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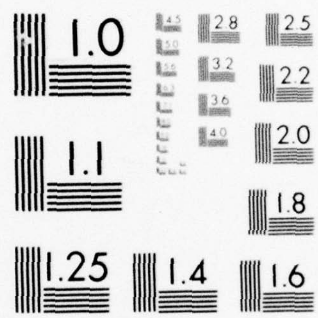

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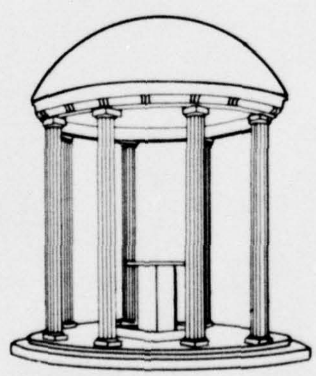

SCHOOL OF BUSINESS ADMINISTRATION
AND
CURRICULUM IN
OPERATIONS RESEARCH AND SYSTEMS ANALYSIS

OPERATING CHARACTERISTIC APPROXIMATIONS FOR THE ANALYSIS OF (s,S) INVENTORY SYSTEMS

Technical Report #12

Richard Ehrhardt

April 1977

Work Sponsored By

U.S. Army Research Office (DAAG29-76-G-0323)

Decision Control Models in Operations Research

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UNCLASSIFIED (18) ARO (19) 14635.2-M

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Report #12 (14) TR-12	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) OPERATING CHARACTERISTIC APPROXIMATIONS FOR THE ANALYSIS OF (s,S) INVENTORY SYSTEMS.		5. TYPE OF REPORT & PERIOD COVERED (9) Technical rept. 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) Richard/Ehrhardt		8. CONTRACT OR GRANT NUMBER(s) (15) DAAG29-76-G-0323
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of North Carolina at Chapel Hill Chapel Hill, North Carolina 27514		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS ~~P-11639-M~~
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematics Division U.S. Army Research Office Box CM, Duke Station, Durham, N.C. 27706		12. REPORT DATE (11) Apr [redacted] 77 13. NUMBER OF PAGES 109 (12) 125p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Inventory Control, (s,S) Policies, Sensitivity Analysis		

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The operating characteristics of (s,S) inventory systems are often difficult to compute, making sensitivity analysis a tedious and often expensive undertaking. Approximate expressions for operating characteristics are presented with a view towards simplified sensitivity analysis.

The operating characteristics under consideration are the expected values of: Total cost per period, period-end inventory, period-end

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stockout quantity, replenishment cost per period, and backlog frequency. The approximations are obtained by using least-squares regression to fit simple functions to the operating characteristics of a large number of parameter settings. Accuracy to within a few percent of actual values is typical for most of the approximations.

Potential uses of the approximations are illustrated for several idealized design problems.

FOREWORD

As part of the on-going program in "Decision and Control Models in Operations Research," Mr. Richard Ehrhardt has developed a number of formulae that give approximate values for the operating characteristics of several classes of (s,S) inventory policies. The approximations are accurate and require substantially less computational effort than exact calculation of the operating characteristics. Many of the approximations are in a form that make them particularly amenable to sensitivity analysis and systems design studies.

Other related reports dealing with this research program are given below.

Harvey M. Wagner

Principal Investigator

MacCormick, A. (1975), Statistical Problems in Inventory Control, ONR and ARO Technical Report, December 1974, School of Organization and Management, Yale University, 244 pp.

Estey, A. S. and R. L. Kaufman (1975), Multi Item Inventory System Policies Using Statistical Estimates: Negative Binomial Demands (Variance/Mean = 1), ONR and ARO Technical Report 3, September 1975, School of Organization and Management, Yale University, 85 pp.

Ehrhardt, R. (1975), Variance Reduction Techniques for an Inventory Simulation, ONR and ARO Technical Report 4, September 1975, School of Organization and Management, Yale University, 24 pp.

Kaufman, R. (1976), Computer Programs for (s,S) Policies under Independent or Filtered Demands, ONR and ARO Technical Report 5, School of Organization and Management, Yale University, 65 pp.

Kaufman, R. and J. Klincewicz (1976), Multi-Item Inventory System Policies Using Statistical Estimates: Sporadic Demands (Variance/Mean = 9), ONR and ARO Technical Report 6, School of Organization and Management, Yale University, 58 pp.

Ehrhardt, R. (1976), The Power Approximation: Inventory Policies Based on Limited Demand Information, ONR and ARO Technical Report 7, School of Organization and Management, Yale University, 58 pp.

Klincewicz, J. G. (1976), Biased Variance Estimators for Statistical Inventory Policies, ONR and ARO Technical Report 8, School of Organization and Management, Yale University, 24 pp.

Klincewicz, J. G. (1976), Inventory Control Using Statistical Estimates: The Power Approximation and Sporadic Demands (Variance/Mean = 9), ONR and ARO Technical Report 9, School of Organization and Management, Yale University, 52 pp.

Klincewicz, J.G. (1976), The Power Approximation: Control of Multi-item Inventory Systems with Constant Standard-deviation-to-mean Ratio for Demand, ONR and ARO Technical Report 10, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 47 pp.

Kaufman, R. L. (1977), (s,S) Inventory Policies in a Nonstationary Demand Environment, ARO Technical Report 11, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 155 pp.

ABSTRACT

OPERATING CHARACTERISTIC APPROXIMATIONS FOR THE ANALYSIS OF (s,S) INVENTORY SYSTEMS

Richard Ehrhardt
University of North Carolina - 1977

The operating characteristics of (s,S) inventory systems are often difficult to compute, making sensitivity analysis a tedious and often expensive undertaking. Approximate expressions for operating characteristics are presented with a view towards simplified sensitivity analysis. The operating characteristics under consideration are the expected values of: Total cost per period, period-end inventory, period-end stockout quantity, replenishment cost per period, and backlog frequency. The approximations are obtained by using least-squares regression to fit simple functions to the operating characteristics of a large number of parameter settings. Accuracy to within a few percent of actual values is typical for most of the approximations.

Potential uses of the approximations are illustrated for several idealized design problems.

CONTENTS

1. (s,S) INVENTORY POLICY OPERATING CHARACTERISTICS

There are many situations in which an inventory systems designer can use estimates of operating characteristics of the system. For example, management may desire forecasts of inventory-on-hand, or system operating costs. The systems designer may also confront questions such as

(a) What are the cost and service implications of consolidating demand from several distinct warehouses into a single central warehouse?

(b) How do parameter changes affect the system's operating characteristics?

(c) By how much do costs rise when service is increased?

(d) What are the effects of changing the review-period length?

To facilitate analysis of design issues of this type, we seek simple approximations for the following operating characteristics: average holding cost per period, average backlog cost per period, frequency of periods without backlogs, average replenishment cost per period, and average total cost per period. These characteristics are defined mathematically in Section 2.1 of this report.

1.1 The Model

Throughout this report, we deal with a single-item inventory model. We assume periodic review of an item's inventory level and employ a stationary, discrete-time stochastic process to describe the item's demand. The demand sequence $\xi_1, \xi_2, \ldots,$ consists of independent, identically distributed random variables taking non-negative integer values.

Demands are met as long as stock on hand is sufficient; when a stockout occurs, the unfilled demand is completely backlogged until a stock replenishment eventually arrives.

Items kept in inventory are assumed to be conserved, there being no losses by deterioration, obsolescence, or pilferage; disposal is not allowed. Inventory-on-hand at the end of a period is the inventory from the previous period plus any replenishment that arrives, less demand. Negative inventory-on-hand represents the amount of backlogged demand. Replenishments are assumed to be delivered a fixed lead time L periods after being ordered. The time sequence of events in any period is taken to be order, delivery, demand.

We assume no time discounting of costs and postulate an unbounded horizon over which the item is demanded and stocked. We seek to minimize expected total cost per period.

The cost of a replenishment quantity q is assumed linear with fixed ordering cost K and constant unit cost c

$$c(q) = \begin{cases} K + cq & \text{for } q > 0 \\ 0 & \text{for } q = 0 . \end{cases}$$

Items are not lost from inventory, so demand is completely filled. Since costs are not discounted, the constant unit cost c is not a factor in choosing a minimum cost policy, and is suppressed hereafter.

The inventory holding cost is proportional to any stock on hand at unit cost h

$$h(i) = \begin{cases} hi & \text{for } i > 0 \\ 0 & \text{for } i \leq 0 , \end{cases}$$

and the unit penalty cost p is applied to any quantity on backorder at the end of each period

$$p(i) = \begin{cases} 0 & \text{for } i \geq 0 \\ -pi & \text{for } i < 0 . \end{cases}$$

The resulting total cost function, therefore, is linear in K , p , and h and we may scale these parameters so that the value of the unit holding cost h is unity. Non-trivial changes in cost arise only with changes in the ratios K/h and p/h .

1.2 Inventory Replenishment Policies

We postulate that control over replenishment is exercised by an (s,S) policy: whenever inventory x on hand and on order at the start of a period drops below the value "little s," an order is placed for a replenishment of size $S-x$.

Given our assumptions, when the demand distribution and the economic parameters are known, there is an optimal policy that has the (s,S) form [Iglehart (1963a,b), Veinott & Wagner (1965)]. When the demand distribution is not known, even though this is the only assumption relaxed, an optimal policy may no longer be of the (s,S) form. Nevertheless, in this experiment we employ an (s,S) policy, since it is in popular use in the applied situation of incomplete information.

We seek approximations for the operating characteristics of three (s,S) policy rules: optimal policies, Power Approximation policies, and Statistical Power Approximation policies. For each policy rule we develop simplified functions for approximating the operating characteristics. We then fit the parameter values of the functions to the observed characteristics of a multi-item system using least-squares regression.

1.2.1 Optimal Policies

We develop approximate expressions for optimal policy operating characteristics in Sections 2 and 3. The approximations are compared with actual values of the operating characteristics, calculated with the algorithm of Veinott and Wagner (1965).

1.2.2 The Power Approximation

The Power Approximation [Ehrhardt (1976)] is an algorithm for computing approximately optimal values for (s,S) using only the mean μ and variance σ^2 of demand. The algorithm is executed as follows.

Let

(1) $$D_p = (1.463)\mu^{.364}(K/h)^{.498}[(L+1)\sigma^2]^{.0691}$$

and

(2) $$s_1 = (L+1)\,\mu + [(L+1)\mu]^{.416}\,(\sigma^2/\mu)^{.603}\,U(z)$$

$$S_1 = s_1 + D_p\,,$$

where $U(z)$ is given by

(3) $$U(z) = .182/z + 1.142 - 3.466z\,,$$

$$z = \left\{\frac{\mu^{.364}(K/h)^{.498}}{(1+\frac{p}{h})\,[(L+1)\sigma^2]^{.431}}\right\}^{\frac{1}{2}}.$$

If D_p/μ is greater than 1.5, let $s = s_1$ and $S = S_1$. Otherwise compute

(4) $$S_2 = (L+1)\,\mu + v\,[(L+1)\sigma^2]^{\frac{1}{2}}\,,$$

where v is the solution to

(5) $\Phi(v) = p/(p+h)$,

and $\Phi(\cdot)$ is the cumulative distribution function of the unit normal distribution. The policy parameters are then given by

$$s = \text{minimum } \{s_1 , S_2\}$$

(6)

$$S = \text{minimum } \{S_1 , S_2\} .$$

If demands are integer valued, S_1 , D_p , and S_2 are rounded to the nearest integer.

Approximate expressions for Power Approximation operating characteristics are developed in Sections 2 and 3. The approximations are compared with actual values of the characteristics, calculated using the methods of Veinott and Wagner (1965).

1.2.3 The Statistical Power Approximation

Of course, in real applications, the mean and variance of demand are not always known. For the situation where only sample statistics of previous demands are available, the Statistical Power Approximation described below can be implemented. This decision rule implicitly assumes that demand is stationary. Such a rule derived for stationary conditions may be a reasonable approximation to an optimal rule when the demand process is mildly non-stationary, provided that the policy parameters s and S are revised periodically to meet the changing conditions.

We assume in this study that a demand history of fixed length is kept to make each revision, and equal weight is given to each observation. This is not optimal if the demand process is known to be stationary, for then the entire history should be accumulated to give progressively better knowledge and performance. Even when demand is known to be non-stationary, but varying in a regular manner, such as by a trend or periodic cycle, or both, an optimal decision rule would generally utilize the entire history.

The decision-maker usually is not in a position to know, however, *that conditions observed, even over the entire past history, will continue* to prevail. This provides justification for making frequent revisions, placing greater weight on observations from the immediate past and less on earlier history. For this study the admittedly arbitrary choice has been made to keep a history of fixed length and give equal weight to all observations in this history. Let T be the number of periods between policy revisions, which will be termed the revision interval; assume that a history of T periods' demands is kept for use at each revision.

The statistics required by our decision rules are the sample mean and variance of demand, $\bar{\xi}$ and $\bar{v}$, respectively. If t is a period at the beginning of which revision is made, then

$$\bar{\xi} = T^{-1} \Sigma_{\tau=1}^{T} \xi_{t-\tau}$$

$$\bar{v} = (T-1)^{-1} \Sigma_{\tau=1}^{T} (\xi_{t-\tau} - \bar{\xi})^2 .$$

When using the Statistical Power Approximation we periodically obtain values for (s,S) by substituting $\bar{\xi}$ and $\bar{v}$ for μ and σ^2 in equations (1) through (6).

We develop approximate expressions for Statistical Power Approximation operating characteristics in Section 3. The approximations are compared with estimates of actual values of the characteristics, obtained from computer simulation experiments.

1.3 Methods of Approximation

In Section 2 we develop "analytic" approximations for the operating characteristics of optimal and Power Approximation policies. We approximate the exact expressions with simplified functions, generalize these functions, and then fit their parameter values to the observed characteristics of 576 policies using least-squares regression. The resulting approximations tend to be extremely accurate but rather difficult to evaluate.

In Section 3 we develop "multiplicative" approximations for fixed and statistical policies. The approximations are based on simple functional forms that are motivated by empirical observations. The functions are multiplicatively separable into simple expressions that each depend on two variables at most. We use least-squares regression to fit the parameters of these functions to the observed characteristics of 576 fixed policies and 288 statistical policies. The resulting approximations are accurate and easy to evaluate.

In Section 4 we use the approximations to study sensitivity analysis questions that typically confront systems designers. We also use the approximations to draw graphs displaying the operating characteristics as functions of system parameter settings.

2. ANALYTIC APPROXIMATIONS FOR FIXED POLICIES

In this section we develop approximate expressions for the operating characteristics of fixed (specified) policies. We approximate exact analytic expressions with simplified functions, generalize these functions, and then fit their parameter values to the observed characteristics of 576 items using least-squares regression. The 576-item system is formed by using a full-factorial combination of the parameters in Table 2.1.

Table 2.1

System Parameters

Factor	Levels	Number of Levels
Demand distribution	Poisson ($\sigma^2/\mu = 1$) Negative Binomial ($\sigma^2/\mu = 3$) Negative Binomial ($\sigma^2/\mu = 9$)	3
Mean demand	2, 4, 8, 16	4
Replenishment leadtime	0, 2, 4	3
Replenishment setup cost	32, 64	2
Unit penalty cost	4, 9, 24, 99	4
Unit holding cost	1	1
Policy	Optimal policy, Power approximation policy	2

The approximations in this section are functions of only the economic parameters, policy parameters, and the mean and variance of demand. We obtain results for holding cost, backlog cost, backlog protection, replenishment cost, and total cost. On the average, the approximations deviate from the actual values of these characteristics by 0.7%, 4.1%, 0.7%, 0.1%, and 1.9% respectively.

2.1 The Analytic Derivation

Consider the model of Section 1.1 and assume that demand follows a probability density $\phi(\cdot)$ and cumulative distribution $\Phi(\cdot)$. Let $\phi^{*n}(\cdot)$ and $\Phi^{*n}(\cdot)$ be the n-fold convolutions of these functions. We consider the following operating characteristics of fixed, infinite-horizon (s,S) policies

(7)
- $H \equiv$ average holding cost per period
- $B \equiv$ average backlog cost per period
- $P \equiv$ backlog protection, i.e., frequency of periods without backlogs
- $R \equiv$ average replenishment cost per period
- $T \equiv$ average total cost per period.

Let

$$m(\cdot) \equiv \sum_{n=1}^{\infty} \phi^{*n}(\cdot)$$

$$M(\cdot) \equiv \sum_{n=1}^{\infty} \Phi^{*n}(\cdot) .$$

We have, as in Roberts (1962), the exact relationships

$$H = h[1+M(D)]^{-1}\{\int_0^D\int_0^{S-y}(S-y-x)\phi^{*(L+1)}(x)m(y)dxdy + \int_0^S(S-x)\phi^{*(L+1)}(x)dx\}$$

$$B = p[H/h + (L+1)\mu-S] + p[1+M(D)]^{-1}\int_0^D ym(y)dy$$

$$P = [1+M(D)]^{-1}\{\int_0^D\Phi^{*(L+1)}(S-y)m(y)dy + \Phi^{*(L+1)}(S)\} \tag{8}$$

$$R = K[1+M(D)]^{-1}$$

$$T = H + B + R ,$$

where

$$D \equiv S - s .$$

It is difficult to obtain any insights from (8) regarding the sensitivity of the operating characteristics to values of model parameters. Indeed it is exceedingly complicated just to calculate values of the characteristics for a given set of parameter values. We proceed to simplify the form of expressions (8) by introducing approximations for the functions $m(\cdot)$, $M(\cdot)$, and $\phi^{*(L+1)}(\cdot)$.

