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TWO SIDED CONFIDENCE INTERVALS FOR AN EXPONENTIAL PARAMETER FRO--ETC(U)

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University of Missouri-Columbia

The Exact Confidence Intervals  
for an Exponential Parameter

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January 1978

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for estimating the exponential parameter from this combined scheme were first considered by Epstein. Epstein established one sided confidence intervals for this parameter. This report reviews Epstein's work and establishes two sided confidence intervals. The confidence intervals are expressed in terms of time on test and chi-square percentiles with the degrees of freedom depending on the number of observed failures. An expression for the expected length of the confidence intervals is also derived.

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**Two Sided Confidence Intervals for an Exponential  
Parameter from a Life Test with Type I and Type II Censoring**

**by**

**Kenneth B. Fairbanks  
University of Missouri - Columbia**

**Technical Report No. 73**

**January 1978**

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**Office of Naval Research**

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## I. Introduction

Reliability acceptance tests are conducted under a variety of test schemes both in the military and in industry. Various combinations of time and failure censoring are commonly utilized. In time censoring,  $n$  items are placed on test and observed for a pre-chosen time  $t^*$ . In failure censoring,  $n$  items are observed until the occurrence of the  $r$ th failure. These procedures are sometimes denoted type I and type II censoring respectively. These two schemes may be combined so that the test terminates at  $\min(t^*, \tau_r)$ , where  $\tau_r$  is the time of the  $r$ th failure in the sample. Sequential tests are also used where acceptance, rejection, and continuation regions are defined in the time-failure plane according to the theory of Wald [10]. The sequential test may be further modified by time censoring, failure censoring, or both. If a sequential test is censored at  $r_0$  failures and at time  $T_0$ , the resulting test is known as a truncated sequential test scheme. Further variations are possible by monitoring total accumulated time on test rather than actual elapsed time in any of the above schemes.

A reliability test is concerned with a parameter  $\theta$  in the underlying distribution of lifetimes. In addition to an accept or reject decision, it might be of interest to form an interval estimate of  $\theta$  at the time of decision.

The objective of this report is precisely that. If the reliability test terminates at time  $t$ , we wish to form a  $100(1-\alpha)$  percent confidence interval for  $\theta$  at that time.

In this report we restrict our work to an underlying distribution of lifetimes which is exponential with parameter  $\theta$ . This report was motivated by a communication from the Office of Naval Research concerning proposed revisions in MIL-STD-781. This standard gives detailed reliability acceptance procedures for military use, employing schemes which combine types I and II censoring as well as truncated sequential tests. The underlying distribution is assumed to be exponential with parameter  $\theta$ , which is the mean time between failures (MTBF). The communication indicated a desire to form confidence intervals for  $\theta$  directly from the testing procedures in MIL-STD-781.

## II. Literature Review and Preliminaries

Much of the existing theory on reliability testing as well as estimation procedures, in the exponential case was developed by Benjamin Epstein and Milton Sobel in a series of papers and technical reports. Epstein eventually unified these papers in a monograph [4] in 1960.

In what follows it is assumed that all test items are drawn at random and independently from a population where the lifetime,  $\tau$ , of an item is exponentially distributed with density

$$f(\tau; \theta) = \frac{1}{\theta} e^{-\tau/\theta}, \quad \tau \geq 0, \quad \theta > 0 \quad (1)$$

$$0, \quad \text{elsewhere.}$$

Hence  $\theta$  is the expected lifetime of any item on test. Suppose  $n$  items are drawn at random from distribution (1) and placed on

life test. Define  $\tau_i$  to be the  $i^{\text{th}}$  observed failure time. If  $r \leq n$  is preassigned for type II censoring, we have  $\tau_1 \leq \tau_2 \leq \dots \leq \tau_r$ . Failed items may not be replaced (sampling without replacement) or they may be immediately replaced by a new item from the same distribution (sampling with replacement). Under these schemes the following results are well established [4]:

i) In the nonreplacement case the maximum likelihood estimator of  $\theta$  is  $\hat{\theta} = T_r/r$ , where

$$T_r = \sum_{i=1}^r \tau_i + (n-r)\tau_r = \sum_{i=1}^r (n-i+1)(\tau_i - \tau_{i-1}) = \sum_{i=1}^r u_i.$$

Here  $\tau_0 \equiv 0$  and  $u_i \equiv (n-i+1)(\tau_i - \tau_{i-1})$ .  $T_r$  is in fact the total accumulated time on test.

ii) The  $u_i$ ,  $i=1, \dots, r$ , are independent and identically distributed with common probability density function (1).

iii)  $2u_i/\theta$  is distributed as a chi-square random variable with 2 degrees of freedom. Consequently  $2T_r/\theta$  is distributed as chi-square with  $2r$  degrees of freedom ( $\chi_{2r}^2$ ).

iv) In the replacement case the maximum likelihood estimator of  $\theta$  is  $\hat{\theta} = T_r/r$ , where now  $T_r = n\tau_r$ .

v) Also in the replacement case  $2r\hat{\theta}/\theta = 2T_r/\theta \sim \chi_{2r}^2$ .

From these results we can obtain one and two sided confidence intervals for  $\theta$ , with or without replacement. From (iii) and (v) we find that a  $100(1-\alpha)$  percent one sided confidence interval for  $\theta$  is

$$\left( \frac{2T_r}{\chi_{\alpha, 2r}^2}, \infty \right) \quad (2)$$

where  $\chi_{\alpha, 2r}^2$  is such that  $P(\chi_{2r}^2 > \chi_{\alpha, 2r}^2) = \alpha$ . The corresponding two-sided interval is

$$\left( \frac{2T_r}{\chi_{\frac{\alpha}{2}, 2r}^2}, \frac{2T_r}{\chi_{1-\frac{\alpha}{2}, 2r}^2} \right) \quad (3)$$

In type I censoring, where testing continues until a preassigned time  $t^*$ , confidence intervals are not as easily found. In this case, with  $n$  items on test with replacement, we have a Poisson process with parameter  $n(\frac{1}{\theta})$ . Hence, if  $k$  is the number of failures observed in  $(0, t^*)$ ,

$$P(k \leq r | \theta) = \sum_{k=0}^r \frac{e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^k}{k!}.$$

By a well known relationship between the Poisson and chi-square distributions we can write

$$P(k \leq r | \theta) = P(\chi_{2r+2}^2 > 2nt^*/\theta).$$

Clearly, if  $\theta \leq 2nt^*/\chi_{\alpha, 2r+2}^2$ , then  $P(k \leq r) \leq \alpha$ . So, if we do

in fact observe  $k=r$  we are at least  $100(1-\alpha)$  percent confident that  $2nt^*/\chi_{\alpha, 2r+2}^2 < \theta$ . Likewise, by considering  $P(k \geq r | \theta)$ ,

we may be  $100(1-\alpha)$  percent confident that, given we observed  $k=r$ ,

$2nt^*/\chi^2_{1-\alpha, 2r} > \theta$  . As a consequence, one sided and two sided 100(1- $\alpha$ ) percent confidence intervals for  $\theta$  , under this sampling scheme, are given by

$$\left( \frac{2nt^*}{\chi^2_{\alpha, 2r+2}}, \infty \right) \quad (4)$$

and

$$\left( \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2r+2}}, \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2r}} \right) . \quad (5)$$

This approach to confidence intervals for the Poisson parameter was originally discussed by Garwood [8]. Note that if  $r=0$  , only the one-sided interval is used since we have no information on which to base an upper limit.

