AFFDL-TR-77-89

ESTIMATING TIMES TO EARLY FAILURES USING SAMPLE DATA TO ESTIMATE THE WEIBULL SCALE PARAMETER

Structural Integrity Branch
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Robert L. Neulieb

ROBERT L. NEULIEB
Project Engineer

R.M. Bader

ROBERT M. BADER, Chief
Structural Integrity Branch

FOR THE COMMANDER

Howard L. Farmer

HOWARD L. FARMER, Colonel, USAF
Chief, Structural Mechanics Division

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Parameters for a Weibull distribution of times to failure must be estimated from a finite set of test points. Much effort has been expended identifying the shape parameter for different materials. This report investigates determining probabilities of early failures when estimating the scale parameter from a finite number of tests.

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FOREWORD

The research reported herein was conducted in the Structural Integrity Branch of the Structural Mechanics Division, for the Air Force Flight Dynamics Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, under project 1367, "Structural Integrity for Military Aerospace Vehicles," and work unit 13670336, "Life Analysis and Design Methods for Aerospace Structures," by Robert L. Neulieb. The research was conducted from September 1976 to March 1977.

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SECTION I

INTRODUCTION

The two- or three-parameter-Weibull distribution is commonly used to describe times to fatigue failures. Extensive work has been undertaken to identify the shape parameter for several aircraft materials (References 1 and 2). Using these estimates for the shape parameter, it is necessary to obtain the scale parameter. Impellizzeri et al (Reference 3) and Freudenthal (Reference 4) have shown that it is possible to obtain in closed form a distribution for the maximum likelihood estimate of the scale parameter divided by the time to failure. Freudenthal (Reference 4) observed that in terms of estimating the time to first failure in a given size fleet, little change in reliability is obtained for the same scatter factor by conducting more than one test to estimate the scale parameter. This result is extremely important and can be extended to the task of estimating any number of failures as long as the quantity time divided by the scale parameter estimate, all raised to a power given by the shape parameter, is small. This is the most important region in terms of aircraft structural integrity, since only a few failures can be tolerated.

However, care must be taken in interpreting the above result. Although the probability of failure (down to a given ratio of the scale parameter estimate to time) is little changed by conducting additional tests, the value of the scale parameter estimate may vary as the precision of the estimate increases with the number of tests. Hence, the precision and value of the estimate of a time to first or subsequent failure (for a constant ratio) may also vary with the data base.

In this report, for the case when the shape parameter is known, the probability distribution functions and asymptotic functions for times to failure as a ratio of the maximum likelihood estimate of the scale parameter are derived. The influence on the probability of failing due to the size of the data base is investigated.

SECTION II

DISTRIBUTION FUNCTION FOR TIME TO FAILURE

In both Reference 3 and 4 derivations were obtained of distribution functions for scatter factors when the Weibull shape parameter is known. By following the same procedure, it is possible to derive the distribution function for time to failure of an item when given the maximum likelihood estimate of the scale parameter. The times to failure, given the exact value of the scale parameter, will be assumed to be two-parameter-Weibull distributed. That is, it is assumed that times to failure are exactly two-parameter-Weibull distributed; however, the scale parameter is not known because it is estimated from a finite set of sample data. In using the estimate of the scale parameter, the distribution of times to failure is not Weibull and must be derived.

The cumulative distribution function for time to failure will be derived by multiplying the density function for the scale parameter and the Weibull density function for time to failure. This product is the joint density function for time to failure and the scale parameter since the tests for the scale parameter are independent of inservice failures. The joint density function will be integrated over the entire possible range of the scale parameter to obtain the density function for time to failure alone. Then this density function will be integrated from the smallest possible time (zero) to the time of interest to form the cumulative distribution function.

