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COMPUTING ECONOMIC EQUILIBRIA ON AFFINE NETWORKS WITH LEMKE'S A--ETC(U)
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BY

RICHARD ASMUTH, B. CURTIS EAVES,
and ELMOR L. PETERSON

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Richard Asmuth*, B. Curtis Eaves*,

and Elmor L. Peterson**

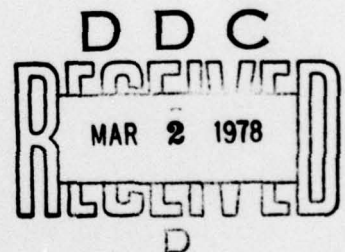
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1. INTRODUCTION

Consider a multicommodity transshipment problem where the prices at each location are an affine function of the supplies and demands at that location and the shipping costs on a link are an affine function of the quantities shipped on that link. A system of prices, supplies, demands, and shipments is defined to be an equilibrium, if there is a balance in the shipments, supplies, and demands of goods at each location, if local prices do not exceed the cost of importing, and if shipments are price efficient. Lemke's algorithm is used to compute an equilibrium in a finite number of steps.

This paper extends the works of Takayama and Judge [4] who utilized quadratic programming. Our approach and results are more direct and more general; we solve the equilibrium conditions directly without passage to an optimization problem. In [3] an even more general equilibrium problem is formulated as a complementarity problem, but in a different manner than that used here and without the results obtained here.

Our problem is conveniently represented with a directed graph $(\mathcal{N}, \mathcal{L})$ with a finite number of nodes $\mathcal{N} = \{1, \dots, n\}$ and a finite number of links $\mathcal{L} = \{1, \dots, \ell\}$. Such a directed graph with $n = 4$

and $\ell = 5$ is illustrated below.

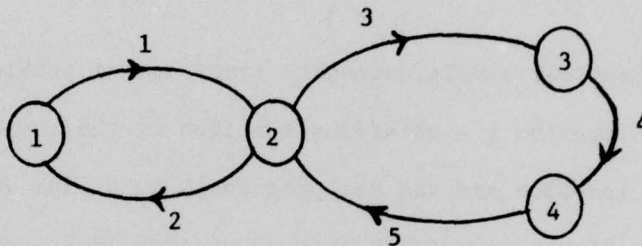
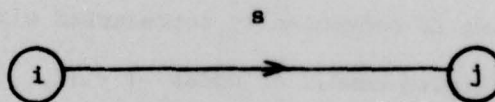


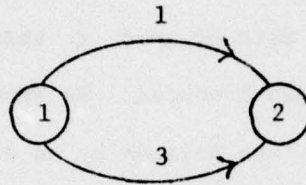
Figure 1

Each node i in \mathcal{N} represents a producer/consumer at a specific spatial/temporal location. Each link s in \mathcal{L} represents a specific transport facility for transferring commodities between nodes, that is, locations, and each link s is oriented to coincide with the direction of a possible transfer activity. For example, if link s has head j and tail i , that is,



then the transfer of goods along s is from node i to node j .

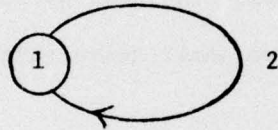
We permit multiple links between the same pair of nodes, e.g.,



$$1, 2 \in \mathcal{N}$$

$$1, 3 \in \mathcal{L}$$

but we do not permit loops, e.g.,



$$1 \in \mathcal{N}$$

$$2 \in \mathcal{L}$$

(Technically speaking, \mathcal{L} indexes a finite subset of $\{(i, j, z) : i, j \in \mathcal{N} ; i \neq j ; z = 1, 2, \dots\}$.)

We are concerned with the supply, demand, and transshipment of m goods $g = 1, \dots, m$ in the network $(\mathcal{N}, \mathcal{L})$. A good g could be a raw material, an intermediate product, or a finished product.

Let the variable $f_s = (f_{s1}, \dots, f_{sm})$ represent the quantities of the various goods shipped along link s and let the per unit shipping cost of the various goods moved along link s be an affine function $C_s f_s + c_s$. Hence, $f_s \cdot (C_s f_s + c_s)$ is the total shipping cost on link s .

