AD A050582

(12)

OFFICE OF NAVAL RESEARCH

CONTRACT N00014-75-C-0196

TECHNICAL REPORT NO.39

ON THE EXISTENCE OF AN ELASTIC POTENTIAL FOR A SIMPLE MATERIAL WITHOUT MEMORY

See 1473 in back

BY

ELI STERNBERG AND J. K. KNOWLES

DIVISION OF ENGINEERING AND APPLIED SCIENCE

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA

JANUARY 1978

Office of Naval Research
Contract N00014-75-C-0196

Technical Report No. 39

On the existence of an elastic potential for a simple material without memory

by

Eli Sternberg and J. K. Knowles

Division of Engineering and Applied Science
California Institute of Technology
Pasadena, California

January 1978

On the existence of an elastic potential for a simple material without memory*

by

Eli Sternberg and J. K. Knowles
California Institute of Technology

Summary

It is shown that a compressible elastic body—not necessarily homogeneous or isotropic—is hyperelastic provided the work done by all external forces acting on an arbitrary part of the body vanishes for every sufficiently smooth cyclic motion, in which each material point returns to its initial position with a velocity equal to its initial velocity.

Introduction

Adhering to the terminology of Truesdell and Noll [1], we consider a body composed of a—possibly non-homogeneous—compressible elastic material (simple material without memory) in the context of the purely mechanical theory of continuous media. If such a material is hyperelastic, and thus possesses a stored-energy function (elastic potential), it is seen at once from the power-identity appropriate to dynamical processes in continuum mechanics that the total work done by the actual surface tractions and body forces external with respect to an essentially arbitrary part of the body vanishes over any "cyclic motion". The latter term connotes a suitably

---

*The results communicated in this paper were obtained in the course of an investigation supported by Contract N00014-75-C-0196 with the Office of Naval Research in Washington, D. C.

regular motion in which the initial and terminal positions of the body, as well as the initial and final velocity fields, are coincident. It is the aim of the present paper to establish a converse of this proposition. Specifically, we seek to show that the vanishing of the work done by <u>all</u> external forces acting on an arbitrary portion of a <u>not necessarily homogeneous</u> elastic body in any cyclic motion is also <u>sufficient</u> for the existence of an elastic potential. In proving this claim we are incidentally led to a rather simple explicit integral representation for the stored-energy function of a hyperelastic material.

It should be made clear how the present result differs in its purpose and underlying hypotheses from a related result that is an immediate consequence of the first work theorem introduced by Truesdell and Noll in Article 83 of [1]. In this theorem the authors of [1] confine their attention to <u>homogeneous</u> elastic bodies. They consider a suitably smooth one-parameter family of <u>homogeneous deformations</u>, and deal with the <u>work done by the actual surface tractions alone</u> during such a "homogeneous deformation process". Their conclusions, which follow from familiar properties of line integrals applied to paths in the space of second-order tensors, in particular justify the following assertion: a homogeneous elastic body is hyperelastic if and only if the traction-work vanishes over any homogeneous deformation process that is closed in the sense of starting from and terminating in the same position of the body. The power-identity mentioned earlier, or indeed the equation of motion, play no role in the setting of the work theorem to which we are referring. Further, no condition analogous to the required coincidence of the initial and terminal <u>velocity</u> fields in a cyclic motion needs to be imposed. The inevitable presence of this velocity condition among the hypotheses of the present theorem, is a source of analytical complications.

1. Notation and other preliminaries.

In this section we explain the notation used in what follows and assemble various continuum-mechanical ingredients[1] that will be needed in the subsequent analysis.

Throughout the present paper $\mathcal{E}$ stands for a three-dimensional Euclidean point-space (reference space). $\mathcal{L}$ is the space of all nonsingular second-order tensors, while $\mathcal{L}_+$ and $\mathcal{L}_-$ denote the set of all second-order tensors with positive or negative determinant, respectively. Next, $\mathcal{S}$ is the space of all symmetric tensors of the second order and $\mathcal{S}^+$ the set of all such tensors that are positive definite. Finally, $\mathcal{O}$ stands for the collection of all orthogonal second-order tensors.

Letters in boldface denote tensors of positive order in three dimensions. Further, if $\mathbf{v}$ is a vector and $\mathbf{T}$ a tensor of order two, the same symbols will also be employed – in the appropriate context – to denote the column matrix $[v_i]$ and the square matrix $[T_{ij}]$ of scalar components of $\mathbf{v}$ and $\mathbf{T}$ in the underlying orthogonal cartesian coordinate frame. Here and in the sequel Latin subscripts are understood to range over the integers (1,2,3). Summation over repeated subscripts is taken for granted.

Let $\mathcal{B}$ be a body which in an arbitrary fixed reference configuration occupies a closed region $\mathcal{R}$ (reference region) in $\mathcal{E}$. A motion of $\mathcal{B}$ is a one-parameter family of mappings of $\mathcal{R}$ into $\mathcal{E}$,

$$\boldsymbol{\chi} = \hat{\boldsymbol{\chi}}(\mathbf{x}, t) = \mathbf{x} + \mathbf{u}(\mathbf{x}, t) \ \forall\ \mathbf{x} \in \mathcal{R} \ (t_0 \le t \le t_1) , \qquad (1.1)$$

[1] See [1], as well as Truesdell and Toupin [2], in connection with this expository summary of prerequisites.

depending on the time t as parameter, $\underset{\sim}{u}(\cdot, t)$ being the instantaneous displacement field. The position vector $\underset{\sim}{x}$ of a point in $\mathcal{R}$ may be regarded as a particle label; thus x_i are material and y_i spatial coordinates. A motion of $\mathcal{B}$ will be called an <u>admissible motion</u>, provided $\hat{\underset{\sim}{y}}$ satisfies the following regularity requirements:

$$\text{(i)} \qquad \hat{\underset{\sim}{y}} \in C^2(\mathcal{R} \times [t_o, t_1]) \ ,$$

i. e., $\hat{\underset{\sim}{y}}$ is twice continuously differentiable on its domain of definition;[1]

$$\text{(ii)} \qquad \hat{\underset{\sim}{y}}(\cdot, t) \text{ is one-to-one on } \mathcal{R} \ \forall \ t \in [t_o, t_1] \ ,$$

so that if $\hat{\underset{\sim}{x}}(\cdot, t)$ is the inverse of the mapping $\hat{\underset{\sim}{y}}(\cdot, t)$, one has

$$\hat{\underset{\sim}{y}}(\hat{\underset{\sim}{x}}(\underset{\sim}{y}, t), t) = \underset{\sim}{y} \ \forall \ \underset{\sim}{y} \in \mathcal{R}_t = \hat{\underset{\sim}{y}}(\mathcal{R}, t) \ \ (t_o \le t \le t_1) \ ; \qquad (1.2)$$

$$\text{(iii)} \qquad \hat{\underset{\sim}{x}}(\cdot, t) \in C^1(\mathcal{R}_t) \ \forall \ t \in [t_o, t_1] \ .$$

