{1}^{i} & \Gamma_{1}Q_{1} \\ C_{1}^{i} & Q_{1}\Gamma_{1}^{i} & Q_{1} \end{bmatrix} = M_{3\times3}$$

$$\begin{bmatrix} R_{1}^{\epsilon} & K_{1}^{i} & C_{1} \\ K_{1} & P_{1+1}+K_{1}R_{1}^{-\epsilon}K_{1}^{i} & \Gamma_{1}Q_{1} \\ C_{1}^{i} & Q_{1}\Gamma_{1}^{i} & Q_{1} \end{bmatrix} = M_{3\times3}$$

$$\begin{bmatrix} S_{1}^{i} & Q_{1}\Gamma_{1}^{i} & Q_{1} \\ S_{2}^{i} & S_{2}^{i} \end{bmatrix} = (S_{1}^{i}) = S_{2}^{i} + S_{2}^$$

The following factorizations into triangular components are made:

$$a_{3x3} = C_1 C_1' = C_2 C_2'$$
 (6)

where  $C_1$  is an upper triangular square-root of the LHS<sup>1</sup> of (5) and  $C_2$  a lower triangular square-root of the RHS matrix.

A geometrical interpretation accompanies this representation (cf. [1]). Indeed, letting

$$\tilde{k}_{1}^{i} \stackrel{Q}{=} [\tilde{y}'_{1}|_{1-1}, \tilde{x}_{1+1}'_{1-1}, \tilde{u}_{1}'_{1-1}]$$

then:

$$C_1^{-1} \underline{\tilde{x}}$$
 and  $C_2^{-1} \underline{\tilde{x}}$ 

are two orthonormal sets ( $\underline{\xi}_1$  and  $\underline{\xi}_2$  respectively) for the (m + n + p)-dimensional space spanned by  $\underline{\tilde{x}}$ . In other words, geometrically an orthonormalization process (Gram-Schmidt) is going on, on  $\underline{\tilde{x}}$ .  $\underline{\xi}_1$  is obtained by orthonormalization of the set  $\{y'_1, x'_{1+1}, u'_1\}$ , conditioned on the data y(0, i-1), sequentially from right to left.

$$[y'_{i},x'_{i+1},u'_{i}] = \underline{\xi}_{1}C'_{1} + [\hat{y}'_{i}|_{i-1},\hat{x}'_{i+1}|_{i-1},\hat{u}'_{i}|_{i-1}]$$
(7)

$$\bar{x} = C_2 \xi_2$$

is obtained by orthonormalizing from left to right

$$[y'_{1},x'_{i+1},u'_{1}] = \underline{\xi}_{2}^{*}C_{2}^{*} + [\hat{y}'_{1}|_{1-1},\hat{x}'_{i+1}|_{1-1},\hat{u}'_{1}|_{1-1}] (8)$$

and the  $\xi$  variables are (see [1] for more details)

$$\underline{\xi_{1}}^{i} = (\tilde{y}_{1}^{i}, \tilde{X}_{1}^{i}, u_{1}^{i})$$

$$\underline{\xi_{2}}^{i} = (v_{1}^{i}, \tilde{X}_{1+1}^{i}, \overline{u}_{1}^{i})$$

In essence  $C_k C'_k$  is the covariance of the random variables x(k = 1, 2). Now we could also work with the inverse relations:

Similarly

$$\underline{\xi}_{k} = C_{k}^{-1} \underline{\tilde{x}} (k = 1, 2)$$

leading to the so-called information filter. These two representations are dual; we can distinguish two types: inversion duality versus the standard duality [1]. Note that this approach leads to one-step predicted estimates  $\hat{x}_{i+1|i}$  and errors (thus innovations). Alternatively we could work with filtered estimates [1]:  $\hat{x}_{i|i}$ , or with  $\lambda$  step predicted/smoothed estimate [4].

Suppose now that we do not know the actual covariances as given in (3) and (4); we use a model containing the <u>estimated</u> covariances:

$$P_{0} \text{ and } \begin{bmatrix} Q_{i} & C_{i}^{\prime} \\ C_{i} & R_{i} \end{bmatrix}$$
(9)