Let the variable $p_i = (p_{i1}, \dots, p_{im})$ represent the prices of the various goods at node i . Let the variable $h_i = (h_{i1}, \dots, h_{im})$ represent the net exports from node i of the various goods; that is, h_{ig} is positive when node i produces more of good g than it consumes, and is negative when more is consumed than produced. We assume that the prices p_i and exports h_i at node i are related by an affine function $p_i = A_i h_i + a_i$.

Our network is completely specified by the graph $(\mathcal{N}, \mathcal{L})$, the $(A_i | a_i)$ for i in \mathcal{N} , and the $(C_s | c_s)$ for s in \mathcal{L} . For the graph of Figure 1 we could have, for example, the data of Figure 2 corresponding to two goods; in Section 4 we shall solve this example.

$$\begin{aligned}
 (A_1 | a_1) &= \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 2 & 1 & 2 \end{array} \right) & (A_2 | a_2) &= \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & -1 \end{array} \right) \\
 (A_3 | a_3) &= \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 1 & -1 \end{array} \right) & (A_4 | a_4) &= \left(\begin{array}{cc|c} 0 & -2 & 2 \\ 2 & 0 & 2 \end{array} \right) \\
 (C_1 | c_1) &= \left(\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & 1 & -1 \end{array} \right) & (C_2 | c_2) &= \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -2 \end{array} \right) & (C_3 | c_3) &= \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 1 \end{array} \right) \\
 (C_4 | c_4) &= \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array} \right) & (C_5 | c_5) &= \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 2 & 1 & 1 \end{array} \right)
 \end{aligned}$$

Figure 2

To define an equilibrium in a precise manner we need some additional notation. For a link s , let ns be the node at the tail of s , and let sm be the node at the head of s . For example, with respect to Figure 1, we have $1l = 1$ and $5m = 2$. For a node i , let $\rightarrow i$ be the set of links s entering i , and $i \rightarrow$ be the set of links leaving i . Hence, link s is in $i \rightarrow$ or $\rightarrow i$ if and only if $ns = i$ or $sm = i$, respectively. With respect to Figure 1, we have $1 \rightarrow = \{1\}$ and $\rightarrow 2 = \{1, 5\}$.

A system of prices $p = (p_1, \dots, p_n)$, exports $h = (h_1, \dots, h_n)$, and shipments $f = f_1, \dots, f_\ell$ is by definition an equilibrium if the following five conditions hold.

$$(1) \left\{ \begin{array}{ll} (a) & f_s \geq 0 \quad s \in \mathcal{L} \\ (b) & h_i = \sum_{i \rightarrow} f_s - \sum_{\rightarrow i} f_s \quad i \in \mathcal{N} \\ (c) & p_i = A_i h_i + a_i \quad i \in \mathcal{N} \\ (d) & p_{ns} + C_s f_s + c_s \geq p_{sm} \quad s \in \mathcal{L} \\ (e) & f_s \cdot (p_{ns} + C_s f_s + c_s - p_{sm}) = 0 \quad s \in \mathcal{L} \end{array} \right.$$

Condition (1a) requires that the shipments or flows be nonnegative, however, note that the prices and exports, may be positive, negative,

or zero. Condition (1b) represents the conservation of goods at node i . Condition (1c) expresses the fact that node i produces and/or consumes according to the prices. Condition (1d) is a price stability condition requiring that the local price must not exceed a neighboring price plus transportation costs. Finally, condition (1e) requires that goods only be shipped (in positive amounts) on price efficient links, that is, if f_{sg} is positive, then

$$(p_{ns} + C_s f_{sg} + c_s)g = (p_{sn})g$$

the price of g at ns plus transportation costs along s to sn must equal the price of g at sn .

In the following sections we prove and illustrate the following theorems. Theorem 1 provides conditions under which Lemke's algorithm will find an equilibrium or show that none exists. Theorems 2 and 3 give additional conditions under which Lemke's algorithm will always find an equilibrium.