Since the density function for a function of the scale parameter is known, the density function for the scale parameter will be obtained by transforming the random variable. The density function for a transformed random variable is related to the original density function by the expression given in Reference 5. This expression is

$$f_X(x) = f_W(w(x)) \left| \frac{dw}{dx} \right| \qquad (1)$$

The probability density function for the two-parameter-Weibull distribution is

$$f(y) = \frac{\alpha}{\beta}\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-(y/\beta)^{\alpha}} \tag{2}$$

The maximum likelihood point estimator, $\hat{\beta}$, of the scale parameter β from n tests for times, y, to failure is

$$\hat{\beta} = \left(\frac{1}{n}\sum_{i=1}^{n} y_i^{\alpha}\right)^{1/\alpha} \tag{3}$$

where α is the known shape parameter. It has been shown (Reference 6) that $2n\left(\hat{\beta}/\beta\right)^{\alpha}$ has a chi-square distribution with 2n degrees of freedom. Therefore the probability density function is given by

$$f\left(\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha}\right) = \frac{n^n}{\Gamma(n)}\left(\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha}\right)^{n-1} e^{-n(\hat{\beta}/\beta)^{\alpha}} \tag{4}$$

where $\Gamma(n)$ is a complete Gamma Function.

The probability density function for the random variable, β, can be obtained by a transformation using Equation 1. Hence,

letting $w = \left(\frac{\hat{\beta}}{\beta}\right)^{\alpha}$ and observing that $\left|\frac{dw}{d\beta}\right| = \alpha\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha-1}\frac{\hat{\beta}}{\beta^2}$,

we can obtain

$$f(\beta) = \frac{\alpha n^n}{\beta\Gamma(n)}\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha n} e^{-n(\hat{\beta}/\beta)^{\alpha}} \tag{5}$$

This is the probability density function for the scale parameter given $\hat{\beta}$ and n. Hence, if $\hat{\beta}$ is determined from n tests, the joint probability density function for time to failure and β is

$$f(y,\beta) = \frac{\alpha}{\beta}\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-(y/\beta)^{\alpha}} \frac{\alpha n^{n}}{\beta\Gamma(n)}\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha n} e^{-n(\hat{\beta}/\beta)^{\alpha}} \tag{6}$$

since the determination of $\hat{\beta}$ and the failure of a given item are independent. The density function for times to failure is given by

$$f(y) = \int_{0}^{\infty} f(y,\beta)\, d\beta \tag{7}$$

or

$$f(y) = \frac{\alpha^{2} y^{\alpha-1} n^{n}}{\Gamma(n)} \hat{\beta}^{\alpha n} \int_{0}^{\infty}\left(\frac{1}{\beta}\right)^{\alpha n+\alpha+1} e^{-\left(\frac{y^{\alpha}+n\hat{\beta}^{\alpha}}{\beta^{\alpha}}\right)} d\beta \tag{8}$$

Letting $u = \beta^{-\alpha}$ and hence $-\alpha\left(\frac{1}{\beta}\right)^{\alpha+1} d\beta = du$ we obtain

$$f(y) = \frac{\alpha y^{\alpha-1} n^{n} \hat{\beta}^{\alpha n}}{\Gamma(n)} \int_{0}^{\infty} u^{n}\, e^{-u(y^{\alpha}+n\hat{\beta}^{\alpha})}\, du \tag{9}$$

Letting $v = u\left(y^{\alpha}+n\hat{\beta}^{\alpha}\right)$ and hence $dv = \left(y^{\alpha}+n\hat{\beta}^{\alpha}\right) du$, we find

$$f(y) = \frac{\alpha y^{\alpha-1} n^{n} \hat{\beta}^{\alpha n}}{\Gamma(n)\left(y^{\alpha}+n\hat{\beta}^{\alpha}\right)^{n+1}} \int_{0}^{\infty} v^{n}\, e^{-v}\, dv \tag{10}$$