The particle velocity and acceleration relative to $\mathcal{E}$ associated with an admissible motion of $\mathcal{B}$ are defined by

$$\underset{\sim}{v}(\underset{\sim}{x}, t) = \dot{\underset{\sim}{u}}(\underset{\sim}{x}, t) \ , \ \underset{\sim}{a}(\underset{\sim}{x}, t) = \ddot{\underset{\sim}{u}}(\underset{\sim}{x}, t) \ \forall \ \underset{\sim}{x} \in \mathcal{R} \ \ (t_o \le t \le t_1) \qquad (1.3)^2$$

and we adopt the notation

$$\bar{\underset{\sim}{v}}(\underset{\sim}{y}, t) = \underset{\sim}{v}(\hat{\underset{\sim}{x}}(\underset{\sim}{y}, t), t) \ , \ \bar{\underset{\sim}{a}}(\underset{\sim}{y}, t) = \underset{\sim}{a}(\hat{\underset{\sim}{x}}(\underset{\sim}{y}, t), t) \ \forall \ \underset{\sim}{y} \in \mathcal{R}_t \ \ (t_o \le t \le t_1) \qquad (1.4)$$

for the spatial velocity and acceleration fields. By a <u>cyclic motion</u> of $\mathcal{B}$ we mean an admissible motion subject to the conditions

---

[1] If A and B are sets, we write A×B for the cartesian product of A and B.

[2] A dot placed above a functional symbol indicates differentiation with respect to the time.

$$\hat{y}(x, t_o) = \hat{y}(x, t_1) \; , \; v(x, t_o) = v(x, t_1) \; \forall \; x \in \mathcal{R} \; . \tag{1.5}$$

Finally, we call F and J the deformation-gradient field and the Jacobian determinant (relative to $\mathcal{R}$) of an admissible motion, whence

$$F = \nabla \hat{y} = [\hat{y}_{i,j}] \; , \; J = \det F \neq 0 \text{ on } \mathcal{R} \times [t_o, t_1] \; . \qquad (1.6)^1$$

Let $\rho_o \in C(\mathcal{R})$ and $\rho(\cdot, t) \in C(\mathcal{R}_t)$ $(t_o \leq t \leq t_1)$, respectively, designate the mass density per unit reference volume and per unit current volume of a body $\mathcal{B}$ undergoing an admissible motion. In accordance with the postulate of mass balance

$$\rho_o(x) = |J(x, t)| \rho(\hat{y}(x, t), t) \; \forall \; (x, t) \in \mathcal{R} \times [t_o, t_1] \; . \tag{1.7}$$

Further, suppose that $\tau(\cdot, t) \in C^1(\mathcal{R}_t)$ $(t_o \leq t \leq t_1)$ is the actual (Cauchy) stress-tensor field induced by such a motion. If $\mathcal{E}$ is a Newtonian reference space, the principles of linear and angular momentum then imply the equation of motion

$$\left.\begin{aligned} &\operatorname{div}_y \tau(y, t) + b(y, t) = \rho(y, t) \bar{a}(y, t) \; , \\ &\tau(y, t) = \tau^T(y, t) \; \forall \; y \in \mathcal{R}_t \; (t_o \leq t \leq t_1) \; , \end{aligned}\right\} \qquad (1.8)^2$$

in which $b(\cdot, t) \in C(\mathcal{R}_t)$ is the instantaneous actual body-force density per unit current volume. The associated instantaneous nominal (Piola) stress field $\sigma(\cdot, t)$, relative to the chosen reference configuration and regarded as

---

[1] Subscripts preceded by a comma signify partial differentiation with respect to the corresponding cartesian coordinate.

[2] A superscript T always indicates transposition.

a function of position on $\mathcal{R}$, is defined by

$$\underset{\sim}{\sigma}(\underset{\sim}{x}, t) = |J(\underset{\sim}{x}, t)|\underset{\sim}{\tau}(\hat{\underset{\sim}{y}}(\underset{\sim}{x}, t), t)[\underset{\sim}{F}^T(\underset{\sim}{x}, t)]^{-1} \;\forall\; \underset{\sim}{x} \in \mathcal{R} \;(t_o \le t \le t_1) . \qquad (1.9)^1$$

Thus, $\underset{\sim}{\sigma}(\cdot, t) \in C^1(\mathcal{R})$ $(t_o \le t \le t_1)$ and (1.8), in view of (1.7), are found to be equivalent to

$$\left.\begin{aligned} \operatorname{div}_{\underset{\sim}{x}} \underset{\sim}{\sigma}(\underset{\sim}{x}, t) + \underset{\sim}{f}(\underset{\sim}{x}, t) &= \rho_o(\underset{\sim}{x})\underset{\sim}{a}(\underset{\sim}{x}, t) , \\ \underset{\sim}{\sigma}(\underset{\sim}{x}, t)\underset{\sim}{F}^T(\underset{\sim}{x}, t) &= \underset{\sim}{F}(\underset{\sim}{x}, t)\underset{\sim}{\sigma}^T(\underset{\sim}{x}, t) \;\forall\; \underset{\sim}{x} \in \mathcal{R} \;(t_o \le t \le t_1) , \end{aligned}\right\} \qquad (1.10)$$

where

$$\underset{\sim}{f}(\underset{\sim}{x}, t) = J(\underset{\sim}{x}, t)\underset{\sim}{b}(\hat{\underset{\sim}{y}}(\underset{\sim}{x}, t), t) \;\forall\; \underset{\sim}{x} \in \mathcal{R} \;(t_o \le t \le t_1) \qquad (1.11)$$

is the <u>nominal body-force density</u> per unit reference volume.

If S is an oriented regular surface element[2] and $S_t = \hat{\underset{\sim}{y}}(S, t)$ its instantaneous motion image, while $\underset{\sim}{N}(\underset{\sim}{x}) \;\forall\; \underset{\sim}{x} \in S$ and $\underset{\sim}{n}(\underset{\sim}{y}, t) \;\forall\; \underset{\sim}{y} \in S_t$ $(t_o \le t \le t_1)$ are the corresponding unit normal vectors, one has the traction-stress relations

$$\left.\begin{aligned} \underset{\sim}{s}(\underset{\sim}{x}, t) &= \underset{\sim}{\sigma}(\underset{\sim}{x}, t)\underset{\sim}{N}(\underset{\sim}{x}) \;\forall\; \underset{\sim}{x} \in S \;(t_o \le t \le t_1) , \\ \underset{\sim}{t}(\underset{\sim}{y}, t) &= \underset{\sim}{\tau}(\underset{\sim}{y}, t)\underset{\sim}{n}(\underset{\sim}{y}, t) \;\forall\; \underset{\sim}{y} \in S_t \;(t_o \le t \le t_1) , \end{aligned}\right\} \qquad (1.12)$$

where $\underset{\sim}{s}(\cdot, t)$ is the instantaneous <u>nominal traction vector field</u> on S and

---

[1] It should be emphasized that while $\underset{\sim}{\tau}(\cdot, t)$ is independent of the choice of the reference configuration, $\underset{\sim}{\sigma}(\cdot, t)$ is not. Since we deal here with but a single fixed reference configuration, we refrain from making this dependence explicit in the notation used for the nominal stress field.

