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## CRITICAL REVIEW OF ONE-DIMENSIONAL TUBE FLOW EQUATIONS

Aivars K. R. Celmins

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flows. In cases of more complicated flows some terms in the traditional momentum and energy equations can be in error by up to 50%. It is also shown that the popular approximation of the energy dissipation function by the product of the average velocity and average shear stress is appropriate for the simplest flows only. The paper reveals shortcomings of traditional methods of derivations of tube flow equations and provides explicit formulas for correction terms which should be used in the governing equations. ~~An interesting~~ property of the new equations for average flow properties is that the momentum equation and energy equation cannot be combined with the continuity equation to yield simple equations for velocity components and specific energy, respectively. Consequently a divergence form of the equations can be obtained only if momentum components and energy per unit volume are used as unknowns.

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## 1. INTRODUCTION

We consider in this paper the derivation of governing equations for fluid flows through ducts. Such flows are important elements in many mechanical systems. Most fluid mechanics textbooks present, therefore, a simple derivation of the governing equations, which reduces the general three-dimensional equations to a set of equations for one-dimensional flow. Experience has shown that these equations are adequate for many applications. Probably because of this success researchers sometimes tend to disregard the limits of applicability of the one-dimensional flow equations. In order to derive governing equations for more complicated flows they duplicate the steps used for simple duct flows. The resulting equations are not always adequate, e.g., in case of certain non-steady flows. Some textbooks discuss limitations of the usual tube flow equations. Often, however, the discussion is rather general, or limited to examples and exercise problems, and easily overlooked by casual readers. In this paper we will concentrate on the limitations. We will keep the discussions simple by considering in detail only a one-phase flow in a straight duct with a constant cross-section. The discussion of the example will provide a methodical approach to the derivation of flow equations for more general cases.

The starting point of our discussion is the set of general three-dimensional flow equations. In order to make this paper self-contained, we list the equations in Section 2. In Section 3 we specialize the equations for the case of a duct flow using a standard procedure, which is found in textbooks. In order to establish limits for the validity of the specialized equations, we carry out in Section 4 a more careful derivation of the duct flow equations. This derivation provides quantitative information about the errors which are introduced by the specialization of the equations. A comparison of the derivations and results of Sections 3 and 4 reveals that in standard derivations of the equations some non-zero terms are neglected. In Section 5 two examples are presented: a steady flow and an approximation to an interior ballistics flow. Quantitative estimates are given for some usually neglected terms in the governing equations. Section 6 contains some conclusions which can be drawn from the discussions of the equations.

## 2. BASIC GOVERNING EQUATIONS

We consider flows which satisfy conservation laws for mass, momentum and energy. Governing equations for such flows are derived and

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discussed, e.g., by Tsien in Reference 1 and Batchelor in Reference 2. In this section we summarize the equations in order to make this paper self-contained. We use, in general, the same notation as Tsien, including the convention about the summation over equal indexes.

First we will consider the equations in integral form. In these equations the volume integrals are for an arbitrary control volume  $V$ , which need not be simply connected. We assume, however, for simplicity that its surface  $S$  has everywhere an outward pointing normal  $n_j$ . The conservation of mass can then be expressed by the equation

$$\frac{\partial}{\partial t} \int \rho \, dV + \oint \rho u_j n_j \, dS = 0 \quad (2.1)$$

The momentum equations are

$$\frac{\partial}{\partial t} \int \rho u_k \, dV + \oint \rho u_k u_j n_j \, dS + \int \frac{\partial p}{\partial x_k} \, dV = \int F_k \, dV. \quad (2.2)$$

The specific kinetic energy of the fluid is

$$k = \frac{1}{2} u_j u_j. \quad (2.3)$$

Combining eqs. (2.1) and (2.2) we obtain for the kinetic energy the equation

$$\frac{\partial}{\partial t} \int \rho k \, dV + \oint \rho k u_j n_j \, dS + \int u_j \frac{\partial p}{\partial x_j} \, dV = \int u_j F_j \, dV. \quad (2.4)$$

<sup>1</sup>H.S. Tsien, "The Equations of Gas Dynamics," in Fundamentals of Gas Dynamics, edited by H.W. Emmons, Princeton University Press, 1958.

<sup>2</sup>G.K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, 1967.

The first law of thermodynamics is

$$\frac{\partial}{\partial t} \int \rho e \, dV + \oint \rho e u_j n_j \, dS + \int p \frac{\partial u_j}{\partial x_j} \, dV = \int \left( Q - \frac{\partial q_j}{\partial x_j} \right) \, dV + \int \Phi \, dV. \quad (2.5)$$

By adding eqs. (2.4) and (2.5) we obtain an equation for the specific total internal energy  $e + k$ :

$$\frac{\partial}{\partial t} \int \rho (e + k) \, dV + \oint \rho (e + k) u_j n_j \, dS + \oint p u_j n_j \, dS = \int \left( Q - \frac{\partial q_j}{\partial x_j} \right) \, dV + \int (\Phi + u_j F_j) \, dV. \quad (2.6)$$

The last integral in eq. (2.6) is the contribution of external and viscous forces to the changes of the total internal energy. Its first part,  $\int \Phi \, dV$ , is the contribution of viscous forces to the internal energy  $e$ . The integrand  $\Phi$ , i.e., the heat dissipation function, can be expressed in terms of the viscous stress tensor  $\tau_{kj}$ :

$$\Phi = \tau_{kj} \frac{\partial u_k}{\partial x_j} \quad (2.7)$$

We assume that the viscous stress tensor is related by Stokes formula to the strain rate tensor  $\epsilon_{kj}$  (Reference 1, page 13, Reference 3, page 132)

$$\tau_{kj} = 2\mu \epsilon_{kj} + \left( \mu' - \frac{2}{3}\mu \right) \delta_{kj} \epsilon_{ii} \quad (2.8)$$

This definition is not restricted to constant viscosities  $\mu$  and  $\mu'$ , i.e., to homogenous fluids. However, it restricts the considerations to isotropic fluids. The viscosities  $\mu$  and  $\mu'$  must be positive or zero.

3G. Hamel, *Mechanik der Kontinua*, R.G. Teubner, Stuttgart, 1956.

The strain rate tensor  $\epsilon_{kj}$  is defined by (Reference 2, page 80)

$$\epsilon_{kj} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right) \quad (2.9)$$

Substituting (2.8) and (2.9) into eq. (2.7) we obtain the following expressions for  $\Phi$ :

$$\begin{aligned} \Phi &= 2\mu \epsilon_{kj} \epsilon_{kj} + \left( \mu' - \frac{2}{3}\mu \right) \delta_{kj} \epsilon_{ii} \epsilon_{kj} = \\ &= 2\mu \{\epsilon^2\}_{\text{trace}} + \left( \mu' - \frac{2}{3}\mu \right) \{\epsilon\}^2_{\text{trace}} = \\ &= \frac{1}{2} \mu \left( \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right)^2 + \left( \mu' - \frac{2}{3}\mu \right) \left( \frac{\partial u_k}{\partial x_k} \right)^2. \end{aligned} \quad (2.10)$$

Eq. (2.10) shows that the heat dissipation function  $\Phi$  is always positive for a stress tensor of the form (2.8).

The term  $\int u_j F_j dV$  in eq. (2.6) is the contribution of viscous and body forces to the changes of the kinetic energy  $k$ . The force (per unit volume)  $F_j$  is a sum of body forces  $\rho X_j$  and viscous forces  $T_j$ . The latter can be expressed in terms of the viscous stress tensor  $\tau_{kj}$ . We thus have the equation

$$F_j = \rho X_j + T_j = \rho X_j + \frac{\partial \tau_{jk}}{\partial x_k} \quad (2.11)$$

Combining eqs. (2.7) and (2.11) we obtain

$$\begin{aligned} \Phi + u_j F_j &= \Phi + u_j T_j + \rho u_j X_j = \\ &= \frac{\partial}{\partial x_k} (u_j \tau_{jk}) + \rho u_j X_j \end{aligned} \quad (2.12)$$

In this form we have subdivided the contributions of forces to the changes of the total internal energy into contributions by viscous and by body forces. The corresponding volume integral in eq. (2.6) is

$$\begin{aligned} \int (\Phi + u_j F_j) dV &= \int (\Phi + u_j T_j + \rho u_j X_j) dV = \\ &= \oint \tau_{jk} u_j n_k dS + \int \rho u_j X_j dV \end{aligned} \quad (2.13)$$

We note that the surface integral in eq. (2.13) contributes to the internal as well as to the kinetic energy of the flow. It represents the viscous forces acting on the surface of the control volume. The volume integral over the body forces contributes to the kinetic energy only.

The governing equations (2.1), (2.2) and (2.4) through (2.6) can also be expressed in differential form as follows

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0, \quad (2.14)$$

$$\frac{\partial}{\partial t} (\rho u_k) + \frac{\partial}{\partial x_j} (\rho u_k u_j) + \frac{\partial p}{\partial x_k} = F_k, \quad (2.15)$$

$$\frac{\partial}{\partial t} (\rho k) + \frac{\partial}{\partial x_j} (\rho k u_j) + u_j \frac{\partial p}{\partial x_j} = u_j F_j, \quad (2.16)$$

$$\frac{\partial}{\partial t} (\rho e) + \frac{\partial}{\partial x_j} (\rho e u_j) + p \frac{\partial u_j}{\partial x_j} = Q - \frac{\partial q_j}{\partial x_j} + \Phi, \quad (2.17)$$

$$\frac{\partial}{\partial t} [\rho(e+k)] + \frac{\partial}{\partial x_j} [\rho(e+k)u_j] + \frac{\partial}{\partial x_j} (p u_j) = Q - \frac{\partial q_j}{\partial x_j} + \Phi + u_j F_j. \quad (2.18)$$

Eq. (2.16) is a consequence of eqs. (2.14) and (2.15), because  $k$  is defined by eq. (2.3). Also, eq. (2.18) is the sum of eqs. (2.16) and (2.17). We have, therefore, only five independent differential equations for the six quantities  $\rho$ ,  $u_k$ ,  $p$ , and  $e$ . To complete the system of equations we need another equation, which is provided by the equations of state for the fluid under consideration. We assume that such an equation is available, e.g., in the form

$$\Theta(e, p, \rho) = 0, \quad (2.19)$$

and that eq. (2.19) can be solved explicitly for either of the arguments. For example, in case of an ideal gas with constant specific heats eq. (2.18) is

$$\frac{p}{\rho} - (\gamma-1)e = 0 . \quad (2.20)$$

For the discussions in the rest of this paper we will not make use of eq. (2.19) or (2.20). The assumption of the existence of such an equation is made here only to close the set of governing equations.

### 3. APPROXIMATE GOVERNING EQUATIONS FOR DUCT FLOWS

In this section we derive approximate governing equations for duct flows. The dominant component of such flows is usually in the axial direction. Also, in many cases only the dependence of flow properties on the axial coordinate is of practical interest. Duct flows are therefore usually treated by one-dimensional equations which are derived from the general flow equations of Section 2.

A standard procedure for the derivation of these equations is to consider a control volume which consists of a length  $\Delta z$  of the duct. The integral forms of the governing equations are applied to this control volume and corresponding differential equations obtained by letting  $\Delta z$  approach zero. This method is used, e.g., in References 4 and 5, and we will follow these references closely.

Another possible approach is to start with governing equations for one-dimensional flow, i.e., a flow which depends on only one coordinate and which has a velocity component in the direction of that coordinate only. Three dimensional effects, e.g., from the wall friction, are then added to the equations by ad hoc procedures. We will not pursue this approach here because the former approach can be generalized more easily.

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<sup>4</sup>A.H. Shapiro, *The Dynamics and Thermodynamics of Compressible Fluid Flow, Vol. I and II*, Roland Press Company, New York, 1954.

<sup>5</sup>L. Crocco, "One-Dimensional Treatment of Steady Gas Dynamics" in *Fundamentals of Gas Dynamics II*, edited by H.W. Emmons, Princeton University Press, 1958.