If the type I scheme proceeds without replacement, corresponding results are not easily obtained. In this case the distribution of the maximum likelihood estimator was shown by Bartholomew [2] to be a weighted sum of chi-square integrals. A closed form expression for confidence intervals from this is unlikely. Epstein used a non-parametric approach [4] to this problem by considering the number of failures observed in  $(0, t^*)$ , but ignoring the actual times of failure. We may improve our position considerably here if we choose to monitor total time on test rather than actual elapsed time. If at some time  $t$  ,  $k$  failures have been observed, the total time on test is seen to be

$$\sum_{i=0}^k \tau_i + (n-k)t .$$

Instead of an actual time of truncation  $t^*$ , we preassign a total time truncation point  $t'$ . Again, Epstein [4] has considered this approach and many resultant properties. Under this scheme it can be shown that we are observing a Poisson process with parameter  $\lambda = \frac{1}{\theta}$  for a length of actual time  $t'$ . Therefore, following the same reasoning that lead to (4) and (5), one-sided and two-sided  $100(1-\alpha)$  percent confidence intervals for  $\theta$  are

$$\left( \frac{2t'}{\chi_{\alpha, 2r+2}^2}, \infty \right) \quad (6)$$

and

$$\left( \frac{2t'}{\chi_{\frac{\alpha}{2}, 2r+2}^2}, \frac{2t'}{\chi_{1-\frac{\alpha}{2}, 2r}^2} \right) . \quad (7)$$

With this scheme, the objective of terminating the test before an excessive amount of time has elapsed may still be achieved.

A more complex situation arises when types I and II are combined. In a combined scheme, the test may terminate either with  $r_0$  failures or at time  $t^*$ , whichever occurs first. If we are testing  $H_0: \theta = \theta_0$  against  $H_1: \theta = \theta_1$  where  $\theta_1 < \theta_0$ , we would accept if  $\min(\tau_{r_0}, t^*) = t^*$  and reject if  $\min(\tau_{r_0}, t^*) = \tau_{r_0}$ . It may be that  $r_0$  and  $t^*$  are prechosen to satisfy  $\alpha, \beta$  level requirements.

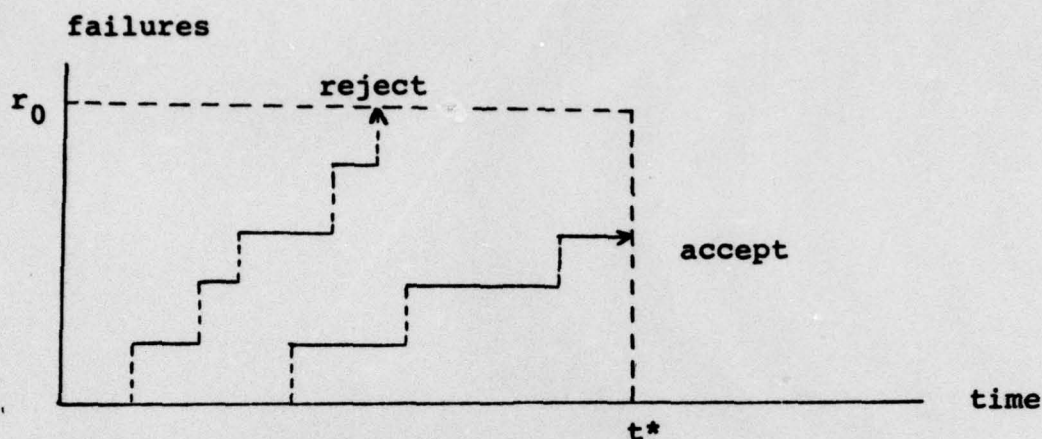


Figure 1. Possible sample paths in the combined test scheme.

Additional complexity in the stopping rule complicates the formulation of a confidence interval at the time the test terminates. The natural inclination here is to combine the confidence intervals from the types I and II. Epstein has done this and showed [4] that the following rule gives, at least, 100(1- $\alpha$ ) percent one-sided confidence intervals when sampling with replacement:

$$\left( \frac{2nt^*}{\chi^2_{\alpha, 2k+2}}, \infty \right) \text{ if } \tau_{r_0} > t^*, \quad k = 0, 1, \dots, r_0 - 1 \quad (8)$$

$$\left( \frac{2n\tau_{r_0}}{\chi^2_{\alpha, 2r_0}}, \infty \right) \text{ if } \tau_{r_0} \leq t^* .$$

If we sample without replacement we face the same problems as above when monitoring actual elapsed time. If, instead, we monitor total time on test and use total time truncation  $t'$ , the following one sided  $100(1-\alpha)$  percent confidence intervals for  $\theta$  were also given by Epstein [4]:

$$\left( \frac{2t'}{\chi_{\alpha, 2k+2}^2}, \infty \right) \quad \text{if } k = 0, 1, \dots, r_0 - 1$$

$$\left( \frac{2t_{r_0}}{\chi_{\alpha, 2r_0}^2}, \infty \right) \quad \text{if } k = r_0.$$
(9)

Here  $k$  is the number of failures observed when total accumulated test time is  $t'$ , and

$$t_{r_0} = \sum_{i=1}^{r_0} \tau_i + (n-r_0)\tau_{r_0} = \text{total time on test at } r_0^{\text{th}} \text{ failure.}$$

Two sided intervals are considered in the next section.

### III. Extension to Two Sided Intervals

#### IIIa. Testing with Replacement

It would seem that an extension from one sided to two sided intervals should now be simple and direct. In fact the proof is not a simple extension from the one sided case. Theorem 1 gives two sided intervals.