Similarly, the system parameters may be incorrectly known, so that we propagate the Kalman filter (and thus the array's  $C_1$  and  $C_2$ ) as if the system model were:

$$\mathbf{x}_{i+1} = \Phi_i \mathbf{x}_i + \Gamma_i \mathbf{u}_i \quad \Phi_i : \mathbf{n} \times \mathbf{n} \quad \Gamma_i : \mathbf{n} \times \mathbf{p} \quad (10)$$

$$y_i = H_i x_i + v_i \quad H_i : m \times n \tag{11}$$

Therefore, x as computed by the filter is not the minimum variance estimate, and there will be a discrepancy between the computed and the actual error covariances. This deterioration of filter accuracy can ultimately lead to filter divergence. As a measure for this discrepancy we define the variance-ratios  $\Sigma_p$ 

$$\Sigma_{\mathbf{p}} \stackrel{\Delta}{=} (\mathbf{P}^{\mathbf{a}})^{-1/2} \mathbf{p}^{1/2} \quad (\mathbf{P}^{\mathbf{a}} = \text{actual error covariance.})$$
(12)

In our geometric picture, the variables  $C_k^{-1} \underbrace{\underline{X}}_{\underline{x}} \underbrace{\underline{A}}_{\underline{n}_k}$  for k = 1,2, (we reserve  $\underline{\xi}$  for truly orthonormal variables in this paper) will no longer be orthonormal.

 $\tilde{x}_i = x_i - \tilde{x}_i|_{i-1}$  will still equal  $P_1^{1/2}X_i$ , where  $P_1^{1/2}$  is the square root of the error covariance as calculated by the filter, but  $\tilde{X}_i$ will not have unit covariance. The square root of cov  $\tilde{X}$  will be the variance ratio

$$L_{i} = P_{i}^{-1/2} (P_{i}^{a})^{1/2}$$

We present algorithms in which L, is propagated as an entry in arrays which express the <u>n</u>-random variables in terms of truly orthonormal sets,

$$n_k = M_k \xi_k k = 1,2$$
 (13)

Obviously, just as with the filter itself, we can implement this propagation in either the covariance from given in (13) or in information form

$$\xi_{1} = \Lambda_{1} \eta_{1} \quad k = 1,2$$
 (14)

in this case we propagate the  $\Sigma_p = L^{-1}$ . Either

filter implementation may be combined with either error analysis implementation, leading to four possibilities. Virtually, all square-root filter/error analysis algorithms proposed to date can be viewed as one of these four types. This is summarized in the following tableau:

Filter Imple- mentation Error Analysis Imple- mentation	Covariance Form	Information Filter form
Covariance Type Form	Thornton (1976) (Propagates x- $\hat{x}$ rather than normalized $\tilde{x}$ and uses UDU <sup>T</sup> rather than $P^{1/2}P^{T/2}$ ), Sec- tion 3 -	Bierman (1974 Nishimura & Nead (1973) Curkendall (1972) Inversion Duality Morf- Kailath (1975)
Information Type Form	Section 2 of this paper	Inversion Duality [6]

The remainder of the paper is organized as follows: Section II examines in detail the filtering algorithm for the special case  $\{\Phi_i, \Gamma_i, H_i\} = \{\Phi_i^a, \Gamma_i^a, H_i^a\}$  and  $C = C^a = 0$ , as in the rest of the paper. The derivation is given in covariance-information form, and the covariance-covariance arrays are quoted (for C and  $C^a \neq 0$ , see [6]).

Section III extends these ideas to the general case of incorrect statistics, as well as model. Finally, Section IV mentions the fixed-interval smoothing problem.

# II. ERROR ANALYSIS OF FILTERING WITH INCORRECT STATISTICS ONLY

### The Pre-Array C1

Defining

$$\frac{\tilde{X}' = [y_1' - \hat{y}_1'][x_1, u_1, y[0, i-1]] \cdot x_1' - \hat{x}_1'][u_1, y[0, i-1]]}{u'i^{-\hat{u}}i|y[0, i-1]}$$

then it is readily shown that

$$\widetilde{\underline{x}} = \begin{bmatrix} \mathbf{I} & \mathbf{H}_{i} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\phi}_{i} & \boldsymbol{\Gamma}_{i} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \underbrace{\widetilde{\underline{X}}}_{i}$$
(15)

and

cov 
$$\tilde{X}_{i}$$
 = Block diag{ $R_{i}^{a}, P_{i}^{a}, Q_{i}^{a}$ }. (16)

The "filter algorithm" rescales the  $\frac{X}{1}$ -variables, setting  $\frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{2}$ 

 $\tilde{\underline{X}}_{i} = \text{Block diag}\{R_{1}^{1/2}, P_{1}^{1/2}, Q_{1}^{1/2}\}\underline{n}_{1} = \underline{\mathbb{P}}^{1/2} \cdot \underline{n}_{1}_{i}.$ 