Let  $z$  be the axial coordinate and let for simplicity the cross-sectional area  $A$  of the duct be constant. The continuity equation (2.1) is then for the control volume

$$\frac{\partial}{\partial t} \left\{ \int_z^{z+\Delta z} \bar{\rho} A dz \right\} + \left[ \bar{\rho} \bar{u} A \right]_z^{z+\Delta z} = 0 \quad (3.1)$$

The bars on  $\rho$  and  $u$  in eq. (3.1) indicate that we are dealing with average density and velocity, respectively. We apply now the mean value theorem to the first term in eq. (3.1) and use a Taylor series expansion for the second term. The result is

$$\frac{\partial}{\partial t} \{ \bar{\rho}(\hat{z}) A \} \cdot \Delta z + \Delta z \cdot A \frac{\partial(\bar{\rho} \bar{u})}{\partial z} = 0(\Delta z^2) , \quad (3.2)$$

where  $z \leq \hat{z} \leq z + \Delta z$ . Letting  $\Delta z$  in eq. (3.2) approach zero we obtain the continuity equation

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial(\bar{\rho} \bar{u})}{\partial z} = 0 . \quad (3.3)$$

The momentum balance equation is considered in the  $z$ -direction only. First we obtain as above from eq. (2.2)

$$\frac{\partial}{\partial t} \{ \bar{\rho}(\hat{z}) \bar{u}(\hat{z}) A \} \cdot \Delta z + \Delta z \cdot A \frac{\partial(\bar{\rho} \bar{u}^2)}{\partial z} + \Delta z \cdot A \frac{\partial \bar{p}}{\partial z} = \bar{F} \cdot A \cdot \Delta z + 0(\Delta z^2) . \quad (3.4)$$

The momentum equation for the average flow properties is obtained from eq. (3.4) by letting  $\Delta z$  approach zero. The result is

$$\frac{\partial}{\partial t} (\bar{\rho} \bar{u}) + \frac{\partial}{\partial z} (\bar{\rho} \bar{u}^2) + \frac{\partial \bar{p}}{\partial z} = \bar{F} . \quad (3.5)$$

The force per unit volume,  $\bar{F}$ , can be expressed as a sum of two components in analogy to eq. (2.11). The momentum equation is then

$$\frac{\partial}{\partial t} (\bar{\rho} \bar{u}) + \frac{\partial}{\partial z} (\bar{\rho} \bar{u}^2) + \frac{\partial \bar{p}}{\partial z} = \bar{\rho} \bar{X} + \bar{T} . \quad (3.6)$$

The quantity  $\bar{T}$  in eq. (3.6) is obtained from the resultant of the viscous boundary forces on the surface of the control volume. For simple tubes  $\bar{T}$  can be expressed in terms of the pipe friction coefficient or, by experimental correlations, in terms of the surface roughness and perimeter of the tube. The term  $\bar{\rho} \bar{X}$  usually represents the gravity force component in the axial direction of the tube.

A combination of eqs. (3.3) and (3.6) yields

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \bar{\rho} \bar{u}^2 \right) + \frac{\partial}{\partial z} \left( \frac{1}{2} \bar{\rho} \bar{u}^3 \right) + \bar{u} \frac{\partial \bar{p}}{\partial z} = \bar{u} \bar{F} = \bar{u} \bar{\rho} \bar{X} + \bar{u} \bar{T} . \quad (3.7)$$

Eq. (3.7) can be considered as an equation for the kinetic energy, if the latter is approximated by

$$\bar{k} = \frac{1}{2} \bar{u}^2 . \quad (3.8)$$

However, eq. (3.7) is a mathematical consequence of the continuity and momentum equations, i.e., eqs. (3.3) and (3.6), and is independent of any assumptions about the kinetic energy.

Next we consider the energy balance. Following general practice (see, e.g., Reference 4) we start with the eq. (2.6) instead of using the first law of thermodynamics, i.e., eq. (2.5). For the control volume we obtain first

$$\begin{aligned} \frac{\partial}{\partial t} \{ \bar{\rho} \cdot (\bar{e} + \bar{k}) \cdot A \} \cdot \Delta z + \Delta z \cdot A \frac{\partial}{\partial z} \{ \bar{\rho} \cdot \bar{u} \cdot (\bar{e} + \bar{k}) + \bar{u} \cdot \bar{p} \} = \\ = \left( \bar{Q} - \frac{\partial \bar{q}}{\partial z} \right) \cdot A \cdot \Delta z + (\bar{\Phi} + \bar{u} \cdot \bar{F}) \cdot A \cdot \Delta z + 0(\Delta z^2) . \end{aligned} \quad (3.9)$$

At the limit  $\Delta z \rightarrow 0$  eq. (3.9) yields the energy equation

$$\begin{aligned} \frac{\partial}{\partial t} \{ \bar{\rho} (\bar{e} + \bar{k}) \} + \frac{\partial}{\partial z} \{ \bar{\rho} \bar{u} (\bar{e} + \bar{k}) + \bar{u} \bar{p} \} = \\ = \bar{Q} - \frac{\partial \bar{q}}{\partial z} + \bar{\Phi} + \bar{u} \bar{F} . \end{aligned} \quad (3.10)$$

The equation of state, such as eq. (2.19), contains usually the internal energy  $\bar{e}$  and not the total internal energy  $\bar{e} + \bar{k}$ . Therefore, eq. (3.10) is modified to eliminate  $\bar{k}$ . To this end it is assumed that the approximation (3.8) holds, and eq. (3.7) is subtracted from eq. (3.10). The result is

$$\frac{\partial}{\partial t} (\bar{\rho} \bar{e}) + \frac{\partial}{\partial z} (\bar{\rho} \bar{u} \bar{e}) + \bar{p} \frac{\partial \bar{u}}{\partial z} = \bar{Q} - \frac{\partial \bar{q}}{\partial z} + \bar{\Phi} . \quad (3.11)$$

Eq. (3.11) is, of course, the first law of thermodynamics and could have been obtained directly from eq. (2.5) without any assumptions about the kinetic energy.

In order to use eqs. (3.3), (3.7), and (3.11) for computations we need among other data estimates for the forces  $\bar{\rho} \bar{X}$  and  $\bar{T}$  and for the heat dissipation function  $\bar{\Phi}$ . The latter is often expressed in terms of  $\bar{T}$  by the following arguments. (See, e.g., Reference 4, page 39 ff. and 972 ff.)

The last integral on the right hand side of the energy equation (2.6) is according to eq. (2.13)

$$W = \int (\bar{\Phi} + u_j F_j) dV = \oint \tau_{jk} u_j n_k dS + \int \rho u_j X_j dV . \quad (3.12)$$

The surface integral in eq. (3.12) represents the work rate of viscous forces acting on the surface  $S$  of the control volume. We subdivide this surface into material boundaries (e.g., duct walls) and flow-through surfaces  $S_0$ . The work done on material boundaries is called shaft work. The work by viscous forces on the flow-through boundaries is called shear work. Let the corresponding work rates be  $W_{\text{shaft}}$  and  $W_0$ , respectively. In these terms eq. (3.12) is

$$W = \int (\bar{\Phi} + u_j F_j) dV = W_{\text{shaft}} + W_0 + \int \rho u_j X_j dV . \quad (3.13)$$

The integral  $W_0$  over the open boundaries is usually assumed to be negligible. For example, in case of a tube flow it is argued that integration over the core flow region contributes very little to the integral because  $\tau_{jk}$  is small in that region. Integration over the boundary layer region also contributes little because the velocity  $u_j$  is small in the boundary layer. Hence  $W_0$  must be small.

If we carry out the derivation of eq. (3.10) using the relation (3.13) we obtain

$$\frac{\partial}{\partial t}[\bar{\rho}(\bar{e} + \bar{k})] + \frac{\partial}{\partial z}[\bar{\rho} \bar{u}(\bar{e} + \bar{k}) + \bar{u} \bar{p}] = \bar{Q} - \frac{\partial \bar{q}}{\partial z} + \bar{W}_{\text{shaft}} + \bar{W}_0 + \bar{\rho} \bar{u} \bar{X}. \quad (3.14)$$

Combining eqs. (3.14), (3.7), and (3.8) we obtain as the first law of thermodynamics instead of eq. (3.11) the equation

$$\frac{\partial}{\partial t}(\bar{\rho} \bar{e}) + \frac{\partial}{\partial z}(\bar{\rho} \bar{u} \bar{e}) + \bar{p} \frac{\partial \bar{u}}{\partial z} = \bar{Q} - \frac{\partial \bar{q}}{\partial z} + \bar{W}_{\text{shaft}} + \bar{W}_0 - \bar{u} \bar{T}. \quad (3.15)$$

In this equation the heat dissipation function  $\bar{\Phi}$  is approximated by

$$\bar{\Phi} = \bar{W}_{\text{shaft}} + \bar{W}_0 - \bar{u} \bar{T}. \quad (3.16)$$

At the material boundaries the velocity of the fluid is equal to the velocity of the boundary. Therefore,  $W_{\text{shaft}}$  is non-zero only if the boundaries are moving. If the tube does not contain moving boundaries and  $\bar{W}_0$  is neglected, then eq. (3.16) becomes

$$\bar{\Phi} = -\bar{u} \bar{T}, \quad (3.17)$$

which is the usual approximation of  $\bar{\Phi}$  for tube flows (Reference 4, page 972 ff.)

In case of two-phase flows, e.g. particles submerged into the fluid,  $\bar{W}_{\text{shaft}}$  is assumed to be the work of drag forces. Let the average particle velocity be  $\bar{u}_{\text{particle}}$  and the drag force be  $\bar{T}_{\text{drag}}$ . Then

$$\bar{W}_{\text{shaft}} = \bar{u}_{\text{particle}} \bar{T}_{\text{drag}}. \quad (3.18)$$

The resultant  $\bar{T}$  of the viscous forces is in this case the sum of particle drag and wall friction forces

$$\bar{T} = \bar{T}_{\text{drag}} + \bar{T}_{\text{wall}} \quad (3.19)$$

The equation for  $\bar{\Phi}$  becomes then

$$\bar{\Phi} = (\bar{u}_{\text{particle}} - \bar{u}) \bar{T}_{\text{drag}} - \bar{u} \bar{T}_{\text{wall}} \quad (3.20)$$

This equation is sometimes modified by an ad hoc factor, see Reference 6, page 81.

In summary, either eq. (3.17), or eq. (3.20) provides a convenient estimate for  $\bar{\Phi}$  i.e., for the right-hand side of the energy equation (3.15). Estimates of  $\bar{T}_{\text{wall}}$  and  $\bar{T}_{\text{drag}}$  are also needed to express the terms on the right-hand sides of the momentum equation (3.6). It appears from the derivation that no further estimates of flow properties are needed under quite general conditions.

Some limitations of the approximation (3.17) become obvious if we consider non-steady fluctuating flows. In such flows it is possible that the signs of  $\bar{T}$  and  $\bar{u}$  are temporarily equal. In these cases  $\bar{W}_0$  cannot be neglected, because otherwise we would have a negative heat dissipation function. Thus it seems appropriate to ask how accurate is the energy equation (3.15). Our derivation does not provide any clues to an answer to this question. We will therefore rederive the duct flow equations more carefully in the next section, keeping track of all approximations involved.

#### 4. PRECISE GOVERNING EQUATIONS FOR DUCT FLOWS

In this section we will derive complete one-dimensional governing equations for flows through constant area ducts, including formulas for quantities which were neglected in Section 3. We will then discuss the differences between the complete equations and those of the previous

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<sup>6</sup>G.B. Wallis, *One-Dimensional Two-Phase Flow*, McGraw-Hill Co, New York, 1969.

section, indicating where the previous derivation of the equations is insufficient.

One-dimensional duct flow equations are relations between average flow properties. The equations depend therefore, among other things, on the definitions of the averages. For steady duct flows certain averages and corresponding governing equations have been discussed by Crocco in Reference 5. Because the averages defined by Crocco cannot be used for non-steady flows, our analysis will be different. The results of our analysis can be applied to steady as well as non-steady flows.