[2] This term is to be interpreted in the sense of Kellogg [3].

$\underset{\sim}{t}(\cdot, t)$ the instantaneous actual traction field on S_t. Moreover,

$$\int_{S} \underset{\sim}{s}(\underset{\sim}{x}, t)\, dA_{\underset{\sim}{x}} = \int_{S_t} \underset{\sim}{t}(\underset{\sim}{y}, t)\, dA_{\underset{\sim}{y}} \quad (t_o \le t \le t_1) . \tag{1.13}$$

Consider now an admissible motion $\hat{\underset{\sim}{y}}(\cdot, t)$ on $\mathcal{R}$ $(t_o \le t \le t_1)$ of a body $\mathcal{B}$, let $\mathcal{P}$ be a regular subregion[1] of the region $\mathcal{R}$ occupied by $\mathcal{B}$ in the reference configuration and let $\mathcal{P}_t = \hat{\underset{\sim}{y}}(\mathcal{P}, t)$ $(t_o \le t \le t_1)$. Next, define the power (rate of work) of the actual body forces and surface tractions acting on the part of $\mathcal{B}$ that occupies $\mathcal{P}$ in the reference configuration through

$$r_{\mathcal{P}}(t) = \int_{\mathcal{P}_t} \underset{\sim}{b}(\underset{\sim}{y}, t) \cdot \overline{\underset{\sim}{v}}(\underset{\sim}{y}, t)\, dV_{\underset{\sim}{y}} + \int_{\partial \mathcal{P}_t} \underset{\sim}{t}(\underset{\sim}{y}, t) \cdot \overline{\underset{\sim}{v}}(\underset{\sim}{y}, t)\, dA_{\underset{\sim}{y}} \quad (t_o \le t \le t_1) . \tag{1.14}$$

Then the second of (1.12), together with (1.8) and the divergence theorem, readily yield the familiar power identity, valid for $t_o \le t \le t_1$,

$$r_{\mathcal{P}}(t) = \int_{\mathcal{P}_t} \underset{\sim}{\tau}(\underset{\sim}{y}, t) \cdot \nabla_{\underset{\sim}{y}} \overline{\underset{\sim}{v}}(\underset{\sim}{y}, t)\, dV_{\underset{\sim}{y}} + \frac{d}{dt} \int_{\mathcal{P}_t} \frac{1}{2} \rho(\underset{\sim}{y}, t) \overline{\underset{\sim}{v}}^2(\underset{\sim}{y}, t)\, dV_{\underset{\sim}{y}} . \tag{1.15}$$ [2]

The first volume integral in (1.15) is the "stress power", the second the kinetic energy associated with $\mathcal{P}_t$. From (1.14), with the aid of (1.4), (1.9), (1.11), and (1.12), follows the alternative representation of $r_{\mathcal{P}}(t)$, in terms of the nominal body-force density and surface traction,

---

[1] The term "regular region" is employed in the sense of Kellogg [3]. Note, in particular, that $\mathcal{P}$ is necessarily a bounded closed region to which the divergence theorem applies.

[2] If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are second-order tensors, we write $\underset{\sim}{A} \cdot \underset{\sim}{B}$ for the inner product $\operatorname{tr}(\underset{\sim}{A}^T \underset{\sim}{B}) = A_{ij} B_{ij}$.

$$r_{\mathcal{P}}(t) = \int_{\mathcal{P}} \underset{\sim}{f}(\underset{\sim}{x},t)\cdot\underset{\sim}{v}(\underset{\sim}{x},t)dV_{\underset{\sim}{x}} + \int_{\partial\mathcal{P}} \underset{\sim}{s}(\underset{\sim}{x},t)\cdot\underset{\sim}{v}(\underset{\sim}{x},t)dA_{\underset{\sim}{x}} \ (t_o \le t \le t_1) . \qquad (1.16)$$

Similarly, (1.4), (1.7), (1.9) enable one to deduce the following alternative representations for the stress power and the kinetic energy:

$$\int_{\mathcal{P}_t} \underset{\sim}{\tau}(\underset{\sim}{y},t)\cdot\underset{\sim}{\nabla}_{y}\bar{\underset{\sim}{v}}(\underset{\sim}{y},t)dV_{\underset{\sim}{y}} = \int_{\mathcal{P}} \underset{\sim}{\sigma}(\underset{\sim}{x},t)\cdot\underset{\sim}{\nabla}_{x}\underset{\sim}{v}(\underset{\sim}{x},t)dV_{\underset{\sim}{x}} \ (t_o \le t \le t_1) , \qquad (1.17)$$

$$\int_{\mathcal{P}_t} \frac{1}{2}\rho(\underset{\sim}{y},t)\bar{\underset{\sim}{v}}^2(\underset{\sim}{y},t)dV_{\underset{\sim}{y}} = \int_{\mathcal{P}} \frac{1}{2}\rho_o(\underset{\sim}{x})\underset{\sim}{v}^2(\underset{\sim}{x},t)dV_{\underset{\sim}{x}} \ (t_o \le t \le t_1) . \qquad (1.18)$$

We turn now to the constitutive law and in this connection limit our attention to a body $\mathcal{B}$ composed of an <u>elastic material</u> (simple material without memory). Accordingly, we assume there is an <u>actual stress-response function</u> $\underset{\sim}{g}(\cdot,\underset{\sim}{x})$, defined on $\mathcal{L}$ for every $\underset{\sim}{x}\in\mathcal{R}$ and having values in $\mathcal{S}$, such that for every admissible motion of $\mathcal{B}$,

$$\underset{\sim}{\tau}(\hat{\underset{\sim}{y}}(\underset{\sim}{x},t),t) = \underset{\sim}{g}(\underset{\sim}{F}(\underset{\sim}{x},t);\underset{\sim}{x}) \ \forall \ \underset{\sim}{x}\in\mathcal{R} \ (t_o \le t \le t_1) . \qquad (1.19)$$