This leads to:

$$\tilde{\mathbf{x}} = C_1 \underline{n}_1 \tag{17}$$

(time index is suppressed), however, by virtue of (16), we see that  $\cot n_1 \neq I$  in general. Let  $\$_1\$'_1$  be a decomposition of  $\cot \tilde{x}_i$ , then

$$\underline{\xi}_{1} = \boldsymbol{\xi}^{-1} \underline{\tilde{X}} = \boldsymbol{\xi}^{-1} \underline{\boldsymbol{y}}^{1/2} \underline{n}_{1} \,\underline{\hat{\boldsymbol{\lambda}}} \,\boldsymbol{\Lambda}_{1} \underline{n}_{1} \tag{18}$$

is again an orthonormal set. Taking the upper triangular factor for \$ yields finally

$$[\underline{\tilde{x}}', \underline{\xi}_{1}'] = \underline{\eta}_{1}'[C_{1}', \Lambda_{1}']$$
(19)

where

$$\Lambda_{1} = \text{Block diag}\{(R^{a})^{-i_{5}}R^{i_{5}}, (P^{a})^{-i_{5}}P^{i_{5}}, (Q^{a})^{-i_{5}}Q^{i_{5}}\}$$
  
$$\Lambda_{1} = \text{Block diag}\{\Sigma_{R}, \Sigma_{P}, \Sigma_{Q}\}$$
(20)

Remarks

- 1. In case  $C^a \neq 0$ , and/or  $C \neq 0$ ,  $\Lambda_1$  contains off-diagonal blocks. The 2-2 block entry remains  $\Sigma_p$  however.
- 2. Having obtained  $P_1$  (the computed covariance) and  $\Sigma_{P_1}$ , we are able to find the actual covariance  $P_1^a$ .
- 3. Effect of the propagation algorithm:

$$[\underline{x}', \underline{\xi}'_{1}] = [\underline{n}'_{1}T][T'C'_{1}, T'\Lambda'_{1}]$$
$$= [n'_{1}T][C'_{1}, T'\Lambda'_{1}]$$
(21)

Thus, in the process of making  $C_1$  lower triangular, we perform the <u>same</u> operations to  $\Lambda_1$ .

The Post-Array C2

Similarly, defining  $\tilde{y}$  by:

$$\underline{\tilde{y}}' = [\varepsilon_1', \tilde{x}_{i+1}' | 1', \tilde{u}_1' | [y[0, 1-1], \varepsilon_1', \tilde{x}_{i+1}' | 1]]$$

where Ei are the innovations, we can find:

$$\operatorname{cov} \tilde{\underline{y}} = \begin{bmatrix} R_{\varepsilon}^{a} & R_{\varepsilon}^{a} \cdot (\Delta K_{g})' - R_{\varepsilon}^{a} \Delta K_{g}' P_{i+1}^{-1} \Gamma Q \\ (\Delta K_{g}) \cdot R_{\varepsilon}^{a} & P_{i+1} | i & P_{i+1}^{a} i \cdot \Delta' \\ 'Q \Gamma' P_{i+1}^{-1} \Delta K_{g} R_{\varepsilon}^{a} & \Delta \cdot P_{i+1}^{a} | i & \bar{Q}^{a} + \Delta P_{i+1}^{a} | i^{\Delta'} \end{bmatrix},$$
(22)

where

$$R_{\varepsilon_{i}}^{a} = \Psi_{i}(P_{i|i-1}^{a}H_{i}^{\prime} - P_{i|i-1}H_{i}^{\prime}R_{i}^{-\varepsilon}R_{\varepsilon_{i}}^{a})$$

$$R_{\varepsilon_{i}}^{a} = H_{i}P_{i|i-1}^{a}H_{i}^{\prime} + R_{i}^{a}$$
(23)

$$\bar{Q}_{1}^{a} = Q_{1}^{a} - Q_{1}^{a} \Gamma_{1}^{i} P_{1+1}^{-a} \Gamma_{1} Q_{1}^{a}$$
$$\Delta = Q_{1}^{a} \Gamma_{1}^{i} P_{1+1}^{-a} - Q_{1} \Gamma_{1}^{i} P_{1+1}^{-1}$$

The  $\Delta$ -terms in (22) vanish when correct statistics are used. Rescaling according to the assumed covariances yields

$$\tilde{\underline{y}}_{1} = \text{Block diag}\{R_{1}^{\varepsilon/2}, P_{1+1}^{1/2} | 1, Q_{1}^{1/2}\} \underline{n}_{2}$$
(24)

and finally

$$\tilde{\mathbf{x}} = \mathbf{C}_{2} \mathbf{n}_{2} \quad . \tag{25}$$

Now again  $\cot n$ ,  $\neq$  I in general, and reorthonormalization is needed to yield the orthonormal set

$$\underline{\xi}_2 \stackrel{\Delta}{=} \Lambda_2 \underline{n}_2 , \qquad (26)$$

thus we have

$$[\underline{\tilde{x}}',\underline{\xi}_2'] = \underline{n}_2'[C_2',\Lambda_2'] . \qquad (27)$$