But

$$\int_0^\infty v^n e^{-v}\,dv = n\Gamma(n) \tag{11}$$

so that

$$f(y) = \frac{\alpha n^{n+1} y^{\alpha-1} \hat{\beta}^{\alpha n}}{(y^\alpha + n\hat{\beta}^\alpha)^{n+1}} \tag{12}$$

It is convenient to convert this density function to a cumulative distribution function.

$$F(y) = \int_0^y f(q)\,dq \tag{13}$$

$$F(y) = \alpha n^{n+1} \hat{\beta}^{\alpha n} \int_0^y \frac{q^{\alpha-1}}{(q^\alpha + n\hat{\beta}^\alpha)^{n+1}}\,dq \tag{14}$$

Letting $z = q^\alpha + n\hat{\beta}^\alpha$ and hence $dz = \alpha q^{\alpha-1}\,dq$, we obtain

$$F(y) = n^{n+1} \hat{\beta}^{\alpha n} \int_{n\hat{\beta}}^{y^\alpha + n\hat{\beta}^\alpha} \frac{dz}{z^{n+1}} \tag{15}$$

$$F(y) = 1 - \left(\frac{n}{(y/\hat{\beta})^\alpha + n}\right)^n \tag{16}$$

This function is the probability distribution function for times to failure when the population of times to failure is Weibull, the shape parameter, α , is known, but a finite data sample of size, n, is used to obtain the scale parameter estimate, $\hat{\beta}$. That is, this function is the probability that an item will fail between time zero and time, y, under the conditions given above. This distribution function can be used to determine the probabilities of the first or subsequent failures occurring before time y. Methods of determining the probability of the first and subsequent failures given the probability of an item failing in some time increment are given in Reference 7.

An expression for the distribution of times to first failure in "m" items can be developed directly. The probability of zero failures in "m" independent items experiencing identical loadings is given by the probability that one item does not fail when raised to the power given by the number of items, m. The probability of an item not failing before time, y, is one minus the probability that the item fails. Hence,

$$P_0(y) = \left(\frac{n}{(y/\hat{\beta})^{\alpha} + n} \right)^{nm} \qquad (17)$$

where $P_0(y)$ is the probability of zero failures in "m" items up to time, y, or equivalently, the probability that the first failure will occur after time, y. Equation 17 is comparable to but not identical to the reliability function developed by Freudenthal in Reference 4. The difference arises in Reference 4 from the use of a function of time, the scatter factor, instead of time itself in the derivation and application. Classically, the definition of reliability involves time directly, as in Equation 17. See Reference 8 for a more complete discussion. Differences between the two formulations are not expected to be significant, at least for aircraft applications.

SECTION III

DISCUSSION OF THE DISTRIBUTION FUNCTION

The cumulative distribution function F(y) in Equation 16 gives the probability that the fatigue life of a specific item is less than or equal to y when $\hat{\beta}$ is the maximum likelihood estimate of the scale parameter obtained from n tests and α is the known value of the shape parameter. The failures are assumed to be two-parameter-Weibull distributed.

The distribution differs from the Weibull distribution because the Weibull scale parameter is not known exactly. As the number of tests conducted to determine β increases, this distribution should approach the Weibull distribution.

By rewriting F(y) = 1 - G(y), the limit of G(y) as $n \to \infty$ can be obtained.

$$
\begin{aligned}
\lim_{n\to\infty} G(y) &= \lim_{n\to\infty} \left(\frac{n}{(y/\hat{\beta})^{\alpha} + n} \right)^{n} \\
&= \lim_{n\to\infty} \left(\frac{1}{1/n(y/\hat{\beta})^{\alpha} + 1} \right)^{n} \\
&= \lim_{n\to\infty} e^{-n\,\ell n\left(1/n(y/\hat{\beta})^{\alpha} + 1\right)} \\
&= \lim_{n\to\infty} e^{-\frac{\ell n(1/n(y/\hat{\beta})^{\alpha} + 1}{n^{-1}}}
\end{aligned}
\qquad (18)
$$