Theorem 1. If each A_i is positive semidefinite and each C_s is copositive plus, then Lemke's algorithm generates an equilibrium or demonstrates that no equilibrium exists. \square

Theorem 2. If each A_i is positive semidefinite and each C_s is strictly copositive, then Lemke's algorithm generates an equilibrium. \square

Theorem 3. If each A_i is positive definite and if for each link s , C_s is copositive plus and $f_s \geq 0$, $f_s \cdot C_s f_s = 0$, $c_s \cdot f_s < 0$ has no solution, then Lemke's algorithm generates an equilibrium. \square

2. THE LINEAR COMPLEMENTARITY PROBLEM

Here we recast the conditions for equilibrium into the form of the linear complementarity problem, namely, into the form

$$(2) \quad \begin{cases} w = Mf + q \\ w \geq 0 \quad f \geq 0 \quad w \cdot f = 0 \end{cases}$$

It is to this system that Lemke's algorithm applies.

With regard for (1d) we introduce slack variables

$w_s = (w_{s1}, \dots, w_{sm})$ for s in \mathcal{L} by setting

$$w_s = p_{ns} + C_s f_s + c_s - p_{sn} \quad s \in \mathcal{L}$$

For notational convenience let us define \bar{c}_s to be $c_s + a_{ns} - a_{sn}$.

Now use (1b) and (1c) to eliminate h_i and p_i in (1d) and

(1e) to obtain

$$(3) \quad \begin{cases} (a) \quad w_s = A_{ns} \left(\sum_{ns \rightarrow} f_t - \sum_{\rightarrow ns} f_t \right) \\ \quad \quad \quad - A_{sn} \left(\sum_{sn \rightarrow} f_t - \sum_{\rightarrow sn} f_t \right) + C_s f_s + \bar{c}_s & s \in \mathcal{L} \\ (b) \quad f_s \cdot w_s = 0 & s \in \mathcal{L} \\ (c) \quad f_s \geq 0 \quad w_s \geq 0 & s \in \mathcal{L} \end{cases}$$

Note that $s\pi\rightarrow$ is the set of all links whose tails are the head of s , etc. Hence, if we solve (3) and use (1b) and (1c) to compute h_i and p_i we have solved (1) and have an equilibrium; conversely, any solution to (1) yields a solution to (3).

Equation (3a) can be rewritten as

$$\begin{aligned}
 w_s &= (A_{\pi s} + A_{s\pi} + C_s) f_s \\
 &+ (A_{\pi s} + A_{s\pi}) \sum_{\alpha} f_t - (A_{\pi s} + A_{s\pi}) \sum_{\beta} f_t \\
 &+ A_{\pi s} \sum_{\gamma} f_t - A_{\pi s} \sum_{\delta} f_t \\
 &+ A_{s\pi} \sum_{\xi} f_t - A_{s\pi} \sum_{\eta} f_t + \bar{c}_s
 \end{aligned}
 \tag{4}$$

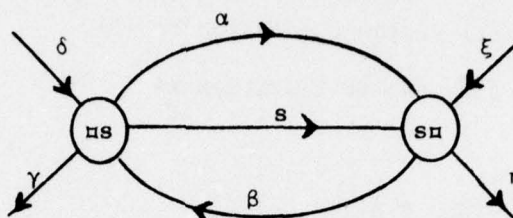
where

$$\alpha = ((\pi s \rightarrow) \cap (\rightarrow s \pi)) \sim s \quad \beta = (s \pi \rightarrow) \cap (\rightarrow \pi s)$$

$$\gamma = (\pi s \rightarrow) \sim (\rightarrow s \pi) \quad \delta = (\rightarrow \pi s) \sim (s \pi \rightarrow)$$

$$\xi = (\rightarrow s \pi) \sim (\pi s \rightarrow) \quad \eta = (s \pi \rightarrow) \sim (\rightarrow \pi s)$$

Note that the index sets α through η are pairwise disjoint; the following schema illustrates the partition:



Now let us cast the system (3) into the linear complementarity problem. Define an $lm \times lm$ partitioned matrix M by $M = (M_{st})$ with $s, t = 1, \dots, l$ and M_{st} is an $m \times m$ matrix defined by

$M_{st} =$	$A_{ns} + A_{sn} + C_s$	$A_{ns} + A_{sn}$	$-A_{ns} - A_{sn}$	A_{ns}	$-A_{ns}$	A_{sn}	$-A_{sn}$	0
if	$t = s$	$t \in \alpha$	$t \in \beta$	$t \in \gamma$	$t \in \delta$	$t \in \xi$	$t \in \eta$	Otherwise

Define an lm vector q to be $(\bar{c}_1, \dots, \bar{c}_l)$. Using (4) it is now evident that solving (2) is equivalent to solving (3).