Replenishment frequency in (8) is given by $[1+M(D)]^{-1}$. To approximate $M(\cdot)$ we use the following result of Smith (1954).

$$M(x) = x/\mu + \sigma^2/(2\mu^2) - 1/2 + o(1) , \quad x\to\infty .$$

This yields the approximate value for replenishment frequency

$$(9) \qquad [1+M(D)]^{-1} \doteq \mu/[D + (\mu+\sigma^2/\mu)/2] \equiv \rho .$$

We identify the quantity (S-y) in (8) as inventory-on-hand-plus-on-order (after ordering), with stationary distribution $F(\cdot)$ given by

$$(10) \qquad F(S-y) = \begin{cases} M(y)/[1+M(D)] , & s \leq S-y<S \\ 1 , & S-y = S . \end{cases}$$

The probability density $f(\cdot)$ of inventory-on-hand plus-on-order (after ordering) on the open interval [s,S) is

$$f(S-y) = m(y)/[1+M(D)] .$$

We approximate $f(\cdot)$ by a constant c on the interval [s,S) . To fix the value of c we normalize the approximated distribution

$$(11) \qquad \int_s^S c\, dx + \Pr\{(S-y) = S\} = 1 \; .$$

Using (10) we find that

$$\Pr\{(S-y) = S\} = [1+M(D)]^{-1} \; ,$$

and we solve (11) for c , yielding

$$(12) \qquad c = \{1 - [1+M(D)]^{-1}\}/D \; .$$

Finally, we use (9) in (12) to obtain

$$(13) \qquad c \equiv m(y)/[1+M(D)] \doteq (1-\rho)/D \; , \; y \in (o,D] \; .$$

We use (9) and (13) in (8) to get

$$H \doteq h\{[(1-\rho)/D] \int_o^D \int_o^{S-y} (S-y-x)\phi^{*(L+1)}(x)dxdy$$

$$+ \rho \int_o^S (S-x)\phi^{*(L+1)}(x)dx\}$$

$$(14) \qquad B \doteq p[H/h + (L+1)\mu - S + (1-\rho)D/2]$$

$$P \doteq \rho \; \Phi^{*(L+1)}(S) - [(1-\rho)/D] \int_o^D \Phi^{*(L+1)}(S-y)dy$$

$$R \doteq \rho K \; .$$

Finally we approximate the demand distribution with a gamma distribution having density $g(\cdot|\alpha,\beta)$. Let

$$(15) \qquad \begin{aligned} \phi^{*(L+1)}(x) &\doteq g(x|\alpha,\beta) \equiv \begin{cases} x^{\alpha-1}\exp(-x/\beta)/[\Gamma(\alpha)\beta^{\alpha}] \; , & x \geq 0 \\ 0 \; , & x < 0 \end{cases} \\ \Phi^{*(L+1)}(x) &\doteq G(x|\alpha,\beta) \equiv \int_{-\infty}^{x} g(y|\alpha,\beta)dy \; , \end{aligned}$$

where

$$\alpha \equiv (L+1)\mu^2/\sigma^2$$

$$\beta \equiv \sigma^2/\mu \; .$$

We define the notation

$$\{f(x) \,\Big|_a^b \equiv f(b) - f(a)$$

and use (15) in (14) to yield

$$H \doteq \rho h[SG(S|\alpha,\beta) - \alpha\beta G(S|\alpha+1,\beta)]$$

$$+ [h(1-\rho)/2D]\,\{x^2 G(x|\alpha,\beta) - 2\alpha\beta x G(x|\alpha+1,\beta)$$

$$+ (\alpha+1)\alpha\beta^2\, G(x|\alpha+2,\beta)\,\Big|_s^S$$

(16)

$$B \doteq p[H/h + (L+1) - S + (1-\rho)D/2]$$

$$P \doteq G(S|\alpha,\beta) - [(1-\rho)/D]\{xG(x|\alpha,\beta)$$

$$- \alpha\beta G(x|\alpha+1,\beta)\,\Big|_s^S$$

$$R \doteq \rho K \; .$$

Observe that the approximations (16) depend on the policy parameters, economic parameters, and the mean and variance of demand. The G-functions must be calculated by a numerical procedure. We use a series expansion for $G(x|\alpha,\beta)$ when x is less than 1 or $\alpha\beta$, and a continued-fraction expansion otherwise. The procedure is part of a package of computer programs entitled "The IMSL Library" which is marketed by the International Mathematical and Statistical Libraries, Inc., Houston, Texas.

Despite the effort required to compute G , the expressions in (16) are an enormous computational simplification over (8). In Section 2.8 we discuss the possibility of using a Normal distribution function in lieu of the G-function; employing the Normal distribution would facilitate manual computations of the approximations we derive below.

In the following sections we use expressions (16) to develop regression models for the operating characteristics. We fit the parameters of the regression models to the observed characteristics of 576 items (Table 2.1). The approximations we obtain are labeled with subscript "a" when they are used for the entire 576-item system. Subscripts "a,p" and "a,o" are used to label expressions for Power Approximation, and optimal policies, respectively.

2.2 An Approximation for Replenishment Cost

We use (9) in (16) to obtain the expression for replenishment cost

$$R \doteq \mu K/[D + (\mu+\sigma^2/\mu)/2] \ .$$

We manipulate the expression to form a linear regression model

$$(\mu K/R) = A_0 + A_1 D + A_2 \mu + A_3 (\sigma^2/\mu) + \varepsilon \ ,$$

where $A_0,\ldots,A_3$ are constants to be fit and ε is the error term. We use least-squares regression to fit the model to the system of 576 inventory policies in Table 2.1. That is, for each of these policies we use as data the actual values of $\mu K/R$, D , μ , and σ^2/μ . The result is the following numerical approximation for R

(17) $$R_a = K\mu/(1.003D+.4942\mu+.4990\sigma^2/\mu-.5339) ,$$

which has a coefficient of determination (fraction of variance explained) of 0.9999 for the quantity $\mu K/R$.

To characterize the quality of the approximation we define

(18) $$\Delta(R_a) \equiv 100\% \times |R_a-R|/R .$$

In other words, $\Delta(R_a)$ is a percentage measure of the approximation error for a particular item. The system of 576 items yields an average value for $\Delta(R_a)$ of 0.1%. Table 2.2 gives the distribution of values for $\Delta(R_a)$. The approximation is clearly an accurate one for the system.

Table 2.2

Frequencies of $\Delta(R_a)$ in a 576-item System

Range for $\Delta(R_a)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	574	100^-%
[2%,4%)	2	100%

The largest value of $\Delta(R_a)$ is 2.5% and it is attained by two optimal policies: each has a variance-to mean ratio of 1, leadtime 0, mean demand 16 and setup cost ratio 32. The unit penalty cost ratios are 24 and 99, yielding the optimal policies (16,39) and (20,43), respectively.

Since the coefficients of D, μ, and σ^2/μ in (17) are so close to the theoretical values of 1, 1/2, and 1/2, respectively, we consider the following regression model.

$$[\mu K/R - D - 1/2(\mu + \sigma^2/\mu)] = A + \varepsilon,$$

where A is a single constant to be fit and ε is the error term. Using least-squares regression to fit the model to our 576-item system we obtain the approximation

$$R_a' = K\mu/[D + (\mu + \sigma^2/\mu)/2 - .5121] .$$

Approximation R_a' provides essentially the same accuracy as (17). The system of 576 items yields an average value of $\Delta(R_a')$ of 0.1% and the distribution of values for $\Delta(R_a')$ is the same as that given in Table 2.2 for $\Delta(R_a)$.

2.3 An Approximation for Holding Cost

We can treat the unit holding cost as a redundant (normalizing) parameter in our model, and so we divide the holding cost expression in (16) by h yielding

$$H/h \doteq \rho[SG(S|\alpha,\beta) - \alpha\beta\, G(S|\alpha+1,\beta)]$$

$$+ [(1-\rho)/2D]\{x^2G(x|\alpha,\beta) - 2\alpha\beta xG(x|\alpha+1,\beta)$$

$$+ (\alpha+1)\alpha\beta^2G(x|\alpha+2,\beta)\ |_s^S\ .$$

We take advantage of our improved estimate of replenishment frequency from (17) and replace ρ with

$$(19) \qquad r \equiv R_a/K=\mu/(1.003D+.4942\mu+.4990\sigma^2/\mu-.5339)\ .$$

The result is a quantity that we denote as W, given by

$$(20) \qquad W \equiv r[SG(S|\alpha,\beta) - \alpha\beta G(S|\alpha+1,\beta)]$$

$$+ [(1-r)/2D]\{x^2G(x|\alpha,\beta) - 2\alpha\beta xG(x|\alpha+1,\beta)$$

$$+ (\alpha+1)\alpha\beta^2\ G(x|\alpha+2,\beta)\ |_s^S\ .$$

We calculated values of W in the 576-item system and in comparing them with the actual values of H/h, we found a systematic variation with respect to μ and σ^2/μ. This motivates the linear regression model

$$H/h = A_0 + A_1\ W + A_2\ \mu + A_3\ (\sigma^2/\mu) + \varepsilon\ ,$$

where $A_0,\ldots,A_3$ are constants to be fit and ε is the error term. As in Section 2.2, we use least-squares regression to fit the model to the system of 576 items. The result is a coefficient of determination of 0.9999 for the approximation

(21) $$H_a = h(W - .1512\,\mu + .1684\sigma^2/\mu + .0689)\ .$$

We define $\Delta(H_a)$ as in (18). The system of 576 items yields an average value of 0.7% for $\Delta(H_a)$. The distribution of values for $\Delta(H_a)$ is summarized in Table 2.3. Our approximation appears to be an excellent one.

Table 2.3

Frequencies of $\Delta(H_a)$ in a 576-item System

Range for $\Delta(H_a)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	553	96%
[2%,4%)	17	99%
[4%,6%)	5	100^-%
[6%,8%)	0	100^-%
[8%,10%)	1	100%

The largest value of $\Delta(H_a)$ is 9.2%, attained by an optimal policy with variance-to-mean ratio 9, leadtime 0, mean demand 2, unit penalty cost ratio 4, and setup cost ratio 32. In general, the largest errors appear to occur for high values of variance-to-mean ratio and low values of all other parameters.

2.4 An Approximation for Backlog Protection

Backlog protection is the frequency of periods without backlogs, that is, one minus the backlog frequency. Since it is a critical measure of service, it is of central interest to the inventory systems designer.

In Section 2.1 we obtained the following approximation for backlog protection

$$P \doteq G(S|\alpha,\beta) - [(1-\rho)/D] \{SG(S|\alpha,\beta) - sG(s|\alpha,\beta) - \alpha\beta G(S|\alpha+1,\beta) + \alpha\beta G(s|\alpha+1,\beta)\} .$$

We replace ρ with r, as given by (19), to obtain

$$P \doteq V \equiv rG(S|\alpha,\beta) - [(1-r)/D] \{SG(S|\alpha,\beta) - sG(s|\alpha,\beta) - \alpha\beta G(S|\alpha+1,\beta) + \alpha\beta G(s|\alpha+1,\beta)\} .$$

We form a linear regression model by adding terms for mean demand and the variance-to-mean ratio of demand, as we did in our approximation for holding cost

(22) $$P = A_0 + A_1 V + A_2 \mu + A_3 (\sigma^2/\mu) + \varepsilon .$$

This model yields a coefficient of determination of only 0.669 when fit to the 576-item system. We also fit (22) separately for Power Approximation and optimal policies, yielding coefficients of variation of 0.660 and 0.681, respectively. These poor results forced us to seek an alternative approach.

We can revise the regression model by using a theoretical result. When demand is continuously distributed, an optimal policy yields $(p/h) / (1+p/h)$ for backlog protection. When the demand distributions are discrete, $(p/h) / (1+p/h)$ is a lower bound on P for optimal policies. It was observed in Ehrhardt (1976) that the Power Approximation and optimal policies differed in their backlog frequency performance. Therefore we decided to fit the two policy rules separately.

We develop the revised linear regression model by multiplying (22) by (1+p/h) to yield

$$(1+p/h)\ P = A_0 + A_1\ (p/h) + A_2\ V + A_3\ (Vp/h) + A_4\ \mu$$

$$+ A_5\ (\mu p/h) + A_6\ (\sigma^2/\mu) + A_7\ [(\sigma^2/\mu)(p/h)] + \varepsilon\ .$$

This model dramatically improves the fit after only one step of a stepwise-regression computer program. For optimal policies, the simple expression

(23) $$P_{a,o} = (0.0857 + p/h)\ /\ (1+p/h)$$

yields a coefficient of determination of 0.99999 for $(1+p/h)P$. We have the same coefficient of determination for Power Approximation policies with

(24) $$P_{a,p} = (0.0695 + p/h)\ /\ (1+p/h)\ .$$

We define $\Delta(P_a)$ as the absolute value of the percentage error of approximation when (23) and (24) are used. Table 2.4 is a summary of the distribution of $\Delta(P_a)$ for the 576-item system. The average value of $\Delta(P_a)$ in the system is 0.7%. All items with $\Delta(P_a)$ larger than 4% have Power Approximation policies with p/h equal to 4. The approximation (23) is somewhat better than that in (24) as 40 of the items with deviations larger than 2% are for the Power Approximation. All the items with $\Delta(P_a)$ larger than 4% have Power Approximation policies with p/h equal to 4. Of those items with deviations larger than 2%, 7 have p/h equal to 9 and 41 have p/h equal to 4. Our approximations are especially accurate for large p/h .

Table 2.4

Frequencies of $\Delta(P_a)$ in a 576-item System

Range for $\Delta(P_a)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	528	92%
[2%,4%)	39	98%
[4%,6%)	8	100^{-}%
[6%,8%)	1	100%

The errors of approximations (23) and (24) display a mild dependence upon the parameter K/h. We therefore seek an approximation of the form

$$P = (a_0 + a_1 h/K + p/h)/(1 + p/h) ,$$

which suggests the regression model

$$Y \equiv [P - p/(h + p)] = A_0 + A_1(h/K) + \varepsilon .$$

The model is fit to minimize <u>relative</u> <u>squared</u> <u>error</u> by first dividing by the left hand side, yielding

$$1 = A_0(1/Y) + A_1(h/KY) + \varepsilon/Y .$$

The transformed model is then fit using least-squares regression; i.e., the dependent variable is set equal to one for each item and the intercept is fixed at zero.

We obtain the following approximations for optimal and Power Approximation policies, respectively

$$P'_{a,0} = (.0543 + .9958\ h/K + p/h)/(1 + p/h)$$

$$P'_{a,p} = (-.0737 + 5.851\ h/K + p/h)/(1 + p/h)\ .$$

We define $\Delta(P'_a)$ as the absoulte value of the percentage error of approximation. The average value of $\Delta(P'_a)$ in the 576-item system is 0.6%. The distribution of $\Delta(P'_a)$ is summarized in Table 2.4a. Although approximation P'_a does not appear significantly better than P_a on the average, it reduces

Table 2.4a

Frequencies of $\Delta(P'_a)$ in a 576-item System

Range for (P_a)	Number of Items	Cumulative Percentage of Items
[0%, 2%)	529	92%
[2%, 4%)	42	99%
[4%, 6%)	5	100%
[6%, 8%)	0	100%

the number of items with errors in excess of 4% from 9 to 5. We recommend approximation P'_a for Power Approximation items with small values of $p/h (\le 9)$.

2.5 An Approximation for Total Cost

We obtain an expression for total cost by summing cost components H, B, and R, and using approximations (16) for B and R

$$T = H + B + R$$

$$\doteq (1 + p/h)\ H + p[(L+1)\mu - S + (1-\rho)\ D/2] + \rho K\ .$$

We divide by h , replace ρ with r , as given by (19), and use approximation (20) for H to obtain

$$(25) \qquad T/h \doteq (1+p/h)W + p/h\,[(L+1)\mu - S + (1-r)D/2] + rK/h\ .$$

As we discovered in obtaining a fit for holding cost, a group of related terms should be added to (25) to obtain a good fit to the system's data. The linear regression model we employed is

$$
\begin{aligned}
T/h = {} & A_0 + A_1 W + A_2 (Wp/h) + A_3 [(L+1)\mu p/h] + A_4 (Sp/h) \\
& + A_5 (Dp/h) + A_6 (rDp/h) + A_7 (rK/h) + A_8 (p/h) \\
& + A_9 (rp/h) + A_{10} [(L+1)\mu] + A_{11} S + A_{12} D + A_{13} (Dr) \\
& + A_{14} r + A_{15}\mu + A_{16} (\sigma^2/\mu) + A_{17} (\mu p/h) \\
& + A_{18}[(\sigma^2/\mu)(p/h)] + \varepsilon .
\end{aligned}
$$

We fit the model to the system of 576 items using stepwise least-squares regression. The following expression yields a coefficient of determination of 0.998

(26) $$
\begin{aligned}
T_a = {} & 1.110\ hW - .001049\ pW + .3364\ Kr \\
& - .2234\ h + .3274\ hD + .4476\ h\ \sigma^2/\mu + .003062\ p\ \sigma^2/\mu .
\end{aligned}
$$

We define $\Delta(T_a)$ as in (18), namely, the absolute value of the percentage difference between T_a and T. The average value of $\Delta(T_a)$ for the 576-item system is 1.9%. A summary of the distribution of $\Delta(T_a)$ for the system is given in Table 2.5. The fit is rather good with virtually all of the large errors occurring for large values of σ^2/μ, and small values of L, μ, and p/h. Of the 63 items having $\Delta(T_a)$ greater than 4%, 52 are for the Power Approximation. We list the four items with $\Delta(T_a)$ exceeding 10% in Table 2.6.

Table 2.5

Frequencies of $\Delta(T_a)$ in a 576-item System

Range for $\Delta(T_a)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	383	66%
[2%,4%)	130	89%
[4%,6%)	45	97%
[6%,8%)	13	99%
[8%,10%)	1	99%
[10%,12%)	2	100^-%
[12%,14%)	0	100^-%
[14%,16%)	1	100^-%
[16%,18%)	0	100^-%
[18%,20%)	1	100%

Table 2.6

Items with $\Delta(T_a)$ exceeding 10%

Policy	σ^2/μ	L	μ	p/h	K/h	Error
Power Approximation	9	0	2	4	64	10.3%
Power Approximation	9	0	2	9	64	11.5%
Power Approximation	9	0	2	9	32	14.6%
Power Approximation	9	0	2	4	32	18.2%

The parameter settings contributing most often to large errors in (26) are Power Approximation items with $\mu = 2$ and $p/h = 4$. We deleted all 18 items having their combination of parameter values and refit (26) to obtain

$$T'_a = 1.111\ hw - .00105\ pw + .3370\ Kr$$

$$- .2190\ h + .3269\ hD + .4775\ h\sigma^2/\mu + .00306\ p\sigma^2/\mu\ ,$$

which has an average value of $\Delta(T'_a)$ of 1.8% in the 558-item system. A summary of the distribution of $\Delta(T'_a)$ is given in Table 2.6a.

Table 2.6a

Frequencies of $\Delta(T'_a)$ in a 558-item System

Range for $\Delta(T'_a)$	Number of Items	Cumulative Percentage of Items
[0%, 2%)	377	68%
[2%, 4%)	126	90%
[4%, 6%)	42	98%
[6%, 8%)	11	100%
[8%, 10%)	0	100%
[10%, 12%)	1	100%
[12%, 14%)	0	100%
[14%, 16%)	1	100%

Although the approximation appears to be accurate, we believe that it will degrade for significantly non-optimal policies. The differences between (25) and (26) and the pattern of values for $\Delta(T_a)$ suggest that the economics of optimal policies is intrinsic to the approximation obtained. The robustness of (26) is discussed explicitly in Section 2.7.

2.6 An Approximation for Backlog Cost

In Section 2.1 we obtained the approximation for backlog cost

$$B \doteq p[H/h + (L+1)\mu - S + (1-\rho)D/2] .$$

Using the approach of Section 2.3, we obtain the linear regression model

$$B/h = A_0 + A_1 (Wp/h) + A_2 [(L+1)\mu p/h] + A_3 (Sp/h)$$
$$+ A_4 (\mu p/h) + A_5 [(\sigma^2/\mu)(p/h)] + A_6 (Dp/h)$$
$$+ A_7 (rDp/h) + \varepsilon ,$$

where r is given by (19), and ε is the error term. When fit to the system of 576 items the result is a low coefficient of determination of 0.444. Splitting the 576-item system into a Power Approximation system and an optimal policy system, and then fitting separately for each value of p/h, does not yield a significant improvement. The two systems also were fit separately for each value of σ^2/μ without success.

We obtain a better fit by employing the identity

$$B = T - H - R .$$

Our first attempt used (17), (21), and (26) in

$$B' \equiv T_a - H_a - R_a .$$

Unfortunately, B' deviates from B by an average of 18%.