First we consider the continuity equation (2.1). For a control volume which consists of a length of  $\Delta x_3$  of the duct, eq. (2.1) is

$$\frac{\partial}{\partial t} \int_{x_3}^{x_3 + \Delta x_3} \left\{ \int_A \rho \, ds \right\} dx_3 + \left[ \int_A \rho u_3 \, ds \right]_{x_3}^{x_3 + \Delta x_3} = 0 \quad (4.1)$$

The integrals  $\int \rho \, ds$  and  $\int \rho u_3 \, ds$  are functions of  $x_3$ . We expand these functions in Taylor series, interchange the order of integration over  $x_3$  and differentiation with respect to  $t$  in the first term of eq. (4.1), and apply the mean value theorem to that term. The result is

$$\Delta x_3 \left\{ \frac{\partial}{\partial t} \int_A \rho \, ds \right\}_{x_3 + \theta \Delta x_3} + \Delta x_3 \left\{ \frac{\partial}{\partial x_3} \int_A \rho u_3 \, ds \right\}_{x_3} = 0(\Delta x_3^2), \quad (4.2)$$

with  $0 \leq \theta \leq 1$ .

At the limit  $\Delta x_3 \rightarrow 0$  eq. (4.2) yields

$$\frac{\partial}{\partial t} \int_A \rho \, ds + \frac{\partial}{\partial x_3} \int_A \rho u_3 \, ds = 0. \quad (4.3)$$

We now define for each cross-section  $x_3 = \text{const.}$  an average fluid density  $\bar{\rho}$  by

$$\bar{\rho} = \frac{1}{A} \int_A \rho \, ds \quad (4.4)$$

and an average fluid velocity  $\bar{u}$  by

$$\bar{u} = \frac{1}{A \bar{\rho}} \int_A \rho u_3 ds . \quad (4.5)$$

The continuity eq. (4.3) can then be expressed in terms of the average density and velocity as

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_3} (\bar{\rho} \bar{u}) = 0 . \quad (4.6)$$

Eq. (4.6) is identical to the continuity equation (3.3). However, we have now established that the continuity equation is of this form only if the average quantities  $\bar{\rho}$  and  $\bar{u}$  are defined by eqs. (4.4) and (4.5), respectively. Thus, if we chose an alternate definition of the average velocity, e.g., the simple spatial average

$$\tilde{u} = \frac{1}{A} \int_A u_3 ds , \quad (4.7)$$

then the corresponding continuity equation would be

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_3} (\bar{\rho} \tilde{u}) = \frac{\partial}{\partial x_3} [\bar{\rho} (\tilde{u} - \bar{u})] . \quad (4.8)$$

The right-hand side of eq. (4.8) is non-zero in general.

The momentum equation (2.2) yields for a duct flow in analogy to eq. (4.3)

$$\frac{\partial}{\partial t} \int_A \rho u_k ds + \frac{\partial}{\partial x_3} \int_A \rho u_k u_3 ds + \int_A \frac{\partial \rho}{\partial x_k} ds = \int_A F_k ds . \quad (4.9)$$

If the flow is axially symmetric, then for  $k = 1$  and  $k = 2$  all terms in eq. (4.9) are identically zero. In cases of non-axisymmetric flows all three momentum equations are needed to describe the flow, e.g., in the case of a non-vertical tube in a gravity field. We will consider

for simplicity only the momentum equation in the axial  $x_3$ -direction, thus restricting the analysis to axisymmetric flows. Eq. (4.9) thus becomes

$$\frac{\partial}{\partial t} \int_A \rho u_3 ds + \frac{\partial}{\partial x_3} \int_A \rho u_3^2 ds + \int_A \frac{\partial \bar{p}}{\partial x_3} ds = \int_A F_3 ds . \quad (4.10)$$

Eq. (4.10) contains two new flow variables for which averages have to be defined. We chose the following definitions:

$$\bar{p} = \frac{1}{A} \int_A p ds \quad (4.11)$$

and

$$\bar{F}_3 = \frac{1}{A} \int_A F_3 ds . \quad (4.12)$$

Expressing the momentum equation (4.10) in terms of average quantities we then obtain

$$\frac{\partial}{\partial t} (\bar{p} \bar{u}) + \frac{\partial}{\partial x_3} (\bar{p} \bar{u}^2) + \frac{\partial \bar{p}}{\partial x_3} = \bar{F}_3 + C_m \quad (4.13)$$

with

$$C_m = \frac{\partial}{\partial x_3} \left\{ \bar{p} \bar{u}^2 - \frac{1}{A} \int_A \rho u_3^2 ds \right\} = \frac{\partial}{\partial x_3} \left\{ \frac{1}{A^2 \bar{p}} \left[ \left( \int_A \rho u_3 ds \right)^2 - \int_A \rho ds \int_A \rho u_3^2 ds \right] \right\} \quad (4.14)$$

Comparing the momentum equation (4.13) with the corresponding eq. (3.5), we see that the latter equation is in error. The reason for this error is that eq. (3.4) should have contained the term  $\Delta z \cdot C_m \cdot A$ . Eq. (4.14) shows that this term is non-zero in general. The expression in square brackets in eq. (4.14) is negative or zero according to Schwarz's inequality. It is zero if and only if  $u_3 = \text{const.}$  across the duct. Hence the correction term  $C_m$  is zero only in case of a slug flow or if the term is independent of  $x_3$ . The latter is the case for steady incompressible flows through constant area ducts. In all more interesting cases  $C_m$  is non-zero and its magnitude should be estimated to justify the neglect of  $C_m$ , or  $C_m$  should be included in the momentum equation.

The average force per unit volume,  $\bar{F}_3$ , is defined by eq. (4.12). For later reference we note that according to equation (2.11)  $F_3$  is a sum of body forces and viscous forces. We define the corresponding averages by

$$\bar{T}_3 = \frac{1}{A} \int_A \frac{\partial \tau_{3k}}{\partial x_k} ds \quad (4.15)$$

and

$$\bar{X}_3 = \frac{1}{\bar{\rho}A} \int_A \rho X_3 ds . \quad (4.16)$$

With these definitions we have

$$\bar{F}_3 = \bar{\rho} \bar{X}_3 + \bar{T}_3 . \quad (4.17)$$

We now consider the first law of thermodynamics, eq. (2.5). First we obtain for the duct flow in analogy to eq. (4.9)

$$\frac{\partial}{\partial t} \int_A \rho e ds + \frac{\partial}{\partial x_3} \int_A \rho e u_3 ds + \int_A p \frac{\partial u_k}{\partial x_k} ds = \int_A \left( Q - \frac{\partial q_k}{\partial x_k} \right) ds + \int_A \Phi ds . \quad (4.18)$$

In order to express this equation in terms of averages we define

$$\bar{e} = \frac{1}{\bar{\rho}A} \int_A \rho e ds , \quad (4.19)$$

$$\bar{\Phi} = \frac{1}{A} \int_A \Phi ds = \frac{1}{A} \int_A \tau_{kj} \frac{\partial u_k}{\partial x_j} ds \quad (4.20)$$

and

$$\bar{H} = \frac{1}{A} \int_A \left( Q - \frac{\partial q_k}{\partial x_k} \right) ds . \quad (4.21)$$

With these definitions eq. (4.18) becomes

$$\frac{\partial}{\partial t} (\bar{\rho} \bar{e}) + \frac{\partial}{\partial x_3} (\bar{\rho} \bar{e} \bar{u}) + \bar{p} \frac{\partial \bar{u}}{\partial x_3} = \bar{H} + \bar{\Phi} + C_{e1} + C_{e2}, \quad (4.22)$$

where

$$C_{e1} = \bar{p} \frac{\partial \bar{u}}{\partial x_3} - \frac{1}{A} \int_A p \frac{\partial u_k}{\partial x_k} ds \quad (4.23)$$

and

$$C_{e2} = \frac{\partial}{\partial x_3} (\bar{\rho} \bar{e} \bar{u}) - \frac{1}{A} \frac{\partial}{\partial x_3} \int_A \rho e u_3 ds. \quad (4.24)$$

The nature of the correction terms  $C_{e1}$  and  $C_{e2}$  is similar to that of the correction term  $C_m$  in the momentum equation. They are zero for slug flow and should be estimated in other cases. If we compare eq. (4.22) with the corresponding eq. (3.11), we see that the latter is in error. The reason for the error is an oversight of a term  $\Delta z(C_{e1} + C_{e2}) \cdot A$  which should have been introduced in eq. (3.9). The correction terms enter the equations because a product of function averages is in general not equal to the average of the product of the functions. Or, differently expressed, multiplications of functions and averaging of functions are not commutative operations.

We mentioned in Section 3 that  $\bar{\Phi}$  is usually approximated by  $-\bar{u} \bar{T}_3$ . It was also shown that such an approximation is based on the assumption that a term  $W_0$  can be neglected. We will now investigate the approximation more carefully. By the definition (4.20) we have

$$\begin{aligned} \bar{\Phi} &= \frac{1}{A} \int_A \tau_{kj} \frac{\partial u_k}{\partial x_j} ds = \\ &= \frac{1}{A} \int_A \frac{\partial}{\partial x_j} (\tau_{kj} u_k) ds - \frac{1}{A} \int_A u_k \frac{\partial \tau_{kj}}{\partial x_j} ds = \\ &= \frac{1}{A} \frac{\partial}{\partial x_3} \int_A \tau_{k3} u_3 ds - \frac{1}{A} \int_A u_k T_k ds. \end{aligned} \quad (4.25)$$

The first term on the right-hand side of eq. (4.25) we recognize as  $\bar{W}_0$ , i.e., the average of the gradient of the work rate of viscous forces on the cross section A. The second term may be approximated by  $-\bar{u} \bar{T}_3$ . The final formula for  $\bar{\Phi}$  is then

$$\bar{\Phi} = -\bar{u} \bar{T}_3 + \bar{W}_0 + C_\phi, \quad (4.26)$$

where

$$\bar{W}_0 = \frac{1}{A} \frac{\partial}{\partial x_3} \int_A \tau_{k3} u_k ds \quad (4.27)$$

and

$$C_\phi = \bar{u} \bar{T}_3 - \frac{1}{A} \int_A u_k T_k ds. \quad (4.28)$$

Combining eqs. (4.26) and (4.22), we obtain for the energy equation (first law of thermodynamics) the expression

$$\frac{\partial}{\partial t} (\bar{\rho} \bar{e}) + \frac{\partial}{\partial x_3} (\bar{\rho} \bar{e} \bar{u}) + \bar{p} \frac{\partial \bar{u}}{\partial x_3} = \bar{H} - \bar{u} \bar{T}_3 + C_{e1} + C_{e2} + \bar{W}_0 + C_\phi. \quad (4.29)$$

The first two correction terms,  $C_{e1}$  and  $C_{e2}$ , appear in eq. (4.29) because of the averaging of some terms. The last two correction terms,  $\bar{W}_0$  and  $C_\phi$ , are due to the approximation of  $\bar{\Phi}$  by  $-\bar{u} \bar{T}_3$ .

The equation for the kinetic energy can be treated formally in the same manner as the equation for the internal energy. One can introduce error terms, corresponding to  $C_{e1}$  and  $C_{e2}$ , either in the kinetic energy equation or in the equation for the total internal energy. Since typically only the equation for internal energy is needed for computations, the other equations are not formally derived.

In summary, we have shown that the one-dimensional governing equations for average flow properties in duct flows are not the same as equations for locally one-dimensional flows. If the medium is compressible then the additional terms in the governing equations vanish only for slug flow. For other flows the magnitudes of the terms should be estimated for each case to check their significance. Formulas given in this section may be used for that purpose.

If the duct is axially symmetric, it is more convenient to use cylindrical coordinates than the cartesian coordinates of this section. We give therefore in Appendix A all pertinent formulas in cylindrical coordinates.