From Remark 3 of the previous paragraph:

$$\underline{n}_1 T = \underline{n}_2$$

 $\underline{n_1}$  and  $\underline{n_2}$  span the same space, and  $\underline{\xi_1}$ , k = 1,2 are both orthonormal bases for this space. They are therefore related by an orthogonal transformation  $\overline{T}$ . Thus

$$\overline{T} \underline{\xi_1} = \underline{\xi_2}$$

$$\bar{T}\Lambda_1 T = \Lambda_2$$

Since we are basically interested in  $\Sigma_P$ , we can consider factoring (22) in a (permuted) triangular form SS' with

$$\mathbf{s} = \begin{bmatrix} \Delta_{11} & \mathbf{x} & \mathbf{x} \\ \mathbf{0} & \Delta_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} & \Delta_{33} \end{bmatrix}$$

then the (2,2)-element of  $\Lambda_2$  is exactly  $(\Sigma_p)$ and can be obtained from S similarly to (18). Therefore, we do not need to perform the complete transformation  $\overline{T}$  on the (m+n+p)-size matrix  $(\Lambda_1 T)$ , but a reduced  $\overline{T}^r$  of size n suffices to yield  $(\Sigma_p)_{i+1}$ .

Remark.

and

1. From the identity  $C_1 \text{ cov } \underline{n}_1 C'_1 = C_2$ cov  $\underline{n}_2 C'_2 = \text{ cov } \underline{\tilde{x}}$  we obtain the identities

$$R_{\varepsilon_{1}}^{a} = R_{1}^{a} + H_{1}P_{1}^{a}H_{1}^{T}$$

$$P_{1+1}^{a} = (\Phi_{1} - K_{g_{1}}H_{1})P_{1}^{a}(\Phi_{1} - K_{g_{1}}H_{1})' + \Gamma_{1}Q_{1}^{a}\Gamma_{1} + K_{g_{1}}R_{1}^{a}K_{g_{1}}'$$

- We could have started from these equations to get the array's without going to the geometrical description.
- For the covariance-covariance form we simply invert (18) and (26) to get:

$$\tilde{\mathbf{x}} = \mathbf{C}_{2} \underline{\mathbf{n}}_{k}, \ \underline{\mathbf{n}}_{k} = \Lambda_{k}^{-1} \underline{\xi}_{k}, \ k = 1,2$$

#### III. GENERAL MODELING ERRORS

We now consider errors in our assumption in the statistics and the model parameters. For simplicity however, we assume here that  $C = C^a = 0$ ; nonzero case. see [6]. In order to reduce the notational burden we omit the explicit time dependence of the model parameters. As a first step in obtaining a representation of  $\underline{\eta}_1$  in terms of an orthonormal set, we invert the equation  $\underline{\tilde{x}} = C_1 \underline{\eta}_1$  to obtain

$$\tilde{y}_{1} = R^{-l_{2}} [(\Delta H - H\Phi^{-1}\Delta\Phi)x_{1} - H\Phi^{-1}\Delta\Gamma u_{1} + v_{1}]$$

$$\tilde{X}_{1}^{+} = P_{1}^{-l_{2}} [(x_{1} - \hat{x}_{1|1-1}) + \Phi^{-1}\Delta\Phi x_{1} + \Phi^{-1}\Delta\Gamma u_{1}]$$

$$|_{1-1} = \tilde{X}_{1} + P^{-l_{2}}\Phi^{-1}\Delta\Phi x_{1} + P_{1}^{-l_{2}}\Phi^{-1}\Delta\Gamma u_{1}$$

Here

u,

 $\Delta \phi \stackrel{\Delta}{=} \phi^{a} - \phi; \ \Delta H \stackrel{\Delta}{=} H^{a} - H; \ \Delta \Gamma \stackrel{\Delta}{=} \Gamma^{a} - \Gamma .$ 

A new subtle point arises, when the system matrices are mismodeled,  $\tilde{X}_{1}^{+}$ , the pre-array variable above, is no longer equal to  $\tilde{X}_{1}$ , the post-array variable of the previous time step.

Furthermore, we see that to obtain a self-contained recursion, we must augment the  $\underline{\eta}_1$  variables with the actual state  $x_1$ .