We form, instead, a linear regression model for B using all the variables appearing in our models for T, H, and R

$$B/h = A_0 + A_1 W + A_2 (Wp/h) + A_3 [(L+1)\mu p/h] + A_4 (Sp/h) + A_5 (Dp/h) + A_6 (rDp/h) + A_7 (rK/h) + A_8 (p/h) + A_9 (rp/h) + A_{10} [(L+1)\mu] + A_{11} S + A_{12} D + A_{13} (Dr) + A_{14} r + A_{15} \mu + A_{16} (\sigma^2/\mu) + A_{17} (\mu p/h) + A_{18} [(p/h)(\sigma^2/\mu)] + \varepsilon . \quad (27)$$

When fit to the 576-item system, (27) yields a coefficient of determination of 0.957. To obtain a reasonably accurate approximation, we further split the data set into Power Approximation and optimal policies, and fit (27) separately for each value of p/h .

We obtain average coefficients of determination of 0.999, and 0.998 for optimal, and Power Approximation policies, respectively. The fits are of the form

$$B_a/h = a_1 W + a_2 (L+1)\mu + a_3 S + a_4 D + a_5 Dr + a_6 r + a_7 Kr + a_8 \mu + a_9 \sigma^2/\mu + a_{10} . \quad (28)$$

We list the fitted values of $\{a_i, i=1,10\}$ in Table 2.7a for each of the eight regressions.

Table 2.7a

Coefficient Values for Backlog Cost Expression (28)

Policy	p/h	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
Optimal	4	3.231	3.212	-3.210	1.680	-1.809	.1968	-.1658	.2600	-.0127	-.3447
	9	7.452	7.473	-7.470	3.854	-4.046	.4851	.2325	.3638	-.0437	-.4972
	24	17.39	17.37	-17.37	8.788	-9.026	1.034	-.2005	.3445	.0336	-.7617
	99	62.86	62.85	-62.84	31.56	-31.74	.9851	-.2572	.3412	.0927	-1.084
Power Approx.	4	4.248	4.276	-4.278	2.141	-1.740	-1.272	-.0758	-.1326	-.1099	-.0900
	9	8.214	8.192	-8.194	4.152	-3.813	-.3054	-.1565	-.1044	-.0635	-.4659
	24	19.18	19.14	-19.14	9.635	-9.228	-.3749	-.1844	-.1606	-.0059	-.6567
	99	70.16	70.18	-70.18	35.22	-34.55	1.428	-.2483	-.3159	.1841	-1.327

As the high coefficients of determination indicate, the fits are excellent in terms of absolute errors of approximation. To illustrate, the average absolute error of approximation for 288 optimal policies is 0.17 as compared with the corresponding average backlog cost 5.69. Nevertheless, because many of the backlog costs are numerically small values and the magnitude of the approximation errors does not seem to depend strongly on the magnitude of the backlog cost, the approximation displays some large *relative* errors. We define $\Delta(B_a)$ as in (18), the absolute value of the percentage difference between B_a and B for a given item. The average value of $\Delta(B_a)$ in the 576-item system is 4.1%; more specifically, the average is 4.3% for optimal policies and 3.9% for Power Approximation policies. We summarize the distribution of $\Delta(B_a)$

in Table 2.8. The largest values of $\Delta(B_a)$ occur when σ^2/μ equals 1 and p/h equals 99. All 16 items having errors in excess of 20% have σ^2/μ equal to 1, and 13 of them have p/h equal to 99. When we consider only those items with σ^2/μ greater than 1, the average value of $\Delta(B_a)$ drops to 2.5%, but there are still 11 out of 384 items with deviations exceeding 10%. When we consider only those items with p/h less than 99, the average value of $\Delta(B_a)$ drops to 2.4%.

To improve on the approximation when p/h equals 99, we exhibit in Table 2.7b an approximation for this value of the unit penalty cost ratio that also depends on σ^2/μ. The average value of $\Delta(B_a)$ for this approximation is 4.2%, with 16 of the 144 items having deviations in excess of 10%.

We attempted to improve the approximation by fitting to minimize <u>relative squared error</u> as in Section 2.4 (p.19). The resulting approximation was less accurate than those reported above.

Since this approximation is somewhat less accurate than the others we have derived, we note that we have a good alternate measure of backlog performance in our approximation for P, (23) and (24).

Table 2.7b

Coefficient Values for Backlog Cost Expression (28) with p/h = 99

Policy	σ^2/μ	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
Optimal	1	71.44	71.37	-71.38	35.76	-35.99	2.246	-.1238	.2124	-.4770	-1.084
	3	83.12	83.15	-83.14	41.63	-41.96	3.309	-.0362	.2961	-.1453	-1.084
	9	75.13	75.14	-75.13	37.62	-37.50	1.764	-.1130	.0516	.0657	-1.084
Power Approx.	1	59.44	59.42	-59.41	29.71	-29.23	1.090	-.0428	-.3647	.4862	-1.327
	3	80.31	80.34	-80.33	40.21	-40.02	2.363	-.0853	-.0832	.0467	-1.327
	9	86.35	86.34	-86.35	43.20	-42.69	3.326	-.0577	-.2866	-.0224	-1.327

Table 2.8

Frequencies of $\Delta(B_a)$ in a 576-item System

Range for $\Delta(B_a)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	275	48%
[2%,4%)	147	73%
[4%,6%)	51	82%
[6%,8%)	27	87%
[8%,10%)	18	90%
[10%,20%)	42	97%
[20%,30%)	8	99%
[30%,40%)	6	100^-%
[40%,50%)	1	100^-%
[50%,60%)	0	100^-%
[60%,70%)	0	100^-%
[70%,80%)	1	100%

2.7 Measures of Robustness

We summarize our results by listing the accurate approximations derived in this section.

$$H_a = hW - .1512\ h\mu + .1684\ h\sigma^2/\mu + .0689\ h$$

$$P_{a,o} = (0.0857 + p/h)/(1+p/h)$$

$$P_{a,p} = (0.0695 + p/h)/(1+p/h)$$

$$R_a = K\mu/(1.003D + .4942\mu + .4990\ \sigma^2/\mu - .5339)$$

$$B_a = h[a_1\ W + a_2\ (L+1)\mu + a_3\ S + a_4\ D + a_5\ Dr + a_6\ r + a_7\ Kr + a_8\ \mu + a_9\ \sigma^2/\mu + a_{10}]$$

$$T_a = h(1.110\ W + .3274\ D + .4476\ \sigma^2/\mu - .2234)$$
$$+ p(-.001049W + .003062\ \sigma^2/\mu) + .3364\ rK\ ,$$

where

$$r = R_a / K$$

$$W = r[SG(S|\alpha,\beta) - \alpha\beta G(S|\alpha+1,\beta)]$$
$$+ [(1-r)/2D]\ \{x^2 G(x|\alpha,\beta) - 2\alpha\beta x G(x|\alpha+1,\beta)$$
$$+ (\alpha+1)\alpha\beta^2 G(x|\alpha+2,\beta)\ |_s^S$$

$$G(x|\alpha,\beta) = \begin{cases} [\Gamma(\alpha)\beta^\alpha]^{-1} \int_o^x y^{\alpha-1} \exp(-y/\beta)dy\ , & x > o \\ 0 & , x \leq o \end{cases}$$

$$\alpha = (L+1)\mu^2/\sigma^2$$
$$\beta = \sigma^2/\mu\ ,$$

and the coefficients for B_a are given in Tables 2.7a and 2.7b. Recall that H_a, R_a, and T_a can be applied to both optimal and Power Approximation policies.

In Tables 2.9 and 2.10 we summarize the multi-item system accuracy of our approximations as a function of the parameter settings (see Table 2.1). Each entry is the percentage excess of the sum of approximate values over the sum of actual values for all the items having the given parameter setting. The approximation we use for backlog cost includes the improved fit for p/h equal 99 (see Table 2.7b). Observe that when the approximations are aggregated over several parameter settings that the errors cancel to a large extent so that most of the relative

Table 2.9

Percentage Errors of Analytic Approximations for a 288-item System Controlled with Optimal Policies

OPERATING CHARACTERISTIC	VARIANCE/ MEAN	TOTAL	DISTRIBUTIONS BY VALUES OF INPUT PARAMETERS											
			C(OUT)/C(IN)			C(FIX)/C(IN)		LEADTIME			MEAN			
			4	9	99	32	64	0	2	4	2	4	8	16
Holding Cost	1	0.2	-0.0	0.2	0.3	-0.6	0.8	0.4	0.1	0.1	-0.6	0.4	0.9	-0.1
	3	-0.2	-0.3	-0.2	-0.2	-0.6	0.1	-0.3	-0.2	-0.1	-1.1	-0.3	0.2	-0.1
	9	0.1	0.6	0.1	-0.1	0.1	0.0	0.2	0.1	0.0	0.6	0.1	0.1	-0.2
Backlog Cost	1	-1.2	0.1	1.2	-6.9	-1.9	-0.6	0.1	-1.7	-1.5	-0.4	-1.8	-1.5	-1.0
	3	-1.8	-0.9	-0.5	-4.4	-1.5	-2.1	-1.8	-2.1	-1.5	-0.7	-1.6	-1.8	-2.4
	9	0.1	-0.1	0.2	0.1	0.1	-0.0	-0.1	0.1	0.1	0.0	0.4	0.1	-0.2
Backlog Protection	1	-0.4	-0.4	-0.6	-0.1	-0.5	-0.3	-0.5	-0.5	-0.1	-1.2	-0.5	0.1	0.1
	3	-0.1	-0.3	-0.1	0.1	0.2	-0.0	-0.4	-0.0	0.1	-0.8	-0.2	0.1	0.4
	9	0.3	0.5	0.3	0.1	0.3	0.3	0.1	0.3	0.5	-0.0	0.3	0.4	0.6
Replenishment Cost	1	0.0	-0.3	-0.0	0.4	0.1	0.0	0.3	-0.0	-0.1	0.0	0.0	0.0	0.1
	3	0.0	0.0	0.0	0.0	0.1	-0.0	0.0	0.0	-0.0	0.0	0.0	-0.0	0.0
	9	-0.0	-0.0	-0.0	-0.0	0.0	-0.0	-0.0	-0.0	-0.0	-0.2	0.0	-0.0	0.0
Total Cost	1	-0.6	0.2	0.5	-2.0	-2.1	0.6	0.2	-0.6	-1.1	0.3	0.5	0.5	-2.2
	3	-0.0	1.1	0.6	-1.2	-0.5	0.3	0.6	-0.1	-0.3	0.1	0.2	0.4	-0.4
	9	-0.4	0.0	-0.6	-0.5	-0.9	-0.0	1.0	-0.6	-1.2	0.2	-0.3	-0.3	-0.7

Table 2.10

Percentage Errors of Analytic Approximations for a 288-item System Controlled with Power Approximation Policies

OPERATING CHARACTERISTIC	VARIANCE/ MEAN	TOTAL	DISTRIBUTIONS BY VALUES OF INPUT PARAMETERS											
			C(OUT)/C(IN)			C(FIX)/C(IN)		LEADTIME			MEAN			
			4	9	99	32	64	0	2	4	2	4	8	16
Holding Cost	1	0.1	0.1	0.2	0.1	-0.6	0.7	0.2	0.1	0.1	-0.6	0.3	0.8	-0.2
	3	-0.2	-0.5	-0.2	-0.1	-0.5	0.1	-0.3	-0.2	-0.2	-1.1	-0.3	0.1	-0.1
	9	0.0	0.5	0.1	-0.1	0.1	-0.0	0.1	0.0	0.0	0.5	0.1	0.1	-0.2
Backlog Cost	1	-0.7	0.4	1.5	-5.5	0.9	-2.0	-1.8	0.3	-0.8	-2.5	-2.9	-2.9	3.2
	3	-2.3	-0.4	-1.0	-6.6	-2.7	-2.0	-2.6	-2.4	-2.1	-0.3	-1.7	-3.0	-3.2
	9	-0.3	0.0	0.1	-1.0	-0.8	0.1	-1.3	-0.1	0.2	-0.3	0.4	-0.3	-0.8
Backlog Protection	1	-0.6	-1.3	-0.6	-0.1	-0.9	-0.3	-1.1	-0.7	-0.1	-0.8	-0.1	-0.3	-1.3
	3	0.9	2.3	0.5	0.0	0.4	1.4	0.8	0.7	1.1	0.7	1.2	1.0	0.5
	9	0.0	0.4	-0.4	0.1	-0.7	0.8	0.1	-0.1	0.1	-0.8	0.0	0.5	0.5
Replenishment Cost	1	-0.1	-0.1	-0.1	-0.1	-0.4	0.0	-0.0	-0.2	-0.2	0.0	0.0	0.0	-0.4
	3	0.0	0.0	0.0	0.0	0.1	-0.0	0.0	0.0	0.0	0.0	-0.0	-0.0	0.0
	9	-0.0	-0.0	-0.0	-0.0	0.0	-0.0	-0.0	-0.0	-0.0	-0.2	-0.1	-0.0	0.0
Total Cost	1	-0.3	-1.8	0.3	0.2	-0.1	-0.5	-0.9	0.3	-0.4	-0.3	-1.3	-0.8	0.5
	3	-0.8	-3.6	-0.6	0.8	-0.5	-1.1	0.1	-0.6	-1.7	-0.7	-1.8	-1.4	0.1
	9	0.1	-0.8	0.3	0.4	0.8	-0.5	1.6	-0.0	-0.8	1.4	0.3	-0.7	0.1

errors in these tables are well under 1%. Even for the backlog cost approximation, the relative errors for multi-item aggregates are tolerable.

We test the robustness of the approximations by using them in a multi-item system with the parameter settings of Table 2.11. Note that all the numerical parameters have values not found in the 576-item system.

Table 2.11

A 64-item System with New Parameter Settings

Factor	Levels	Number of Levels
Demand distribution	Negative Binomial ($\sigma^2/\mu = 5$) Negative Binomial ($\sigma^2/\mu = 15$)	2
Mean demand	0.5, 7.0	2
Replenishment leadtime	1, 6	2
Replenishment setup cost	16, 48	2
Unit penalty cost	49, 132	2
Unit holding cost	1	1
Policy	Optimal policy, Power Approximation policy	2

Each parameter has one interpolated value and one extrapolated value. A full factorial combination of the values is used, yielding 64 items. The system is a rather severe test of robustness since only two items have all parameters with values within the ranges used to derive the approximations. There are 10 items with one extrapolated parameter, 20 items with two extrapolated parameters, 20 with three extrapolations, 10 with four extrapolations, and 2 items with all five parameters extrapolated.

We compare actual values of H, P, R, and T for the 64 items with our analytic approximations. Backlog cost B is not considered because of the complexity of our approximation and the absence of an explicit dependence on p/h in (28). The average percent deviations from actual values of H, P, R, and T are 1.6%, 0.2%, 1.4%, and 2.6%, respectively. The distributions of percent deviations are summarized in Table 2.12. Our approximations are quite accurate considering the wide range of parameters spanned by the system.

The holding cost approximation is extremely accurate for all cases with μ greater than 0.5 or σ^2/μ less than 15. All items with deviations greater than 4% have μ equal 0.5 and σ^2/μ equal 15. If we consider only the items with fewer than two parameters extrapolated, the average $\Delta(H_a)$ is 0.4%.

Table 2.12

Percentage Deviations of Analytic Approximations in a 64-item System

Range of Deviation	Holding Cost	Backlog Protection	Replenishment Cost	Total Cost
[0%,2%)	48 (75%)	64 (100%)	48 (75%)	30 (47%)
[2%,4%)	6 (84%)		8 (88%)	22 (81%)
[4%,6%)	5 (92%)		0 (88%)	6 (91%)
[6%,8%)	3 (97%)		6 (97%)	4 (97%)
[8%,10%)	2 (100%)		2 (100%)	1 (98%)
[10%,12%)				1 (100%)

The backlog protection approximation is excellent, with only one item having a deviation in excess of 0.7%.

Our approximation for replenishment cost is also robust. All items with deviations in excess of 4% have μ equal 0.5, σ^2/μ equal 15, and K/h equal 16. Items with fewer than two extrapolated parameters have an average $\Delta(R_a)$ of 0.1%.

Low μ and high σ^2/μ are also sources of large errors for our total cost approximation. All items with deviations in excess of 4% have either μ equal 0.5 or σ^2/μ equal 15, or both. Items with fewer than two extrapolated parameters have an average deviation of 1.2%.

We commented in Section 2.5 that the approximation for total cost may degrade for significantly non-optimal policies. The remark is equally valid for the backlog protection expressions (23) and (24), since they are based on a theoretical result for optimal policies. We now proceed to examine the accuracy of the approximations for non-optimal policies.

Consider the following system of non-optimal policies. We choose a base-case item with σ^2/μ equal 5, μ equal 9, L equal 2, p/h equal 49, and K/h equal 48. The optimal policy for this item is (43,73). We now use this policy on items with different parameter values. The new parameters are obtained by increasing and decreasing each base-case parameter value, one at a time, yielding 10 items; the parameter values of the system are displayed in Table 2.13. For each item we compare the actual and approximate values of H, P, R, and T.

Table 2.13

Percentage Errors of Approximation for Non-optimal Policies

	Percentage Errors of Approximations			
Changed Value	Holding Cost	Backlog Protection	Replenishment Cost	Total Cost
σ^2/μ = 4 (-20%)	.07%	-.6%	-.00%	6.0%
6 (+20%)	-.05%	.7%	.00%	-5.0%
μ = 7 (-22%)	-.11%	-1.3%	-.03%	13.7%
11 (+22%)	-.04%	2.9%	.03%	-22.2%
L = 1 (-50%)	-.04%	-1.6%	.00%	12.6%
3 (+50%)	.02%	5.5%	.00%	-36.2%
p/h = 39 (-20%)	-.01%	-.5%	.00%	3.9%
59 (+20%)	-.01%	.3%	.00%	-2.2%
K/h = 38 (-21%)	-.01%	.0%	.00%	4.0%
58 (+21%)	-.01%	.0%	.00%	-2.2%
Average of Absolute Values	.04%	1.3%	.01%	10.8%

The approximations for holding cost and replenishment cost are very accurate, with average percentage deviations of 0.04% and 0.01%, respectively. The approximation for backlog protection is somewhat less accurate, with the largest errors occurring for large values of leadtime and mean demand. The total cost approximation does not perform well in the system, deviating by an average of 10.8%. Thus we conclude that the approximations for backlog protection and total cost should be used with caution for significantly non-optimal policies. An approach to reducing the errors might be gleaned from the pattern of deviations in Table 2.13. Notice that when each parameter is larger than in the base case, the approximation underestimates the total cost, and when the parameter is smaller than in the base case, the approximation overestimates the total cost. The reverse is true for backlog protection.

2.8 Topics for Future Research

In this final portion of Section 2 we suggest several extensions of our research on analytic approximations.