## 5. EXAMPLES OF TUBE FLOWS

### 5.1 Incompressible Steady Flow Through Cylindrical Tubes

In the case of an incompressible steady duct flow the flow velocity is constant along the duct and dependent on the radial coordinate  $r$  only. Also, only the axial coordinate  $u$  of the velocity is non-zero. Therefore, of all the correction terms given in Appendix A, only  $C_\phi$  can be non-zero in this case. It is given by eq. (A.35), which reduces to

$$C_\phi = \bar{u} \bar{T}_z - \frac{2}{R^2} \int_0^R u \frac{\partial}{\partial r} (r \tau_{rz}) dr . \quad (5.1.1)$$

The shearing stress  $\tau_{rz}(r)$  is in the present case a linear function of  $r$ . This is a consequence of the second momentum equation (A.13) which reduces to

$$\frac{\partial p}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) . \quad (5.1.2)$$

The left-hand side of eq. (5.1.2) is constant. Therefore,  $\tau_{rz}$  must be linear in  $r$ :

$$\tau_{rz}(r) = \frac{r}{R} \tau_{rz}(R) . \quad (5.1.3)$$

Substituting eq. (5.1.3) into eq. (5.1.1), we obtain

$$\begin{aligned} C_\phi &= \bar{u} \bar{T}_z - \frac{4\tau_{rz}(R)}{R^3} \int_0^R u r dr = \\ &= \bar{u} \bar{T}_z - 2 \bar{u} \tau_{rz}(R) \frac{1}{R} . \end{aligned} \quad (5.1.4)$$

The average shear stress  $\bar{T}_z$  is, according to eqs. (A.17) and (A.21),

$$\bar{T}_z = \frac{2}{R^2} \int_0^R \frac{\partial}{\partial r} (r \tau_{rz}) dr = 2\tau_{rz}(R) \frac{1}{R}. \quad (5.1.5)$$

Substituting eq. (5.1.5) into eq. (5.1.4) we see that the correction term  $C_\phi$  is zero.

Hence the average flow equations are exact for incompressible steady flows through circular tubes. This is essentially a consequence of eq. (5.1.3) and the result is valid for either turbulent or laminar flows. Also, we have not made use of Stokes equations for the stress tensor, nor made any assumptions about the viscosity of the fluid.

## 5.2 Lagrange's Interior Ballistics Flow

As an example for non-steady tube flows we consider Lagrange's approximation to interior ballistics flow (Reference 7). The approximation is obtained by postulating that the average axial velocity  $\bar{u}$  of the gas in a gun tube is at any time a linear function of the axial distance  $z$ , i.e.,

$$\bar{u}(z,t) = \frac{z}{z_p(t)} u_p(t), \quad (5.2.1)$$

where  $z_p(t)$  and  $u_p = dz_p/dt$  are the location and velocity of the projectile, respectively. We assume that the local velocity can have axial as well as radial components which may depend on  $z$ ,  $t$ , and on the radial coordinate  $r$ .

Some consequences of the assumption (5.2.1) are discussed in the Appendix B. In summary, the discussion shows that this assumption, complemented with a second Lagrange's assumption

$$\rho = \rho_0 \frac{z_p(0)}{z_p(t)}, \quad (5.2.2)$$

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<sup>7</sup>J. Corner, *Theory of the Interior Ballistics of Guns*, John Wiley and Sons, New York, 1950.

is consistent with the average continuity equation (4.6) for the flow. One can also assume that the local velocity vector has the form

$$u^* = \begin{pmatrix} \bar{u}(z,t) \cdot f(r) \\ \bar{v}(z,t) \cdot h(r) \end{pmatrix}. \quad (5.2.3)$$

For any reasonable functions  $\bar{u}(z,t)$  and  $f(r)$  one can determine corresponding functions  $\bar{v}(z,t)$  and  $h(r)$  such that the local continuity equation is satisfied. (The necessary formulas are given in Appendix B.) However, a flow characterized by eqs. (5.2.1) through (5.2.3) in general does not satisfy the local momentum equations if constant viscosities are assumed. Hence Lagrange's approximation, (5.2.1) and (5.2.2), and a local velocity field of the type (5.2.3) can be consistent only for inhomogeneous media, i.e., media with variable viscosity.

Because an exact solution of the viscous tube flow equations is not available, we cannot obtain exact values for all correction terms. However, the correction term in the momentum equation is independent of the viscosities and can be computed exactly for any flow profile. In contrast, the correction terms in the energy equation can be computed only if additional information is available about the stress tensor and the internal energy profile. These terms we will estimate by computing their values for constant viscosities and for a number of "reasonable" flow profiles. We expect by such calculations to obtain at least order-of-magnitude estimates of the correction terms.

Particularly we will consider flow profiles of two types. First we will assume a flow field which is described by

$$\left. \begin{aligned} u(r,z,t) &= z \frac{u_p(t)}{z_p(t)} \frac{n+2}{n} \left[ 1 - \left( \frac{r}{R} \right)^{\bar{n}} \right] \\ \text{and} \\ v(r,z,t) &= - R \frac{u_p(t)}{z_p(t)} \frac{1}{n} \left( \frac{r}{R} \right) \left[ 1 - \left( \frac{r}{R} \right)^{\bar{n}} \right] \end{aligned} \right\} (5.2.4)$$

This flow field has a Hagen-Poiseuille profile for  $n = 2$ . For larger values of  $n$  it approximates turbulent flow profiles or profiles with thin boundary layers.

As a second example we will consider a flow profile which approximates the universal profile for steady turbulent tube flow.

The flow field defined by eq. (5.2.4) satisfies the local continuity equation, if the density is given by eq. (5.2.2). We note the interesting fact that local continuity requires the radial flow component to be directed toward the center of the tube. This is due to the higher mass flow rate at the center and due to the assumed increase of the average axial velocity  $\bar{u}(z,t)$  with  $z$ .

The correction term  $C_m$  of the momentum equation is given for our flow by eq. (B.48)

$$C_m = \left\{ 1 - \frac{2}{R^2} \int_0^R f^2 r \, dr \right\} \frac{\partial}{\partial z} (\rho u^2) . \quad (5.2.5)$$

Substituting

$$f(r) = \frac{n+2}{n} \left[ 1 - \left( \frac{r}{R} \right)^{\bar{n}} \right] \quad (5.2.6)$$

into eq. (5.2.5), we obtain

$$C_m = - \frac{1}{n+1} \frac{\partial}{\partial z} (\bar{\rho} \bar{u}^2) . \quad (5.2.7)$$

The momentum equation is therefore in terms of the average axial velocity

$$\frac{\partial(\bar{\rho} \bar{u})}{\partial t} + \left( 1 + \frac{1}{n+1} \right) \frac{\partial}{\partial z} (\bar{\rho} \bar{u}^2) + \frac{\partial \bar{p}}{\partial z} = \bar{T}_z . \quad (5.2.8)$$

Eq. (5.2.8) shows that in the case of a Hagen-Poiseuille profile the momentum transport term in the momentum equation should be increased by about 33%. Even for a rather flat profile with, say,  $n = 10$  the correction term is 9% in this example.

The first correction term  $C_{e1}$  of the energy equation is zero in our example because the density  $\rho$  is independent of  $r$  and  $z$ . (See Appendix B for a discussion of this term.)

The second correction term  $C_{e2}$  of the energy equation is (see eq. (B.54))

$$\begin{aligned}
 C_{e2} &= \frac{\partial}{\partial z} \left\{ \bar{\rho} \bar{u} \frac{2}{R^2} \int_0^R (1-f) e r dr \right\} = \\
 &= \rho \frac{u_p}{z_p} \frac{2}{R^2} \frac{\partial}{\partial z} \left\{ z \int_0^R (1-f) e r dr \right\}. \quad (5.2.9)
 \end{aligned}$$

In order to compute this term we would need to make an assumption about the internal energy function  $e$ . For the present discussion we will not make any assumptions and leave eq. (5.2.9) unchanged.

The average heat dissipation function  $\bar{\Phi}$ , which appears on the right hand-side of the energy equation can be computed by the formulas (B.56) and (B.57). The result of the computation is

$$\bar{\Phi} = 2\mu \bar{u}^2 \frac{1}{R^2} \frac{(n+2)^2}{n} + \left( \frac{u_p}{z_p} \right)^2 \left( z \frac{n+3}{n+1} - \frac{2}{3} \mu + \mu' \right). \quad (5.2.10)$$

The equation for the average internal energy (first law of thermodynamics) is in our case

$$\frac{\partial(\bar{\rho} \bar{e})}{\partial t} + \frac{\partial}{\partial z} (\bar{\rho} \bar{e} \bar{u}) + \bar{p} \frac{\partial \bar{u}}{\partial z} = \bar{H} + \bar{\Phi} + C_{e2}. \quad (5.2.11)$$

Substituting eqs. (5.2.9) and (5.2.10) into eq. (5.2.11), we obtain

$$\begin{aligned}
 \frac{\partial(\bar{\rho} \bar{e})}{\partial t} + \frac{\partial}{\partial z} \left[ \bar{p} \bar{e} \bar{u} \left( 1 - \frac{1}{\bar{e}} \frac{2}{R^2} \int_0^R (1-f) e r dr \right) \right] + \bar{p} \frac{\partial \bar{u}}{\partial z} = \\
 = \bar{H} + 2\mu \bar{u}^2 \frac{1}{R^2} \left[ \frac{(n+2)^2}{2n} + \left( \frac{2}{3} \frac{n+4}{n+1} + \frac{\mu'}{2\mu} \right) \left( \frac{R}{z} \right)^2 \right]. \quad (5.2.12)
 \end{aligned}$$

In eq. (5.2.12) we have included the correction term  $C_{e2}$  into the energy flux term on the left hand side. It is readily apparent from the form of the term that the correction is zero, if the specific internal energy  $e$  is independent of  $r$ .

In the heat dissipation function on the right-hand side of eq. (5.2.12) the term with the factor  $(R/z)^2$  can generally be neglected, because  $(R/z)^2$  is of the order  $10^{-2}$ . ( $R/z$  is large in the vicinity of the breach, where the one-dimensional approximation should not be used anyway.) The other term in the square brackets is usually replaced by  $-\bar{u} \bar{T}_z$ . If this is done, then two additional correction terms should be included in the equation. The general formulas for these terms are given by eqs. (B.61) and (B.62). They are in our case

$$\begin{aligned} \bar{W}_0 &= \left(\frac{u}{z_p}\right)^2 \left\{ \frac{2}{R^2} \int_0^R \mu \left[ R^2 h'^2 + \left(\frac{R}{r}\right)^2 h^2 + 2f^2 \right] r \, dr + \mu' - \frac{2}{3} \mu \right\} = \\ &= \left(\frac{u}{z_p}\right)^2 \cdot \left[ 2\mu \frac{n+5/2}{n+1} + \mu' - \frac{2}{3} \mu \right] \end{aligned} \quad (5.2.13)$$

and

$$\begin{aligned} C_\phi &= \bar{u} \bar{T}_z + \left(\frac{u}{z_p}\right)^2 \mu \frac{2}{R^2} \int_0^R \left[ R^2 h'^2 + \left(\frac{R}{2}\right)^2 h^2 + z^2 f'^2 \right] r \, dr = \\ &= \bar{u} \bar{T}_z + \left(\frac{u}{z_p}\right)^2 2\mu \frac{1}{2(n+1)} + 2\mu \bar{u}^2 \frac{1}{R^2} \frac{(n+2)^2}{2n} \end{aligned} \quad (5.2.14)$$

The first term  $\bar{u} \bar{T}_z$  in eq. (5.2.14) is according to eq. (B.63)

$$\bar{u} \bar{T}_z = 2\mu \bar{u}^2 \frac{1}{R} f'(R) = -2\mu \bar{u}^2 \frac{1}{R^2} (n+2) = -\left(\frac{u}{z_p}\right)^2 \left(\frac{z}{R}\right)^2 2\mu (n+2). \quad (5.2.15)$$

Comparing eqs. (5.2.13) and (5.2.15), we see that the term  $\bar{W}_0$  is indeed small relative to the magnitude of  $\bar{u} \bar{T}_z$ . In Section 3 such a ratio of magnitudes was anticipated based on plausibility arguments.

The total correction is the sum of  $C_\phi$  and  $\bar{W}_0$ . Combining eqs. (5.2.13) through (5.2.15) we obtain for the sum

$$C_\phi + \bar{W}_0 = \bar{u} \bar{T}_z \left[ \frac{n-2}{2n} - \left(\frac{R}{z}\right)^2 \frac{2}{3} \frac{n+4}{(n+1)(n+2)} \left( 1 + \frac{\mu'}{\mu} \frac{3(n+1)}{4(n+4)} \right) \right] \quad (5.2.16)$$

The right-hand side of the energy equation (5.2.12) is thus

$$\bar{H} + \bar{\Phi} = \bar{H} - \bar{u} \bar{T}_z + C_\phi + \bar{W}_0 = \bar{H} - \bar{u} \bar{T}_z (1-a), \quad (5.2.17)$$

where  $a$  is a relative correction which is to be applied to  $\bar{u} \bar{T}_z$ . It is given by

$$a = \frac{n-2}{2n} - \left(\frac{R}{z}\right)^2 \frac{2}{3} \frac{n+4}{(n+1)(n+2)} \left(1 + \frac{\mu'}{\mu} \frac{3}{4} \frac{n+1}{n+4}\right). \quad (5.2.18)$$

The second term in this formula can in general be neglected, because  $(R/z)^2$  is of the order  $10^{-2}$ . The first term is zero only for  $n = 2$ , i.e., for a Hagen-Poiseuille flow profile. In this case the shear stress is a linear function of  $r$ , which causes certain correction terms to vanish, as shown in Section 5.1. For a flat flow profile with, say,  $n = 10$ , the relative correction is  $a = 0.4$ . Clearly such a 40% approximation error will be seldom tolerable. Hence for flat flow profiles and constant viscosities the approximation of  $\bar{\Phi}$  by  $-\bar{u} \bar{T}_z$  is not realistic for calculations in interior ballistics.