The argument is in an indeterminate form $\frac{0}{0}$ as $n \rightarrow \infty$, hence L' Hospital's rule may be applied. Let the argument equal A. Then

$$\lim_{n \rightarrow \infty} A = -\lim_{n \rightarrow \infty} \frac{\frac{d}{dn}\left(\ell n \; 1/n \left(y/\beta\right)^{\alpha} + 1\right)}{\frac{d}{dn}\left(n^{-1}\right)}$$

$$= -\lim_{n \rightarrow \infty} \frac{\dfrac{-n^{2}\left(y/\hat{\beta}\right)^{\alpha}}{1/n\left(y/\hat{\beta}\right)^{\alpha} + 1}}{-n^{-2}} \tag{19}$$

$$= \left(y/\beta\right)^{\alpha}$$

Hence $\lim_{n \rightarrow \infty} G(y) = e^{-\left(y/\hat{\beta}\right)^{\alpha}}$ and $F(y) = 1 - e^{\left(y/\hat{\beta}\right)^{\alpha}}$

Thus, as more information about the scale parameter is obtained, this distribution does approach the Weibull as expected, and $\hat{\beta}$ goes to β .

The region where $\left(y/\hat{\beta}\right)^{\alpha} \ll 1$ is important to analyze, because this implies that there is a small probability of any given item failing and hence, that only a few failures occur in a fleet. Since tests on either full scale or large scale items are very expensive, the information obtainable from one test should be analyzed. Thus, let F(y) for n = 1 be $F_1(y)$.

$$F_1(y) = \frac{1}{\left(y/\hat{\beta}\right)^{\alpha} + 1} \tag{20}$$

If

$$(y/\hat{\beta})^{\alpha} \ll 1$$

$$F_1(y) \approx (y/\hat{\beta})^{\alpha} \qquad (21)$$

Now condsider the maximum information which can be obtained from conducting tests to determine $\hat{\beta}$.

$$\lim_{n\to\infty} F(y) = 1 - e^{-(y/\hat{\beta})^{\alpha}} \qquad (22)$$

Expanding $\lim_{n\to\infty} F(y)$ in a power series in $(y/\hat{\beta})^{\alpha}$, we find that for small values of $(y/\hat{\beta})^{\alpha}$,

$$\lim_{n\to\infty} F(y) \approx (y/\hat{\beta})^{\alpha} \qquad (23)$$

For application to aircraft failures, $(y/\hat{\beta})^{\alpha}$ is almost always small. The asymptotic distribution seems to be the same, independent of the number of tests conducted to estimate the scale parameter. This is consistent with the observations made by Freudenthal (Reference 4). However, the quantity $\hat{\beta}$ need not be identical for different numbers of tests used to estimate the scale parameter. Hence, for a given probability, y need not be identical. In addition to Equation 5, the distribution of the statistic

$\dfrac{F_n(y)}{F(y)}$ where $F_n(y)$ is given by Equation 16 and $F(y)$ is Weibull can indicate the significance of possible variations in $\hat{\beta}$.

Let

$$X = \frac{F_n(y)}{F(y)}$$

$$x = \frac{F_n(y)}{1-e^{-(y/\hat{\beta})^{\alpha}}} \qquad (24)$$

implying that

$$\beta = y\left(-\ell n\left(1-\frac{F_n(y)}{x}\right)\right)^{-1/\alpha} \qquad (25)$$

as $F_n(y)$ is independent of both β and x. By using the transformation of a random variable, Equation 1, the probability density function for the ratio X can be obtained from Equation 5. Observing that

$$\frac{d\beta}{dx} = \frac{y}{\alpha}\left(-\ell n\left(1-\frac{F_n(y)}{x}\right)\right)^{-1/\alpha-1}\left(1-\frac{F_n(y)}{x}\right)^{-1}\frac{F_n(y)}{x^2} \qquad (26)$$

we find that

$$r(x) = \left(\frac{\hat{\beta}}{y}\right)^{\alpha n} n^n \frac{F_n(y)}{\Gamma(n)}\left(-\ell n\left(1-\frac{F_n(y)}{x}\right)\right)^{n-1}\frac{\left(1-\frac{F_n(y)}{x}\right)^{n(\hat{\beta}/y)^{\alpha}-1}}{x^2} \qquad (27)$$