Notice in the summary of the approximations in Section 2.7 that the G-function is used directly only in the calculation of W. As we mentioned in Section 2.1, the computation of G is complex, even on a digital computer. As an alternative to W for manual computations, we could calculate

$$\begin{aligned} W' = {} & r[SN(S_o) - \alpha\beta N(S_1)] \\ & + [(1-r)/2D]\,[S^2N(S_o) - 2\alpha\beta SN(S_1) + (\alpha+1)\alpha\beta^2N(S_2) \\ & - s^2N(s_o) + 2\alpha\beta sN(s_1) - (\alpha+1)\alpha\beta^2N(s_2)] \end{aligned}$$

where $N(\cdot)$ is the unit Normal distribution function, and

$$S_i = [S - (\alpha+i)] / [(\alpha+i)\beta^2]^{\frac{1}{2}}$$

$$s_i = [s - (\alpha+i)] / [(\alpha+i)\beta^2]^{\frac{1}{2}} .$$

Further research is needed to investigate how much degradation in the approximations occurs if W' is used instead of W. Also, it would be possible to refit the coefficients in the approximations for H, B, and T utilizing W' instead of W.

We now discuss an approach for improving the approximations for P and B. Recall that these fits have some items with large relative errors despite very high coefficients of determination. That is, the regressions were very effective in minimizing absolute squared errors, but several items with small values of P or B were left with large relative errors. Additional research is needed to devise a method to fit these approximations to data such that relative squared error is minimized.

A final research topic involves the potential use of the approximations in forecasting the future performance of inventory systems. The method of forecasting discussed in Ehrhardt (1976) is based on a retrospective simulation of system performance using a sequence of past demands. In actual practice, this method can be very expensive since a time series of demand data must be stored for every item involved in the forecasting. If the approximations of this chapter are substituted for the retrospective simulation, only the sample mean and sample variance of demand must be stored. Additional research is needed to evaluate the accuracy of this method of forecasting.

3. MULTIPLICATIVE APPROXIMATIONS FOR FIXED AND STATISTICAL POLICIES

In Section 2 we derived analytically-motivated approximations for operating characteristics of fixed (s,S) policies. The approximations are accurate but they lack the ease of interpretation and calculation that we desire for sensitivity analysis of systems design issues. Furthermore, they inherently do not treat statistically-derived policies, because they are based on theory that is valid only for fixed policies.

In this chapter we develop approximate expressions for the operating characteristics (7) of both fixed and statistical policies. We postulate a functional form for the expressions from examining empirical observations and derive explicit approximations using least-squares regression. The expressions are particularly attractive for sensitivity analysis because they are products of simple-to-compute functions, each of which depends on, at most, two parameters. Further, they do not require the calculation of the replenishment policy itself.

As in Section 2, we obtain good approximations for holding cost, backlog protection, replenishment cost, and total cost. On the average, the fixed policy approximations deviate from the actual values of these characteristics by 4%, 1%, 2%, and 3%, respectively. The statistical policy expressions have average errors of 4%, 1%, 2%, and 3%, respectively. Using the approach in this chapter, we have not found a good approximation for backlog cost.

3.1 Methodology

Our motivation here derives from the empirical observations found in MacCormick (1974), Estey and Kaufman (1975), and Ehrhardt (1976). In all these instances, multi-item systems were studied with a full factorial setting of parameters, and constant demand variance-to-mean ratio. For each of the systems, the average total cost of an item appears approximately to follow a multiplicatively separable function of the economic and demand parameters. The relationship has been observed when the systems are controlled with optimal policies, approximately optimal policies, and statistically-derived policies.

For an item controlled with an optimal or approximately optimal policy, we postulate that the average total cost per period can be approximated by a function of the form

$$T/h \doteq cf_1(L,\sigma^2/\mu)f_2(\mu,\sigma^2/\mu)f_3(p/h,\sigma^2/\mu)f_4(K/h,\sigma^2/\mu) \ . \tag{29}$$

Expression (29) is consistent with the empirically-observed property of multiplicative separability for a given value of σ^2/μ. We allow σ^2/μ to enter all four functions because we have no reason to expect separability in this parameter. For subsequent analysis, we limit consideration to functions of the form

$$f_i(x,\sigma^2/\mu) = x^{\gamma_i(x,\sigma^2/\mu)} \exp[\delta_i(x,\sigma^2/\mu)], \ i = 1,\ldots,4 \ , \tag{30}$$

where $\gamma_i(x,\sigma^2/\mu)$ and $\delta_i(x,\sigma^2/\mu)$ are linear combinations of the variables

$$\begin{gathered} 1,\ x,\ 1/x,\ \sigma^2/\mu,\ \mu/\sigma^2 \ , \\ x^2,\ 1/x^2,\ (\sigma^2/\mu)^2,\ (\mu/\sigma^2)^2 \ , \\ x\sigma^2/\mu,\ x\mu/\sigma^2,\ \sigma^2/\mu x,\ \mu/\sigma^2 x \ . \end{gathered} \tag{31}$$

As in Section 2, we transform the characteristics (7) to H/h, B/h, P(1+p/h), R/h, and T/h . We seek approximations for all five quantities using linear least-squares regression on models of the form (29), (30), (31). Given the multiplicative form in (29), we make a logarithmic transformation before applying least-squares computations; hence, in reporting below the resulting coefficients of determination, we always shall be referring to the proportion of variance explained for the logarithmically transformed variables. The 576-item system of Table 2.1 provides the data source for separate fits for the characteristics of optimal and Power Approximation policies.

For the statistical policy characteristics (see Section 1), we use the 72-item systems of Table 3.1. We simulated the system with σ^2/μ equal 3 using a 26-period revision interval. For the system with σ^2/μ equal 9, we utilized revision intervals of 13, 26, and 52 periods. We

Table 3.1

72-item System Parameters

Factor	Levels	Number of levels in each system
Demand distribution	Negative Binomial ($\sigma^2/\mu=9$) or Negative Binomial ($\sigma^2/\mu=3$)	1
Mean demand	2, 4, 8, 16	4
Unit holding cost	1	1
Unit backlog penalty cost	4, 9, 99	3
Replenishment setup cost	32, 64	2
Replenishment leadtime	0, 2, 4	3

seek approximations for statistical policy operating characteristics by performing separate fits for each of the 72-item systems. We use regression models of the form (29), (30) with γ_i and δ_i being linear combinations of the variables

$$(32) \qquad 1,\ x,\ 1/x,\ x^2,\ 1/x^2\ .$$

For several operating characteristics we seek an explicit functional relationship with revision-interval length T. These operating characteristics are fit in the σ^2/μ equal 9 system using γ_i and δ_i that are linear combinations of all the variables

$$(33) \qquad 1,\ x,\ 1/x,\ T,\ 1/T,\ x^2,\ 1/x^2\ .$$

As we shall observe, several fits for individual γ_i and δ_i do not require the presence of all these variables.

3.2 Approximations for Fixed Policies

In this section we describe approximations for fixed policy operating characteristics. Since optimal and Power Approximation policies are fit separately, we adopt the convention that a subscript "o" labels an optimal policy approximation, and a subscript "p" labels a Power Approximation quantity. In some instances we use summaries that combine results for optimal and Power Approximation systems. We use the subscript "f" to label combined fixed-policy results.

3.2.1 Average Holding Cost per Period

The following approximation for optimal holding cost yields a coefficient of determination of 0.992 for the 288 data points

$$H_o = 2.022\ h\ \exp(.09835\sigma^2/\mu)\mu^{.4788}(L+1)^{.2509+.01470\sigma^2/\mu-.1179\mu/\sigma^2}$$

$$(p/h)^{-2.424h/p-.04409\mu/\sigma^2+.01552\sigma^2/\mu}$$

$$(K/h)^{.3660-.02692\sigma^2/\mu}\ .$$

On the average, H_o is within 4.2% of H for the system.

We obtain the following approximation for Power Approximation holding cost

$$H_p = 2.598\ h\ \exp(.1204\sigma^2/\mu)\mu^{.5076-.004964\sigma^2/\mu}$$

$$(L+1)^{.2287+.01472\sigma^2/\mu-.0900\mu/\sigma^2}$$

$$(p/h)^{-3.040h/p+.01137\sigma^2/\mu-.07292\mu/\sigma^2+.03355(\sigma^2h/p\mu)}$$

$$(K/h)^{.3057-.02667\sigma^2/\mu+.03058\mu/\sigma^2}\ ,$$

which yields a coefficient of determination of 0.994 and is within 3.9% of H, on the average.

The distribution of errors for the optimal and Power Approximation fits are so alike that we summarize them as one system. Define $\Delta(H_f)$ as in (18), the absolute percentage difference between H_o or H_p and H for an item. Table 3.2 is a summary of the distribution of $\Delta(H_f)$ in the system. The largest values of $\Delta(H_f)$ occur consistently for a variance-to-mean ratio 9, leadtime of 0, unit setup cost ratio of 32, and penalty cost ratios of 4 and 9. This is a pattern we noted also in Section 2.3 for the analytic approximation of holding cost.

We attempted to improve the approximation by refitting separately for each value of leadtime L. The resulting approximations were not significantly better than H_0 and H_p.

Table 3.2

Frequencies of $\Delta(H_f)$ in a 576-item System

Range for $\Delta(H_f)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	164	28%
[2%,4%)	181	60%
[4%,6%)	101	77%
[6%,8%)	57	87%
[8%,10%)	39	94%
[10%,12%)	20	98%
[12%,14%)	7	99%
[14%,16%)	3	99%
[16%,18%)	2	100^-%
[18%,20%)	0	100^-%
[20%,22%)	1	100^-%
[22%,24%)	0	100^-%
[24%,26%)	0	100^-%
[26%,28%)	0	100^-%
[28%,30%)	1	100%

3.2.2 Average Backlog Cost per Period

We have not found a very good multiplicative fit for backlog cost. The result is not surprising since a multiplicative form is not observed empirically for this characteristic.

In fitting a multiplicative model of the form (29), (30), (31) to backlog cost, we obtained coefficients of determination of 0.976 and 0.952, respectively, for optimal and Power Approximation policies. An improvement resulted when we ran the regressions separately for each value of p/h. We obtained average coefficients of determination of 0.988 and 0.981 for optimal and Power Approximation policies with a function of the following form

$$B_f/h = a(\sigma^2/\mu)^{b_1 - b_2\sigma^2/\mu}\, \mu^{c_1 + c_2\sigma^2/\mu + c_3\mu/\sigma^2}\, (L+1)^{d_1 + d_2\mu/\sigma^2}\, (K/h)^{e_1\sigma^2/\mu + e_2\mu/\sigma^2} .$$

Values of the coefficients $a, b_1, \ldots, e_2$ are listed in Table 3.3 for each policy and each value of p/h .

Table 3.3

Values of Coefficients in B_f

Policy	p/h	a	b_1	b_2	c_1	c_2	c_3	d_1	d_2	e_1	e_2
Optimal	4	.3398	1.003	-.0100	.5240	-.0119	.0056	.4239	-.1792	-.0008	.2891
	9	.5800	.7796	.0034	.4909	-.0189	-.0321	.2999	.0528	.0034	.1063
	24	.2297	1.706	-.0350	.3310	-.0104	.1854	.2318	.1799	-.0033	.2646
	99	.3222	1.580	-.0303	.2786	-.0104	.1669	.1739	.2939	-.0020	.1378
Power Approx.	4	.3704	1.167	-.0806	.3775	.0070	-.0296	.3174	-.0074	.0280	.2748
	9	.3316	1.226	-.0596	.3711	.0043	.0168	.3870	-.0018	.0162	.2557
	24	.2539	1.520	-.0456	.3759	-.0111	.1135	.3237	.1619	.0077	.2296
	99	.1532	2.235	-.0875	.2144	-.0258	.4537	-.0614	.7401	.0240	.1297

On the average, approximation B_f deviates from actual values of B by 6.2%. In Table 3.4 we summarize the distribution of percentage deviations in the multi-item system. We see that the approximation is not accurate, with 18% of the deviations exceeding 10%, and 5% of the deviations larger than 20%. The pattern of errors is similar to that

Table 3.4

Frequencies of $\Delta(B_f)$ in a 576-item System

Range for $\Delta(B_f)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	164	28%
[2%,4%)	116	49%
[4%,6%)	85	63%
[6%,8%)	53	73%
[8%,10%)	52	82%
[10%,20%)	78	95%
[20%,30%)	22	99%
[30%,40%)	5	100^-%
[40%,50%)	0	100^-%
[50%,60%)	1	100%

noted in Section 2.6 for the analytic approximation for backlog cost. The largest errors occur for items having high p/h and low σ^2/μ . We tried improving the approximation by fitting separately for each combination of values of p/h and σ^2/μ , but the results did not improve significantly.

We improve the approximation by fitting to minimize relative squared error as in Section 2.4 (p.19). We obtain a separate fit of the following form for each combination of values of p/h and σ^2/μ

$$B_f/h = a\mu^b (L+1)^c \ (K/h)^d \ .$$

The results did not improve for items with $\sigma^2/\mu = 1$ but the average percentage errors dropped to 5.7% and 3.3%, for items with σ^2/μ equal to 3 and 9, respectively. Values of the coefficients a,b,c and d for σ^2/μ equal to 3 and 9 are listed in Table 3.4a and the distribution of percentage errors is summarized in Table 3.4b.

Table 3.4a

Values of Coefficients in B_f'

		$\sigma^2/\mu = 3$				$\sigma^2/\mu = 9$			
Policy	p/h	a	b	c	d	a	b	c	d
Optimal	4	.7164	.5123	.3958	.1614	3.029	.4356	.4267	-.0369
	9	1.747	.4236	.3115	-.0175	3.550	.3113	.3072	.0345
	24	1.455	.3525	.3001	.0563	4.870	.2512	.2355	.0067
	99	2.031	.2851	.2660	-.0079	5.643	.1975	.1986	.0035
Power	4	.8282	.3927	.2867	.2349	.9887	.4456	.3352	.2714
Approx.	9	.4450	.4280	.4255	.3302	1.722	.4233	.3892	.1332
	24	1.171	.3775	.3938	.0932	3.115	.2864	.3250	.0814
	99	1.015	.2790	.1934	.1870	4.237	.0288	-.0093	.2013

Table 3.4b

Frequencies of $\Delta(B_f')$ in a 384-item System

Range for $\Delta(B_f')$	Number of Items			Cumulative Percentage of Items		
	Items with $\sigma^2/\mu=3$	Items with $\sigma^2/\mu=9$	All Items	Items with $\sigma^2/\mu=3$	Items with $\sigma^2/\mu=9$	All Items
[0%,2%)	34	74	108	18%	39%	28%
[2%,4%)	42	55	97	40%	67%	53%
[4%,6%)	38	34	72	59%	85%	72%
[6%,8%)	25	14	39	72%	92%	82%
[8%,10%)	29	7	36	88%	96%	92%
[10%,12%)	10	6	16	93%	99%	96%
[12%,14%)	5	2	7	95%	100%	98%
[14%,16%)	5		5	98%		99%
[16%,18%)	3		3	99%		100%
[18%,20%)	1		1	100%		100%

We examined other ways of approximating backlog cost with multiplicative expressions. The quantity $[1-B/(\mu p)]$ was fit to our data without success. Even with separate fits for each value of p/h our coefficients of determination were only of the order 0.93. We also examined the quantity

$$B_f'' \equiv T_f - R_f - H_f \ ,$$

where T_f, R_f, and H_f are our fixed-policy multiplicative expressions for total cost, replenishment cost, and holding cost, respectively. This approach also was unsuccessful; B_f'' deviates from B by an average of 17%.

3.2.3 Backlog Protection

To obtain a multiplicative expression for backlog protection, we used the quantity $(1+p/h)P$ as the dependent variable in a regression model of the form (29), (30), (31). For optimal policies we obtain a coefficient of determination of 0.99995, and, dividing by $(1+p/h)$, we have

$$(34) \qquad P_o = 49.47\ (p/h)^{-7.324h/p}\exp(.01045p/h)/(1+p/h)\ .$$

For Power Approximation policies we have a coefficient of determination of 0.99986 with the expression

$$(35) \qquad P_p = 49.74\ (p/h)^{-7.353h/p}\exp(.01041p/h)/(1+p/h)\ .$$

We define $\Delta(P_f)$ as the percentage deviation of (34) or (35) from the actual value of P for an item. Table 3.5 is a summary of the distribution of $\Delta(P_f)$ in the 576-item system. The average value of $\Delta(P_f)$ in the system is 0.7% with fewer than 1% of the items deviating

Table 3.5

Frequencies of $\Delta(P_f)$ in a 576-item System

Range for $\Delta(P_f)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	525	91%
[2%,4%)	43	99%
[4%,6%)	7	100^-%
[6%,8%)	1	100%

by more than 4%. The pattern of errors is quite similar to that noted for the analytic expressions P_a in Section 2.4. The largest deviations occur for items with small p/h. Items with p/h equal to 4 account for all errors greater than 4%, and 44 of 51 items with deviations greater than 2%.

The pattern of errors for P_f is nearly identical to the pattern noted for P_a (23), (24) in Section 2.4 Since P_a is much easier to compute, we suggest that it be used rather than P_f.

3.2.4 Average Replenishment Cost per Period

The following approximation for optimal replenishment cost yields a coefficient of determination of 0.996 and an average deviation of 2.2% from actual values of R

$$R_o = .6050h \exp(-.08549\sigma^2/\mu)\mu^{.5126+.004333\sigma^2/\mu}$$
$$(L+1)^{-.03847-.006147\sigma^2/\mu}(p/h)^{-.09095h/p+.02130\mu/\sigma^2}$$
$$(K/h)^{.5107+.01492\sigma^2/\mu} .$$

Table 3.6 is a summary of the distribution of percentage errors for R_o

Table 3.6

Frequencies of $\Delta(R_o)$ in a 288-item System

Range for $\Delta(R_o)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	163	57%
[2%,4%)	88	87%
[4%,6%)	21	94%
[6%,8%)	11	98%
[8%,10%)	3	99%
[10%,12%)	0	99%
[12%,14%)	1	100^{-}%
[14%,16%)	1	100%

in the 288-item system. The two items with values of $\Delta(R_o)$ exceeding 10% have a variance to-mean ratio 9, leadtime 0, mean 2, and unit penalty cost ratio and unit setup cost ratio of 4,64 and 99,32, respectively.

We obtain an even better fit for Power Approximation replenishment cost. The following approximation has a coefficient of determination of 0.999 and an average percentage deviation of 1.4% from actual values of R

$$R_p = .5763h(\sigma^2/\mu)^{-.2774}\mu^{.4959+.007906\sigma^2/\mu-.003856\mu/\sigma^2}$$

$$(L+1)^{-.05744}(K/h)^{.6046-.06563\mu/\sigma^2} .$$

There are no significant outliers in the distribution of $\Delta(R_p)$. We find 74% of the 288 items with $\Delta(R_p)$ in [0%,2%) and all the remaining items have values of $\Delta(R_p)$ in [2%,4%) .

The reason for R_p having smaller errors than R_o is that the Power Approximation has a simple multiplicative formula for setting (S-s). The multiplicative nature of the formula and its independence of the parameter p/h make the associated cost a natural candidate for a regression fit of this type.

3.2.5 Average Total Cost per Period

We would expect to obtain good multiplicative approximations for total cost because of the empirical observations described in Section 3.1 We also would expect the Power Approximation and optimal policy fits to be similar because actual total cost values for these policies are nearly equal. Both of these conjectures are correct.