The flow profile which is defined by eq. (5.2.4) does not have the characteristic form of a fully developed turbulent flow profile for any  $n$ . We may therefore ask whether the correction terms are possibly smaller for such a profile. In order to investigate this question we approximate the universal turbulent profile (see, e.g., Reference 8, page 512) by defining

$$f(r) = 0.456 \left\{ 1 - \left(\frac{r}{R}\right)^{1.5} + 2 \left[ 1 - \left(\frac{r}{R}\right)^{15} \right] \right\}. \quad (5.2.19)$$

The corresponding function  $h(r)$  is

$$h(r) = -0.456 \frac{r}{R} \left\{ \frac{1}{3.5} \left[ 1 - \left(\frac{r}{R}\right)^{1.5} \right] + \frac{2}{17} \left[ 1 - \left(\frac{r}{R}\right)^{15} \right] \right\}. \quad (5.2.20)$$

<sup>8</sup>H. Schlichting, *Boundary Layer Theory*, McGraw-Hill, New York (4th Edition), 1960.

The correction term  $C_m$  of the momentum equation can now be computed using eq. (5.2.5). The result is

$$C_m = - 0.090 \frac{\partial}{\partial z} (\bar{\rho} \bar{u}^2) . \quad (5.2.21)$$

In analogy to eq. (5.2.8) we conclude from eq. (5.2.21) that the momentum flux term in the momentum equation should be increased by 9% in the present case.

Assuming as before constant viscosities, we obtain for the average heat dissipation function

$$\bar{\Phi} = 2\mu \bar{u}^2 \frac{1}{R^2} 7.840 + \left( \frac{u}{z} \frac{p}{p} \right)^2 (1.699 \mu + \mu') . \quad (5.2.22)$$

For the product  $-\bar{u} \bar{T}_z$  we obtain

$$-\bar{u} \bar{T}_z = - 2\mu \bar{u}^2 \frac{1}{R^2} f'(R) = 2\mu \bar{u}^2 \frac{1}{R^2} \cdot 14.364 . \quad (5.2.23)$$

The right-hand side of the energy equation (5.2.12) is therefore

$$\bar{H} + \bar{\Phi} = \bar{H} - \bar{u} \bar{T}_z \left[ 1 - 0.454 + \left( \frac{R}{z} \right)^2 0.059 \left( 1 + 0.588 \frac{\mu'}{\mu} \right) \right] \quad (5.2.24)$$

The error which is introduced by replacing  $\bar{\Phi}$  by  $-\bar{u} \bar{T}_z$  is about 45% of  $|\bar{u} \bar{T}_z|$ . As in the previously treated case, such errors will be seldom tolerable.

We may conclude from these examples that the magnitudes of correction terms are essentially the same for flow profiles described by eq. (5.2.4) as for profiles described by eqs. (5.2.19) and (5.2.20). Using conventional tube flow equations, e.g. from Reference 4, for interior ballistics calculations, one introduces errors in the momentum and energy equations which are of the order of 9-50% of several of the terms. The examples indicate that an investigation of magnitudes of the correction terms is necessary whenever average flow equations are used to describe non-steady tube flows.

## 6. CONCLUSIONS

Tube flow governing equations for average properties differ from one-dimensional flow equations. The differences are caused by the fact that averaging of functions and multiplication of functions are not commutative operations. The magnitudes of the differences depend on the particular problem. If the unsteady tube flow is of a type which is encountered in interior ballistics, then several terms in the equations can be in error by up to 50%.

One consequence of the various correction terms in the equations is that the continuity and momentum equations cannot be combined to yield a simple equation for the average axial velocity component. Instead, the original equation for the average axial momentum component is the simplest form. Correspondingly, the energy equation should be formulated for the internal energy per unit volume instead of using the specific internal energy.

The popular approximation of the heat dissipation function by the product of average velocity and average shear stress is appropriate only in the simplest cases, e.g., for steady flows or flows with a Hagen-Poiseuille velocity profile. In other cases the approximation can be off by up to 50%. In cases of more complicated flows even the sign of the approximation can be wrong. Hence the approximation should not be used unless one can demonstrate its validity in the particular case of application.

Formulas for the correction terms in the governing equations can be derived for other than simple tube flows following the outline of this paper. The derivations which are presented in some engineering textbooks neglect important first-order terms. The apparent success of the inaccurate equations for the treatment of tube flows is probably due to the fact that the neglected terms are small or vanish for steady flows, for which most comparisons between calculation and experiments are made.

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## APPENDIX A

### FORMULAS IN CYLINDRICAL COORDINATES

In Section 2 through 4 a convenient cartesian tensor notation was used to derive all formulas. If the results are to be used for axially symmetric tube flows, then it is more convenient to use cylindrical coordinates. In this appendix we express the important formulas in these coordinates.

Stokes equation for the stress tensor of an isotropic fluid can be expressed in coordinate independent form as follows (Reference 3, page 132; Reference 2, page 144)

$$\tau = 2\mu \epsilon + (\mu' - \frac{2}{3}\mu) \text{div } u^* \cdot I, \quad (\text{A.1})$$

where  $\tau$  is the stress tensor,  $\epsilon$  is the strain rate tensor,  $u^*$  is the velocity vector of the fluid, and  $I$  is the unit tensor. The viscosities  $\mu$  and  $\mu'$  in eq. (A.1) need not be constant, i.e., the fluid under consideration need not be homogeneous. However,  $\mu$  as well as  $\mu'$  must be positive or zero.

Next we compute the work rate of the viscous forces acting within an arbitrary volume. To this end we compute the inner product of the viscous forces  $\nabla \cdot \tau$  with the velocity vector  $u^*$  and integrate over the volume. The result can be expressed as follows:

$$\int (u^* \cdot (\nabla \cdot \tau)) dV = \oint (u^* \cdot (\tau \cdot n)) dS - \int \Phi dV. \quad (\text{A.2})$$

In eq. (A.2)  $n$  is a unit vector, orthogonal to the surface of the volume  $V$  and pointing inward, and  $\Phi$  is the heat dissipation function defined by

$$\Phi = \{\tau\epsilon\}_{\text{trace}} = 2\mu \{\epsilon^2\}_{\text{trace}} + (\mu' - \frac{2}{3}\mu) \text{div } u^* \{\epsilon\}_{\text{trace}}. \quad (\text{A.3})$$

Because  $\text{div } u^* = \{\epsilon\}_{\text{trace}}$ , eq. (A.3) can be also expressed as follows:

$$\Phi = 2\mu \{\epsilon^2\}_{\text{trace}} + (\mu' - \frac{2}{3}\mu) \{\epsilon\}_{\text{trace}}^2. \quad (\text{A.4})$$

Eq. (A.4) corresponds to eq. (2.10) in cartesian coordinates. Eq. (A.2) corresponds to eq. (2.13) in cases where the body forces  $X_j$  are absent.

We now express the various quantities appearing in the equations using cylindrical coordinates. Let the coordinates be  $r$ ,  $\phi$  and  $z$ . Components of vectors and tensors we denote by attaching corresponding indexes to the quantities. Thus, the velocity vector  $u^*$  is

$$u^* = (u_r, u_\phi, u_z) . \quad (A.5)$$

The strain rate tensor  $\epsilon$  has the following components (Reference 2, Page 602)

$$\left. \begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \epsilon_{\phi\phi} &= \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{1}{r} u_r, & \epsilon_{zz} &= \frac{\partial u_z}{\partial z}, \\ \epsilon_{r\phi} &= \frac{1}{2} \left[ r \frac{\partial}{\partial r} \left( \frac{1}{r} u_\phi \right) + \frac{1}{r} \frac{\partial u_r}{\partial \phi} \right], \\ \epsilon_{rz} &= \frac{1}{2} \left[ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right], \\ \epsilon_{\phi z} &= \frac{1}{2} \left[ \frac{1}{r} \frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z} \right]. \end{aligned} \right\} \quad (A.6)$$

The vector  $\nabla \cdot \tau$  has the components  $T_r$ ,  $T_\phi$  and  $T_z$ , representing the viscous forces acting in the three coordinate directions. The components are

$$\begin{aligned} (\nabla \cdot \tau)_r &= T_r = \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r} (\tau_{rr} - \tau_{\phi\phi}), \\ (\nabla \cdot \tau)_\phi &= T_\phi = \frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\partial \tau_{\phi z}}{\partial z} + 2 \frac{1}{r} \tau_{r\phi}, \\ (\nabla \cdot \tau)_z &= T_z = \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\phi z}}{\partial \phi} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r} \tau_{rz}. \end{aligned} \quad (A.7)$$

We specialize these equations for the case of an axisymmetric flow without swirl through a circular tube. The flow is then independent of the coordinate  $\phi$ , and the  $\phi$ -component of the velocity,  $u_\phi$ , is zero. In order to simplify the notation we denote the non-zero velocity components as follows:

$$u_z = u(t, z, r) , \quad (A.8)$$

$$u_r = v(t, z, r) .$$

Let  $R$  be the radius of the tube. The average density is then defined by

$$\bar{\rho}(t, z) = \frac{2}{R^2} \int_0^R \rho(t, z, r) r dr . \quad (A.9)$$

The average axial velocity is

$$\bar{u}(t, z) = \frac{2}{R^2 \bar{\rho}(t, z)} \int_0^R \rho(t, z, r) u(t, z, r) r dr . \quad (A.10)$$

The local continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v) + \frac{\partial}{\partial z} (\rho u) = 0 . \quad (A.11)$$

The corresponding equation for the averages is

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial z} (\bar{\rho} \bar{u}) = 0 . \quad (A.12)$$

The local balance of momentum is expressed by the following two differential equations:

$$\left. \begin{aligned} \frac{\partial(\rho v)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r v^2) + \frac{\partial}{\partial z} (\rho u v) + \frac{\partial p}{\partial r} &= T_r \\ \frac{\partial(\rho u)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r u v) + \frac{\partial}{\partial z} (\rho u^2) + \frac{\partial p}{\partial z} &= T_z \end{aligned} \right\} \text{(A.13)}$$

The third momentum equation is satisfied identically because of our symmetry assumptions. The right-hand sides of eqs. (A.13) depend on the strain rate tensor  $\epsilon$  by eq. (A.7) and (A.1). In our case the strain rate tensor has the following components:

$$\left. \begin{aligned} \epsilon_{rr} &= \frac{\partial v}{\partial r}, \quad \epsilon_{\phi\phi} = \frac{1}{r} v, \quad \epsilon_{zz} = \frac{\partial u}{\partial z}, \\ \epsilon_{r\phi} &= 0, \quad \epsilon_{\phi z} = 0, \\ \epsilon_{rz} &= \frac{1}{2} \left[ \frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right]. \end{aligned} \right\} \text{(A.14)}$$

The divergence of the velocity vector is

$$\begin{aligned} \text{div } u^* &= \{\epsilon\}_{\text{trace}} = \epsilon_{rr} + \epsilon_{\phi\phi} + \epsilon_{zz} = \\ &= \frac{1}{r} \frac{\partial}{\partial r} (r v) + \frac{\partial u}{\partial z}. \end{aligned} \quad \text{(A.15)}$$