Theorem 1. Let  $\tau$  be the lifetime of an item having distribution  $f(\tau) = (1/\theta)e^{-\tau/\theta}$ ,  $\tau \geq 0$ . If  $n$  items from this distribution are placed on test with replacement, and if the test terminates at  $\min(t^*, \tau_{r_0})$ , then the following rule gives a two sided  $100(1-\alpha)$  percent confidence interval for  $\theta$  :

$$\begin{aligned}
 & \left( \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2}}, \infty \right) && \text{if } k = 0, \\
 & \left( \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2k+2}}, \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2k}} \right) && \text{if } k = 1, 2, \dots, r_0 - 1, \\
 & \left( \frac{2n\tau_{r_0}}{\chi^2_{\frac{\alpha}{2}, 2r_0}}, \frac{2n\tau_{r_0}}{\chi^2_{1-\frac{\alpha}{2}, 2r_0}} \right) && \text{if } \tau_{r_0} \leq t^*,
 \end{aligned} \tag{10}$$

Here  $k$  is the number of failures in  $(0, t^*)$ . Epstein proposed intervals nearly identical to these [4] with the exception that

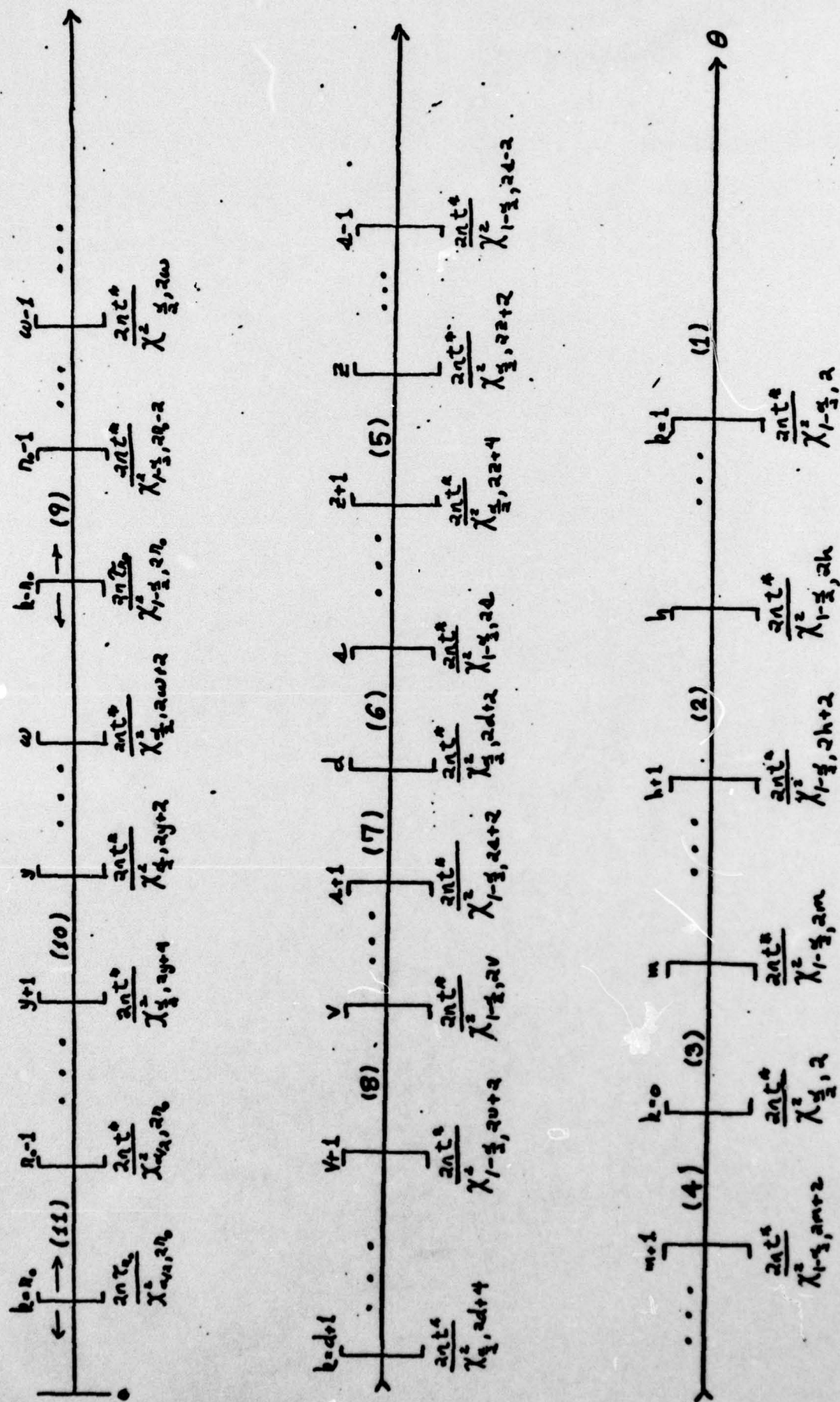
he used  $\chi_{\alpha,2}^2$  in the denominator when  $k=0$ . He was unable to provide a proof of his conjecture. To prove the intervals (1) are valid we need to show that, wherever  $\theta$  be in  $(0,\infty)$ , the probability is at least  $1-\alpha$  that  $\theta$  is covered by one of the intervals (10). The proof is somewhat tedious and requires attention to figure 2. The endpoints of the intervals for  $k=0,1,\dots,r_0-1$  form a partition of the parameter space. The proof consists of establishing the required probability of  $1-\alpha$  for each subset of the partition. Note first that the lower endpoints of the intervals when  $k=0,1,2,\dots,r_0-1$  are ordered among themselves, the smallest when  $k=r_0-1$ , the largest when  $k=0$ . This is a result of the fact that  $\chi_{\alpha,k+1}^2 > \chi_{\alpha,k}^2$  for any fixed  $\alpha$ , and any integer  $k$ . Likewise the upper endpoints when  $k=1,2,\dots,r_0-1$  are also ordered, the smallest when  $k=r_0-1$ , the largest when  $k=1$ . The endpoints of the interval when  $k=r_0$  are random since they are functions of  $\tau_{r_0}$ , the time of the  $r_0^{\text{th}}$  failure. However,  $\tau_{r_0} < t^*$  implies this random lower endpoint is bounded above by the fixed lower endpoint of the interval corresponding to  $k=r_0-1$ , specifically

$$2n\tau_{r_0}/\chi_{\alpha/2,2r_0}^2 \leq 2nt^*/\chi_{\alpha/2,2r_0}^2. \text{ Also this random upper endpoint}$$

is bounded above by the fixed upper endpoint when  $k=r_0$ , i.e.,

$$2n\tau_{r_0}/\chi_{1-\alpha/2,2r_0}^2 \leq 2nt^*/\chi_{1-\alpha/2,2r_0}^2. \text{ Note that if the smallest}$$

upper endpoint ( $k=r_0-1$ ) was always greater than the largest



**Figure 2. General alignment of interval endpoints in the replacement case**

lower endpoint ( $k=0$ ), there would then be a convenient overall ordering. Unfortunately this is not necessarily the case. The partitioning of the parameter space may be complicated by what we shall refer to as a "cross-over". A cross-over occurs when an upper limit for one fixed interval crosses below the lower limit of another fixed interval. The first such cross-over will occur when  $2nt^*/\chi_{1-\frac{\alpha}{2}, 2r_0-2}^2$  becomes less than  $2nt^*/\chi_{\frac{\alpha}{2}, 2}^2$ ,

that is when  $\chi_{1-\frac{\alpha}{2}, 2r_0-2}^2 > \chi_{\frac{\alpha}{2}, 2}^2$ . This can occur if  $\alpha$  is large enough or, more likely, if  $r_0$  is large enough for a fixed  $\alpha$  level. Clearly a vast number of possible configurations exist where upper endpoints are intermixed with lower endpoints. One such configuration is illustrated in figure 3 below. In this illustration  $\alpha = .10$  and  $r_0 = 13$ .