Moreover, consistent with our previous agreement regarding the smoothness of the actual stress field $\underset{\sim}{\tau}(\cdot,t)$, we suppose henceforth that $\underset{\sim}{g}\in C^1(\mathcal{L}\times\mathcal{R})$. The dependence of $\underset{\sim}{g}(\cdot;\underset{\sim}{x})$ upon $\underset{\sim}{x}$ allows for the possible non-homogeneity of the material, as far as its stress response is concerned. On defining an associated <u>nominal stress-response function</u> $\underset{\sim}{h}(\cdot,\underset{\sim}{x})$ by means of

$$\underset{\sim}{h}(\underset{\sim}{F};\underset{\sim}{x}) = |\det \underset{\sim}{F}|\underset{\sim}{g}(\underset{\sim}{F};\underset{\sim}{x})(\underset{\sim}{F}^T)^{-1} \;\forall\; (\underset{\sim}{F},\underset{\sim}{x})\in\mathcal{L}\times\mathcal{R}\,, \tag{1.20}$$ [1]

one draws from the assumed symmetry of $\underset{\sim}{g}(\underset{\sim}{F};\underset{\sim}{x})$ that

$$\underset{\sim}{h}(\underset{\sim}{F};\underset{\sim}{x})\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{h}^T(\underset{\sim}{F};\underset{\sim}{x}) \;\forall\; (\underset{\sim}{F},\underset{\sim}{x})\in\mathcal{L}\times\mathcal{R}\,. \tag{1.21}$$

Further, (1.19), (1.20), and (1.9) assure that the constitutive law for an elastic body $\mathcal{B}$ may also be written as

$$\underset{\sim}{\sigma}(\underset{\sim}{x},t) = \underset{\sim}{h}(\underset{\sim}{F}(\underset{\sim}{x},t);\underset{\sim}{x}) \;\forall\; \underset{\sim}{x}\in\mathcal{R}\; (t_0\le t\le t_1)\,, \tag{1.22}$$

which must hold true for every admissible motion of $\mathcal{B}$. Because of the stipulated smoothness of $\underset{\sim}{g}$, evidently $\underset{\sim}{h}\in C^1(\mathcal{L}\times\mathcal{R})$.

At this stage we recall the definition of a <u>hyperelastic material</u> ("conservative" simple material without memory). We shall say that $\mathcal{B}$ is a hyperelastic body if $\mathcal{B}$ is elastic and there exists a scalar-valued function $W(\cdot;\underset{\sim}{x})$, defined on $\mathcal{L}$ for every $\underset{\sim}{x}\in\mathcal{R}$, such that $W\in C^2(\mathcal{L}\times\mathcal{R})$ and

$$\underset{\sim}{h}(\underset{\sim}{F};\underset{\sim}{x}) = W_{\underset{\sim}{F}}(\underset{\sim}{F};\underset{\sim}{x}) \;\forall\; (\underset{\sim}{F},\underset{\sim}{x})\in\mathcal{L}\times\mathcal{R} \tag{1.23}$$

or, equivalently,

$$h_{ij}(\underset{\sim}{F};\underset{\sim}{x}) = \partial W(\underset{\sim}{F};\underset{\sim}{x})/\partial F_{ij} \;\forall\; (\underset{\sim}{F},\underset{\sim}{x})\in\mathcal{L}\times\mathcal{R}\,. \tag{1.24}$$

Here W is the stored-energy function (elastic potential), which represents the strain-energy density per unit reference volume. The space $\mathcal{L}$ consists of the two disjoint domains $\mathcal{L}_+$ and $\mathcal{L}_-$, which are separated by the set of all

---

[1] Note that here $\underset{\sim}{F}$ denotes an arbitrary nonsingular second-order tensor, rather than the function of position and time representing the deformation-gradient tensor field of an admissible motion.

second-order tensors with a vanishing determinant. In order to render W unique for a hyperelastic body with a given nominal stress-response function $\underset{\sim}{h}$, we adopt the convenient normalization

$$W(\underset{\sim}{1};\underset{\sim}{x}) = W(-\underset{\sim}{1};\underset{\sim}{x}) = 0 \ \forall \ \underset{\sim}{x} \in \mathcal{R} \ . \tag{1.25}$$ [1]

By virtue of (1.17), (1.23) and (1.1), (1.3), (1.6), the stress-power in an admissible motion of a hyperelastic body is given by

$$\int_{\mathcal{P}_t} \underset{\sim}{\tau}(\underset{\sim}{y},t) \cdot \nabla_{\underset{\sim}{y}} \underset{\sim}{\bar{v}}(\underset{\sim}{y},t)\, dV_{\underset{\sim}{y}} = \frac{d}{dt}\int_{\mathcal{P}} W(\underset{\sim}{F}(\underset{\sim}{x},t);\underset{\sim}{x})\, dV_{\underset{\sim}{x}} \quad (t_o \le t \le t_1) \ , \tag{1.26}$$

the integral on the right being the total strain-energy stored in the region $\mathcal{P}_t$. Thus, and on account of (1.18), the power-identity (1.15) in the present circumstances takes the form

$$r_{\mathcal{P}}(t) = \frac{d}{dt}\int_{\mathcal{P}} W(\underset{\sim}{F}(\underset{\sim}{x},t);\underset{\sim}{x})\, dV_{\underset{\sim}{x}} + \frac{d}{dt}\int_{\mathcal{P}} \frac{1}{2}\rho_o(\underset{\sim}{x})\underset{\sim}{v}^2(\underset{\sim}{x},t)\, dV_{\underset{\sim}{x}} \quad (t_o \le t \le t_1) \ . \tag{1.27}$$

We conclude this compilation of prerequisites by citing certain implications of the <u>principle of material frame indifference</u> (objectivity). In what follows let $\underset{\sim}{Q}$ be an arbitrary orthogonal second-order tensor. Necessary and sufficient for the objectivity of the constitutive law appropriate to an elastic body is that

$$\underset{\sim}{g}(\underset{\sim}{Q}\underset{\sim}{F};\underset{\sim}{x}) = \underset{\sim}{Q}\underset{\sim}{g}(\underset{\sim}{F};\underset{\sim}{x})\underset{\sim}{Q}^T \ \forall \ (\underset{\sim}{F},\underset{\sim}{x}) \in \mathcal{L} \times \mathcal{R} \tag{1.28}$$

or alternatively,

---

[1] Here and in the sequal $\underset{\sim}{1}$ is the idem tensor.

$$\underset{\sim}{h}(\underset{\sim}{Q}\underset{\sim}{F};\underset{\sim}{x}) = \underset{\sim}{Q}\underset{\sim}{h}(\underset{\sim}{F};\underset{\sim}{x}) \ \forall \ (\underset{\sim}{F},\underset{\sim}{x}) \in \mathcal{L} \times \mathcal{R} \ . \tag{1.29}$$

On the other hand, for a hyperelastic body one has

$$W(\underset{\sim}{Q}\underset{\sim}{F};\underset{\sim}{x}) = W(\underset{\sim}{F};\underset{\sim}{x}) \ \forall \ (\underset{\sim}{F},\underset{\sim}{x}) \in \mathcal{L} \times \mathcal{R} \ , \tag{1.30}$$

provided the normalization (1.25) is in force, as has been assumed already.