Suppose we have available a representation for the augmented state in terms of orthonormal variables  $\xi_k$ 

$$\tilde{X}_{i} \\ \mathbf{x}_{i} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{\Delta} & \mathbf{0} \\ \mathbf{L}_{\mathbf{x}\Delta} & \mathbf{L}_{\mathbf{x}} \end{bmatrix} \begin{bmatrix} \xi_{1} \\ \xi_{2} \end{bmatrix} .$$
(28)  
$$\mathbf{u}_{i} = (\mathbf{Q}^{a})^{\mathbf{1}_{i}} \xi_{3}, \mathbf{v}_{i} = (\mathbf{R})^{\mathbf{1}_{i}} \xi_{4}$$

are certainly orthogonal to the augmented state. Therefore using (28) and the equation

$$\mathbf{x}_{i+1} = \Phi^{a} \mathbf{x}_{i} + \Gamma^{a} \mathbf{u}_{i}$$

we can set up the following representation

$$\begin{bmatrix} \tilde{y}_{1} \\ \tilde{\chi}_{1}^{*} \\ \tilde{u}_{1} \\ \mathbf{x}_{1+1} \end{bmatrix} = \begin{bmatrix} R^{-l_{2}} (\Delta H - H\bar{\Phi}^{-1} \Delta \bar{\Phi}) L_{\mathbf{x} \Delta} & R^{-l_{2}} (\Delta H - H\bar{\Phi}^{-1} \Delta \bar{\Phi}) L_{\mathbf{x}} \\ L_{\Delta} + P_{1}^{-l_{2}} \bar{\phi}^{-1} \Delta \Phi L_{\mathbf{x} \Delta} & P_{1}^{-l_{2}} \bar{\phi}^{-1} \Delta \Phi L_{\mathbf{x}} \\ 0 & 0 \\ \Phi^{a} L_{\mathbf{x} \Delta} & \Phi^{a} L_{\mathbf{x}} \end{bmatrix}$$

$$\begin{bmatrix} -R^{-l_{2}}H \bar{\phi}^{-l} \Delta \Gamma (Q^{a})^{l_{2}} & R^{-l_{2}} (R^{a})^{l_{2}} \end{bmatrix} \begin{bmatrix} \xi_{1} \\ \xi_{2} \\ \xi_{1} \end{bmatrix}$$

$$\begin{bmatrix} P_{1}^{-l_{2}} \phi_{1}^{-l} \Delta \Gamma (Q^{a})^{l_{2}} & 0 \\ Q^{-l_{2}} (Q^{a})^{l_{2}} & 0 \\ \xi_{3} \end{bmatrix} \begin{bmatrix} \xi_{3} \\ \xi_{4} \end{bmatrix}$$

$$\begin{bmatrix} (29) \\ \xi_{4} \end{bmatrix}$$

$$\begin{bmatrix} Q^{a} Q^{l_{2}} & Q^{a} \end{bmatrix} \begin{bmatrix} \xi_{1} \\ \xi_{2} \\ \xi_{3} \\ \xi_{4} \end{bmatrix}$$

This corresponds (in augmented form) to Eq. (18) of Section II. The above array corresponds to  $(\Lambda_1^*)^{-1}$ .

We apply on the left of the above array the orthogonal transformation which updates  $\underline{n}_1$  to  $\underline{n}_2$ ; that is, we apply the transpose of the transformation which updates the filter array  $C_1$  to  $C_2$ , thus

$$\begin{bmatrix} \mathbf{x} & , & \mathbf{x} & , & \mathbf{x} & , & \mathbf{x} \\ \mathbf{\tilde{X}_{i+1}} \\ \mathbf{\tilde{u}_{i}} \\ \mathbf{x} \\ \mathbf{x}_{i+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x} & , & \mathbf{x} & , & \mathbf{x} & , & \mathbf{x} \\ \mathbf{s}_{21} & \mathbf{s}_{22} & \mathbf{s}_{23} & \mathbf{s}_{24} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{\phi}^{\mathbf{a}}_{\mathbf{L}_{\mathbf{x}\Delta}} & \phi^{\mathbf{a}}_{\mathbf{L}_{\mathbf{x}}} & \Gamma^{\mathbf{a}}(\mathbf{Q}^{\mathbf{a}})^{\mathbf{s}_{2}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_{1} \\ \boldsymbol{\xi}_{2} \\ \boldsymbol{\xi}_{3} \\ \boldsymbol{\xi}_{4} \end{bmatrix}$$
(30)

(The X's denote entries of no particular interest.) We now apply orthogonal transformation on the right of the above array to bring the second and fourth block rows to the form

$$\begin{bmatrix} \tilde{X}_{i+1} \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} L_{\Delta} & 0 & 0 & 0 \\ L_{X\Delta} & L_{X} & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}$$

This yields  $L_{\Delta}$ , the square-root covariance of  $\tilde{X}_{i+1}$  and updates (28) to time i + 1. We can easily initialize (28) at time 0, when  $\tilde{X}_0 = x_0 = P_0^a \xi$ .