The viscous stress tensor  $\tau$  has the components

$$\begin{aligned}
 \tau_{rr} &= 2\mu \frac{\partial v}{\partial r} + (\mu' - \frac{2}{3}\mu) \operatorname{div} u^* , \\
 \tau_{r\phi} &= 0 , \\
 \tau_{rz} &= \mu \left[ \frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right] , \\
 \tau_{\phi\phi} &= 2\mu \frac{1}{r} v + (\mu' - \frac{2}{3}\mu) \operatorname{div} u^* , \\
 \tau_{\phi z} &= 0 , \\
 \tau_{zz} &= 2\mu \frac{\partial u}{\partial z} + (\mu' - \frac{2}{3}\mu) \operatorname{div} u^* .
 \end{aligned}
 \tag{A.16}$$

The right-hand side of the local momentum equations (A.13) is

$$\begin{aligned}
 (\nabla \cdot \tau)_r = T_r &= \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rr}) + \frac{\partial \tau_{rz}}{\partial z} - \frac{1}{r} \tau_{\phi\phi} \\
 (\nabla \cdot \tau)_z = T_z &= \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}
 \end{aligned}
 \tag{A.17}$$

Substituting (A.16) into (A.17), we obtain

$$\tau_r = \frac{1}{r} \frac{\partial}{\partial r} \left( 2\mu r \frac{\partial v}{\partial r} \right) - 2\mu \frac{v}{r^2} + \frac{\partial}{\partial r} \left[ \left( \mu' - \frac{2}{3} \mu \right) \text{div } u^* \right] + \frac{\partial}{\partial z} \left[ \mu \frac{\partial v}{\partial z} + \mu \frac{\partial u}{\partial r} \right],$$

$$\tau_z = \frac{1}{r} \frac{\partial}{\partial r} \left[ \mu r \frac{\partial v}{\partial z} + \mu r \frac{\partial u}{\partial r} \right] + \frac{\partial}{\partial z} \left[ 2\mu \frac{\partial u}{\partial z} + \left( \mu' - \frac{2}{3} \mu \right) \text{div } u^* \right] \quad (\text{A.18})$$

The momentum equation for the averages is

$$\frac{\partial(\bar{\rho} \bar{u})}{\partial t} + \frac{\partial}{\partial z} (\bar{\rho} \bar{u}^2) + \frac{\partial \bar{p}}{\partial z} = \bar{\tau}_z + C_m, \quad (\text{A.19})$$

where the average pressure  $\bar{p}$  is defined by

$$\bar{p} = \frac{2}{R^2} \int_0^R p(t, z, r) r \, dr \quad (\text{A.20})$$

and the average viscous force  $\bar{\tau}_z$  by

$$\bar{\tau}_z = \frac{2}{R^2} \int_0^R \tau_z(t, z, r) r \, dr. \quad (\text{A.21})$$

The correction term  $C_m$  in eq. (A.19) is

$$C_m = \frac{\partial}{\partial z} \left\{ \bar{\rho} \bar{u}^2 - \frac{2}{R^2} \int_0^R \rho u^2 r \, dr \right\}. \quad (\text{A.22})$$

The equation for the local internal energy is

$$\frac{\partial(\rho e)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho e v) + \frac{\partial}{\partial z} (\rho e u) + p \left[ \frac{1}{r} \frac{\partial}{\partial r} (r v) + \frac{\partial u}{\partial z} \right] = Q - \text{div } q + \Phi \quad (\text{A.23})$$

The heat dissipation function  $\Phi$  is given by eq. (A.4). Substituting the strain rate tensor components from eq. (A.6) into (A.4), we obtain

$$\begin{aligned} \Phi = 2\mu & \left[ \left( \frac{\partial v}{\partial r} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right)^2 + \left( \frac{v}{r} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right] + \\ & + (\mu' - \frac{2}{3}\mu) \left[ \frac{1}{r} \frac{\partial}{\partial r} (r v) + \frac{\partial u}{\partial z} \right]^2 . \end{aligned} \quad (\text{A.24})$$

The energy equation for the average internal energy is

$$\frac{\partial(\bar{\rho} \bar{e})}{\partial t} + \frac{\partial}{\partial z} (\bar{\rho} \bar{e} \bar{u}) + \bar{p} \frac{\partial \bar{u}}{\partial z} = \bar{H} + \bar{\Phi} + C_{e1} + C_{e2} , \quad (\text{A.25})$$

where

$$\bar{e} = \frac{1}{\bar{\rho}} \cdot \frac{2}{R^2} \int_0^R \rho e r dr , \quad (\text{A.26})$$

$$\bar{H} = \frac{2}{R^2} \int_0^R (Q - \text{div } q) r dr , \quad (\text{A.27})$$

$$\bar{\Phi} = \frac{2}{R^2} \int_0^R \Phi r dr , \quad (\text{A.28})$$

$$C_{e1} = \bar{p} \frac{\partial \bar{u}}{\partial z} - \frac{2}{R^2} \int_0^R p \left[ \frac{\partial u}{\partial z} r + \frac{\partial}{\partial r} (r v) \right] dr \quad (\text{A.29})$$

and

$$C_{e2} = \frac{\partial}{\partial z} (\bar{\rho} \bar{u} \bar{e}) - \frac{2}{R^2} \int_0^R \frac{\partial}{\partial z} (\rho u e) r dr . \quad (\text{A.30})$$

The term  $-\bar{u} \bar{T}_z$  is often used instead of  $\bar{\Phi}$  in eq. (A.25). In that case the equation becomes

$$\frac{\partial(\bar{p} \bar{\theta})}{\partial t} + \frac{\partial}{\partial z} (\bar{p} \bar{\theta} \bar{u}) + \bar{p} \frac{\partial \bar{u}}{\partial z} = \bar{H} - \bar{u} \bar{T}_z + C_{e1} + C_{e2} + \bar{W}_0 + C_\phi. \quad (\text{A.31})$$

The additional correction terms  $\bar{W}_0$  and  $C_\phi$  are

$$\bar{W}_0 = \frac{2}{R^2} \int_0^R \text{div}(\tau u^*) r dr \quad (\text{A.32})$$

and

$$C_\phi = \bar{u} \bar{T}_z - \frac{2}{R^2} \int_0^R u^* \cdot (\nabla \cdot \tau) r dr. \quad (\text{A.33})$$

In eq. (A.32) we have

$$\tau u^* = \begin{pmatrix} \tau_{rr} v + \tau_{rz} u \\ 0 \\ \tau_{rz} v + \tau_{zz} u \end{pmatrix}$$

and

$$\text{div}(\tau u^*) = \frac{1}{r} \frac{\partial}{\partial r} \left[ r (\tau_{rr} v + \tau_{rz} u) \right] + \frac{\partial}{\partial z} \left[ \tau_{rz} v + \tau_{zz} u \right].$$

Therefore

$$\begin{aligned} \bar{W}_0 &= \frac{2}{R^2} \int_0^R \frac{\partial}{\partial z} \left[ \tau_{rz} v + \tau_{zz} u \right] r dr = \\ &= \frac{2}{R^2} \int_0^R \frac{\partial}{\partial z} \left[ \mu v \left( \frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right) + 2\mu u \frac{\partial u}{\partial z} + \left( \mu' - \frac{2}{3} \mu \right) u \text{div} u^* \right] r dr, \quad (\text{A.34}) \end{aligned}$$

where  $\text{div} u^*$  is given by (A.15).

The integrand in (A.33) can be obtained from eq. (A.17). Carrying out the substitutions, we obtain for the second correction term

$$\begin{aligned}
 C_{\phi} &= \bar{u} \bar{T}_z - \frac{2}{R^2} \int_0^R (v T_r + u T_z) r dr = \\
 &= \bar{u} \bar{T}_z - \frac{2}{R^2} \int_0^R \left\{ v \left[ \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rr}) + \frac{\partial \tau_{rz}}{\partial z} - \frac{1}{r} \tau_{\phi\phi} \right] + \right. \\
 &\quad \left. + u \left[ \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z} \right] \right\} r dr . \tag{A.35}
 \end{aligned}$$

The separate expressions for  $C_{\phi}$  and  $\bar{W}_0$  might be of interest for the discussion of approximations. Usually  $C_{\phi}$  is neglected completely and  $\bar{W}_0$  is assumed to be small by plausibility arguments. The total correction, which is caused by replacement of  $\bar{\Phi}$  by  $-\bar{u} \bar{T}_z$ , is the sum of  $\bar{W}_0$  and  $C_{\phi}$ . The sum is, of course,

$$\begin{aligned}
 \bar{W}_0 + C_{\phi} &= \bar{u} \bar{T}_z + \bar{\Phi} = \\
 &= \bar{u} \bar{T}_z + \frac{2}{R^2} \int_0^R \left\{ 2\mu \left[ \left( \frac{\partial v}{\partial r} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{v}{r} \right)^2 \right] + \right. \\
 &\quad \left. + \left( \mu' - \frac{2}{3} \mu \right) (\text{div } u^*)^2 \right\} r dr . \tag{A.36}
 \end{aligned}$$

Eq. (A.36) may be more advantageous for actual calculations than (A.34) and (A.35) because it does not contain derivatives of the viscosities.

APPENDIX B

LAGRANGE'S APPROXIMATION TO INTERIOR BALLISTICS FLOW

It is plausible to assume that in a gun tube the average axial velocity  $\bar{u}(z,t)$  of each cross-section is a linear function of the distance  $z$  from the breech of the weapon. Let  $z_p(t)$  be the location of the projectile and  $u_p(t) = dz_p/dt$  be its velocity. The above-mentioned Lagrange's approximation is then

$$\bar{u}(z,t) = \frac{z}{z_p(t)} u_p(t) . \quad (B.1)$$

In the classical Lagrange's approximation (B.1) is supplemented with the assumption that the gas density in the tube is a function of time only.

In this appendix we shall investigate some consequences of these assumptions. Particularly we are interested in finding if there is a three-dimensional viscous tube flow which satisfies Lagrange's assumptions.

First we will consider flows in which the gas density is a separable function of  $z$ ,  $t$ , and the radial coordinate  $r$ :

$$\rho(r,z,t) = g(r) \cdot P(z) \cdot K(t) . \quad (B.2)$$

Later we will specialize our considerations to the classical Lagrange's approximation, where  $P(z)$  and  $g(r)$  are constants.

We assume that  $g(r)$  is non-dimensional and normalized by

$$\frac{2}{R^2} \int_0^R g(r) r dr = 1 . \quad (B.3)$$

The product of the other two functions in eq. (B.2) is then the average density

$$\bar{\rho}(z,t) = P(z) K(t) = \frac{2}{R^2} \int_0^R \rho(r,z,t) r dr . \quad (B.4)$$

In eq. (B.1) the variables  $z$  and  $t$  are already separated. We assume that the dependence of  $u$  on  $r$  can be separated also, such that

$$u(r,z,t) = f(r) \cdot \bar{u}(z,t) . \quad (\text{B.5})$$

It was shown in Section 4 that a reasonable definition of the average axial velocity  $\bar{u}$  in terms of the local velocity  $u$  is

$$\bar{u} = \frac{1}{\rho} \cdot \frac{2}{R^2} \int_0^R u \rho r \, dr . \quad (\text{B.6})$$

With this definition of  $\bar{u}$  we have the following relation between the nondimensional functions  $f(r)$  and  $g(r)$ :

$$\frac{2}{R^2} \int_0^R f(r) g(r) r \, dr = 1 . \quad (\text{B.7})$$

The functions  $\bar{u}(z,t)$  and  $\bar{\rho}(z,t)$  satisfy the continuity equation (4.6), i.e.,

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial z} (\bar{\rho} \bar{u}) = 0 . \quad (\text{B.8})$$

Substituting the product  $P \cdot K$  for  $\bar{\rho}$  into eq. (B.8) and the expression (B.1) for  $\bar{u}$  we obtain

$$P(z) \cdot K'(t) + K(t) \frac{u_p(t)}{z_p(t)} \frac{d}{dz} (z P(z)) = 0 . \quad (\text{B.9})$$

This equation has solutions of the form

$$\bar{\rho} = \rho_0 \left( \frac{z}{z_p(t)} \right)^m (m+1) \frac{z_p(0)}{z_p(t)} \quad (\text{B.10})$$

with arbitrary  $m$ . In eq. (B.10)  $\rho_0$  is the average density of the gas in the tube at time  $t = 0$ . For  $m = 0$  we obtain the classical Lagrange's

solution. More generally we may assume  $\bar{\rho}$  to be, e.g., of the form

$$\bar{\rho} = \frac{\rho_0}{A_0 + \frac{1}{m+1} A_m} \left[ A_0 + A_m \left( \frac{z}{z_p(t)} \right)^m \right] \frac{z_p(0)}{z_p(t)} \quad (\text{B.11})$$

with arbitrary  $m$ ,  $A_0$  and  $A_m$ . For physical reasons  $m \geq 0$ ,  $A_0 \geq 0$  and  $A_m \geq -A_0$ .