The following approximation for optimal total cost yields a coefficient of determination of 0.994 and an average percentage deviation of 2.7%

(36) $$T_o = 2.663h \exp(.09784\sigma^2/\mu)\mu^{.4947-.004792\sigma^2/\mu}$$
$$(L+1)^{.1479+.01268\sigma^2/\mu-.06757\mu/\sigma^2}$$
$$(p/h)^{-1.143h/p-.02488\mu/\sigma^2+.01302\sigma^2/\mu}$$
$$(K/h)^{.4093-.02376\sigma^2/\mu} .$$

The following approximation for Power Approximation total cost yields a coefficient of determination of 0.994 and an average percentage deviation of 2.8%

(37) $$T_p = 2.438h(\sigma^2/\mu)^{.3836}\mu^{.4964-.005209\sigma^2/\mu}(p/h)^{-.9230h/p+.01508\sigma^2/\mu}$$
$$(L+1)^{.1498+.01231\sigma^2/\mu-.07050\mu/\sigma^2}$$
$$(K/h)^{.3095-.01310\sigma^2/\mu+.1073\mu/\sigma^2}$$

The distributions of percentage deviations for the two approximations are nearly identical. We summarize them as one system in Table 3.7 The

Table 3.7

Frequencies of $\Delta(T_f)$ in a 576-item System

Range for $\Delta(T_f)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	272	47%
[2%,4%)	176	77%
[4%,6%)	69	90%
[6%,8%)	31	95%
[8%,10%)	22	99%
[10%,12%)	1	99%
[12%,14%)	5	100%

largest errors are consistently associated with small leadtime and large variance-to-mean ratio.

We attempted to improve the approximation by fitting separately for each value of σ^2/μ , but the approach was unsuccessful.

The approximation does improve when we fit separately for each value of leadtime. We obtain coefficients of variation averaging 0.996 and an average percentage deviation of 2.1% with expressions of the form

(38) $$T_f'/h = a \exp(b_1\sigma^2/\mu+b_2\mu/\sigma^2)\mu^{c_1+c_2\sigma^2/\mu}$$

$$(p/h)^{d_1h/p+d_2\sigma^2/\mu+d_3\mu/\sigma^2}(K/h)^{e_1+e_2\sigma^2/\mu} .$$

Values of the coefficients in (38) are given in Table 3.7 for each policy and each value of L . We summarize the distribution of errors for (38)

Table 3.7

Coefficient Values in Expression (61) for T_f'

Policy	L	a	b_1	b_2	c_1	c_2	d_1	d_2	d_3	e_1	e_2
Optimal	0	2.084	.0798	-.0153	.4929	-.0033	-.8398	.0158	-.0198	.4601	-.0218
	2	3.212	.1196	-.0584	.4955	-.0054	-1.226	.0124	-.0282	.4024	-.0249
	4	4.216	.1289	-.1042	.4958	-.0057	-1.377	.0108	-.0280	.3652	-.0246
Power Approx.	0	2.095	.0791	-.0198	.4951	-.0041	-.8052	.0160	-.0159	.4552	-.0212
	2	3.216	.1198	-.0606	.4976	-.0058	-1.211	.0125	-.0266	.4003	-.0248
	4	4.216	.1296	-.1098	.4968	-.0058	-1.358	.0109	-.0258	.3644	-.0248

in Table 3.8. The number of items having deviations within 2% has increased by more than a third, and the number of items with deviations exceeding 6% has been nearly halved. The four items with deviations exceeding 10% are listed in Table 3.9.

3.3 Approximations for Statistical Policies

We consider four 72-item systems (see Table 3.1) controlled with the Statistical Power Approximation (see Section 1). One system has a

Table 3.8

Frequencies of $\Delta(T_f')$ in a 576-item System

Range for $\Delta(T_f')$	Number of Items	Cumulative Percentage of Items
[0%,2%)	364	63%
[2%,4%)	145	88%
[4%,6%)	36	95%
[6%,8%)	13	97%
[8%,10%)	14	99%
[10%,12%)	4	100%

Table 3.9

Items with $\Delta(T_f')$ Exceeding 10%

Policy	σ^2/μ	L	μ	p/h	K/h	Error
Optimal	9	0	2	99	32	-11.8%
Power Approximation	9	0	2	99	32	-11.5%
Power Approximation	9	0	2	4	32	10.9%
Optimal	9	0	2	4	32	10.9%

variance-to-mean ratio of 3, and revision interval length equal to 26 periods. The other three systems all have a variance-to-mean ratio of 9, with revision interval lengths of 13, 26, and 52 periods.

We develop separate operating characteristic approximations for each of the systems using regression models of the form (29), (30), (32). We label a quantity with subscript (x,y) when it pertains to a system with σ^2/μ equal x, and a y-period revision interval. For example $H_{3,26}$ is the holding-cost approximation for the system with σ^2/μ equal 3, and revision interval equal 26 periods.

We also develop approximations that explicitly include revision interval length T as a variable. For these approximations we use regression models of the form (29), (30), (33) on the systems with variance-to-mean equal 9. We label these approximations with the subscript (9,T).

When we summarize error distributions we combine the systems into one 288-item system. The subscript "s" is used to label the system.

Recall that our statistical policy data contain random simulation errors. Hence, we can expect that the quality of the regression fits will degrade as compared with those for fixed policies.

3.3.1 <u>Average Holding Cost per Period</u>

We obtain the following approximations for statistical policy holding cost

$$H_{3,26} = 4.968h\mu^{.4881}(L+1)^{.2679}(p/h)^{-3.394h/p}(K/h)^{.1934}$$

$$H_{9,13} = 10.76h\mu^{.4499}(L+1)^{.4187}(p/h)^{-3.691h/p}(K/h)^{.1028}$$

$$H_{9,26} = 10.40h\mu^{.4824}(L+1)^{.3722}(p/h)^{-3.730h/p}(K/h)^{.09987}$$

$$H_{9,52} = 10.27h\mu^{.4958}(L+1)^{.3645}(p/h)^{-3.743h/p}(K/h)^{.09743} .$$

The approximations have coefficients of determination of 0.989, 0.990, 0.992, and 0.994, respectively. Average percentage deviations from actual values of H are 5.0%, 5.4%, 4.7%, and 4.3%, respectively, as compared with 3.9% for Power Approximation fixed policies.

The percentage deviations are distributed similarly for the four systems. We summarize the deviations in Table 3.10. Unlike the fixed-policy fits, we find no general pattern in the parameter settings of

Table 3.10

Frequencies of $\Delta(H_s)$ in a 288-item System

Range for $\Delta(H_s)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	63	22%
[2%,4%)	76	50%
[4%,6%)	59	66%
[6%,8%)	43	81%
[8%,10%)	23	89%
[10%,12%)	11	97%
[12%,14%)	12	100⁻%
[14%,16%)	1	100%

outliers. Nearly half of the items with deviations larger than 10% have p/h equal to 99.

We also performed separate fits for each value of p/h, and explicitly included revision interval length T as a variable in the σ^2/μ equal 9 system. The average percent deviation decreases to 3.8% for the approximations

$$H'_{(3,26)} = \begin{cases} 1.365h\mu^{.4905}(L+1)^{.2076}(K/h)^{.2319}, & p/h = 4 \\ 2.156h\mu^{.4861}(L+1)^{.2474}(K/h)^{.2082}, & p/h = 9 \\ 4.792h\mu^{.4877}(L+1)^{.3488}(K/h)^{.1400}, & p/h = 99 \end{cases}$$

$$H'_{(9,T)} = \begin{cases} 2.770h\mu^{.4655+.00033T}(L+1)^{.2902+1.409/T}(K/h)^{.1155}, & p/h = 4 \\ 4.000h\mu^{.4587+.00042T}(L+1)^{.2994+1.250/T}(K/h)^{.1280}, & p/h = 9 \\ 9.698h\mu^{.4654+.00047T}(L+1)^{.3935+1.183/T}(K/h)^{.0566}, & p/h = 99 . \end{cases}$$

Although the stepwise regression program permitted the variables T and 1/T to enter every exponent, notice that T and $1/T$ each enter only once in H'.

We summarize the distribution of errors for the approximations in Table 3.11. The distribution is closer to 0% than in Table 3.10, with 27% more items with deviations less than 4%.

3.3.2 Average Backlog Cost per Period

We encountered serious difficulties in fitting multiplicative expressions to statistical policy backlog costs. When we use the model (29), (30), (32) on the four statistical systems we obtain coefficients of determination averaging 0.80.

Table 3.11

Frequencies of $\Delta(H_s')$ in a 288-item System

Range for $\Delta(H_s')$	Number of Items	Cumulative Percentage of Items
[0%,2%)	96	33%
[2%,4%)	80	61%
[4%,6%)	48	78%
[6%,8%)	36	90%
[8%,10%)	18	97%
[10%,12%)	5	98%
[12%,14%)	4	100^{-}%
[14%,16%)	1	100%

We obtain improved results by fitting each system separately for each value of p/h, yielding expressions of the form

$$B_s = a\,h\,\exp(b_1/\mu)\mu^{b_2-b_3/\mu}(L+1)^{c_1+c_2(L+1)}(K/h)^{dh/K} .$$

In Table 3.12 we list values of the constants $a,\ldots,d$ for each statistical system and each value of p/h. The quality of the fit is

Table 3.12

Values of Constants in B_s

System	p/h	a	b_1	b_2	b_3	c_1	c_2	d
(3,26)	4	6.355	-.6755	.2258	-1.067	.3691	.0087	-3.989
	9	2.599	-.3709	.3581	.0952	.4952	.0066	-2.697
	99	.4842	1.568	.6172	2.856	.3656	-.0132	-1.380
(9,13)	4	1.983	-.1610	.5620	1.012	.6015	-.0171	-1.609
	9	3.319	-.3796	.4052	.4183	.5499	.0070	-.3021
	99	.1781	2.833	1.294	6.461	.3405	-.0008	-.6803
(9,26)	4	7.427	.8536	.2346	-.7821	.5433	-.0105	-2.791
	9	1.380	.4098	.6079	1.464	.5070	.0066	-.2855
	99	4.251	.8241	.2755	2.513	.1585	.0207	-1.120
(9,52)	4	4.428	-.4380	.3681	-.1355	.5166	-.0126	-2.763
	9	4.376	-.2043	.3079	-.0763	.4799	-.0031	-1.333
	99	7.242	.5117	.0630	1.690	.0555	.0082	-1.030

rather good for p/h equal 4 and 9 , with an average deviation of 4% from actual values of B . The approximation for p/h equal 99 , however, deviates by an average of 10%. We summarize the distribution of deviations in Table 3.13.

Table 3.13

Frequencies of $\Delta(B_s)$ in a 288-item System

Range for $\Delta(B_s)$	Number of Items			Cumulative Percentage of Items		
	Items with p/h<99	Items with p/h=99	All Items	Items with p/h<99	Items with p/h=99	All Items
[0%,2%)	72	10	82	38%	10%	28%
[2%,4%)	48	11	59	62%	22%	49%
[4%,6%)	37	8	45	82%	30%	65%
[6%,8%)	16	9	25	90%	40%	73%
[8%,10%)	6	12	18	93%	52%	80%
[10%,20%)	13	36	49	100%	90%	97%
[20%,30%)		9	9		99%	100^-%
[30%,40%)		1	1		100%	100%

We caution that the approximations may not be robust. One reason is that we used only 24 items to fit the seven constants in each regression. Another reason is that most of the coefficients in Table 3.12 do not vary monotonically with the three values for p/h ; interpolating coefficient values for other settings of p/h would not be warranted.

3.3.3 Backlog Protection

Recall our observation in Section 3.2.3 that for fixed-policy backlog protection, the easy-to-calculate expressions (23) and (24) are as accurate as the complicated multiplicative expressions (34) and (35). Therefore we fit functions of the form (23) and (24) to the statistical-policy data.

We have average percent deviations of only 1.3% for the following backlog protection approximations

$$P_{(3,26)} = (-.0925+.9995p/h)/(1+p/h)$$

$$P_{(9,13)} = (-.0057+.9853p/h)/(1+p/h)$$

$$P_{(9,26)} = (.0167+.9904p/h)/(1+p/h)$$

$$P_{(9,52)} = (.0511+.9943p/h)/(1+p/h) .$$

We summarize the distribution of errors in Table 3.14. We see the same pattern of errors as with the approximations for backlog protection in Sections 2.4 and 3.2.3. The largest errors are due to items with low values of p/h, for example, fifteen of the twenty items with deviations greater than 4% have p/h equal four or nine.

Table 3.14

Frequencies of $\Delta(P_s)$ in a 288-item System

Range for $\Delta(P_s)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	223	77%
[2%,4%)	45	93%
[4%,6%)	18	99%
[6%,8%)	2	100%

The approximation is improved by fitting to minimize relative squared error as in Section 2.4. Revision interval length T is also introduced as a variable in the systems having $\sigma^2/\mu = 9$. The result is the following approximation which has an average percentage error of 1.3%

$$P'_{(3,26)} = (-.272 + 5.66\ h/K + 1.003\ p/h)/(1 + p/h)$$

$$P'_{(9,T)} = (-.081 + 5.42\ h/K - 1.119/T + .994\ p/h)/(1 + p/h)\ .$$

We summarize the distribution of errors for this approximation in Table 3.14a. Although P'_s has the same average percentage error in the 288-item

Table 3.14a

Frequencies of $\Delta(P'_s)$ in a 288-item System

Range for $\Delta(P'_s)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	220	76%
[2%,4%)	53	95%
[4%,6%)	15	100%

system, there are fewer items with large percentage errors using approximation P'_s.

3.3.4 Average Replenishment Cost per Period

We obtain the following approximations for statistical policy replenishment cost

$$R_{3,26} = .4313h\mu^{.5174}(L+1)^{-.06109}(K/h)^{.5818}$$

$$R_{9,13} = .2676h\mu^{.5688}(L+1)^{-.1084}(K/h)^{.6470}$$

$$R_{9,26} = .2949h\mu^{.5526}(L+1)^{-.07235}(K/h)^{.6269}$$

$$R_{9,52} = .3012h\mu^{.5563}(L+1)^{-.06014}(K/h)^{.6165} .$$

The approximations have coefficients of determination of 0.999, 0.989, 0.998, and 0.999, respectively. Average percentage deviations from actual values of R are 1.1%, 4.0%, 1.6%, and 1.2%, respectively, as compared with 2.0% for Power Approximation fixed policies.

We summarize the distribution of errors in Table 3.15. Of the 34 items with deviations larger than 4%, 28 are from the (9,13) system. All of the items with deviations larger than 10% are from this system, and also have L equal 4 , and p/h equal 99 .

We also performed separate fits for each value of p/h , and explicitly included revision interval length T and its reciprocal $1/T$ as variables in the σ^2/μ equal 9 system. The average percent deviation decreases from 2.0% to 1.6% for the approximations

$$R'_{(3,26)} = \begin{cases} .4427h\mu^{.5155}(L+1)^{-.0586}(K/h)^{.5761} , & p/h = 4 \\ .4364h\mu^{.5175}(L+1)^{-.0593}(K/h)^{.5790} , & p/h = 9 \\ .4155h\mu^{.5194}(L+1)^{-.0653}(K/h)^{.5903} , & p/h = 99 \end{cases}$$

Table 3.15

Frequencies of $\Delta(R_s)$ in a 288-item System

Range for $\Delta(R_s)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	202	70%
[2%,4%)	52	88%
[4%,6%)	20	95%
[6%,8%)	6	97%
[8%,10%)	4	99%
[10%,12%)	1	99%
[12%,14%)	0	99%
[14%,16%)	2	100^{-}%
[16%,18%)	0	100^{-}%
[18%,20%)	0	100^{-}%
[20%,22%)	0	100^{-}%
[22%,24%)	0	100^{-}%
[24%,26%)	1	100%

$$R'_{(9,T)} = \begin{cases} .2903h\mu^{.5470}(L+1)^{-.0638-.1269/T}(K/h)^{.6344}, & p/h = 4 \\ .2872h\mu^{.5540}(L+1)^{-.0568-.3930/T}(K/h)^{.6333}, & p/h = 9 \\ .2852h\mu^{.5766}(L+1)^{-.0363-1.351/T}(K/h)^{.6227}, & p/h = 99 . \end{cases}$$

Observe that in the stepwise least-squares fitting procedure, the revision interval entered only through a single term in $1/T$, and was not selected for any of the other coefficients.

We summarize the distribution of errors for the approximations in Table 3.16. The distribution is better than in Table 3.15, with outliers still coming from the (9,13) system. All the items with deviations greater than 6% come from this system. They also tend to have L equal 4 , p/h equal 99 , and small μ .

Table 3.16

Frequencies of $\Delta(R_s')$ in a 288-item System

Range for $\Delta(R_s')$	Number of Items	Cumulative Percentage of Items
[0%,2%)	208	72%
[2%,4%)	63	94%
[4%,6%)	11	98%
[6%,8%)	2	99%
[8%,10%)	2	99%
[10%,12%)	1	100^-%
[12%,14%)	0	100^-%
[14%,16%)	1	100%

3.3.5 Average Total Cost per Period

We obtain the following approximations for statistical policy total cost

$$T_{3,26} = 2.370h\mu^{.4702}(L+1)^{.1958}(p/h)^{.1560}(K/h)^{.3022}$$

$$T_{9,13} = 3.776h\mu^{.4387}(L+1)^{.3346}(p/h)^{.2727}(K/h)^{.1843}$$

$$T_{9,26} = 3.798h\mu^{.4309}(L+1)^{.3024}(p/h)^{.2550}(K/h)^{.1917}$$

$$T_{9,52} = 3.758h\mu^{.4405}(L+1)^{.2779}(p/h)^{.2397}(K/h)^{.1964} .$$

The approximations have coefficients of determination of 0.989, 0.987, 0.983, and 0.985, respectively. Average percentage deviations from actual values of T are 3.8%, 5.1%, 5.4%, and 4.9%, respectively, as compared to 2.8% for the Power Approximation fixed policies.

We summarize the distribution of percentage deviations in Table 3.17. The patterns of percentage errors are quite similar in all four systems, with large errors tending to occur for items with large p/h and small μ.

We also performed separate fits for each value of p/h, and explicitly included revision interval length T as a variable in the σ^2/μ equal 9 system. The average percent deviation drops from

Table 3.17

Frequencies of $\Delta(T_s)$ in a 288-item System

Range for $\Delta(T_s)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	83	29%
[2%,4%)	69	53%
[4%,6%)	39	66%
[6%,8%)	43	81%
[8%,10%)	19	88%
[10%,12%)	16	93%
[12%,14%)	11	97%
[14%,16%)	7	100^-%
[16%,18%)	0	100^-%
[18%,20%)	1	100%

4.8% to 2.6% for the following approximations

$$T'_{(3,26)} = \begin{cases} 2.299h\mu^{.4922}(L+1)^{.1544}(K/h)^{.3641}, & p/h = 4 \\ 3.183h\mu^{.4842}(L+1)^{.1858}(K/h)^{.3179}, & p/h = 9 \\ 6.607h\mu^{.4343}(L+1)^{.2470}(K/h)^{.2246}, & p/h = 99 \end{cases}$$

$$T'_{(9,T)} = \begin{cases} 3.790h\mu^{.4961}(L+1)^{.2403+.8479/T}(K/h)^{.2572+.0400/T}, & p/h = 4 \\ 6.002h\mu^{.4680}(L+1)^{.2659+.9762/T}(K/h)^{.2008+.1115/T}, & p/h = 9 \\ 19.20h\mu^{.3461}(L+1)^{.2757+1.131/T}(K/h)^{.0881+.4356/T}, & p/h = 99 . \end{cases}$$

We summarize the distribution of errors for the approximations in Table 3.18. The distribution is markedly better than that in Table 3.17. The number of items with deviations smaller than 4% has increased by more than half, and all 35 items that previously had deviations greater than 10% now are within 10%.