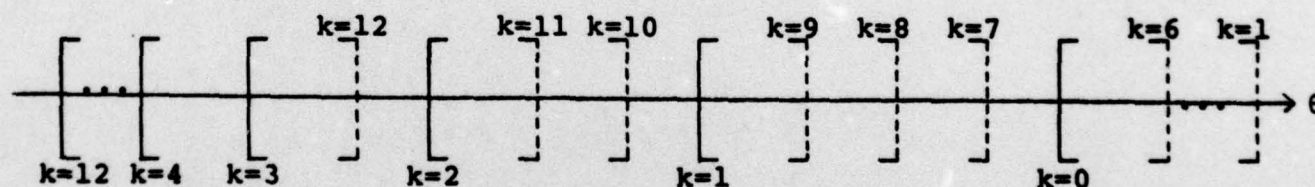


Figure 3. Example of interval endpoints intermixed,  $\alpha = .10$ ,  $r_0 = 13$ .

All of these possible configurations may be reduced to the consideration of a relatively few cases as indicated in figure 2. Clearly we cannot consider, individually, all possible subsets of every possible configuration of endpoints in the partition. In each

conceivable case, however, the proof would utilize only a subset of the cases below.

We first prove two short lemmas that will be used in the general proof.

Lemma 1. If  $x \geq 0$ , then  $f(x) = \sum_{k=0}^n e^{-x} x^k / k!$  is monotonically decreasing in  $x$ .

The proof is direct. We have  $f'(x) = e^{-x} \sum_{k=0}^n (kx^{k-1} - x^k) / k!$ , which is a telescoping sum. Hence  $f'(x) = -e^{-x} x^n / n!$ . Since  $f'(x) \leq 0$  for all  $x \geq 0$ , the result follows.

Lemma 2. Let  $\tau_1, \dots, \tau_{r_0}$  be the first  $r_0$  failure times out of  $n$  items on test, where each item has an exponential lifetime with parameter  $\theta$ . That is  $f(\tau_i) = (1/\theta)e^{-\tau_i/\theta}$ ,  $\tau_i \geq 0$ . Then  $Y = 2n\tau_{r_0}/\theta$  has a chi-square distribution with  $2r_0$  degrees of freedom.

To prove this we write  $\tau_{r_0} = \tau_1 + (\tau_2 - \tau_1) + \dots + (\tau_{r_0} - \tau_{r_0-1})$ . Since we are observing a Poisson process with parameter  $\lambda = n/\theta$  it follows that  $\tau_{r_0}$  is the sum of  $r_0$  independent, identically distributed random variables with common exponential distribution  $f(x) = \frac{n}{\theta} e^{-nx/\theta}$ ,  $x \geq 0$ . Consequently  $\tau_{r_0}$  has a gamma distribution with parameters  $\alpha = r_0$ ,  $\beta = \theta/n$ . If we let  $Y = 2n\tau_{r_0}/\theta$ , the distribution of  $Y$  is easily found to be  $\chi^2_{2r_0}$  from the transformation on  $\tau_{r_0}$ .

Proof of Theorem 1. Case (i) corresponds to region i in figure 2.

Let  $I$  represent the family of intervals (10), and  $k$  be the number of failures observed in  $(0, t^*)$ .

$$\text{Case (1): } \theta \geq \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2}}.$$

Under this restriction on  $\theta$ ,  $P(\theta \in I) = P(\text{no failures before } t^*) = e^{-\frac{nt^*}{\theta}}$ , since we are observing a Poisson process with parameter  $\frac{n}{\theta}$ . By the case (1) assumption  $nt^*/\theta \leq \chi^2_{1-\frac{\alpha}{2}, 2}/2$  which gives

$$e^{-\frac{nt^*}{\theta}} \geq e^{-\chi^2_{1-\frac{\alpha}{2}, 2}/2}.$$

Now we can make use of the well known relationship

$$P(\chi^2_{2r+2} > y) = \sum_{k=0}^r \frac{e^{-y/2} (y/2)^k}{k!}. \quad (11)$$

Thus, with  $r=0$ ,  $P(\chi^2_2 > \chi^2_{1-\frac{\alpha}{2}, 2}) = e^{-\chi^2_{1-\frac{\alpha}{2}, 2}/2} = 1 - \frac{\alpha}{2}$ . So

we have  $P(\theta \in I) \geq 1 - \frac{\alpha}{2} > 1 - \alpha$  which was to be proved for case (1).

$$\text{Case (2): } \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2h+2}} \leq \theta < \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2h}}.$$

$$\begin{aligned} P(\theta \in I) &= P\left[\bigcup_{r=0}^h (r \text{ failures before } t^*)\right] = \sum_{r=0}^h P(r \text{ failures in } t^*) \\ &= \sum_{r=0}^h \frac{e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r}{r!}. \end{aligned}$$

By case (2) assumption,  $nt^*/\theta \leq \chi^2_{1-\frac{\alpha}{2}, 2h+2}/2$  which implies

$$\sum_{r=0}^h \frac{e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r}{r!} \geq \sum_{r=0}^h \frac{e^{-\chi^2_{1-\frac{\alpha}{2}, 2h+2}/2} (\chi^2_{1-\frac{\alpha}{2}, 2h+2}/2)^r}{r!} = 1 - \frac{\alpha}{2} .$$

The inequality follows from Lemma 1 and the equality from (11).

Hence, again,  $P(\theta \in I) > 1 - \alpha$ .

$$\text{Case (3): } \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2}} \leq \theta < \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2m}} .$$

This is included in the more general case (6) with  $d=0$  and  $s=m$ .

$$\text{Case (4): } \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2m+2}} \leq \theta < \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2}} .$$

Again, this is included in case (7) with  $d=0$ ,  $s=m$ .

$$\text{Case (5): } \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2z+4}} \leq \theta < \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2z+2}} .$$

Note here that we necessarily have  $s \geq d+1$ . Otherwise, if

$$s=d, \text{ we would have } \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2d}} < \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2d}} \text{ or } \chi^2_{1-\frac{\alpha}{2}, 2d} > \chi^2_{\frac{\alpha}{2}, 2d}$$

which is clearly not possible. Now within case (5),

$$\begin{aligned}
 P(\theta \in I) &= P\left[\bigcup_{r=z+1}^{s-1} (r \text{ failures in } t^*)\right] = \sum_{r=z+1}^{s-1} P(r \text{ failures in } t^*) = \\
 &= \sum_{r=z+1}^{s-1} e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! = \sum_{r=0}^{s-1} e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! - \sum_{r=0}^z e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! .
 \end{aligned}$$

By case (5) assumption  $\theta > \frac{2nt^*}{\chi_{1-\frac{\alpha}{2}, 2s}^2}$  and so  $\frac{nt^*}{\theta} < \frac{\chi_{1-\frac{\alpha}{2}, 2s}^2}{2}$ .