Next we investigate the radial velocity component  $v(r, z, t)$ . The local continuity equation is

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v) + \frac{1}{r} \frac{\partial}{\partial \phi} (\rho w) + \frac{\partial}{\partial z} (\rho u) = 0, \quad (\text{B.12})$$

where  $w$  is the angular velocity component. Let  $w = 0$  (no swirl) and

$$v = \bar{v}(z, t) \cdot h(r). \quad (\text{B.13})$$

Eq. (B.12) can then be expressed by

$$\frac{\partial \bar{\rho}}{\partial t} \cdot g(r) + \bar{\rho} \bar{v} \frac{1}{r} \frac{d}{dr} [r g(r) h(r)] + g(r) f(r) \frac{\partial}{\partial z} [\bar{\rho} \bar{u}] = 0. \quad (\text{B.14})$$

Eliminating  $\partial \bar{\rho} / \partial t$  from eq. (B.14) with the aid of eq. (B.8), we obtain

$$[-1+f(r)] g(r) \frac{\partial(\bar{\rho} \bar{u})}{\partial z} + \bar{\rho} \bar{v} \frac{1}{r} \frac{d}{dr} [r g(r) h(r)] = 0. \quad (\text{B.15})$$

This equation is satisfied by the functions

$$\bar{v}(z, t) = R \frac{1}{\bar{\rho}} \frac{\partial(\bar{\rho} \bar{u})}{\partial z} \quad (\text{B.16})$$

and

$$h(r) = \frac{1}{R r g} \int_0^r (1-f) g r dr. \quad (\text{B.17})$$

Eqs. (B.13), (B.16), and (B.17) give the local radial velocity  $v(r,z,t)$  for any flow profile specified by  $\bar{\rho}$ ,  $\bar{u}$ ,  $g$ , and  $f$ . Clearly, the factor  $h(r)$  is not normalized in the same manner as  $f(r)$ . Therefore,  $\bar{v}(z,t)$  is not an "average" radial velocity. If  $\bar{\rho}$  is given by eq. (B.11), then

$$\bar{v}(z,t) = \frac{A_0 + (m+1) \left( \frac{z}{z_p(t)} \right)^m A_m u_p(t)}{A_0 + \left( \frac{z}{z_p(t)} \right)^m A_m} \frac{z_p(t)}{z_p(t)} \quad (B.18)$$

It is interesting to note that  $\bar{v}$  is not zero for  $z = 0$  and  $z = z_p$ . This is an indication that the assumption (B.5) about separation of variables for the axial velocity is not valid in the vicinities of the breech and the projectile. These regions we will therefore exclude from our considerations.

In summary we have found a flow field in a cylindrical tube which satisfies the local continuity equation and Lagrange's assumption (B.1). The flow field is described by the following functions

$$\left. \begin{aligned} u &= \frac{z}{z_p(t)} u_p(t) \cdot f(r) \\ \rho &= \bar{\rho}(z,t) \cdot g(r) \\ v &= \bar{v}(z,t) \cdot h(r) \end{aligned} \right\} (B.19)$$

If one specifies  $u$ , then  $\bar{\rho}$  is given by eq. (B.11) and  $\bar{v}$  is given by eq. (B.18). The dependence of the flow field on  $r$  can be specified by two functions,  $g(r)$  and  $f(r)$ , from which  $h(r)$  is then computed by eq. (B.17). The function  $g(r)$  has to be positive for  $0 < r < R$  and normalized by eq. (B.3). We assume also that  $g'(0) = 0$ . The function  $f(r)$  has to satisfy the conditions

$$\text{and} \quad \left. \begin{aligned} f'(0) &= 0 \\ f(R) &= 0 \end{aligned} \right\} (B.20)$$

It is normalized by eq. (B.7). Hence we have a total of three conditions which restrict the choice of  $f(r)$ .

Instead of specifying  $f(r)$  we may also specify  $h(r)$ . The function  $f(r)$  is then given by

$$f(r) = 1 - \frac{R}{r g(r)} [r g(r) h(r)]' . \quad (B.21)$$

The function  $h(r)$  has to satisfy the following four conditions

$$\left. \begin{aligned} h(0) &= 0 , \\ h''(0) &= 0 , \\ h(R) &= 0 , \\ h'(R) &= 1/R . \end{aligned} \right\} \quad (B.22)$$

The flow field also has to satisfy the momentum equations. The analysis of these equations is more complicated because it involves, in addition to the velocity and density functions, the pressure function  $p(r,z,t)$  and the viscosities  $\mu$  and  $\mu'$ , which in general are variable. We have tried to restrict our considerations to the special case with constant viscosities and classical Lagrange's approximation (i.e.,  $g(r) \equiv 1$ ). We have found that the flow field, as defined by eq. (B.19), does not satisfy the momentum equations in this special case. The formulas for correction terms, which we shall derive at the end of this appendix, are therefore to be considered as approximations only.

If  $g(r) \equiv 1$ , then the flow field is given by

$$u = \bar{u}(z,t) f(r) = z \frac{u_p(t)}{z_p(t)} f(r) , \quad (B.23)$$

$$v = R \frac{\partial \bar{u}}{\partial z} h(r) = R \frac{u_p(t)}{z_p(t)} h(r) = \frac{u_p}{z_p} \frac{1}{r} \int_0^r (1-f) r dr , \quad (B.24)$$

$$\rho = \bar{\rho}(t) = \rho_0 \frac{z_p(0)}{z_p(t)} . \quad (B.25)$$

Note that according to eq. (B.24) the radial velocity component  $v$  is independent of  $z$  for the classical Lagrange's approximation.

The divergence of the velocity  $u^*$  of this flow is (see eq. (A.15))

$$\operatorname{div} u^* = \frac{1}{r} \frac{\partial(rv)}{\partial r} + \frac{\partial u}{\partial z} = \frac{\partial \bar{u}}{\partial z} = \frac{u_p(t)}{z_p(t)}. \quad (\text{B.26})$$

The components of the forces caused by the viscous stress tensor are given by eq. (A.18). In our case, assuming constant viscosities, we obtain for the  $r$ -component

$$\begin{aligned} T_r &= 2\mu \bar{v} \frac{1}{r} [(r h')' - \frac{1}{r} h] + \mu \frac{\partial \bar{u}}{\partial z} f' = \\ &= [-2\mu f' + \mu f'] \frac{\partial \bar{u}}{\partial z} = -\mu \frac{u_p}{z_p} f'. \end{aligned} \quad (\text{B.27})$$

The  $z$ -component of the force is

$$T_z = \frac{1}{r} \mu \bar{u} [r f']' = \mu z \frac{u_p}{z_p} \frac{1}{r} [r f']'. \quad (\text{B.28})$$

The local momentum equations are according to eqs. (A.13) and (B.12)

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial r} + \rho u \frac{\partial v}{\partial z} + \frac{\partial p}{\partial r} = T_r, \quad (\text{B.29})$$

$$\rho \frac{\partial u}{\partial t} + \rho v \frac{\partial u}{\partial r} + \rho u \frac{\partial u}{\partial z} + \frac{\partial p}{\partial z} = T_z. \quad (\text{B.30})$$

For a flow field described by eqs. (B.23) through (B.24) eq. (B.29) is

$$\rho R \frac{d}{dt} \left( \frac{u_p}{z_p} \right) h + \rho R^2 \left( \frac{u_p}{z_p} \right)^2 h h' + \frac{\partial p}{\partial r} = -\mu \frac{u_p}{z_p} f'. \quad (\text{B.31})$$

Differentiating eq. (B.31) with respect to  $z$ , we obtain

$$\frac{\partial^2 p}{\partial r \partial z} = 0 . \quad (\text{B.32})$$

The function  $p(r, z, t)$  is therefore of the form

$$p(r, z, t) = p_1(r, t) + p_2(z, t) . \quad (\text{B.33})$$

Eq. (B.31) might be used to determine the function  $p_1(r, t)$  if the other terms in the equation are given.

Eq. (B.30) is in the present case, i.e., for the flow described by eqs. (B.23) through (B.25)

$$\rho z f \frac{d}{dt} \left( \frac{u_p}{z_p} \right) + \rho R \left( \frac{u_p}{z_p} \right)^2 z h f' + \rho z \left( \frac{u_p}{z_p} \right)^2 f^2 + \frac{\partial p_2}{\partial z} = \mu z \frac{u_p}{z_p} \frac{1}{r} (r f')' \quad (\text{B.34})$$

or

$$\rho f \frac{z_p}{u_p} \frac{d}{dt} \left( \frac{u_p}{z_p} \right) (R f' h + f^2) + \frac{z_p}{u_p} \frac{1}{z} \frac{\partial p_2}{\partial z} - \mu \frac{1}{2} (r f')' = 0 . \quad (\text{B.35})$$

From eq. (B.35) we can conclude that the expression  $(\partial p_2 / \partial z) / z$  is independent of  $z$ . The various terms in this equation are products of functions of  $r$  and  $t$  and the equation has the form

$$g_1(t) f_1(r) + g_2(t) f_2(r) + g_3(t) + f_3(r) = 0 . \quad (\text{B.36})$$

Such an equation can be satisfied identically only if either all  $g_i(t)$  are constant or all  $f_i(r)$  are constant. The case with all  $f_i(r) = \text{const.}$  corresponds to a slug flow in which we are not interested. Assuming the time functions  $g_i(t)$  to be constant, we obtain first

$$g_2(t) = \rho \frac{u_p}{z_p} = A$$

or

$$\rho_0 z_p(0) \frac{dz_p}{dt} \frac{1}{z_p} = A . \quad (\text{B.37})$$

Eq. (B.37) can be integrated to yield

$$z_p(t) = z_p(0) \frac{1}{1 - At/\rho_0} . \quad (\text{B.38})$$

The corresponding velocity of the projectile is

$$u_p(t) = z_p(0) \frac{A}{\rho_0} \frac{1}{(1 - At/\rho_0)^2} \quad (\text{B.39})$$

The density as a function of time is

$$\rho(t) = A \frac{z_p}{u_p} = \rho_0 \left(1 - A \frac{t}{\rho_0}\right) . \quad (\text{B.40})$$

The first time function in eq. (B.35) is then

$$g_1(t) = \rho \frac{z_p}{u_p} \frac{d}{dt} \left(\frac{u_p}{z_p}\right) = \rho_0 \left(1 - A \frac{t}{\rho_0}\right)^2 \frac{d}{dt} \left(\frac{1}{1 - At/\rho_0}\right) \equiv A \quad (\text{B.41})$$

Let the value of the third time function be B. We obtain then

$$\frac{\partial p_2}{\partial z} = B \cdot z \frac{u_p}{z_p} = B z \frac{A}{\rho_0} \frac{1}{1 - At/\rho_0} = A B \rho_0 \frac{z}{z_p(0)} z_p(t) \quad (\text{B.42})$$

Eq. (B.35) takes now the form

$$A(f + f'^2 + R h f') + B - \mu \frac{1}{r} (r f')' = 0 . \quad (\text{B.43})$$

The functions  $f(r)$  and  $h(r)$  are related by eq. (B.21). We can therefore express (B.43) in terms of  $h(r)$  only. We also multiply the equation by  $R^2/\mu$ . The result is

$$A^* \left[ 1 - \frac{R}{r} (r h)' + \left( 1 - \frac{R}{2} (r h)' \right)^2 - R h \left( \frac{R}{r} (r h)' \right)' \right] + \frac{R^2}{r} \left\{ r \left[ \frac{R}{2} (r h)' \right]' \right\}' + B^* = 0, \quad (\text{B.44})$$

with the constants

$$A^* = \frac{R^2}{\mu} A = \frac{R u_p \rho}{\mu} \cdot \frac{R}{z_p} \quad (\text{B.45})$$

and

$$B^* = \frac{R^2}{\mu} B = \left( \frac{\partial p}{\partial z} \cdot \frac{R^2}{\mu u_p} \right) \frac{z_p}{z}. \quad (\text{B.46})$$

The first factor in eq. (B.45) is a Reynolds number of the projectile. It is typically of the order  $10^6$ . The first factor in eq. (B.46) for  $B^*$  is a Poiseuille number of the flow. Its magnitude is of the order  $10^4$ . The function  $h(r)$  has to satisfy the differential equation (B.44) and the four boundary conditions (B.22). Since eq. (B.44) is of third order only, the function  $h(r)$  will in general not satisfy all boundary conditions. We conclude, therefore, that Lagrange's approximation is not consistent with a flow field which can be described by separation of variables, eqs. (B.23) through (B.25).