Table 3.18

Frequencies of $\Delta(T'_s)$ in a 288-item System

Range for $\Delta(T'_s)$	Number of Items	Cumulative Percentage of Items
[0%,2%)	130	45%
[2%,4%)	100	80%
[4%,6%)	49	97%
[6%,8%)	8	100^-%
[8%,10%)	1	100%

3.4 Measures of Robustness

In this section we analyze the robustness of our multiplicative approximations for fixed- and statistical-policy operating characteristics. We use the approach of Section 2.7 to examine the aggregated behavior of the approximations and to investigate their accuracy when parameter values are extrapolated.

In Tables 3.19 and 3.20 we summarize the accuracy of our fixed-policy approximations H_f , B_f , P_f , R_f , and T_f' as a function of the parameter settings in the 576-item system. Each entry is the percentage excess of the sum of approximate values over the sum of actual values for all items having the given parameter setting. As we noted in Section 2.7, most of the large errors tend to cancel when the items are aggregated in this way. Even the large errors we found for backlog cost with p/h equal 99 do not appear in Tables 3.19 and 3.20. The largest systematic errors appear for the Power Approximation holding cost when σ^2/μ equals 1 . This pattern did not appear in Section 3.2 when we examined single-item behavior.

Table 3.21a is a multi-item summary of errors for the statistical-policy approximations H_s , B_s , P_s , R_s , and T_s . We summarize the multi-item behavior of H_s' , R_s' , and T_s' in Table 3.21b. These tables exhibit the same property noted above that the large single-item errors tend to cancel when the items are aggregated. Approximations H_s' , R_s' , and T_s' appear to be more accurate than H_s , R_s , and T_s , respectively, in all systems except for the σ^2/μ equals 9 systems where R_s has smaller errors than R_s' . The largest systematic errors appear in the total cost approximations T_s for p/h equal to 99 and μ equal to 2.

We test the robustness of our fixed-policy approximations by using them in the 64-item system of Table 2.11. The parameter settings are

Table 3.19

Percentage Errors of Multiplicative Approximations for a 288-item System

Controlled with Optimal Policies

(Approximations H_o , B_f , P_o , R_o , and T'_f are used)

OPERATING CHARACTERISTIC	VARIANCE/ MEAN	TOTAL	DISTRIBUTIONS BY VALUES OF INPUT PARAMETERS											
			C(OUT)/C(IN)			C(FIX)/C(IN)		LEADTIME			MEAN			
			4	9	99	32	64	0	2	4	2	4	8	16
Holding Cost	1	-0.9	0.0	-1.6	-0.8	0.5	-1.9	-1.1	0.4	-1.8	-0.8	-1.7	-2.1	0.5
	3	-0.7	0.3	-1.4	-0.7	-3.1	1.4	-1.5	1.0	-1.5	-1.6	1.1	-0.8	0.0
	9	0.6	-1.4	-3.6	3.3	0.0	1.2	-1.7	1.6	1.1	-2.1	-0.8	0.6	2.4
Backlog Cost	1	-0.3	0.5	-0.8	-0.9	0.6	-1.0	-0.3	1.8	-2.0	0.1	-0.2	-4.3	2.6
	3	-0.5	-0.8	-0.1	-0.4	0.4	-1.3	-1.0	-0.1	-0.5	-2.1	1.3	0.5	-1.4
	9	0.1	0.3	0.1	-0.1	0.1	0.1	-0.8	0.3	0.5	-0.4	0.5	1.4	-0.9
Backlog Protection	1	-0.3	-0.3	-0.5	-0.2	-0.4	-0.2	-0.5	-0.4	-0.0	-1.1	-0.5	0.1	0.2
	3	-0.0	-0.3	0.0	-0.0	-0.2	0.0	-0.3	0.0	0.1	-0.8	-0.2	0.2	0.5
	9	0.3	0.5	0.5	0.0	0.3	0.4	0.1	0.4	0.5	0.0	0.3	0.4	0.6
Replenishment Cost	1	-0.1	0.4	-1.6	0.9	-1.4	0.8	1.0	-0.8	-0.5	-0.3	0.4	1.9	-1.6
	3	0.6	2.2	-0.1	-0.2	1.5	0.0	0.3	0.3	1.3	-0.6	-0.6	0.9	1.3
	9	-0.0	0.1	0.1	-0.3	-0.5	0.3	-0.0	0.4	0.4	-0.5	-0.4	-0.3	0.5
Total Cost	1	-0.7	-1.3	-0.6	-0.4	0.1	-1.4	-0.5	-0.7	-0.9	-0.3	-0.7	-1.0	-0.7
	3	-0.4	-0.1	-0.1	-0.7	-1.4	0.4	-0.3	-0.3	-0.5	-1.2	-0.5	-0.2	-0.2
	9	0.0	-1.5	-1.6	1.9	0.2	-0.1	0.1	0.1	-0.1	-1.4	-0.3	0.4	0.5

Table 3.20

Percentage Errors of Multiplicative Approximations for a 288-item System

Controlled with Power Approximation Policies

(Approximations H_p , B_f , P_p , R_p , and T'_f are used.)

OPERATING CHARACTERISTIC	VARIANCE/ MEAN	TOTAL	DISTRIBUTIONS BY VALUES OF INPUT PARAMETERS											
			C(OUT)/C(IN)			C(FIX)/C(IN)		LEADTIME			MEAN			
			4	9	99	32	64	0	2	4	2	4	8	16
Holding Cost	1	-1.5	-2.3	-1.7	-0.9	-1.8	-1.2	-1.2	-1.1	-2.1	-3.3	-0.5	-0.3	-2.2
	3	-1.1	0.9	-2.5	-1.1	-3.1	0.6	-0.9	-0.0	-2.2	-3.6	-0.5	0.0	-1.3
	9	-0.5	0.6	-3.6	0.7	-1.2	0.2	0.1	0.6	-1.6	-1.8	0.6	0.5	-0.5
Backlog Cost	1	-0.4	-0.5	-0.5	-0.1	-0.6	-0.2	-2.3	1.5	-0.9	0.6	-1.3	-1.7	0.8
	3	-0.5	-0.0	-0.4	-1.2	0.5	-1.4	-0.1	0.6	-1.6	0.0	0.9	0.8	-2.6
	9	-0.1	-0.3	0.2	-0.1	-0.5	0.3	0.2	-1.2	0.7	-2.7	2.0	1.5	-1.3
Backlog Protection	1	-0.8	-1.7	-0.6	-0.2	-1.1	-0.5	-1.3	-0.8	-0.3	-1.0	-0.3	-0.5	-1.4
	3	0.7	1.9	0.5	-0.0	0.2	1.2	0.6	0.6	0.9	0.6	1.1	0.9	0.4
	9	-0.1	0.0	-0.5	0.1	-0.9	0.6	-0.0	-0.3	-0.1	-1.0	-0.2	0.3	0.4
Replenishment Cost	1	-0.1	-0.1	-0.1	-0.1	0.6	-0.6	-0.5	-0.0	0.1	0.9	-0.8	-0.8	0.3
	3	0.1	0.1	0.1	0.1	-0.3	-0.3	0.3	-0.2	0.2	1.0	-0.6	-1.0	0.8
	9	-0.0	-0.0	-0.0	-0.0	0.5	-0.4	0.2	0.1	-0.4	0.7	-0.4	-1.7	1.0
Total Cost	1	-0.7	-1.4	-0.6	-0.4	-0.0	-1.2	-0.6	-0.8	-0.8	-0.4	-0.6	-0.7	-0.9
	3	-0.3	0.0	-0.0	-0.8	-1.2	0.4	-0.3	-0.3	-0.3	-1.2	-0.5	-0.1	-0.0
	9	0.1	-1.5	-1.5	1.8	0.1	0.0	0.1	0.1	0.0	-1.4	-0.1	0.5	0.5

Table 3.21a

Percentage Errors of Multiplicative Approximations for a 288-item System Controlled with Statistical Power Approximation Policies

(Approximations H_s, B_s, P_s, R_s, and T_s are used)

OPERATING CHARACTERISTIC	(VARIANCE/MEAN, REVISION INTERVAL)	TOTAL	DISTRIBUTIONS BY VALUES OF INPUT PARAMETERS											
			C(OUT)/C(IN)			C(FIX)/C(IN)		LEADTIME			MEAN			
			4	9	99	32	64	0	2	4	2	4	8	16
Holding Cost	(3,26)	-0.4	2.2	-2.7	-0.1	-1.8	0.9	1.1	2.4	-3.6	-1.3	0.6	0.6	-1.2
	(9,13)	-0.4	0.9	-1.5	-0.2	-1.6	0.8	-0.1	3.3	-3.3	-2.5	1.1	0.4	-0.8
	(9,26)	-0.6	1.4	-1.0	-1.1	-2.0	0.7	1.0	1.6	-3.2	-0.4	0.2	-0.5	-1.2
	(9,52)	-0.6	1.8	-1.7	-0.8	-1.9	0.7	1.3	1.4	-3.1	0.2	-0.2	-0.7	-0.9
Backlog Cost	(3,26)	-0.2	0.0	0.1	-0.7	0.1	-0.4	-0.4	-0.4	0.1	-0.8	-0.4	0.3	-0.1
	(9,13)	-0.5	0.0	-0.1	-0.9	0.1	-1.1	0.0	-1.1	-0.3	0.1	0.0	-0.3	-1.3
	(9,26)	-0.4	0.1	-0.1	-0.8	-0.2	-0.5	-0.3	-0.6	-0.2	-1.7	0.2	-0.1	-0.1
	(9,52)	-0.2	0.2	0.1	-0.7	-0.1	-0.3	-0.3	-0.5	0.1	-1.6	-0.2	0.4	0.2
Backlog Protection	(3,26)	0.1	1.0	-0.5	0.0	-0.3	0.6	-0.8	0.4	0.8	-0.3	0.2	0.5	0.2
	(9,13)	-0.0	-0.3	0.1	0.0	-0.3	-0.3	-1.8	0.5	1.2	-1.5	0.2	0.5	0.7
	(9,26)	0.0	0.0	-0.0	0.0	-0.4	0.5	-1.4	0.4	1.1	-0.9	0.1	0.4	0.4
	(9,52)	0.1	0.6	-0.3	0.0	-0.4	0.6	-0.9	0.4	0.8	-0.6	0.1	0.4	0.4
Replenishment Cost	(3,26)	-0.0	-0.2	-0.2	0.4	0.2	-0.1	0.0	0.0	-0.1	1.0	-0.8	-0.9	0.7
	(9,13)	0.1	-2.2	-1.0	3.6	-0.2	0.3	1.4	-1.1	-0.3	-0.1	-0.6	-0.9	1.1
	(9,26)	0.0	-0.7	-0.2	1.1	-0.1	0.1	0.4	-0.4	0.1	0.4	-0.6	-0.8	0.8
	(9,52)	0.1	0.0	-0.1	0.3	-0.0	0.1	0.1	-0.1	0.1	0.7	-0.6	-1.2	1.0
Total Cost	(3,26)	0.0	0.4	-1.7	1.0	-0.6	0.5	-0.3	2.1	-1.5	-1.1	0.0	0.6	0.0
	(9,13)	0.3	-0.2	-2.4	2.0	-0.7	1.2	0.2	2.6	-1.5	-2.7	-1.5	1.8	1.5
	(9,26)	0.4	-0.3	-2.6	2.4	-0.6	1.2	-0.1	2.5	-1.1	-3.8	-0.6	1.7	1.8
	(9,52)	0.4	-0.2	-2.7	2.5	-0.5	1.2	-0.2	2.1	-0.7	-3.3	-0.2	1.4	1.6

Table 3.21b

Percentage Errors of Multiplicative Approximations for a 288-item System Controlled with Statistical Power Approximation Policies

(Approximations H'_s, R'_s, and T'_s are used.)

COST COMPONENT	(VARIANCE/MEAN, REVISION INTERVAL)	TOTAL	DISTRIBUTIONS BY VALUES OF INPUT PARAMETERS											
			C(OUT)/C(IN)			C(FIX)/C(IN)		LEADTIME			MEAN			
			4	9	99	32	64	0	2	4	2	4	8	16
Holding Cost	(3,26)	-0.1	-0.2	-0.1	-0.1	-0.9	0.6	-1.2	2.8	-1.9	-1.0	0.9	0.8	-1.0
	(9,13)	0.1	-0.2	-0.3	0.4	-0.6	0.7	-2.4	3.8	-1.5	-4.5	0.3	1.0	1.1
	(9,26)	-0.6	-0.3	0.1	-1.1	-1.5	0.2	-1.6	1.6	-1.9	0.6	0.7	-0.5	-1.8
	(9,52)	-0.4	-0.2	-0.3	-0.4	-1.2	0.5	0.6	1.6	-2.4	1.7	0.6	-0.6	-1.4
Replenishment Cost	(3,26)	0.0	0.0	0.0	0.1	0.2	-0.1	0.1	0.1	-0.0	1.0	-0.8	-0.9	0.8
	(9,13)	-0.5	-0.5	-0.5	-0.7	-0.1	-0.8	0.1	-1.5	-0.2	0.7	-0.5	-1.5	-0.2
	(9,26)	-0.2	-0.2	-0.0	-0.4	-0.5	-0.0	0.5	-0.8	-0.4	-0.7	-1.2	-1.0	1.0
	(9,52)	1.0	0.9	0.8	1.2	0.3	1.4	1.4	0.7	0.7	1.2	0.1	-0.3	2.1
Total Cost	(3,26)	-0.0	-0.0	-0.0	-0.1	-0.2	0.1	-1.2	2.0	-1.1	-0.4	0.3	0.5	-0.4
	(9,13)	0.1	0.1	0.2	0.0	-0.6	0.7	-1.0	2.4	-1.2	0.7	-0.2	1.2	-0.8
	(9,26)	-0.5	-0.0	-0.2	-0.9	-0.6	-0.5	-1.2	1.6	-1.9	-2.4	-0.6	0.5	-0.4
	(9,52)	0.4	0.1	0.2	0.6	0.6	0.1	-0.9	2.1	-0.3	0.1	1.2	1.0	-0.5

interpolated and extrapolated in such a way that no item has any one parameter with the same value as in the 576-item system used to derive the approximations. We examine errors of approximations H_f , R_f , and T_f only. We already have a simple, robust approximation for P that was analyzed thoroughly in Section 2.7. Our approximation for B is not tested here because of its inaccuracies and lack of an explicit dependence on the parameter p/h .

We summarize the distributions of percent deviations of H_f , R_f , and T_f in Table 3.22. Optimal and Power Approximation policies display similar patterns of errors, so we summarize them as one 64-item system. The approximations clearly lack the robustness that we noted in Section 2.7 for the analytic approximations. Even if we confine our attention to items with fewer than two extrapolated parameters, we find average errors of 7.8%, 4.5%, and 5.4% for holding cost, replenishment cost, and total cost,

Table 3.22

Percentage Deviations of Multiplicative Approximations in a 64-item System

Range of Deviation	Number of Items (Cumulative Percentage) Holding Cost	Replenishment Cost	Total Cost
[0%,10%)	12 (19%)	31 (48%)	31 (48%)
[10%,20%)	16 (44%)	23 (84%)	12 (67%)
[20%,30%)	21 (77%)	9 (98%)	14 (89%)
[30%,40%)	12 (95%)	1 (100%)	6 (98%)
[40%,50%)	2 (98%)		1 (100%)
[50%,60%)	1 (100%)		

respectively. The worst single-parameter extrapolations for all the approximations are σ^2/μ equal to 15 and μ equal to 0.5 . We conclude that the multiplicative approximations should be used only with great caution when parameters are extrapolated beyond the ranges of the 576-item system (Table 2.1).

We also examine the behavior of the statistical-policy approximations H_s , P_s , R_s , and T_s for interpolated and extrapolated parameter values. We are unable to extrapolate these approximations for values of σ^2/μ so we use the 16-item system of Table 2.11 with σ^2/μ fixed at 9 . The results are similar to the fixed-policy results. In Table 3.23 we summarize the distribution of errors for the 16-item system using the approximations for a 26-period revision-interval length. Remember that the deviations are a composite of the error introduced by using a multiplicative form as well

Table 3.23

Percentage Deviations of Statistical Policy Approximations in a 16-item System

Range of Deviation	Number of Items (Cumulative Percentage)			
	Holding Cost	Backlog Protection	Replenishment Cost	Total Cost
[0%,10%)	8 (50%)	16 (100%)	9 (56%)	7 (44%)
[10%,20%)	4 (75%)		1 (62%)	3 (62%)
[20%,30%)	3 (94%)		2 (75%)	3 (81%)
[30%,40%)	1 (100%)		1 (81%)	2 (94%)
[40%,50%)			1 (88%)	1 (100%)
[50%,60%)			0 (88%)	
[60%,70%)			1 (94%)	
[70%,80%)			1 (100%)	

as a random component from the statistical aspect of the data. Backlog protection has an average deviation of 0.4%. If we consider only items with fewer than two extrapolated parameters, the average deviation is only 0.2%.

The holding cost approximation H_s has an average deviation of 13%, which drops to 4% when items with more than one extrapolation are ignored. The worst single-parameter extrapolation is for μ equal 0.5 .

The replenishment cost approximation R_s has an average deviation of 18%. If we consider only items with fewer than two extrapolated parameters, the average deviation is only 3%. The worst deviations occur for μ equal 0.5 .

The total cost approximation T_s has an average deviation of 16%, which drops to 5% when only single-parameter extrapolations are included. The worst single-parameter extrapolations are for μ equals 0.5 and p/h equal 132.

We conclude that the statistical-policy approximations should not be used for extremely small values of μ , and that multiple-parameter extrapolations should be considered cautiously.

4. THE SENSITIVITY OF OPERATING CHARACTERISTICS TO PARAMETER VALUES

In Sections 2 and 3 we developed approximate expressions for the operating characteristics of fixed-policy and statistical-policy inventory systems. Now that we have the approximations, we are in a position to perform analyses that were hitherto intractable. In this chapter we will make simple statements about the following questions:

1. What are the cost and service implications of a change in mean demand?
2. What are the effects of consolidating demand from several distinct warehouses into a single central warehouse?
3. By how much do costs rise when service is increased?
4. How much is it worth to obtain quicker delivery of replenishment orders?
5. What are the cost implications of a change in unit holding cost?
6. What are the effects of changing the review period length?

For each question we present calculations based on a simplified model to illustrate the uses of the approximations in Section 3. In many cases, our assumptions will be unrealistically simple for the systems planner who may require extensive analyses of such questions. Nevertheless, our results indicate general trends and illustrate the potential uses of the approximations.