This is the probability density function for the ratio X. The values of X may range from $F_n(y)$ to ∞ as the denominator ranges from 1 to 0. This probability density function can be integrated to a cumulative distribution function

$$R(x) = \left(\frac{\hat{\beta}}{y}\right)^{\alpha n} \frac{n^n F_n(y)}{\Gamma(n)} \int_{F_n(y)}^{x} \left(-\ell n\left(1-\frac{F_n(y)}{q}\right)\right)^{n-1} \frac{\left(1-\frac{F_n(y)}{q}\right)^{n(\hat{\beta}/y)^{\alpha}-1}}{q^2} \tag{28}$$

Let $\left(1-\frac{F_n(y)}{q}\right) = g$ and $\frac{F_n(y)}{q^2}\, dq = dg$

$$R(x) = \left(\frac{\hat{\beta}}{y}\right)^{\alpha n} \frac{n^n}{\Gamma(n)} \int_{0}^{1-\frac{F_n(y)}{x}} \left(-\ell n(g)\right)^{n-1} g^{n(\hat{\beta}/y)^{\alpha}-1} \tag{29}$$

This form can readily be integrated for n = 1 and yields

$$R_1(x) = \left(1-F_1(y)\right)^{(\hat{\beta}/y)^{\alpha}} \tag{30}$$

$$R_1(x) \approx e^{-1/x} \text{ when } \left(y/\hat{\beta}\right)^{\alpha} \ll 1 \tag{31}$$

Additional $R_n(x)$'s can be determined by integrating by parts, or by writing a general expression involving the ratio of incomplete and complete Gamma Functions (complement of the Gamma distribution).

Let $g = e^{-p}$ and $dg = -e^{-p}\,dp$. Then

$$R(x) = \left(\frac{\hat{\beta}}{y}\right)^{\alpha n} \frac{n^n}{\Gamma(n)} \int_{-\ell n\left(1-\frac{F_n(y)}{x}\right)}^{\infty} p^{n-1} e^{-np\left(\hat{\beta}/y\right)^{\alpha}} dp \tag{32}$$

Let $z = pn\left(\hat{\beta}/y\right)^{\alpha}$ and $dz = n\left(\hat{\beta}/y\right) dp$ Then

$$R(x) = \frac{1}{\Gamma(n)} \int_{-\left(\hat{\beta}/y\right)\ell n\left(1-\frac{F_n(y)}{x}\right)}^{\infty} z^{n-1} e^{-z} dz \tag{33}$$

$$R(x) = \frac{\Gamma\left(n, -\left(\hat{\beta}/y\right)^{\alpha} \ell n\left(1-\frac{F_n(y)}{x}\right)\right)}{\Gamma(n)} \tag{34}$$

where Γ (n, x) is an incomplete Gamma Function as given in Reference 9.

For n = 2

$$R_2(x) = \left(1-2\left(\hat{\beta}/y\right)^{\alpha} \ell n\left(1-\frac{F_2(y)}{x}\right)\right)\left(1-\frac{F_2(y)}{x}\right)^{2\left(\hat{\beta}/y\right)^{\alpha}} \tag{35}$$

See Figure 1 for plots of $R_1(x)$, $R_2(x)$ and $R_\infty(x)$. Table 1 gives the probability that X will be in specified ranges. There is significant variation in X about the value 1 for n = 1. However, the value of X is more frequently greater than 1. This implies that in these cases the probability of failure is overestimated. As n increases, this variability in X is reduced. Conducting additional tests to estimate the scale parameter does provide increased information. However, the value of this information must be weighed against the high cost of obtaining it.