Consequently the first sum is  $> \sum_{r=0}^{s-1} e^{-\chi_{1-\frac{\alpha}{2}, 2s}^2/2} (\chi_{1-\frac{\alpha}{2}, 2s}^2/2)^r / r! = 1 - \frac{\alpha}{2}$ ,

and  $P(\theta \in I) > 1 - \frac{\alpha}{2} - \sum_{r=0}^z e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r!$ . But

$nt^*/\theta > \chi_{\frac{\alpha}{2}, 2z+2}^2/2$  and so

$$\sum_{r=0}^z e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! < \sum_{r=0}^z e^{-\chi_{\frac{\alpha}{2}, 2z+2}^2/2} (\chi_{\frac{\alpha}{2}, 2z+2}^2/2)^r / r! = \frac{\alpha}{2} .$$

Finally  $P(\theta \in I) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$ .

$$\text{Case (6): } \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2s+2}} < \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2d+2}} \leq \theta < \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2s}} < \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2d}} .$$

$$P(\theta \in I) = P\left[\bigcup_{r=d}^s (r \text{ failures in } t^*)\right] = \sum_{r=d}^s e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^{r/r!} =$$

$$\sum_{r=0}^s e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^{r/r!} - \sum_{r=0}^{d-1} e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^{r/r!} >$$

$$> \sum_{r=0}^s e^{-\chi^2_{1-\frac{\alpha}{2}, 2s+2}/2} (\chi^2_{1-\frac{\alpha}{2}, 2s+2}/2)^{r/r!} - \sum_{r=0}^{d-1} e^{-\chi^2_{\frac{\alpha}{2}, 2d}/2} (\chi^2_{\frac{\alpha}{2}, 2d}/2)^{r/r!} .$$

The last inequality follows from case (6) assumption which implies

$$\frac{nt^*}{\theta} < \frac{\chi^2_{1-\frac{\alpha}{2}, 2s+2}}{2} \quad \text{and} \quad \frac{nt^*}{\theta} > \frac{\chi^2_{\frac{\alpha}{2}, 2d}}{2} .$$

Consequently  $P(\theta \in I) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$ .

$$\text{Case (7): } \frac{2nt^*}{\chi^2_{1-\frac{\alpha}{2}, 2s+2}} \leq \theta < \frac{2nt^*}{\chi^2_{\frac{\alpha}{2}, 2d+2}} .$$

$$P(\theta \in I) = P\left[\bigcup_{r=d+1}^s (r \text{ failures in time } t^*)\right] =$$

$$= \sum_{r=0}^s e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^{r/r!} - \sum_{r=0}^d e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^{r/r!} >$$

$$\sum_{r=0}^s e^{-\chi_{1-\frac{\alpha}{2}, 2s+2}^2/2} (\chi_{1-\frac{\alpha}{2}, 2s+2}^2/2)^r / r!$$

$$- \sum_{r=0}^d e^{-\chi_{\frac{\alpha}{2}, 2d+2}^2/2} (\chi_{\frac{\alpha}{2}, 2d+2}^2/2)^r / r! .$$

Again the inequality follows from the restriction on  $\theta$  which implies  $nt^*/\theta \leq \chi_{1-\frac{\alpha}{2}, 2s+2}^2/2$  and  $nt^*/\theta > \chi_{\frac{\alpha}{2}, 2d+2}^2/2$ . So again  $P(\theta \in I) = 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$ . It is noteworthy here to consider this case when  $d=0$ ,  $s=m$ , which is merely case (4) as previously cited. The proof here relies on the fact that  $\theta < 2nt^*/\chi_{\frac{\alpha}{2}, 2}^2$ . But in Epstein's conjectured intervals, this partition point would be  $2nt^*/\chi_{\alpha, 2}^2$ , which gives  $P(\theta \in I)$

$$1 - \frac{\alpha}{2} - \alpha = 1 - \frac{3\alpha}{2} . \text{ Problems arise if } 2nt^*/\chi_{\alpha/2, 2}^2 < \theta < 2nt^*/\chi_{\alpha, 2}^2 .$$

For example, it can be shown that for the simple case with  $r_0 = 2$ , with  $\theta$  in this small interval,  $P(\theta \in I) < 1 - \alpha$ .

$$\text{Case (8): } \frac{2nt^*}{\chi_{1-\frac{\alpha}{2}, 2v+2}^2} \leq \theta < \frac{2nt^*}{\chi_{1-\frac{\alpha}{2}, 2v}^2} .$$

$$P(\theta \in I) = \sum_{r=d+1}^v P(r \text{ failures before time } t^*) =$$

$$\sum_{r=0}^v e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! - \sum_{r=0}^d e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! >$$

$$> \sum_{r=0}^v e^{-\chi_{1-\frac{\alpha}{2}, 2v+2}^2} (\chi_{1-\frac{\alpha}{2}, 2v+2}^2)^r / r! - \sum_{r=0}^d e^{-\chi_{\frac{\alpha}{2}, 2d+2}^2} (\chi_{\frac{\alpha}{2}, 2d+2}^2)^r / r!.$$

Again this follows from  $nt^*/\theta \leq \chi_{1-\frac{\alpha}{2}, 2v+2}^2$  and  $nt^*/\theta > \chi_{\frac{\alpha}{2}, 2d+2}^2$ .

So, for case (8),  $P(\theta \in I) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$ .

$$\text{Case (9): } \frac{2nt^*}{\chi_{\frac{\alpha}{2}, 2w+2}^2} \leq \theta < \frac{2nt^*}{\chi_{1-\frac{\alpha}{2}, 2r_0-2}^2}.$$

Note that now we must include the possibility of coverage by the random interval corresponding to  $k=r_0$ .

$$P(\theta \in I) = \sum_{r=w}^{r_0-1} P(r \text{ failures before } t^*) + P(r_0 \text{ fail and } \theta \leq \frac{2n\tau_{r_0}}{\chi_{1-\frac{\alpha}{2}, 2r_0}^2})$$

$$= \sum_{r=w}^{r_0-1} e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! + P\left(\frac{\theta \chi_{1-\frac{\alpha}{2}, 2r_0}^2}{2n} \leq \tau_{r_0} \leq t^*\right).$$

It follows from Lemma 2 that we can write this second term as

$$P\left(\frac{\theta \chi_{1-\frac{\alpha}{2}, 2r_0}^2}{2n} \leq \tau_{r_0} \leq t^*\right) = P\left(\chi_{1-\frac{\alpha}{2}, 2r_0}^2 \leq Y \leq \frac{2nt^*}{\theta}\right)$$

$$= F_Y(2nt^*/\theta) - \frac{\alpha}{2} = 1 - \frac{\alpha}{2} - P\left(\chi_{2r_0}^2 > \frac{2nt^*}{\theta}\right) = 1 - \frac{\alpha}{2} - \sum_{r=0}^{r_0-1} e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r!.$$

$$\begin{aligned}
 P(\theta \in I) &= \sum_{r=w}^{r_0-1} e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! + 1 - \frac{\alpha}{2} - \sum_{r=0}^{r_0-1} e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! \\
 &= 1 - \frac{\alpha}{2} - \sum_{r=0}^{w-1} e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! .
 \end{aligned}$$