In Sections 4.1 through 4.5 we discuss the sensitivity of Power Approximation and Statistical Power Approximation operating characteristics to mean demand μ, unit penalty cost ratio p/h, leadtime L, setup cost ratio K/h, unit holding cost h, and review period length. In each section we present a simple analysis of one or more of the above questions. We also include graphs displaying the operating characteristics of Sections 2and 3 as functions of parameter values. Throughout these sections we assume that σ^2/μ equals 9, and in the graphs we examine single parameter variations about the settings μ equals 8, L equals 2, p/h equals 9, and K/h equals 48.

For the Power Approximation we draw graphs using the analytic approximations H_a (Section 2.3), $P_{a,p}$ (Section 2.4), R_a (Section 2.2), and T_a (Section 2.5). We note that approximations H_a, R_a, and T_a are discontinuous functions of the parameters μ, σ^2/μ, L, p, K, and h. This is because the approximations depend on the policy parameters s and D which are rounded to the nearest integer. We have drawn smooth curves for the characteristics by omitting the round-off procedure for s and D in the computation of H_a, R_a, and T_a. We perform sensitivity analyses for Power Approximation policies using approximations H_p (Section 3.2.1), $P_{a,p}$ (Section 2.4), R_p (Section 3.2.4), and T_p (Section 3.2.5). With σ^2/μ equal 9 we have

$$H_p = 7.678\ h\ \mu^{.4629}\ (L+1)^{.3512}(p/h)^{-2.738\ h/p\ +\ .0942}(K/h)^{.0691}$$

$$P_{a,p} = (.0695 + p/h)/(1 + p/h)$$

(39)

$$R_p = .3133\ h\ \mu^{.5666}\ (L+1)^{-.0574}(K/h)^{.5973}$$

$$T_p = 5.663\ h\ \mu^{.4495}\ (L+1)^{.2528}\ (p/h)^{-.9230\ h/p\ +\ .1357}(K/h)^{.203}.$$

We consider the statistical policy with a 26-period revision interval and σ^2/μ equal 9. We draw graphs using two sets of approximations. When p/h is held constant at 9 we use H_s' (Section 3.3.1), R_s' (Section 3.3.4), and T_s' (Section 3.3.5). We draw the remaining graphs and perform sensitivity analyses using the following approximations from Section 3.3

$$H_{(9,26)} = 10.40\ h\ \mu^{.4824}\ (L+1)^{.3722} (p/h)^{-3.730\ h/p} (K/h)^{.0999}$$

$$P_{(9,26)} = (\ .0167 + .9904\ p/h)\ /\ (1 + p/h)$$

(40)

$$R_{(9,26)} = .2949\ h\ \mu^{.5526}\ (L+1)^{-.0724}\ (K/h)^{.6269}$$

$$T_{(9,26)} = 3.798\ h\ \mu^{.4309}\ (L+1)^{.3024}\ (p/h)^{.2550}\ (K/h)^{.1917}\ .$$

We do not use the approximations for backlog cost in this chapter because of their complexity and their inaccuracy for certain parameter settings.

4.1 Sensitivity to Mean Demand

Consider a situation in which mean demand μ_1 changes to the value

(41) $$\mu_2 = a\ \mu_1$$

and all other parameters remain unchanged. From (39) and (40) we see that backlog protection will be approximately unchanged for both policies since the expressions for P are independent of the parameter μ. All other cost expressions are of the form

(42) $$\text{Cost} \doteq c\mu^{\alpha},$$

where c and α are independent of μ. Let Cost_1 and Cost_2 be values of a cost component when μ equals μ_1 and μ_2 respectively. Then (41) and (42) yield

$$Cost_2/Cost_1 \doteq a^{\alpha},$$

that is, the ratio of approximate costs is a function of the ratio of the two mean demands, and the exponent of μ in the cost approximation. In Table 4.1 we display approximate percentage changes in operating characteristics for several values of the parameter a. Similar behavior is exhibited by the fixed policy and the statistical policy. All three cost components are roughly proportional to the square root of μ, with replenishment cost

Table 4.1

Approximate Percentage Changes in Operating Characteristics

Policy	Percentage Increase in μ	Holding Cost	Backlog Protection	Replenishment Cost	Total Cost
Power Approx.	10%	4.5%	0%	5.5%	4.4%
	25%	11%	0%	14%	11%
	50%	21%	0%	26%	20%
Statistical Power Approx.	10%	4.7%	0%	5.4%	4.1%
	25%	11%	0%	13%	10%
	50%	22%	0%	25%	19%

being slightly more sensitive than the other costs to changes in μ.

Now we consider a situation in which demand from n independent and identical warehouses is consolidated into a single central warehouse. Let $Cost_1$ be the sum of values for a cost component over the n warehouses (before consolidation), and let $Cost_2$ be the value of the cost component after consolidation. Using (42) we have

$$Cost_1 \doteq n\, c\, \mu^{\alpha}$$

$$Cost_2 \doteq c\, (n\, \mu)^{\alpha} .$$

Combining these expressions, we have

$$Cost_2 / Cost_1 \doteq n^{\alpha - 1}$$

which is independent of all parameters except for the number of warehouses before consolidation, and the exponent of μ in the cost approximation.

Table 4.2

Approximate Percentage Decreases for Operating Characteristics

Policy	n	Holding Cost	Backlog Protection	Replenishment Cost	Total Cost
Power Approx.	2	31%	0%	26%	32%
	4	52%	0%	45%	53%
	8	67%	0%	59%	68%
Statistical Power Approx.	2	30%	0%	27%	33%
	4	51%	0%	46%	55%
	8	66%	0%	61%	69%

Since the values of α in (39) and (40) are all less than 1, we know that $Cost_2$ will be less than $Cost_1$ for all values of n greater than 1. In n Table 4.2 we list approximate percentage decreases in costs for several values of n. The savings are considerable for both fixed and statistical policies.

Finally we consider consolidating demand from two warehouses that differ in one respect. Let the mean demand in one warehouse be much smaller than that in the other warehouse. That is, the item is a "slow mover" in one location and not in the other location. Let the ratio of mean demand at the two locations be a, with a less than one. We use (41) to calculate the total cost T_1 at the two warehouses before consolidation, yielding

$$T_1 \doteq c[\mu^\alpha + (a\mu)^\alpha] .$$

After consolidation we have total cost T_2 given by

$$T_2 \doteq c\,[(1 + a)\mu]^\alpha .$$

After combining the two expressions we have

$$T_2/T_1 \doteq (1 + a)^\alpha/(1 + a^\alpha) .$$

We also consider the ratio of the marginal total cost of the small demand after consolidation to its total cost before consolidation, given approximately by

$$(T_2 - c\mu^\alpha)/(a\mu)^\alpha \doteq [(1 + a)^\alpha - 1]/a^\alpha$$

In Table 4.3 we list approximate values for total cost reductions for several values of a when the Power Approximation is used. The reduction in

Table 4.3

Approximate Percentage Reductions in
Power Approximation Total Cost

Ratio of Small Demand to Large Demand	Combined System	Marginal Cost of Slow Mover
0.1	23%	88%
0.25	28%	80%
0.50	31%	73%

total costs for the combined system is quite large even when the ratio of mean demands is only 0.1. The percentage reduction in the marginal cost of the slow mover is quite dramatic, especially for the lowest value of a. The results for statistical policies are nearly identical.

In Figures 4.1, 4.2, and 4.3 we present graphs of the approximations for holding cost, replenishment cost, and total cost, respectively, as functions of mean demand. The fixed- and statistical-policy approximations exhibit similar behavior. The statistical policy has higher values and slightly greater sensitivity to μ for holding cost and total cost. Replenishment cost values are nearly identical for both policies.

Figure 4.1

Holding Cost Approximations versus Mean Demand

($\sigma^2/\mu=9$, $L=2$, $p=9$, $K=48$, $h=1$)

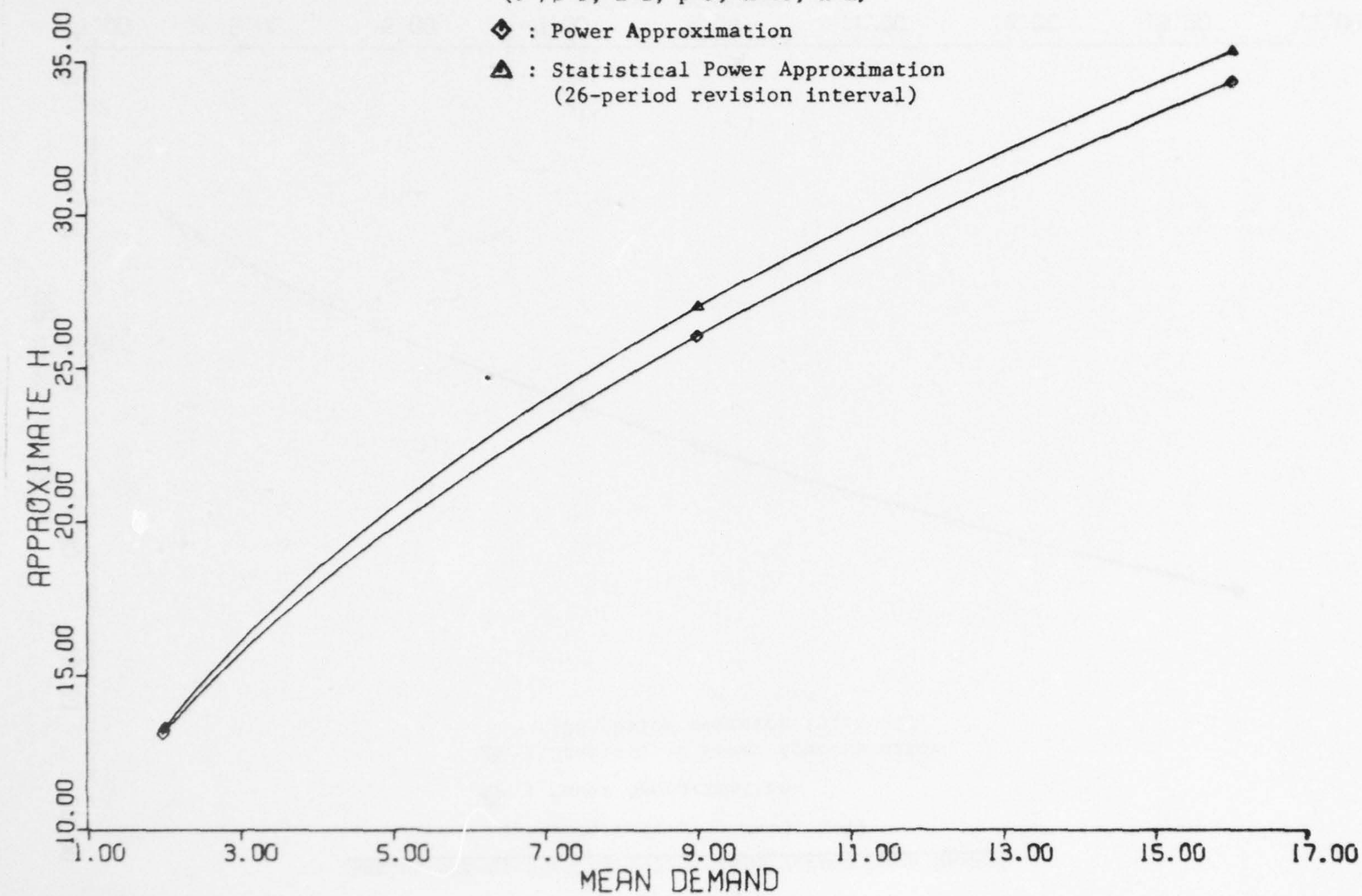


Figure 4.2

Replenishment Cost Approximations versus Mean Demand

($\sigma^2/\mu=9$, $L=2$, $p=9$, $K=48$, $h=1$)

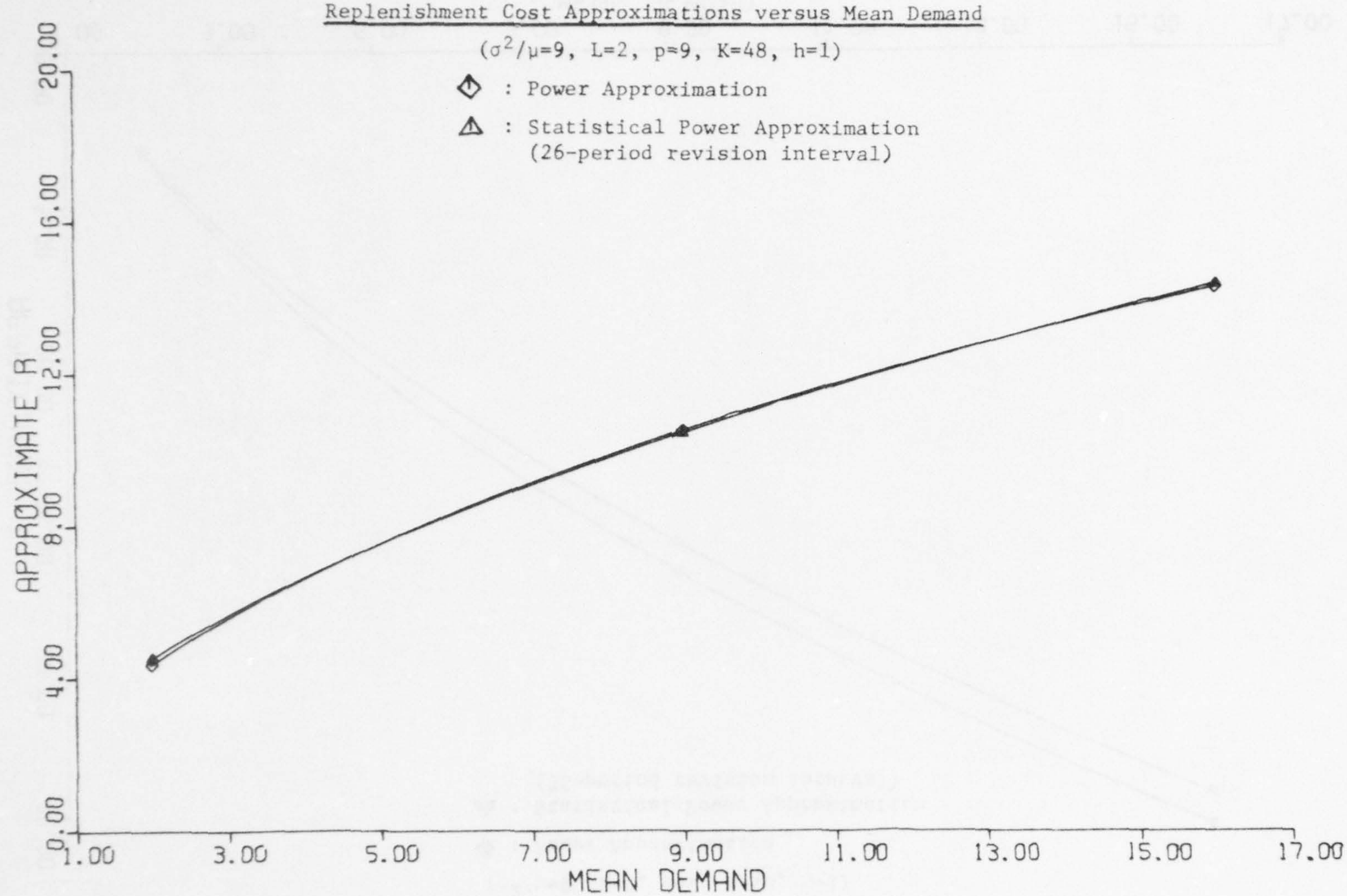
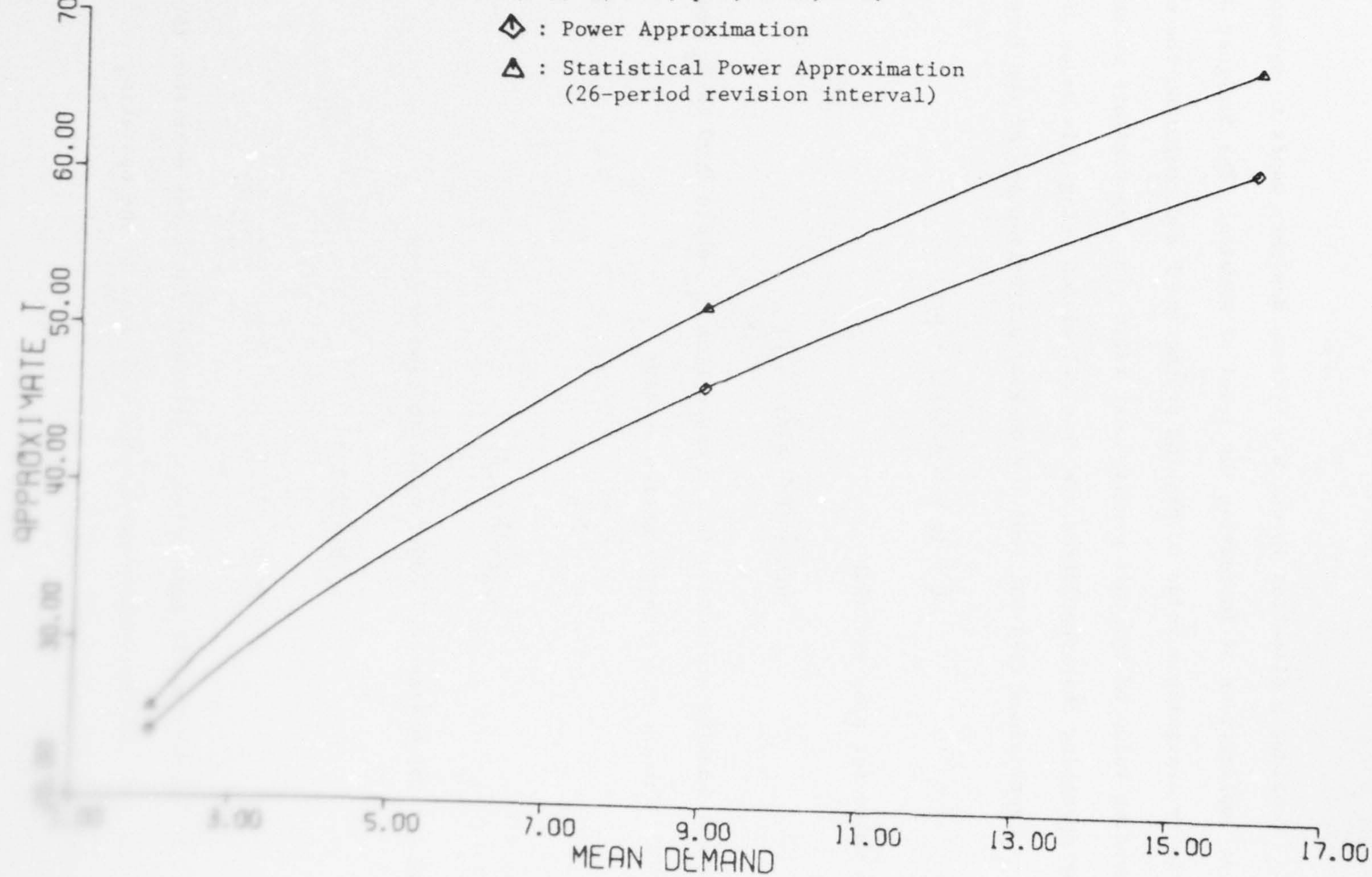


Figure 4.3

Total Cost Approximations versus Mean Demand

($\sigma^2/\mu=9$, L=2, p=9, K=48, h=1)



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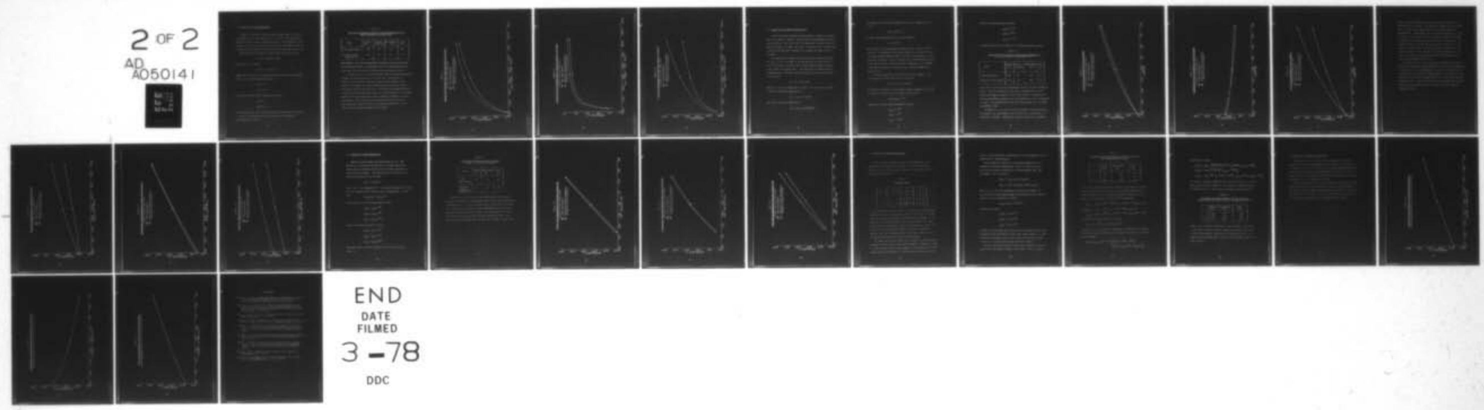

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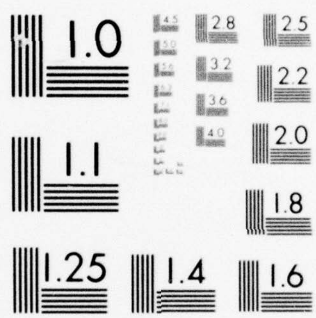

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4.2 Sensitivity to Unit Penalty Cost

Consider a situation in which a systems designer wants to assess the cost implications of increasing the level of service. The designer does this by assigning a value to backlog protection P and computing the corresponding value of the unit penalty cost ratio p/h. Approximate values for the operating characteristics are then calculated using this value of p/h.