Remark.

- 1. There is some redundancy in considering  $\Delta\Gamma \neq 0$  and  $Q^a \neq Q$  together, because  $\Gamma$  and Q only arrive in the combination  $\Gamma Q^{i_1}$ . So, we can always define a  $Q_{equiv} = I$ ,  $\Gamma_{equiv} = \Gamma Q^2$ . In this case.  $\Delta\Gamma \neq 0$ , but  $Q_{equiv} = Q^a_{equiv}$ . Alternatively, if  $(\Phi, \Gamma Q^2)$  forms a controllable pair, then we can set  $Q_{equiv} = \Gamma Q^{\Gamma'}$  and  $\Gamma_{equiv} = I$ thus  $\Delta\Gamma = 0$  if both  $(\Phi^a, \Gamma^a Q^{a_2})$  are controllable.
  - 2. Choosing  $\phi_1 x_1$  as the "augmenting" state to  $\underline{\tilde{x}}$ , and expressing  $x^* = [(\phi_1 x_1)', \tilde{x}']$  in terms of  $\underline{n}^* := [\hat{x}'_1|_{i-1}, \underline{n}'_1]$  gives an augmented prearray  $C_1^*$  which contains  $C_1$ , and is itself in upper triangular form (note that  $\hat{x}_{11} = \frac{1}{x_{11}} = \frac{\tilde{x}_{11}}{1}$  with correct model and statistics). Similarly, we express  $\tilde{x}' = [x', (\phi_1 x_1)']$  in terms of  $\underline{n}_2^* := [\underline{n}_2', x_{1+1}]$ which yields a lower triangular  $C_2^*$  with  $C_2$  contained in it

$$\tilde{\mathbf{x}}^{*} = \begin{bmatrix} \Phi & \Sigma^{\mathbf{i}_{2}} & \mathbf{0} & \Phi \mathbf{P}^{\mathbf{i}_{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{\mathbf{i}_{2}} & \mathbf{H} \mathbf{P}^{\mathbf{i}_{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Phi \mathbf{P}^{\mathbf{i}_{2}} & \Gamma \mathbf{Q}^{\mathbf{i}_{2}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}^{\mathbf{i}_{2}} \end{bmatrix} \tilde{\mathbf{n}}^{*}$$

$$*_{\underline{\mathbf{x}}} = \begin{bmatrix} \mathbf{R}^{\mathbf{E}/2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K} & \mathbf{P}_{\mathbf{1}+1}^{\mathbf{i}_{2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{K} & \mathbf{P}_{\mathbf{1}+1}^{\mathbf{i}_{2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} & \mathbf{Q}^{-\mathbf{i}_{2}} & \mathbf{0} \\ -\Gamma \mathbf{D} & \mathbf{P}_{\mathbf{1}+1}^{\mathbf{i}_{2}} - \Gamma \mathbf{E} & -\Gamma \mathbf{Q}^{-\mathbf{i}_{2}} & \mathbf{I} \end{bmatrix}$$

The augmented  $\Lambda$ -arrays are quite involved however, see [6].

# IV. SMOOTHING

We now briefly describe how to extend the algorithm of Section III to the square-root form of the two-filter fixed-interval smoother. Each filter will have an error analysis addendum propagated according to the idea of Section III. Here we will just point out the modifications needed in the separate filters to facilitate a smoothing solution and demonstrate how to combine information from the two filters to obtain an  $L_{\Delta}$  for the smoothed estimate.

The backward filter is implemented in information form since it is initialized with infinite a priori covariance. However, the  $\underline{m}_k \ k = 1,2$  variables are the same for both information and covariance filters. Therefore, when the only modeling error is in the statistics, the error analysis algorithm is unchanged from Section II. When there are mismodeled system matrices and the  $\underline{n}$ are augmented by  $x_i$ , it must be remembered that  $u_i$ is not orthogonal to  $x_{i+1}$ . Probably the most straightforward way to obtain a representation for  $(\underline{n}_i', \mathbf{x}')'$  in terms of an orthonormal set (that is to obtain the array analogous to the inverse of (18), (26) is to decompose  $u_i$  as

$$u_{i} = Q_{i}^{a} \Gamma_{i}^{a} \Pi_{i+1}^{-1} x_{i+1}^{+} \tilde{u}_{i} | x_{i+1}^{+}, \Pi_{i+1} = cov(x_{i+1}^{-}) \quad (31)$$

where  $\tilde{u}_{i|x_{i+1}}$  is orthogonal to  $u_{i+1}$  and has

covariance

$$\boldsymbol{Q_i^b} = (\boldsymbol{Q_i^a} - \boldsymbol{Q_i^a}\boldsymbol{\Gamma_i^a}\boldsymbol{\Pi_{i+1}^{-1}}\boldsymbol{\Gamma_i^a}\boldsymbol{Q_i^a})$$