But, by assumption,  $\theta < 2nt^*/\chi_{\alpha/2, 2w}^2$  and so  $nt^*/\theta > \chi_{\alpha/2, 2w/2}^2$

which then implies  $P(\theta \in I) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$ .

$$\text{Case (10): } \frac{2nt^*}{\chi_{\alpha/2, 2y+4}^2} \leq \theta < \frac{2nt^*}{\chi_{\alpha/2, 2y+2}^2} .$$

$$P(\theta \in I) = \sum_{r=y+1}^{r_0-1} P(r \text{ fail in } t^*) + P(r_0 \text{ fail and } \theta \leq \frac{2nr_0}{\chi_{1-\frac{\alpha}{2}, 2r_0}^2})$$

$$= \sum_{r=y+1}^{r_0-1} e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! + P(\chi_{1-\frac{\alpha}{2}, 2r_0}^2 \leq Y \leq \frac{2nt^*}{\theta}) \text{ as in case (9).}$$

By the same reasoning as in case (9),

$$P(\theta \in I) = 1 - \frac{\alpha}{2} - \sum_{r=0}^y e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^r / r! \text{ and since,}$$

$$nt^*/\theta > \chi_{\alpha/2, 2y+2}^2 / 2, \quad P(\theta \in I) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha .$$

$$\text{Case (11): } \theta < \frac{2nt^*}{\chi_{\frac{\alpha}{2}, 2r_0}^2} .$$

$$P(\theta \in I) = P\left[ \frac{2n\tau_{r_0}}{\chi_{\frac{\alpha}{2}, 2r_0}^2} \leq \theta \leq \frac{2n\tau_{r_0}}{\chi_{1-\frac{\alpha}{2}, 2r_0}^2} \right]$$

$$= P\left[ \frac{\theta \chi_{1-\alpha/2, 2r_0}^2}{2n} \leq \tau_{r_0} \leq \frac{\theta \chi_{\alpha/2, 2r_0}^2}{2n} \right]$$

$$= P(\chi_{1-\frac{\alpha}{2}, 2r_0}^2 \leq \chi_{2r_0}^2 \leq \chi_{\frac{\alpha}{2}, 2r_0}^2) = 1 - \alpha .$$

These cases cover any possible subset of every partition. A formal proof of this claim is found in Appendix 1. The proof of Theorem 1 is complete.

### IIIb. Expected Interval Length

It is of some interest now to derive an expression for the expected length of the confidence interval under this scheme. There is a positive probability of observing no failures, in which case the interval length is infinite. To avoid this situation we will condition on one or more failures. Denote the length of the interval by  $L$  and consider the following two possibilities:

$$\text{If } \tau_{r_0} > t^* \text{ then } L = 2nt^* \left[ \frac{1}{\chi^2_{1-\frac{\alpha}{2}, 2k}} - \frac{1}{\chi^2_{\frac{\alpha}{2}, 2k+2}} \right] = l_k, \text{ say.}$$

Here  $k$  is the number of failures observed with  $1 \leq k \leq r_0 - 1$ .

$$\text{If } \tau_{r_0} \leq t^* \text{ then } L = \left[ \frac{2n}{\chi^2_{1-\frac{\alpha}{2}, 2r_0}} - \frac{2n}{\chi^2_{\frac{\alpha}{2}, 2r_0}} \right] \tau_{r_0} = c\tau_{r_0}, \text{ say.}$$

Note that in the first case the length is discrete while in the second it is a continuous random variable. Consequently the unconditional distribution of  $L$  is of the mixed type with jumps at  $l_k, k=1, 2, \dots, r_0-1$ . So,

$$\begin{aligned} F(l) &= \int_0^{l/c} f(t) dt + \sum_{k \in A} P(k \text{ failures in } t^*) \text{ if } l < ct^* \\ &= \int_0^{t^*} f(t) dt + \sum_{k \in A} P(k \text{ failures in } t^*) \text{ if } l \geq ct^*, \end{aligned}$$

where  $f$  is the density function of  $\tau_{r_0}$  and  $A = \{k: l_k \leq l\}$ .

From Lemma 2 we have that  $\tau_{r_0} \sim \text{gamma} (\alpha = r_0, \beta = \theta/n)$ , and

$P(k \text{ failures}) = e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^k / k!$ . We want  $E(L|L < \infty)$  which is  $E(L|k \neq 0)$  since  $L = \infty$  if and only if  $k = 0$ . We need now the conditional distribution of  $L$ . We can write

$P(L \leq l | L < \infty) = P(L \leq l \cap L < \infty) / P(k \neq 0)$ . Now if  $l$  is finite,

$(L \leq l) \subset (L < \infty)$  which then implies  $P(L \leq l \cap L < \infty) = P(L \leq l)$ .

On the other hand, if  $l$  is infinite,  $(L < \infty) \subset (L \leq l)$  and  $P(L \leq l \cap L < \infty) = P(L < \infty)$ . Thus

$$F(l|L < \infty) = \frac{F(l)}{P(k \neq 0)} \quad \text{if } l \text{ is finite}$$

$$= 1 \quad \text{if } l \text{ is infinite.}$$

$$\text{Consequently, } E(L|k \neq 0) = \int_0^{\infty} l \, dF(l|k \neq 0) = \frac{1}{P(k \neq 0)} \int_0^{\infty} l \, dF(l),$$

$$\text{where } P(k \neq 0) = 1 - e^{-\frac{nt^*}{\theta}}. \quad \text{Thus,}$$

$$\begin{aligned} E(L|k \neq 0) &= \frac{1}{P(k \neq 0)} \left[ \sum_{k=1}^{r_0-1} l_k P(k \text{ fail in } t^*) + \int_0^{ct^*} l g(l) dl \right] \\ &= \frac{1}{P(k \neq 0)} \left[ \sum_{k=1}^{r_0-1} l_k P(k \text{ fail in } t^*) + \int_0^{t^*} c \tau_{r_0} f(\tau_{r_0}) d\tau_{r_0} \right]. \end{aligned}$$

To evaluate the integral:

$$\begin{aligned} c \int_0^{t^*} y f(y) dy &= c \int_0^{t^*} \frac{y^{r_0} \cdot e^{-\frac{ny}{\theta}}}{\Gamma(r_0) \left(\frac{\theta}{n}\right)^{r_0}} dy \\ &= \frac{c}{\Gamma(r_0) \left(\frac{\theta}{n}\right)^{r_0}} \left[ e^{-\frac{ny}{\theta}} \sum_{k=0}^{r_0-1} \frac{(-1)^k r_0! y^{r_0-k}}{(r_0-k)! \left(-\frac{\theta}{n}\right)^{k+1}} \right]_{y=0}^{t^*} \end{aligned}$$

$$= \frac{c r_0 \theta}{n} \left[ 1 - \sum_{k=0}^{r_0} \frac{e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^k}{k!} \right] .$$