Suppose, finally,

$$\underset{\sim}{F} \in \mathcal{L} \ , \ \underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U} \ , \ \underset{\sim}{R} \in \mathcal{O} \ , \ \underset{\sim}{U} \in \mathcal{S}^{+} \ , \tag{1.31}$$

so that $\underset{\sim}{R}$ and $\underset{\sim}{U}$ are the orthogonal and the symmetric positive-definite factors in the unique right polar decomposition of the nonsingular tensor $\underset{\sim}{F}$. Thus,

$$\underset{\sim}{R}\underset{\sim}{R}^{T} = \underset{\sim}{1} \ , \ \underset{\sim}{R} = \underset{\sim}{R}(\underset{\sim}{F}) = \underset{\sim}{F}\underset{\sim}{U}^{-1} \ , \ \underset{\sim}{U} = \underset{\sim}{U}(\underset{\sim}{F}) = \sqrt{\underset{\sim}{F}^{T}\underset{\sim}{F}} \ . \tag{1.32}$$

Moreover, (1.28), (1.29), (1.30) yield the well-known conclusions:

$$\underset{\sim}{g}(\underset{\sim}{F};\underset{\sim}{x}) = \underset{\sim}{R}\underset{\sim}{g}(\underset{\sim}{U};\underset{\sim}{x})\underset{\sim}{R}^{T} \ \forall \ (\underset{\sim}{F},\underset{\sim}{x}) \in \mathcal{L} \times \mathcal{R} \ , \tag{1.33}$$

$$\underset{\sim}{h}(\underset{\sim}{F};\underset{\sim}{x}) = \underset{\sim}{R}\underset{\sim}{h}(\underset{\sim}{U};\underset{\sim}{x}) \ \forall \ (\underset{\sim}{F},\underset{\sim}{x}) \in \mathcal{L} \times \mathcal{R} \ , \tag{1.34}$$

$$W(\underset{\sim}{F};\underset{\sim}{x}) = W(\underset{\sim}{U};\underset{\sim}{x}) \ \forall \ (\underset{\sim}{F},\underset{\sim}{x}) \in \mathcal{L} \times \mathcal{R} \ . \tag{1.35}$$

2. A work theorem for hyperelastic bodies.

With our present purpose in mind, we first recall a familiar property of hyperelastic materials.

Theorem 1. Let $\mathcal{B}$ be a hyperelastic body. Then the work

$$w_{\mathcal{P}} = \int_{t_o}^{t_1} r_{\mathcal{P}}(t)\,dt = 0 \tag{2.1}$$

for every cyclic motion of $\mathcal{B}$ and for every regular subregion $\mathcal{P}$ of the region $\mathcal{R}$ occupied by $\mathcal{B}$ in the reference configuration. Moreover, if $\underset{\sim}{h}$ is the nominal stress-response function of $\mathcal{B}$, then

$$\operatorname{curl}_{\underset{\sim}{F}} \underset{\sim}{h}(\underset{\sim}{F};\underset{\sim}{x}) = \underset{\sim}{0} \;\; \forall \;(\underset{\sim}{F},\underset{\sim}{x}) \in \mathcal{L} \times \mathcal{R} \tag{2.2}$$

or, equivalently,

$$\partial h_{ij}(\underset{\sim}{F};\underset{\sim}{x})/\partial F_{pq} = \partial h_{pq}(\underset{\sim}{F};\underset{\sim}{x})/\partial F_{ij} \;\; \forall \;(\underset{\sim}{F},\underset{\sim}{x}) \in \mathcal{L} \times \mathcal{R} \,. \tag{2.3}$$ [1]

The conclusion (2.1) is immediate from the power-identity (1.27) for hyperelastic bodies together with (1.5), (1.6), while (2.3) follows at once from (1.24) and the assumed smoothness of the elastic potential W. We now state and prove the following converse of Theorem 1, which constitutes our main objective.

Theorem 2. Let $\mathcal{B}$ be an elastic body. Suppose the total work done by the actual body forces and surface tractions acting on any part of $\mathcal{B}$ that occupies a regular subregion of the region $\mathcal{R}$ occupied by $\mathcal{B}$ in the reference configuration vanishes for every cyclic motion of the body. Then the nominal stress-response function $\underset{\sim}{h}$ satisfies (2.2) and $\mathcal{B}$ is hyperelastic. Further, the elastic potential of $\mathcal{B}$ conforming to the normalization (1.25) is given by

---

[1] Thus the fourth-order tensor $\underset{\sim}{h}_{\underset{\sim}{F}}(\underset{\sim}{F};\underset{\sim}{x})$ is symmetric in the sense that

$$\underset{\sim}{A} \cdot \underset{\sim}{h}_{\underset{\sim}{F}}(\underset{\sim}{F};\underset{\sim}{x})[\underset{\sim}{B}] = \underset{\sim}{B} \cdot \underset{\sim}{h}_{\underset{\sim}{F}}(\underset{\sim}{F};\underset{\sim}{x})[\underset{\sim}{A}]$$

for all second-order tensors $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

$$W(\underset{\sim}{F};\underset{\sim}{x}) = \int_0^1 \underset{\sim}{h}(\alpha \underset{\sim}{U}(\underset{\sim}{F}) + (1-\alpha)\underset{\sim}{1};\underset{\sim}{x})\cdot[\underset{\sim}{U}(\underset{\sim}{F}) - \underset{\sim}{1}]d\alpha \;\forall\; (\underset{\sim}{F},\underset{\sim}{x})\in\mathcal{L}\times\mathcal{R}\;, \tag{2.4}$$

provided $\underset{\sim}{U}(\underset{\sim}{F})$ is the symmetric positive-definite factor of $\underset{\sim}{F}$ in its right polar decomposition.

With a view toward a proof of this theorem we consider a homogeneous motion of $\mathcal{B}$, defined by

$$\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t) = \underset{\sim}{F}(t)\underset{\sim}{x} \;\forall\; \underset{\sim}{x}\in\mathcal{R}\;(t_o \le t \le t_1)\;, \tag{2.5}$$

such that

$$\underset{\sim}{F}(t)\in\mathcal{L}\;(t_o\le t\le t_1)\;,\;\underset{\sim}{F}\in C^2([t_o,t_1])\;,\;\underset{\sim}{F}(t_o) = \underset{\sim}{F}(t_1)\;,\;\dot{\underset{\sim}{F}}(t_o) = \dot{\underset{\sim}{F}}(t_1)\;. \tag{2.6}$$

Clearly, (2.5) *represents* a cyclic *motion* of $\mathcal{B}$ for every choice of the deformation gradient $\underset{\sim}{F}$ satisfying (2.6). From (2.5), (2.6) and the power identity (1.15) in conjuction with (1.17), (1.18), one draws

$$w_{\mathcal{P}} = \int_{t_o}^{t_1} r_{\mathcal{P}}(t)dt = \int_{t_o}^{t_1}\left\{\int_{\mathcal{P}} \underset{\sim}{\sigma}(\underset{\sim}{x},t)\cdot\dot{\underset{\sim}{F}}(t)dV_{\underset{\sim}{x}}\right\}dt\;. \tag{2.7}$$