When there are mismodeled system matrices, the augmented state in both forward and backward error analysis arrays should be represented in the form

$$\begin{bmatrix} \mathbf{X} \\ \tilde{\mathbf{X}} \end{bmatrix} = \begin{bmatrix} \Pi^{\frac{1}{2}} & \mathbf{0} \\ \mathbf{M}_{\Delta \mathbf{X}} & \mathbf{M}_{\Delta} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \text{ rather than } \begin{bmatrix} \mathbf{X} \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{\Delta} & \mathbf{0} \\ \mathbf{L}_{\mathbf{X}\Delta} & \mathbf{L}_{\mathbf{X}} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$
(32)

Decomposing each  $\tilde{X}_1$  into a component correlated with and a component orthogonal to the actual state enables us to get a joint representation for  $\tilde{X}^f$  (from forward filter) and  $\tilde{X}^b$  (from backward filter). We have the separate representations

$$\begin{bmatrix} \mathbf{x}_{1} \\ \mathbf{z}_{1}^{\mathbf{f}} \end{bmatrix} = \begin{bmatrix} \mathbf{\pi}_{1}^{\mathbf{f}_{2}} & \mathbf{0} \\ \mathbf{M}_{\Delta \mathbf{x}}^{\mathbf{f}} & \mathbf{M}_{\Delta}^{\mathbf{f}} \end{bmatrix} \boldsymbol{\xi}_{2}^{\mathbf{f}} \end{bmatrix} \text{ and } \tilde{\mathbf{X}}_{1}^{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \mathbf{\pi}_{2}^{\mathbf{f}_{2}} & \mathbf{0} \\ \mathbf{M}_{\Delta \mathbf{x}}^{\mathbf{b}} & \mathbf{M}_{\Delta}^{\mathbf{b}} \end{bmatrix} \boldsymbol{\xi}_{2}^{\mathbf{b}} \end{bmatrix}$$
(33)

 $\xi_1^f$  must be related to  $\xi_1^b$  by an orthogonal transformation.  $\xi_1^f = \xi_1^b$  if we insist on a unique form for M7, say lower triangular with positive diagonal elements. Some study reveals [6] that  $\xi_1^f$  is a function of  $\{u_0, \ldots, u_{i-1}\}$  and  $\{v_0, \ldots, v_{i-1}\}$ , while  $\xi_2^h$  is a function of  $\{u_1, \ldots, u_N\}$  and  $\{v_1, \ldots, v_N\}$  where the smoothing interval is  $i = 1, \ldots, N$ . Thus  $\xi_2^f$  and  $\xi_2^b$  are orthogonal and we may write

When we have only incorrect modeling of statistics this simplifies to

₹ <sup>b</sup> <sub>1</sub>	=	r <sub>p</sub>	0	ξ <sup>b</sup>
Ĩ, Î		0	$\mathbf{L}_{\Delta}^{\mathbf{f}}$	ξ <sup>f</sup>

The forward filter may be implemented in either covariance or information form. For purposes of illustration, let us assume a covariance forward filter and an information backward filter. To combine the estimates of the two filters to obtain a smoothed estimate we may interpret

 $(P_{1}^{b})^{-l_{2}} \stackrel{h}{\approx} \stackrel{h}{=} d_{1}^{b}$  as an additional measurement with output matrix  $(P_{1}^{b})^{-l_{2}}$  which is to be used to update  $\hat{x}_{1|1-1}^{f}$  [5]. This interpretation is illustrated below with corresponding quantities one under the other

$$y_{i} = H x_{i} + v_{i}$$
  
 $d_{i}^{b} = (P_{i}^{b})^{-b_{i}} x_{i} + \tilde{X}_{i}^{b}$ 
(35)