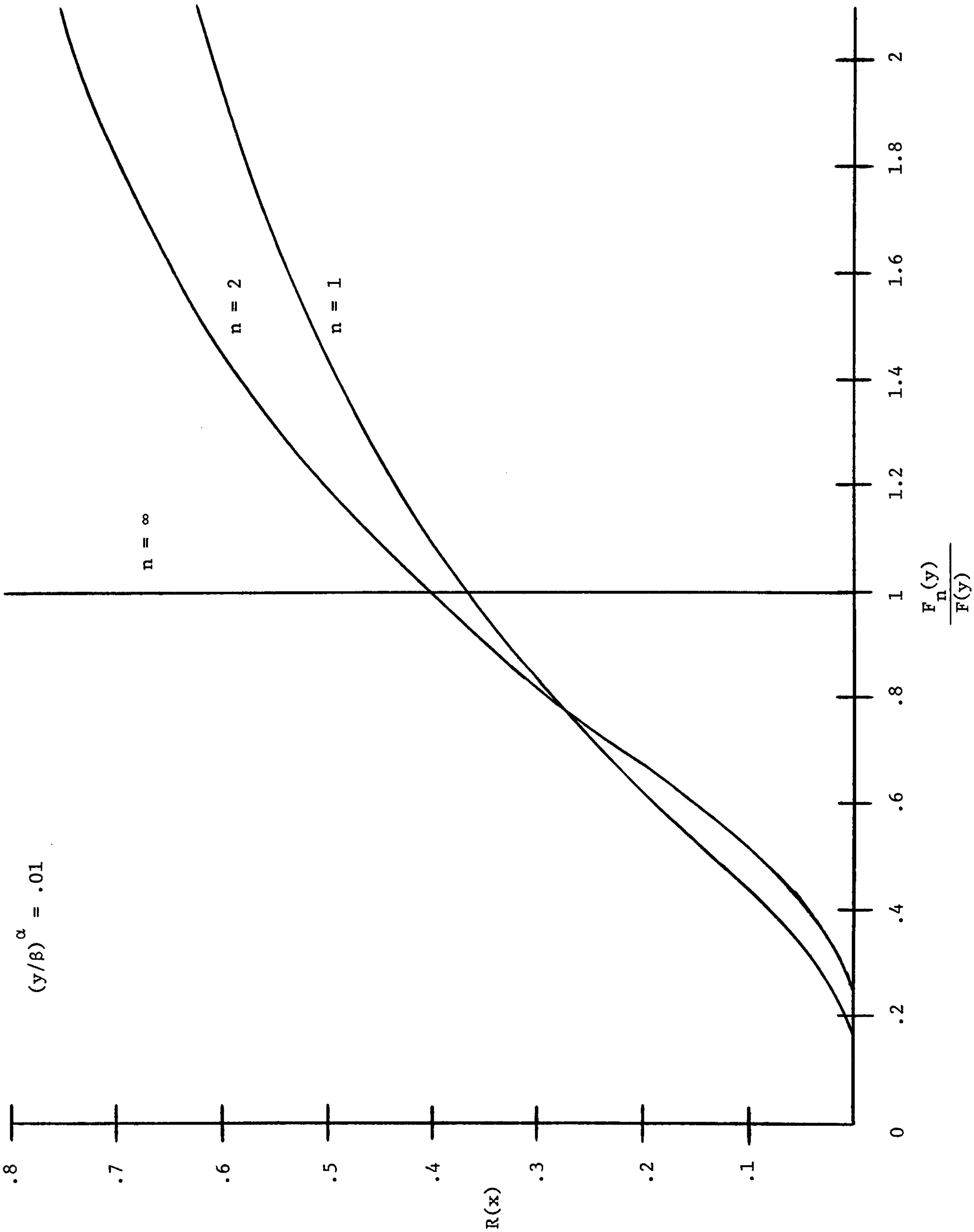

Figure 1. Probability Distribution Function for $\frac{F_n(y)}{F(y)}$

TABLE I

PROBABILITIES OF $F_n(y)/F(y)$ BEING IN SPECIFIED RANGES

$$\left(\frac{y}{\hat{\beta}}\right)^{\alpha} = .01$$

Range	n = 1	n = 2
R(1.25) - R(.8)	.164	.239
R(2) - R(.5)	.473	.647
R(3) - R(1/3)	.669	.840
R(∞) - R(1)	.630	.593

The results can be extended to the case of a three-parameter-Weibull distribution when both the minimum life parameter and the shape parameter are known. When the minimum life parameter is known, a parameter, z, the difference between times to failure and the minimum life parameter is two-parameter-Weibull distributed. However, in terms of times to failure this is the three-parameter-Weibull distribution. Thus if μ is the minimum life

$$z = y - \mu \tag{36}$$

$$F(z) = 1 - e^{-(z/\beta)^{\alpha}} \tag{37}$$

and

$$F(y) = 1 - e^{-\left(\frac{y-\mu}{\beta}\right)^{\alpha}} \tag{38}$$

Hence, when μ is known, a random variable which is three-parameter distributed can be transformed to a variable which is two-parameter-Weibull distributed. All properties of the two-parameter-Weibull distribution thus apply to this transformed random variable.

SECTION IV

CONCLUSIONS AND RECOMMENDATIONS

The probability of failure in the region of small probabilities has the same form whether one or an infinite number of tests is used to estimate the scale parameter when the shape parameter is known. This confirms Freudenthal's observation (Reference 4) that, for the first failure, scatter factors change little with the number of tests conducted.

More information can be obtained, however, by conducting additional tests because the estimate of the scale parameter can vary with the number of tests conducted. Equation 29 along with the scatter factor or probability of failure (Equation 16) should be analyzed to determine the number of tests required.