Since the body is elastic, (1.22) holds and (2.7) thus leads to

$$\begin{aligned} w_{\mathcal{P}} &= \int_{t_o}^{t_1}\left\{\int_{\mathcal{P}} \underset{\sim}{h}(\underset{\sim}{F}(t);\underset{\sim}{x})\cdot\dot{\underset{\sim}{F}}(t)dV_{\underset{\sim}{x}}\right\}dt \\ &= \int_{\mathcal{P}}\left\{\int_{t_o}^{t_1} \underset{\sim}{h}(\underset{\sim}{F}(t);\underset{\sim}{x})\cdot\dot{\underset{\sim}{F}}(t)dt\right\}dV_{\underset{\sim}{x}}\;, \end{aligned} \tag{2.8}$$

the interchange in the order of the two integrations being legitimate because of the assured smoothness of the function $\underset{\sim}{F}$ and of the nominal stress-response function $\underset{\sim}{h}$. But, by hypothesis, the work $w_{\mathcal{P}}$ must vanish for <u>every</u> regular region $\mathcal{P} \subset \mathcal{R}$ and the integrand of the last volume integral in (2.8) is continuous on $\mathcal{R}$. Consequently,

$$\int_{t_o}^{t_1} \underset{\sim}{h}(\underset{\sim}{F}(t);\underset{\sim}{x}) \cdot \dot{\underset{\sim}{F}}(t)dt = 0 \ \forall \ \underset{\sim}{x} \in \mathcal{R} \tag{2.9}$$

for every choice of the function $\underset{\sim}{F}$ consistent with (2.6).

At this stage we take $t_o = 0$, $t_1 = 2\pi$ and specialize $\underset{\sim}{F}$ in (2.9) as follows. Let $\overset{o}{\underset{\sim}{F}}$ be a nonsingular second-order tensor. Define $\underset{\sim}{F}(t)$ $(0 \le t \le 2\pi)$ in terms of its components in an orthogonal cartesian frame by means of

$$\left.\begin{aligned} &F_{11}(t) = \overset{o}{F}_{11} + \epsilon\cos t, \ F_{22}(t) = \overset{o}{F}_{22} + \epsilon\sin t, \\ &F_{ij}(t) = \overset{o}{F}_{ij} \text{ for } (i,j) \ne (1,1), \ (i,j) \ne (2,2). \end{aligned}\right\} \tag{2.10}$$

By continuity, since $\overset{o}{\underset{\sim}{F}}$ is nonsingular, so is $\underset{\sim}{F}(t)$ for $0 \le t \le 2\pi$ and all sufficiently small positive values of the parameter ϵ — say for $0 < \epsilon < \epsilon_*$. The choice of $\underset{\sim}{F}$ adopted in (2.10) thus conforms to (2.6) in this range of ϵ. Next, let

$$I(\epsilon) = \frac{1}{\epsilon^2} \int_0^{2\pi} \underset{\sim}{h}(\underset{\sim}{F}(t);\underset{\sim}{x}) \cdot \dot{\underset{\sim}{F}}(t)dt \ \forall \ \underset{\sim}{x} \in \mathcal{R} \ (0 < \epsilon < \epsilon_*) \tag{2.11}$$

with $\underset{\sim}{F}(t)$ $(0 \le t \le 2\pi)$ specified by (2.10).[1] Equation (2.9) now demands that

[1]Note accordingly that the integrand in (2.11) depends also on ϵ.

$$\left.\begin{aligned} I(\epsilon) = \frac{1}{\epsilon}\int_0^{2\pi}[-h_{11}(\underset{\sim}{F}(t);\underset{\sim}{x})\sin t + h_{22}(\underset{\sim}{F}(t);\underset{\sim}{x})\cos t]dt = 0 \\ \forall\ \underset{\sim}{x}\in\mathcal{R}\ (0<\epsilon<\epsilon_*). \end{aligned}\right\} \tag{2.12}$$

The foregoing identity, upon an appeal to (2.10) and to L'Hôpital's rule, readily leads to

$$I(0+) = \pi\left[\frac{\partial h_{22}}{\partial F_{11}} - \frac{\partial h_{11}}{\partial F_{22}}\right]_{(\overset{\circ}{\underset{\sim}{F}},\underset{\sim}{x})} = 0\ \forall\ \underset{\sim}{x}\in\mathcal{R}\ . \tag{2.13}$$

Since $\overset{\circ}{\underset{\sim}{F}}$ here is <u>any</u> nonsingular tensor of order two, one thus infers that (2.3) holds true for $i = j = 1$, $p = q = 2$. Obvious modifications of (2.10) confirm that (2.3), and hence (2.2), is valid for all admissible choices of i,j and p, q.

It remains to be seen that $\mathcal{B}$ is hyperelastic and that the scalar-valued function W given by (2.4) supplies the normalized elastic potential of $\mathcal{B}$. To this end it suffices to prove that W is twice continuously differentiable on $\mathcal{L}\times\mathcal{R}$ and satisfies (1.23), (1.25).

If one sets

$$\underset{\sim}{S}(\underset{\sim}{F},\alpha) = \alpha\underset{\sim}{U}(\underset{\sim}{F}) + (1-\alpha)\underset{\sim}{1}\ \forall\ \underset{\sim}{F}\in\mathcal{L}\ (0\le\alpha\le 1)\ , \tag{2.14}$$

then (2.4) may be written as

$$W(\underset{\sim}{F};\underset{\sim}{x}) = \int_0^1 \underset{\sim}{h}(\underset{\sim}{S}(\underset{\sim}{F},\alpha);\underset{\sim}{x})\cdot[\underset{\sim}{U}(\underset{\sim}{F}) - \underset{\sim}{1}]d\alpha\ \forall\ (\underset{\sim}{F},\underset{\sim}{x})\in\mathcal{L}\times\mathcal{R}\ . \tag{2.15}$$

We observe first that $W(\mathbf{F};\mathbf{x})$ is indeed defined for every nonsingular second-order tensor $\mathbf{F}$ and every $\mathbf{x}\in\mathcal{R}$ since $\mathbf{h}\in C^1(\mathcal{L}\times\mathcal{R})$ and because $\mathbf{S}(\mathbf{F},\alpha)$ is a symmetric positive-definite tensor[1] for every $\mathbf{F}\in\mathcal{L}$ $(0\le\alpha\le 1)$. We show next that W is at least once continuously differentiable on $\mathcal{L}\times\mathcal{R}$. In order to confirm this claim it is evidently sufficient to prove that the components of $\mathbf{U}(\mathbf{F})$ in an arbitrary orthogonal cartesian coordinate frame are continuously differentiable with respect to the components of $\mathbf{F}$ in the same frame. For this purpose we note on the basis of (1.32) that

$$\mathbf{U} = \mathbf{U}(\mathbf{F}) = \sqrt{\mathbf{C}}\ , \quad \mathbf{C} = \mathbf{C}(\mathbf{F}) = \mathbf{F}^T\mathbf{F} \tag{2.16}$$