The "measurement noise"  $\tilde{\chi}_1^b$  is assumed by the filter to have unit covariance; so, the appropriate measurement update is given by the following arrays:

$$\begin{bmatrix} \mathbf{x}_{i}^{b} - (\mathbf{P}_{i}^{b})^{-l_{i}} \hat{\mathbf{x}}_{i|1-1}^{f} \\ \mathbf{x}_{i} - \hat{\mathbf{x}}_{i|1-1}^{f} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & (\mathbf{P}_{i}^{b})^{-l_{i}} (\mathbf{P}_{i}^{f})^{l_{i}} \\ \mathbf{0} & (\mathbf{P}_{i}^{f})^{l_{i}} \end{bmatrix} \tilde{\mathbf{X}}_{i}^{b} \\ \tilde{\mathbf{X}}_{i}^{f} \end{bmatrix}$$
(36)

Lower triangularizing (36) with orthogonal transformations on the right gives the post-array

The smoothed estimate of x; is

$$\hat{\mathbf{x}}_{i}^{s} = \hat{\mathbf{x}}_{i|i-1}^{f} + \kappa_{i}^{*} v_{i}^{-1} [d_{i}^{b} - (P_{i}^{b})^{-l_{2}} \hat{\mathbf{x}}_{i|i-1}^{f}]$$
(37)

and

$$x_{1} - \hat{x}_{1}^{S} = (P_{1}^{S})^{\frac{1}{2}} \tilde{X}_{1}^{S}$$

The set  $\{\tilde{X}_{j}^{s}, v_{j}^{s}\}$  is obtained from the set  $\{\tilde{X}_{j}^{b}, \tilde{X}_{1}^{f}\}$ through the array update transformation. Applying the same transformation to the representation (34) for  $\{\tilde{X}_{i}^{b}, \tilde{X}_{i}^{f}\}$  yields

$$\begin{bmatrix} \mathbf{v}_{1}^{s} \\ \tilde{\mathbf{x}}_{1}^{s} \end{bmatrix} = \begin{bmatrix} \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{s}_{21} & \mathbf{s}_{22} & \mathbf{s}_{23} \end{bmatrix} \boldsymbol{\xi}_{1}^{t} \\ \boldsymbol{\xi}_{2}^{b} \\ \boldsymbol{\xi}_{2}^{f} \end{bmatrix}$$

An additional orthogonal transformation applied on the right will give

$$\tilde{X}_{1}^{s} = [L_{\Delta}^{s} \quad 0 \quad 0] \quad \bar{\xi}_{1} \\ & \bar{\xi}_{2} \\ & \bar{\xi}_{3} \end{bmatrix}$$
(38)  
$$(P_{s}^{a})^{\frac{1}{2}} = (P_{1}^{s})^{\frac{1}{2}} L_{\Delta}^{s} .$$

#### V. CONCLUSION

By viewing the square-root arrays as representations of the estimate residuals in terms of orthonormal sets of random variables we have expressed a framework for the known square-root error analysis algorithm. This framework led us to a new algorithm presented herein which is compatible with either the covariance or information implementations of the filter.

#### REFERENCES

- [1] M. Morf and T. Kailath, "Square-Root Algorithms for Least-Squares Estimation," <u>IEEE</u> <u>Trans. on Automatic Control</u>, vol. AC-20, no. 4, pp. 487-497, August 1975.
- [2] G.J. Bierman, <u>Factorization Methods for Discrete Sequential Estimation</u>, New York: Academic Press, 1976.
- [3] C.L. Thornton, "Triangular Covariance Factorizations for Kalman Filtering," JPL Tech. Memo 33-798, October 1976.
- [4] M. Gevers and T. Kailath, "An Innovations Approach to Least-Squares Estimation, Part VI: Discrete-Time Innovations Representations and Recursive Estimation," <u>IEEE Trans. on Automatic Control</u>, vol. AC-18, no. 6, pp. 588-600, December 1973.
- [5] P.G. Kaminski, A.E. Bryson and S.S. Schmidt, "Discrete Square-Root Filtering, A Survey of

Current Techniques," <u>IEEE Trans. on Automatic</u> <u>Control</u>, vol. AC-16, no. 6, pp. 727-735, December 1971.

[6] M. Morf, E. Verriest, J. Dobbins and T. Kailath, "Square-Root Algorithms for Model Sensitivity Analysis," submitted to <u>IEEE</u> <u>Trans. on Automatic Control</u>, 1977.

> . We new brackly coverate on the uncest the algorefers of Shellow Alfold The Ale square-rouse form of the rouseks or fixed to be mained and here an effect as rousing to be descent Striken III bere affort as rousing to be descent Striken III bere to separate to are the routilization a second as the separate to are to be to consider a receiver it withe an each life of the total beaches to the separate to are to be to consider a figure it withe a constitute to be to consider a figure it withe a base life of the second as