As an example, consider the multicommodity network of Figure 1, we have displayed the matrix $(M|q)$ in Figure 3.

	1	2	3	4	5	
1	$A_1 + A_2 + C_1$	$-A_1 - A_2$	$-A_2$	0	A_2	\bar{c}_1
2	$-A_1 - A_2$	$A_1 + A_2 + C_2$	A_2	0	$-A_2$	\bar{c}_2
3	$-A_2$	A_2	$A_2 + A_3 + C_3$	$-A_3$	$-A_2$	\bar{c}_3
4	0	0	$-A_3$	$A_3 + A_4 + C_4$	$-A_4$	\bar{c}_4
5	A_2	$-A_2$	$-A_2$	$-A_4$	$A_2 + A_4 + C_5$	\bar{c}_5

Figure 3

3. SOLUTION

We now prove the three theorems stated in Section 1. A square matrix N is said to be positive definite if xNx is positive for all nonzero x . N is said to be positive semidefinite if xNx is nonnegative for all x . N is said to be strictly copositive if xNx is positive for all nonzero nonnegative x . N is said to be copositive plus, if xNx is nonnegative for all nonnegative, x and if xNx equals zero for nonnegative x implies $(N^T + N)x$ equals zero. Incidentally, for clarity, we remark that the A_i , C_s and M are not assumed to be symmetric.

For any system of shipments f and exports h satisfying the conservation equations, we have the equality

$$(5) \quad f \cdot Mf = \sum_{\mathcal{N}} h_i \cdot A_i h_i + \sum_{\mathcal{S}} f_s \cdot C_s f_s$$

From our formula (5) we obtain the following lemma.

Lemma 4. If each A_i is positive semidefinite and each C_s is copositive plus, then M is copositive plus.

Proof. If each C_s is copositive plus, then so is the matrix $C = \text{diag}(C_1, \dots, C_\ell)$. Also, clearly the matrix $M - C$ is positive semidefinite, and, therefore, copositive plus. Finally, the sum of two copositive plus matrices is copositive plus. \square

Towards solving the linear complementarity problem (2) we apply Lemke's algorithm to the system

$$(6) \quad \begin{aligned} &Iw - Mf - ez = q \\ &w \geq 0 \quad f \geq 0 \quad w \cdot f = 0 \quad z \geq 0 \end{aligned}$$

Briefly, Lemke's algorithm proceeds by generating a path of solutions to (6) beginning with the family of solutions for which $f = 0$. A complete description of Lemke's algorithm can be found in Lemke [2], or [1].

Theorem 5 (Lemke). Suppose M is copositive plus. If Lemke's algorithm is applied to (6) where $e = (e_1, e_2, \dots, e_k, \dots)$, $e_k = 1$ if $q_k \leq 0$, and $e_k = 0$ if $q_k > 0$, then either a solution to (6) with $z = 0$ is generated or a vector f is generated satisfying (7).

$$(7) \quad fM \leq 0 \quad f \cdot Mf = 0 \quad f \cdot q < 0 \quad f \geq 0$$

Proof. See [1]. \square

Theorem 1 now follows from Lemma 4 and Theorem 5; note that (2) cannot have a solution if (7) does.