and let $\mu_i > 0$ denote the eigenvalues of $\mathbf{U}$, whence $\mathbf{C}$ has the eigenvalues μ_i^2. Further, we designate the fundamental scalar invariants of $\mathbf{U}$ by φ_i, so that

$$\left.\begin{aligned} \varphi_1 &= \operatorname{tr}\mathbf{U} = \mu_1+\mu_2+\mu_3 > 0\ , \\ \varphi_2 &= \tfrac{1}{2}[(\operatorname{tr}\mathbf{U})^2 - \operatorname{tr}(\mathbf{U}^2)] = \mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1 > 0\ , \\ \varphi_3 &= \det\mathbf{U} = \mu_1\mu_2\mu_3 > 0\ , \end{aligned}\right\} \tag{2.17}$$

and call Φ_i the corresponding scalar invariants of $\mathbf{C}$, which also belongs to $\mathcal{S}$ for $\mathbf{F}\in\mathcal{L}$. From the Cayley-Hamilton theorem one has

$$\mathbf{U}^3 - \varphi_1\mathbf{U}^2 + \varphi_2\mathbf{U} - \varphi_3\mathbf{1} = \mathbf{0}\ , \tag{2.18}$$

[1] Note that the eigenvalues of $\mathbf{S}(\mathbf{F},\alpha)$ are $\alpha\mu_i + 1 - \alpha > 0$, if μ_i are the positive eigenvalues of $\mathbf{U}(\mathbf{F})$, which is in $\mathcal{S}$ for every $\mathbf{F}\in\mathcal{L}$. The set $\mathcal{S}$ of all symmetric positive-definite tensors of order two is thus seen to be star-shaped with respect to the tensor $\mathbf{1}$. In fact, $\mathcal{S}$ can be shown to be convex.

and (2. 18), together with the first of (2. 16), furnishes

$$\underset{\sim}{U} = \underset{\sim}{U}(\underset{\sim}{F}) = \sqrt{\underset{\sim}{C}} = (\underset{\sim}{C} + \varphi_2 \underset{\sim}{1})^{-1}(\varphi_1 \underset{\sim}{C} + \varphi_3 \underset{\sim}{1}) \quad . \qquad (2.19)^1$$

Moreover, (2. 17) and the analogous formulas for the scalar invariants of $\underset{\sim}{C}$ give

$$\Phi_1 = \varphi_1^2 - 2\varphi_2 > 0 \ , \ \Phi_2 = \varphi_2^2 - 2\varphi_1\varphi_3 > 0 \ , \ \Phi_3 = \varphi_3^2 > 0 \ . \qquad (2.20)$$

From (2. 16), the last of (2. 17), as well as the last of (2. 20), follows

$$\varphi_3 = \sqrt{\Phi_3} = |\det \underset{\sim}{F}| \ ; \qquad (2.21)$$

accordingly φ_3 is a continuously differentiable function of the components of $\underset{\sim}{F}$ on $\mathcal{L}_+$ and on $\mathcal{L}_-$, and thus on $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_-$. On the other hand, the first two of (2. 20), with the aid of the implicit-function theorem, can be shown to imply that φ_1, φ_2 are continuously differentiable functions of Φ_1, Φ_2, and φ_3. Therefore (2. 19) now assures the desired smoothness of $\underset{\sim}{U}(\underset{\sim}{F})$ for every $\underset{\sim}{F} \in \mathcal{L}$. According to a remark of John [4][2] the components of $\underset{\sim}{U}(\underset{\sim}{F})$ can even be shown to depend analytically on the components of $\underset{\sim}{F}$.

We are now in a position to deduce the gradient $W_{\underset{\sim}{F}}(\underset{\sim}{F};\underset{\sim}{x})$ of the presumed elastic potential defined in (2. 4). In doing so we shall for the sake of brevity suppress the argument $\underset{\sim}{x}$. If $\underset{\sim}{T}$ is any tensor of the second order, one finds from (2. 14), (2. 15) that

---

[1] The existence of the inverse $(\underset{\sim}{C} + \varphi_2 \underset{\sim}{1})^{-1}$ is assured since $\underset{\sim}{C} \in \overset{+}{\mathcal{S}}$, $\varphi_2 > 0$ imply $\underset{\sim}{C} + \varphi_2 \underset{\sim}{1} \in \overset{+}{\mathcal{S}}$.

[2] See Footnote No. 25 in [4].

$$\underset{\sim}{T} \cdot W_{\underset{\sim}{F}}(\underset{\sim}{F}) = T_{ij} \frac{\partial W(\underset{\sim}{F})}{\partial F_{ij}} =$$

$$\int_0^1 [\alpha \frac{\partial h_{rs}(\underset{\sim}{S})}{\partial F_{pq}} \left(U_{rs} - \delta_{rs}\right) + h_{pq}(\underset{\sim}{S})] T_{ij} \frac{\partial U_{pq}}{\partial F_{ij}} d\alpha \,, \tag{2.22}$$

in which $\underset{\sim}{S}$ and $\underset{\sim}{U}$ abbreviate $\underset{\sim}{S}(\underset{\sim}{F},\alpha)$ and $\underset{\sim}{U}(\underset{\sim}{F})$, respectively, δ_{rs} is Kronecker's delta, while $\partial h_{rs}(\underset{\sim}{S})/\partial F_{pq}$ stands for $\partial h_{rs}(\underset{\sim}{F})/\partial F_{pq}$ evaluated at $\underset{\sim}{F} = \underset{\sim}{S}$. In view of (2.14), $U_{rs} - \delta_{rs} = \partial S_{rs}/\partial\alpha$, so that (2.22) may be written as

$$\underset{\sim}{T} \cdot W_{\underset{\sim}{F}}(\underset{\sim}{F}) = \int_0^1 \Big\{ \alpha \left[\frac{\partial h_{rs}(\underset{\sim}{S})}{\partial F_{pq}} - \frac{\partial h_{pq}(\underset{\sim}{S})}{\partial F_{rs}} \right] \frac{\partial S_{rs}}{\partial \alpha}$$

$$+ \alpha \frac{\partial h_{pq}(\underset{\sim}{S})}{\partial \alpha} + h_{pq}(\underset{\sim}{S}) \Big\} T_{ij} \frac{\partial U_{pq}}{\partial F_{ij}} d\alpha \,. \tag{2.23}$$

The term within brackets in (2.23) vanishes by virtue of (2.3), and an integration by parts — with attention to (2.14) — justifies

$$\underset{\sim}{T} \cdot W_{\underset{\sim}{F}}(\underset{\sim}{F}) = h_{pq}(\underset{\sim}{U}) T_{ij} \frac{\partial U_{pq}}{\partial F_{ij}} \,. \tag{2.24}$$