Thus,

$$E(L|k \neq 0) = \frac{1}{1 - e^{-\frac{nt^*}{\theta}}} \left[ \sum_{k=1}^{r_0-1} l_k \frac{e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^k}{k!} + \frac{\theta c r_0}{n} \sum_{k=r_0+1}^{\infty} \frac{e^{-\frac{nt^*}{\theta}} \left(\frac{nt^*}{\theta}\right)^k}{k!} \right] \quad (12)$$

$$\text{where } l_k = 2nt^* \left[ \frac{1}{\chi_{1-\frac{\alpha}{2}, 2k}^2} - \frac{1}{\chi_{\frac{\alpha}{2}, 2k+2}^2} \right]$$

$$\text{and } c = 2n \left[ \frac{1}{\chi_{1-\frac{\alpha}{2}, 2r_0}^2} - \frac{1}{\chi_{\frac{\alpha}{2}, 2r_0}^2} \right] .$$

Several questions may come to mind concerning the expression for  $E(L|k \neq 0)$ . For instance, if we take the limit as  $r_0 \rightarrow \infty$  in (12), do we obtain the  $E(L|k \neq 0)$  for the type I censoring scheme? If we take the limit as  $t^* \rightarrow \infty$ , do we obtain the  $E(L)$  for the type II censoring scheme? It can be shown quite easily that the answer to both questions is yes. In the preceding derivations we have placed no restrictions on  $t^*$  and  $r_0$  other than to say they are censoring, or truncation, values. In practice the values

of  $t^*$  and  $r_0$  are determined by the test procedure which provides the data for estimation. MIL-STD-781 is concerned with testing  $H_0: \theta = \theta_0$  against  $H_1: \theta = \theta_1$  where  $\theta_1 < \theta_0$ . If a test combining type I and type II censoring with  $P(\text{reject } H_0 | \theta_1) = \alpha$  is desired, Epstein [5] has shown that  $t^* = \theta_0 x_{1-\alpha, 2r_0}^2 / 2n$ .

If in addition we require  $P(\text{accept } H_1 | \theta_1) \leq \beta$ , Epstein has shown this may be achieved if we choose  $r_0$  to be the smallest integer such that

$$\frac{\theta_0}{\theta_1} \geq \frac{x_{\beta, 2r_0}^2}{x_{1-\alpha, 2r_0}^2}.$$

Then we may decrease  $\beta$  if we increase  $r_0$ . It seems reasonable that an increase in  $r_0$ , or a decrease in  $\beta$ , would result in a corresponding decrease in  $E(L | k \neq 0)$ . An attempt at a direct analytic proof using (12) was not successful. We then programmed (12) and generated  $E(L | k \neq 0)$  for various combinations of  $\theta, \theta_0$  and  $\alpha$ . The value of  $r_0$  was increased for each combination. The results, somewhat surprisingly, show that for some combinations of  $\theta, \theta_0$  and  $\alpha$ , the expected length increases initially before it does decrease with increasing  $r_0$ . Partial results of this computer work are given in Table 1. We have, thus far, been unable to provide a satisfactory explanation of this behavior.

Table I. Expected Lengths for Increasing Truncation Values, R

 $\theta = 1.00$  $\theta_0 = 3.00$  $\alpha = 0.05$ 

R = 2	E(L) = 25.7057
R = 3	E(L) = 26.1465
R = 4	E(L) = 15.0962
R = 5	E(L) = 6.9176
R = 6	E(L) = 3.4234
R = 7	E(L) = 2.2408
R = 8	E(L) = 1.8238
R = 9	E(L) = 1.6299
R = 10	E(L) = 1.5034
R = 11	E(L) = 1.4068
R = 12	E(L) = 1.3266
R = 13	E(L) = 1.2590
R = 14	E(L) = 1.2001
R = 15	E(L) = 1.1490

 $\alpha = 0.10$ 

R = 2	E(L) = 14.6718
R = 3	E(L) = 11.1729
R = 4	E(L) = 5.6256
R = 5	E(L) = 2.8950
R = 6	E(L) = 1.9311
R = 7	E(L) = 1.5818
R = 8	E(L) = 1.4094
R = 9	E(L) = 1.2954
R = 10	E(L) = 1.2071
R = 11	E(L) = 1.1351
R = 12	E(L) = 1.0745
R = 13	E(L) = 1.0223
R = 14	E(L) = 0.9771
R = 15	E(L) = 0.9374

 $\theta = 2.00$  $\theta_0 = 3.00$  $\alpha = 0.05$ 

R = 2	E(L) = 33.8920
R = 3	E(L) = 53.1913
R = 4	E(L) = 54.6214
R = 5	E(L) = 44.3260
R = 6	E(L) = 30.9006
R = 7	E(L) = 19.6450
R = 8	E(L) = 12.0050
R = 9	E(L) = 7.4825
R = 10	E(L) = 5.0097
R = 11	E(L) = 3.7223
R = 12	E(L) = 3.0517
R = 13	E(L) = 2.6882
R = 14	E(L) = 2.4717
R = 15	E(L) = 2.3277

 $\alpha = 0.10$ 

R = 2	E(L) = 22.2509
R = 3	E(L) = 28.8162
R = 4	E(L) = 25.6829
R = 5	E(L) = 18.7492
R = 6	E(L) = 12.2808
R = 7	E(L) = 7.7827
R = 8	E(L) = 5.1069
R = 9	E(L) = 3.6492
R = 10	E(L) = 2.8815
R = 11	E(L) = 2.4700
R = 12	E(L) = 2.2326
R = 13	E(L) = 2.0790
R = 14	E(L) = 1.9683
R = 15	E(L) = 1.8805

 $\theta = 4.00$  $\theta_0 = 3.00$  $\alpha = 0.05$ 

R = 2	E(L) = 38.6661
R = 3	E(L) = 73.4801
R = 4	E(L) = 97.9833
R = 5	E(L) = 109.0797
R = 6	E(L) = 108.3810
R = 7	E(L) = 99.3520
R = 8	E(L) = 85.6724
R = 9	E(L) = 70.4874
R = 10	E(L) = 55.8853
R = 11	E(L) = 43.1227
R = 12	E(L) = 32.6525
R = 13	E(L) = 24.4603
R = 14	E(L) = 18.3067
R = 15	E(L) = 13.8143

 $\alpha = 0.10$ 

R = 2	E(L) = 27.0906
R = 3	E(L) = 44.5701
R = 4	E(L) = 54.0796
R = 5	E(L) = 56.0089
R = 6	E(L) = 52.4240
R = 7	E(L) = 45.7010
R = 8	E(L) = 37.8212
R = 9	E(L) = 30.1389
R = 10	E(L) = 23.4156
R = 11	E(L) = 17.9371
R = 12	E(L) = 13.6968
R = 13	E(L) = 10.5338
R = 14	E(L) = 8.2472
R = 15	E(L) = 6.6256