Proof of Theorem 2. Suppose the hypotheses of Theorem 2 are satisfied but Lemke's algorithm does not find a solution. From (5) and (7)

$$0 = f \cdot Mf = \sum h_1 \cdot A_1 h_1 + \sum f_s \cdot C_s f_s$$

Since each term is nonnegative, each term is zero. Since each C_s is strictly copositive, each f_s equals zero. But this contradicts $f \cdot q < 0$. So Lemke's algorithm will generate a solution. \square

Proof of Theorem 3. Suppose the hypotheses of Theorem 3 are satisfied, but Lemke's algorithm does not find a solution. From (5) and (7)

$$0 = f \cdot Mf = \sum h_1 \cdot A_1 h_1 + \sum f_s \cdot C_s f_s$$

Since each term is nonnegative, each term is zero. Each h_1 is zero since each A_1 is positive definite. Also

$$0 > \sum f_s \cdot (c_s + a_{ns} - a_{sn}) = \sum c_s \cdot f_s + \sum a_1 \cdot h_1 = \sum c_s \cdot f_s$$

so for some link s , $c_s \cdot f_s < 0$, $f_s \cdot C_s f_s = 0$, $f_s \geq 0$ which contradicts the hypotheses. So Lemke's algorithm will generate a solution. \square

4. EXAMPLE

We shall now illustrate the preceding results by solving the 2-commodity network problem defined by the graph of Figure 1 and the data of Figure 2. (M, q) is shown in Figure 4, see Figure 3. Lemke's algorithm is applied as described in Theorem 5. In view of Theorem 2 convergence to an equilibrium is guaranteed; the flow generated is

$$\begin{aligned} f_1 &= \begin{pmatrix} .2353 \\ .7059 \end{pmatrix} & f_2 &= \begin{pmatrix} 0 \\ 2.2941 \end{pmatrix} \\ f_3 &= \begin{pmatrix} 1.5294 \\ 0 \end{pmatrix} & f_4 &= \begin{pmatrix} 1.0098 \\ .2451 \end{pmatrix} & f_5 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Hence, (p, h, f) is an equilibrium when the prices p and exports h are computed according to (1b) and (1c).

$$\begin{aligned} h_1 &= \begin{pmatrix} .2353 \\ -1.5882 \end{pmatrix} & h_2 &= \begin{pmatrix} 1.2941 \\ 1.5882 \end{pmatrix} \\ h_3 &= \begin{pmatrix} -.5196 \\ .2451 \end{pmatrix} & h_4 &= \begin{pmatrix} -1.0098 \\ -.2451 \end{pmatrix} \\ p_1 &= \begin{pmatrix} -.7647 \\ .8824 \end{pmatrix} & p_2 &= \begin{pmatrix} -1.2941 \\ .5882 \end{pmatrix} \\ p_3 &= \begin{pmatrix} 1.2353 \\ -1.2745 \end{pmatrix} & p_4 &= \begin{pmatrix} 2.4902 \\ .0196 \end{pmatrix} \end{aligned}$$

4	-1	-2	1	-1	1	0	0	1	-1	-1
2	3	-2	-2	0	-1	0	0	0	1	2
-2	1	3	0	1	-1	0	0	-1	1	-1
-2	-2	2	3	0	1	0	0	0	-1	-5
-1	1	1	-1	3	-3	-1	1	-1	1	-2
0	-1	0	1	2	3	-1	-1	0	-1	1
0	0	0	0	-1	1	2	-2	0	2	0
0	0	0	0	-1	-1	4	2	-2	0	-3
1	-1	-1	1	-1	1	0	2	2	-3	2
0	1	0	-1	0	-1	-2	0	4	2	4

Figure 4

REFERENCES

1. EAVES, B. C., "The Linear Complementarity Problem", Management Science, 17, 9, May 1971, 612-634.
2. LEMKE, C. E., "Bimatrix Equilibrium Points and Mathematical Programming", Management Science, 11, 1965, 681-689.
3. PETERSON, E. L., "The Conical Duality and Complementarity of Price and Quantity for Multicommodity Spatial and Temporal Network Allocation Problems", Center for Math. Studies in Economics and Management Science, Disc. Paper #207, Northwestern University, Evanston, Ill., March 1976.
4. TAKAYAMA, T., and G. G. JUDGE, Spatial and Temporal Price and Allocation Models, 1971, North-Holland Pub. Co.

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19. ABSTRACT (Continue on reverse side if necessary and identify by block number) Economic equilibria between supply and demand on certain affine multi-commodity networks are characterized as solutions to a linear complementarity problem to which Lemke's algorithm is applied.			

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