Next, from (1.31), (1.32) follows

$$\frac{\partial U_{pq}}{\partial F_{ij}} = \frac{\partial R_{kp}}{\partial F_{ij}} F_{kq} + R_{ip}\delta_{qj} = \frac{\partial R_{kp}}{\partial F_{ij}} R_{ks} U_{sq} + R_{ip}\delta_{qj} \,, \tag{2.25}$$

where $\underset{\sim}{R} = \underset{\sim}{R}(\underset{\sim}{F})$ is the orthogonal factor of $\underset{\sim}{F}$ in its two polar decompositions.[1] Upon substitution from (2.25) into (2.24), one arrives at

[1] Since $\underset{\sim}{R} = \underset{\sim}{F}\underset{\sim}{U}^{-1}$, the differentiability of $\underset{\sim}{R}(\underset{\sim}{F})$ is implied by the differentiability of $\underset{\sim}{U}(\underset{\sim}{F})$.

$$\underset{\sim}{T}\cdot W_{\underset{\sim}{F}}(\underset{\sim}{F}) = R_{ip}h_{pj}(\underset{\sim}{U})T_{ij} + T_{ij}\frac{\partial R_{kp}}{\partial F_{ij}}R_{ks}U_{sq}h_{pq}(\underset{\sim}{U}) . \tag{2.26}$$

But, because of the orthogonality of $\underset{\sim}{R}$ and on account of (1.21), one has

$$\frac{\partial R_{kp}}{\partial F_{ij}}R_{ks} = -\frac{\partial R_{ks}}{\partial F_{ij}}R_{kp} , \quad U_{sq}h_{pq}(\underset{\sim}{U}) = U_{pq}h_{sq}(\underset{\sim}{U}) . \tag{2.27}$$

Accordingly, $\left(\partial R_{kp}/\partial F_{ij}\right)R_{ks}$ is skew-symmetric and $U_{sq}h_{pq}(\underset{\sim}{U})$ symmetric with respect to (p, s). The second term in the right member of (2.26) therefore vanishes and thus

$$\underset{\sim}{T}\cdot W_{\underset{\sim}{F}}(\underset{\sim}{F}) = \underset{\sim}{T}\cdot\underset{\sim}{R}\underset{\sim}{h}(\underset{\sim}{U}) = \underset{\sim}{T}\cdot\underset{\sim}{h}(\underset{\sim}{F}) , \tag{2.28}$$

the last of the foregoing equalities being a consequence of the objectivity relation (1.34). Since (2.28) must hold for <u>every</u> choice of the tensor $\underset{\sim}{T}$, it follows that W satisfies (1.23). Further, (1.23) and the hypothesis that $\underset{\sim}{h}\in C^1(\mathcal{L}\times\mathcal{R})$ allow one to infer that $W\in C^2(\mathcal{L}\times\mathcal{R})$, as required. In addition, noting that $\underset{\sim}{U}(\underset{\sim}{1}) = \underset{\sim}{U}(-\underset{\sim}{1}) = \underset{\sim}{1}$, one confirms on the basis of (2.4) at once that W obeys the normalization conditions (1.25). This completes the proof of Theorem 2.

The hypotheses of Theorem 2 can be weakened in one respect without impairment of the conclusions: as is clear from the preceding proof, it suffices to demand that the work $w_{\mathcal{P}}$ of the body forces and surface tractions vanish merely for every "simple-harmonic" cyclic motion of the type presupposed in (2.10).

Finally, we remark on the motivation for the anticipated representation (2.4) of the elastic potential W. According to the objectivity-relation (1.35), one needs to construct $W(\cdot;\underset{\sim}{x})$ only on the set $\mathcal{J}^+$ of all symmetric positive-definite second-order tensors. Now, as α ranges over the interval $[0,1]$, $\alpha\underset{\sim}{U} + (1-\alpha)\underset{\sim}{1}$ traverses the "straight-line segment" Γ from $\underset{\sim}{1}$ to $\underset{\sim}{U}$, which lies wholly in $\overset{+}{\mathscr{J}}$ if $\underset{\sim}{U}$ is in $\overset{+}{\mathscr{J}}$. Bearing in mind also the required normalization (1.25), it is thus natural to expect that

$$W(\underset{\sim}{F};\underset{\sim}{x}) = W(\underset{\sim}{U}(\underset{\sim}{F});\underset{\sim}{x}) = \int_{\underset{\sim}{1}}^{\underset{\sim}{U}} \underset{\sim}{h}(\underset{\sim}{S};\underset{\sim}{x})\cdot d\underset{\sim}{S}\ , \tag{2.29}$$

with $\underset{\sim}{S} = \underset{\sim}{S}(\underset{\sim}{F},\alpha)$ given by (2.14) and Γ as the path of integration for the above line-integral in the six-dimensional inner-product space associated with $\overset{+}{\mathscr{J}}$. Upon converting this line-integral into a Riemann integral, one is led to (2.4). In this connection we observe with the aid of (2.6), (2.9) that

$$\oint \underset{\sim}{h}(\underset{\sim}{F};\underset{\sim}{x})\cdot d\underset{\sim}{F} = 0 \ \forall\ \underset{\sim}{x}\in\mathcal{R}\ , \tag{3.30}$$

for every closed path of integration admitted by (2.6) that lies entirely in either of the two disjoint domains $\mathscr{L}_+$ or $\mathscr{L}_-$. If (3.30) were known to hold for every merely <u>piecewise</u> <u>smooth</u> closed path in $\mathscr{L}_+$ or $\mathscr{L}_-$, one could construct the desired elastic potential directly in terms of the appropriate path-independent line integrals.

References

[1] C. Truesdell and W. Noll, The Non-Linear Field Theories of Mechanics, Encyclopedia of Physics, Volume III/3, Springer, Berlin, 1965.

[2] C. Truesdell and R. A. Toupin, The Classical Field Theories, in Encyclopedia of Physics, Volume III/1, Springer, Berlin, 1960.

[3] O. D. Kellogg, Potential Theory, Frederick Ungar, New York, 1929.

[4] F. John, Plane Elastic Waves of Finite Amplitude. Hadamard Materials and Harmonic Materials, Communications on Pure and Applied Mathematics, Volume XIX (1966), p. 309.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
No. 39 14 TR-39

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
6 On the existence of an elastic potential for a simple material without memory.

5. TYPE OF REPORT & PERIOD COVERED
9 Technical Rep[redacted]t

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
10 Eli Sternberg [redacted] J. K. Knowles

8. CONTRACT OR GRANT NUMBER(s)
15 N00014-75-C-0196

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Division of Engineering and Applied Science
California Institute of Technology
Pasadena, California 91125

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
Office of Naval Research
Washington, D.C.

12. REPORT DATE
11 Jan[redacted]78

13. NUMBER OF PAGES
21 12 24p.

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Elasticity, hyperelasticity, nonlinear, work

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is shown that a compressible elastic body — not necessarily homogeneous or isotropic — is hyperelastic provided the work done by all external forces acting on an arbitrary part of the body vanishes for every sufficiently smooth cyclic motion, in which each material point returns to its initial position with a velocity equal to its initial velocity.

071585

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)