### IIIc. Testing without Replacement

The second testing scheme we are considering terminates at  $\min(\tau_{r_0}, t')$ , where failed items are not replaced. Recall that the truncation value  $t'$  now refers to the total accumulated test time. At the time of decision the following rule gives a  $100(1-\alpha)$  percent confidence interval for  $\theta$  :

$$\begin{aligned} & \left( \frac{2t'}{\chi^2_{\frac{\alpha}{2}, 2}}, \infty \right) && \text{if } k = 0, \\ & \left( \frac{2t'}{\chi^2_{\frac{\alpha}{2}, 2k+2}}, \frac{2t'}{\chi^2_{1-\frac{\alpha}{2}, 2k}} \right) && \text{if } k = 1, 2, \dots, r_0 - 1, \\ & \left( \frac{2T_{r_0}}{\chi^2_{\frac{\alpha}{2}, 2r_0}}, \frac{2T_{r_0}}{\chi^2_{1-\frac{\alpha}{2}, 2r_0}} \right) && \text{if } k = r_0. \end{aligned} \quad (13)$$

Here  $k$  is the number of failures observed when the total time on test reaches  $t'$ , and  $T_{r_0}$  is the total time on test at the time of the  $r_0^{\text{th}}$  failure. This rule is nearly identical to that conjectured by Epstein [4], with again the only change being the use of  $\alpha/2$  rather than  $\alpha$  when  $k=0$ . The proof of (13) becomes simple when we realize that this scheme may be viewed as

a Poisson process with parameter  $1/\theta$  which we observe for actual time  $t'$ . To see this, let  $T_k$  be the total time on test when the  $k^{\text{th}}$  item fails. Then

$$T_k = \sum_{i=1}^k \tau_i + (n-k)\tau_k = \sum_{i=1}^k (n-i+1)(\tau_i - \tau_{i-1}) = \sum_{i=1}^k U_i$$

where we have previously noted the  $U_i$  are independent and identically distributed as exponential  $(\frac{1}{\theta})$ . It follows that

$T_k \sim \text{gamma } (\alpha=k, \beta=\theta)$ . We may write

$P(k \text{ failures in total time } t') = P(T_{k+1} > t') - P(T_k > t')$ . From direct integration we find that

$$P(T_{k+1} > t') = \sum_{r=0}^k e^{-\frac{t'}{\theta}} \left(\frac{t'}{\theta}\right)^{k/k!}$$

$$P(T_k > t') = \sum_{r=0}^{k-1} e^{-\frac{t'}{\theta}} \left(\frac{t'}{\theta}\right)^{k/k!}$$

and  $P(k \text{ failures}) = e^{-\frac{t'}{\theta}} \left(\frac{t'}{\theta}\right)^{k/k!}$ . Hence, to prove (13) we need only duplicate the proof of (10) substituting  $t'$  for  $t^*$  and  $1/\theta$  for  $n/\theta$ . The derivation of  $E(L|k \neq 0)$  for this scheme follows in the same manner. The result is

$$E(L|k \neq 0) = \frac{1}{1 - e^{-t'/\theta}} \left[ \sum_{k=1}^{r_0-1} \frac{e^{-\frac{t'}{\theta}} \left(\frac{t'}{\theta}\right)^k}{k!} + c' r_0 \theta \sum_{k=r_0+1}^{\infty} \frac{e^{-\frac{t'}{\theta}} \left(\frac{t'}{\theta}\right)^k}{k!} \right]$$

$$\text{where } l_k = 2t' \left( \frac{1}{x_{1-\frac{\alpha}{2}, 2k}^2} - \frac{1}{x_{\frac{\alpha}{2}, 2k+2}^2} \right) \quad k = 1, 2, \dots, r_0 - 1,$$

$$\text{and } c' = 2 \left( \frac{1}{x_{1-\frac{\alpha}{2}, 2r_0}^2} - \frac{1}{x_{\frac{\alpha}{2}, 2r_0}^2} \right).$$

Other results from the previous scheme carry over similarly.

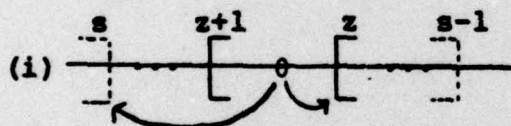
## Appendix I

We now give a proof that the cases considered in Theorem 1 do in fact cover every subset of the possible partitions of the parameter space determined by the fixed interval endpoints.

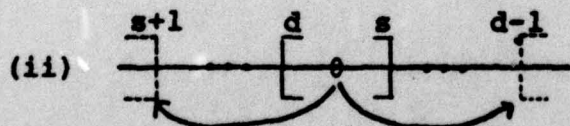
With the exception of cases (1) and (11) (regions 1 and 11 in figure 2), each possible subset in the partition will be bounded above and below by a left or right interval endpoint. Consequently there are only four possible subsets:

$$(i) \quad \left[ \quad \left[ \quad (ii) \quad \left[ \quad \right] \quad (iii) \quad \right] \quad \left[ \quad (iv) \quad \right] \quad \right]$$

We consider each of these and show how each is covered in Theorem 1.

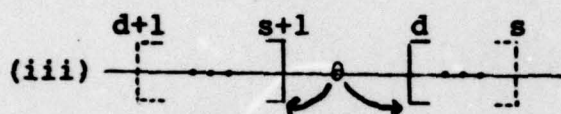


In the most general situation this is case (5) of Theorem 1. The arrows indicate the bounds on  $\theta$  which are necessary for the proof of case (5). It is possible that there is no right endpoint below the left endpoint determined by  $k = z+1$ . In this event we have  $s-1 = r_0-1$  and we are then in case (10) of Theorem 1.

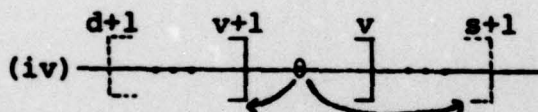


In general this is case (6) of Theorem 1. Again the arrows indicate the boundary points necessary for its proof. If there is no right endpoint corresponding to  $k = s+1$ , it must be that

$s = r_0 - 1$  and we have case (9) with  $d = w$ . If there is no left endpoint indicated by  $d-1$  we have the situation where  $d=0$  and  $s=m$  which is case (3). If neither of the points indicated by  $s+1$  and  $d-1$  are found, we must have the situation where no crossover occurs. Then we are in case (9) with  $w=0$ .



This is case (7) in general and its proof is dependent only on the actual boundaries of the subset. No other considerations are necessary.



In general this is case (8) of Theorem 1. If the endpoint indicated by  $k = s+1$  is not present, we must be in case (2) with  $d+1=0$  and  $v=h$ .

By adding regions (1) and (11) in figure 2, we see that Theorem 1 covers all possible subsets. Cases (3) and (4) are included only for clarity in figure 2 and may be omitted, since they are special cases of (6) and (7).

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