When estimating probabilities of first or early failures from one or more tests when the shape parameter is believed known, Equation 16 or Freudenthal's results (Reference 4) should be used.

REFERENCES

1. Whittaker, I. C., and Besuner P. M., "A Reliability Analysis Approach to Fatigue Life Variability of Aircraft Structures," Air Force Materials Laboratory, AFML-TR-69-65, April 1969.

2. Whittaker, I. C., "Development of Titanium and Steel Fatigue Variability Model for Application of Reliability Analysis Approach to Aircraft Structures," Air Force Materials Laboratory, AFML-TR-72-236, October 1972.

3. Impellizzeri, L. F., Siegel, A. E., and McGinnis, R. A., "Evaluation of Structural Reliability Analysis Procedures as Applied to a Fighter Aircraft," Air Force Materials Laboratory, AFML-TR-73-150, September 1973.

4. Freudenthal, A. M., "The Scatter Factor in the Reliability Assessment of Aircraft Structures," Journal of Aircraft, Vol. 14, No. 2, February 1977, pp. 202-208.

5. Parzen, E., Modern Probability Theory and Its Applications, John Wiley & Sons, Inc., New York, 1960.

6. Harter, H. L., and Moore, A. H., "Point and Internal Estimators, Based on m Order Statistics, for the Scale Parameter of a Weibull Population with Known Shape Parameter," Technometrics, Vol. 7, No. 3, August 1965, pp. 404-422.

7. Neulieb, R. L., "Including Fleet Usage Variability In Reliability Analyses for Both Safety and Economic Limit," Air Force Flight Dynamics Laboratory, AFFDL-TR-77-77, August 1977.

8. Shinozuka, M., private communication, results to be published by the Air Force Flight Dynamics Laboratory in a technical report, 1978.

9. Abramowitz, M., and Stegun, I. A. (Editors), Handbook of Mathematical Functions, Dover Publications, Inc., New York, 1965.

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