In Section 5.2 we have nevertheless used this flow field to obtain estimates of correction terms because we were not able to find an exact three-dimensional solution of Navier-Stokes equations which is also consistent with Lagrange's approximation.

Next we compute the various correction terms for the average flow equations using the formulas of Appendix A and the flow described by eq. (B.19).

The correction term  $C_m$  of the momentum equation is given by eq. (A.22):

$$C_m = \frac{\partial}{\partial z} \left\{ \bar{\rho} \bar{u}^2 - \frac{2}{R^2} \int_0^R \rho u^2 r \, dr \right\}. \quad (\text{B.47})$$

Substituting the expressions (B.23) and (B.25) for  $u$  and  $\rho$  respectively into eq. (B.47), we obtain

$$\begin{aligned} C_m &= \frac{\partial}{\partial z} \left\{ \bar{\rho} \bar{u}^2 - \bar{\rho} \bar{u}^2 \frac{2}{R^2} \int_0^R f^2(r) r \, dr \right\} \\ &= \left\{ 1 - \frac{2}{R^2} \int_0^R f^2 r \, dr \right\} \frac{\partial}{\partial z} (\bar{\rho} \bar{u}^2). \end{aligned} \quad (\text{B.48})$$

Eq. (B.48) is of course valid for any functions  $u = \bar{u}(z, t) \cdot f(r)$  and  $\rho = \bar{\rho}(z, t)$ . In Lagrange's case  $\bar{\rho}$  is independent of  $z$ , and  $\bar{u}$  is linear in  $z$ . We obtain in this case

$$C_m = \left\{ 1 - \frac{2}{R^2} \int_0^R f^2 r \, dr \right\} 2z \bar{\rho}(t) \frac{u_p(t)}{z_p(t)}. \quad (\text{B.49})$$

The energy equation has several correction terms. First we consider the term  $C_{e1}$ , given by eq. (A.29):

$$C_{e1} = \bar{p} \frac{\partial \bar{u}}{\partial z} - \frac{2}{R^2} \int_0^R p \left[ r \frac{\partial u}{\partial z} + \frac{\partial}{\partial r} (r v) \right] dr. \quad (\text{B.50})$$

We substitute the flow field formulas (B.19) into this equation and obtain

$$\begin{aligned} C_{e1} &= \bar{p} \frac{\partial \bar{u}}{\partial z} - \frac{2}{R^2} \int_0^R \left[ r f \frac{\partial \bar{u}}{\partial z} + \bar{v} (r h)' \right] p \, dr = \\ &= \bar{p} \frac{\partial \bar{u}}{\partial z} - \frac{2}{R^2} \int_0^R \left[ \frac{\partial \bar{u}}{\partial z} \left( r - \frac{R}{g} (r h)' \right) + \frac{R}{\bar{\rho}} \frac{\partial (\bar{\rho} \bar{u})}{\partial z} (r h)' \right] p \, dr. \end{aligned} \quad (\text{B.51})$$

This expression can be transformed by partial integration and some algebra into

$$C_{e1} = \frac{2}{R} \frac{\partial \bar{u}}{\partial z} \int_0^R \left( \frac{1}{g} - 1 \right) (r h)' p \, dr - \frac{2}{R} \frac{\bar{u}}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial z} \int_0^R (r h)' p \, dr. \quad (B.52)$$

Eq. (B.52) shows that the correction term  $C_{e1}$  is zero in the following cases:

- (a)  $g \equiv 1$  and  $\frac{\partial \bar{\rho}}{\partial z} = 0$ , i.e., the classical Lagrange's case;
- (b)  $g \equiv 1$  and  $\frac{\partial p}{\partial r} = 0$ , i.e.,  $\rho$  and  $p$  independent of  $r$ ;
- (c)  $h \equiv 0$ , or  $f \equiv 1$ , i.e., slug flow.

The second correction term of the energy equation,  $C_{e2}$ , depends on the local internal energy. According to eq. (A.30),

$$C_{e2} = \frac{\partial}{\partial z} \left\{ \bar{\rho} \bar{u} \bar{e} - \frac{2}{R^2} \int_0^R \rho u e r \, dr \right\}. \quad (B.53)$$

If the flow field is given by eq. (B.19), then

$$\begin{aligned} C_{e2} &= \frac{\partial}{\partial z} \left\{ \bar{\rho} \bar{u} \frac{2}{R^2 \rho} \int_0^R \rho e r \, dr - \frac{2}{R^2} \int_0^R \rho u e r \, dr \right\} = \\ &= \frac{\partial}{\partial z} \left\{ \bar{\rho} \bar{u} \frac{2}{R^2} \int_0^R g (1-f) e r \, dr \right\}. \end{aligned} \quad (B.54)$$

This correction term vanishes if the internal energy  $e$  is independent of the radial coordinate  $r$ .

The remaining correction terms,  $\bar{W}_0$  and  $C_\phi$ , in the energy equation are caused by the replacement of the heat dissipation function  $\bar{\Phi}$  by the product  $-\bar{u} \bar{T}_z$ . The heat dissipation function  $\bar{\Phi}$  is according to eq. (A.24)

$$\begin{aligned} \bar{\Phi} &= 2\mu \left[ \nabla^2 h'^2 + \frac{1}{2} \left( \frac{\partial \nabla}{\partial z} h + \bar{u} f' \right)^2 + \frac{1}{r^2} \nabla^2 h^2 + \left( \frac{\partial \bar{u}}{\partial z} \right)^2 f^2 \right] + \\ &+ \left( \mu' - \frac{2}{3} \mu \right) \left[ \frac{1}{r} \nabla h' + \frac{\partial \bar{u}}{\partial z} f \right]^2. \end{aligned} \quad (B.55)$$

For the flow field described by eqs. (B.23) through (B.25) we obtain

$$\Phi = 2\mu \left[ R^2 h'^2 + \left( \frac{R}{r} \right)^2 h^2 + f^2 + \frac{1}{2} z^2 f'^2 \right] \left( \frac{u_p}{z_p} \right)^2 + \left( \mu' - \frac{2}{3} \mu \right) \left( \frac{u_p}{z_p} \right)^2. \quad (\text{B.56})$$

The average  $\bar{\Phi}$  is by definition

$$\bar{\Phi} = \frac{2}{R^2} \int_0^R \Phi r \, dr. \quad (\text{B.57})$$

The correction term  $\bar{W}_0$  is according to eq. (A.34)

$$\begin{aligned} \bar{W}_0 &= \frac{2}{R^2} \int_0^R \left[ \mu \frac{\partial \bar{u}}{\partial z} f' \bar{v} h + 2\mu \frac{\partial}{\partial z} \left( \bar{u} \frac{\partial \bar{u}}{\partial z} \right) f^2 + \left( \mu' - \frac{2}{3} \mu \right) \frac{\partial \bar{u}}{\partial z} \frac{u_p}{z_p} f \right] r \, dr = \\ &= \frac{2}{R^2} \left( \frac{u_p}{z_p} \right)^2 \int_0^R \left[ \mu (f' h + 2f^2) + \left( \mu' - \frac{2}{3} \mu \right) f \right] r \, dr. \end{aligned} \quad (\text{B.58})$$

The correction term  $C_\phi$  is given by eq. (A.35). In the present case with constant viscosities we obtain by substituting eqs. (B.27) and (B.28) into eq. (A.35)

$$\begin{aligned} C_\phi &= \bar{u} \bar{T}_z - \frac{2}{R^2} \int_0^R \left[ -\mu \bar{v} h \frac{u_p}{z_p} f' + \mu \bar{u} f z \frac{u_p}{z_p} \frac{1}{r} (r f')' \right] r \, dr = \\ &= \bar{u} \bar{T}_z + \frac{2}{R^2} \int_0^R \left[ \mu R h f' + \mu z^2 f'^2 \right] \left( \frac{u_p}{z_p} \right)^2 r \, dr. \end{aligned} \quad (\text{B.59})$$

By partial integration and using eq. (B.21) we can show that

$$\int_0^R R h f' r \, dr = \int_0^R \left( R^2 h'^2 + \left( \frac{R}{r} \right)^2 h^2 \right) r \, dr. \quad (\text{B.60})$$

Making use of eq. (B.60) we can express the correction terms as follows:

$$\bar{W}_0 = \left(\frac{u_p}{z_p}\right)^2 \frac{2}{R^2} \int_0^R \left[ \mu R^2 h'^2 + \mu \left(\frac{R}{r}\right)^2 h^2 + 2\mu f^2 \right] r \, dr + \left(\frac{u_p}{z_p}\right)^2 \left(\mu' - \frac{2}{3}\mu\right) \quad (\text{B.61})$$

and

$$C_\phi = \bar{u} \bar{T}_z + \left(\frac{u_p}{z_p}\right)^2 \frac{2}{R^2} \int_0^R \left[ \mu R^2 h'^2 + \mu \left(\frac{R}{r}\right)^2 h^2 + \mu z^2 f'^2 \right] r \, dr . \quad (\text{B.62})$$

The first term in eq. (B.62) is according to eq. (B.28)

$$\bar{u} \bar{T}_z = \bar{u}^2 \mu \frac{2}{R^2} \int_0^R (r f')' \, dr = 2\mu z^2 \left(\frac{u_p}{z_p}\right)^2 \frac{1}{R} f'(R) . \quad (\text{B.63})$$

The correction terms  $\bar{W}_0$  and  $C_\phi$  vanish for a slug flow. However, for such a flow  $\bar{u} \bar{T}_z$  and  $\bar{\Phi}$  are also zero.

### LIST OF SYMBOLS

$C_{e1}$	Correction term in energy equation ( $J \cdot s^{-1} m^{-3}$ ). Definition by eq. (4.23)
$C_{e2}$	Correction term in energy equation ( $J s^{-1} m^{-3}$ ) Definition by eq. (4.24)
$C_m$	Correction term in momentum equation ( $N/m^3$ ). Definition by eq. (4.14)
$C_\phi$	Correction term in energy equation ( $J s^{-1} m^{-3}$ ) Definition by eq. (4.28)
$e$	Specific internal energy (J/kg)
$F$	Force per volume ( $N/m^3$ )
$\bar{H}$	Heat source and heat flux terms in the energy equation ( $J \cdot s^{-1} m^{-3}$ )
$k$	Specific kinetic energy (J/kg)
$n$	Unit normal vector
$p$	Pressure (Pa)
$q$	Heat flux per volume ( $J \cdot s^{-1} m^{-2}$ )
$Q$	Heat source per volume ( $J \cdot s^{-1} m^{-3}$ )
$r$	Radial coordinate (m)
$R$	Radius of tube (m)
$t$	Time (s)
$l$	Viscous force per volume ( $N/m^3$ )
$u$	Velocity (m/s). (Axial velocity of a tube flow)
$v$	Velocity (m/s). (Radial velocity of a tube flow)
$\bar{W}_0$	Correction term in energy equation ( $J s^{-1} m^{-3}$ ) Definition by eq. (4.27)
$x$	Cartesian coordinate (m)
$X$	Specific body force (N/kg)

LIST OF SYMBOLS (Cont'd)

$\epsilon$	Strain rate tensor ( $s^{-1}$ ). Definition by eq. (2.9) and (A.6)
$\mu$	Ordinary dynamic viscosity ( $Pa \cdot s$ )
$\mu'$	Dilatational dynamic viscosity ( $Pa \cdot s$ )
$\rho$	Density ( $kg/m^3$ )
$\tau$	Viscous stress tensor ( $Pa$ )
$\Phi$	Heat dissipation function ( $J \cdot s^{-1} m^{-3}$ ) Definition by eq. (2.10)

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