Expressions (39) and (40) for backlog protection are of the form

$$P \doteq (a + b\,p/h)/(1 + p/h) \ .$$

We solve for p/h, yielding

$$p/h \doteq (P - a)/(b - P) \ .$$

Suppose backlog protection is increased from P_1 equals 0.90 to P_2 equals 0.95. For the Power Approximation we have

$$p_1/h \doteq 8.2$$

$$p_2/h \doteq 17.3,$$

and for the Statistical Power Approximation we have

$$p_1/h \doteq 9.8$$

$$p_2/h \doteq 23.1 \ .$$

For both policies the unit penalty cost ratio increases more than two-fold.

We list the approximate percentage increases in the operating characteristics in Table 4.4.

Table 4.4

Approximate Percentage Increases in Operating Characteristics When P Increases from 0.90 to 0.95

Policy	Holding Cost	Backlog Protection	Replenishment Cost	Total Cost
Power Approximation	38%	6%	0%	20%
Statistical Power Approximation	44%	6%	0%	25%

Replenishment costs remain approximately unchanged since the expressions for this characteristic are independent of p/h. All other costs rise by 20% to 44%, with the statistical policy displaying larger increases.

In Figures 4.4, 4.5, and 4.6 we present graphs of the approximations for holding cost, backlog protection, and total cost, respectively, as functions of p/h. All the curves are concave over the range of values plotted. The holding cost curves display higher values for the statistical policy than for the fixed policy. The curves for backlog protection are nearly parallel, with larger values of P for the fixed policy. For both policies we see that backlog protection is much more sensitive to p/h for small values of p/h. The total cost curves show that the statistical policy has higher values of total cost and greater sensitivity of total cost to p/h for all values displayed in the figure.

Figure 4.4

Holding Cost Approximations versus Unit Penalty Cost Ratio

($\sigma^2/\mu=9$, $\mu=8$, $L=2$, $K=48$, $h=1$)

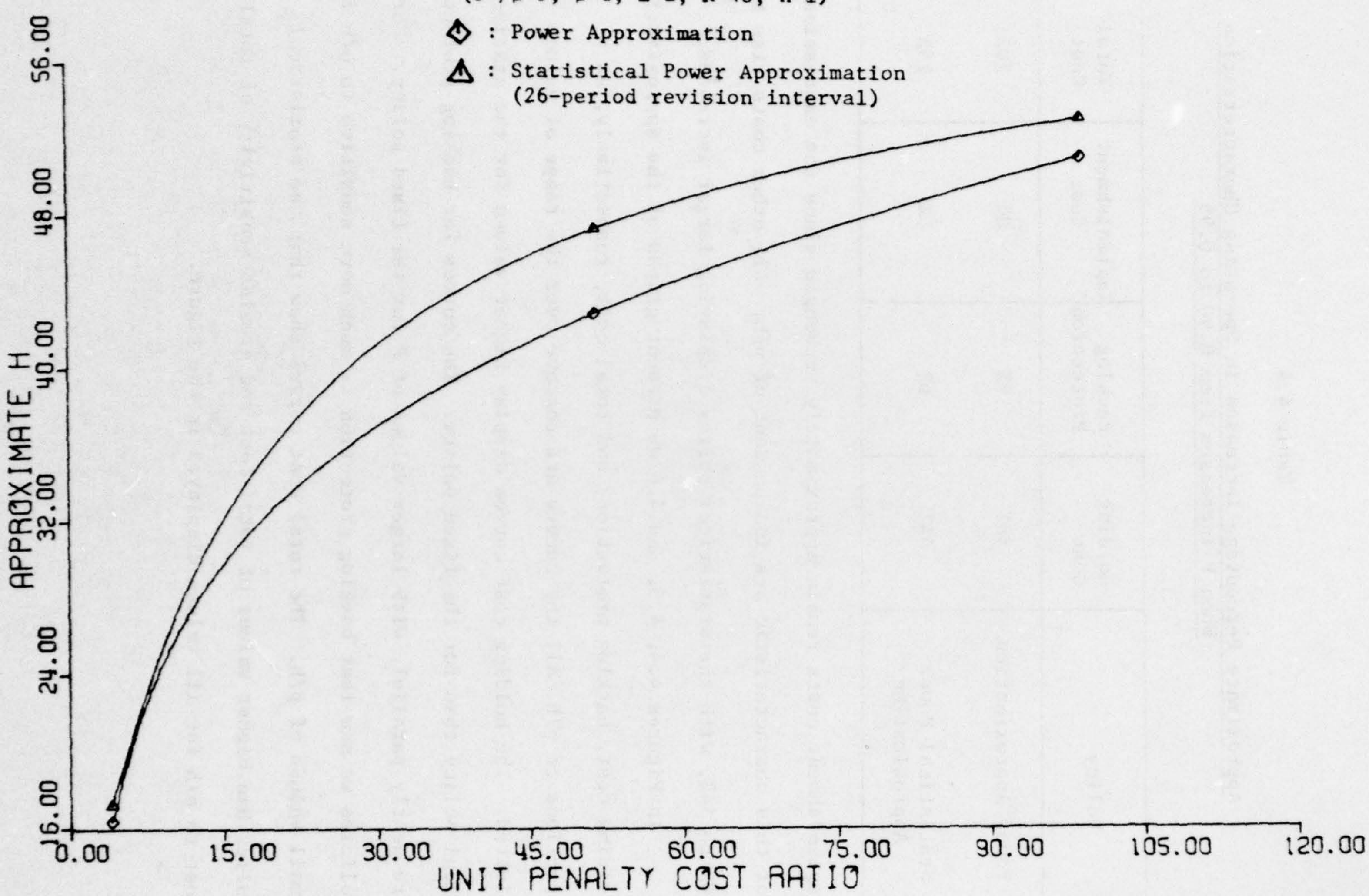


Figure 4.5

Backlog Protection Approximations versus Unit Penalty Cost Ratio

($\sigma^2/\mu=9$, $\mu=8$, $L=2$, $K=48$, $h=1$)

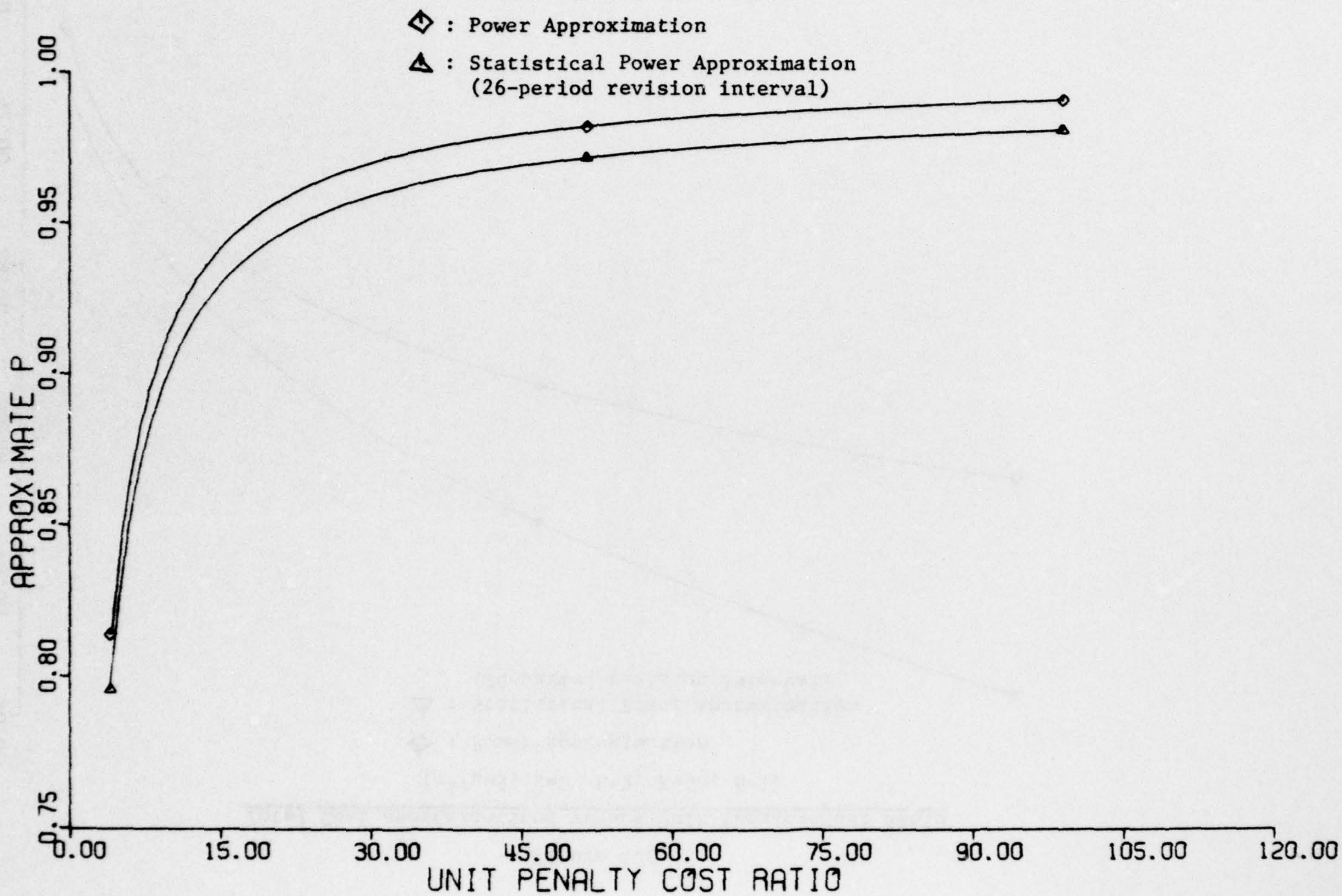
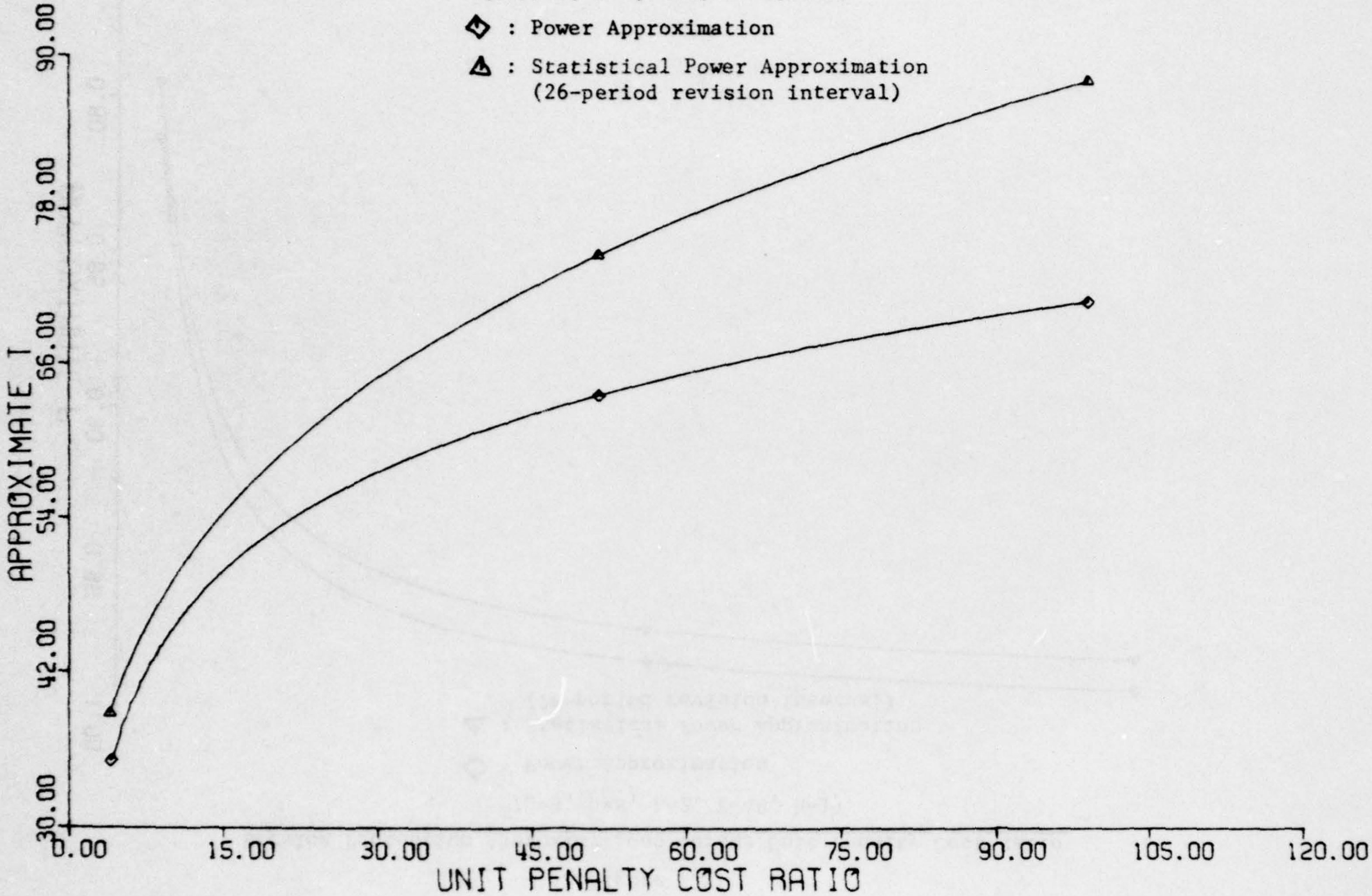


Figure 4.6

<u>Total Cost Approximations versus Unit Penalty Cost Ratio</u>

($\sigma^2/\mu=9$, $\mu=8$, $L=2$, $K=48$, $h=1$)



4.3 Sensitivity to Leadtime and Setup Cost

In this section we consider the fourth question raised at the beginning of the chapter. Suppose a quicker means of replenishment delivery is being considered. What is it worth? We assume that a quicker delivery is paid for by means of a higher setup cost. Therefore we are confronted with a trade-off decision where leadtime L is decreased and setup cost K is increased.

For simplicity, we assume that the time between ordering and delivery $(L_1 + 1)$ is an even number of periods and that it can be reduced to $(L_1 + 1)/2$ periods. The setup cost is simultaneously increased from K_1 to aK_1, where a is greater than 1. We assume that all other parameters remain unchanged. From (39) and (40) we see that total cost per period is of the form

$$T \doteq c\ (L + 1)^{\alpha}\ (K/h)^{\beta}$$

where c, α, and β are independent of L and K. The total cost T_1 before L and K are changed is approximated by

$$T_1 \doteq c\ (L_1 + 1)^{\alpha}\ (K_1/h)^{\beta},$$

and after L and K are changed we have

$$T_2 \doteq c\ [(L_1 + 1)/2]^{\alpha}(aK_1/h)^{\beta}.$$

We estimate that the reduced leadtime will realize a savings in total cost if

$$T_2/T_1 \doteq a^{\beta} 2^{-\alpha} \leq 1 .$$

In other words, approximate total cost is reduced if

$$a \leq a^* = 2^{\alpha/\beta} .$$

We interpret a^* as an approximate break-even point. That is, if the setup cost corresponding to the quicker delivery is less than a^*K, then the quick delivery method has lower total cost. Using values of α and β from (39) and (40) we see that a^* equals 2.4 for the Power Approximation policy and 3.0 for the statistical policy. Therefore, it is worthwhile to pay more than double the setup cost for a delivery scheme that is twice as fast.

We calculate numerical results for the case of a equals 2. The cost components in (39), (40) are all of the form

$$\text{Cost} \doteq c\ (L + 1)^{\alpha}\ (K/h)^{\beta} .$$

Let Cost_1 be the value of a cost component before changing L and K, and let Cost_2 be the value after the change. Then we have

$$\text{Cost}_2/\text{Cost}_1 \doteq 2^{\beta-\alpha} .$$

Specifically, for the Power Approximation we have

$$H_2/H_1 \doteq 2^{-.282}$$

$$R_2/R_1 \doteq 2^{.655}$$

$$T_2/T_1 \doteq 2^{-.049} ,$$

and for the statistical policy we have

$$H_2/H_1 \doteq 2^{-.272}$$

$$R_2/R_1 \doteq 2^{.699}$$

$$T_2/T_1 \doteq 2^{-.111} .$$

We list numerical results in Table 4.5. The quicker delivery reduces

Table 4.5

Approximate Percent Changes in Operating Characteristics
$[\ (L_2 + 1)/(L_1 + 1) = 1/2;\ K_2/K_1 = 2\ ]$

Policy	Holding Cost	Backlog Protection	Replenishment Cost	Total Cost
Power Approximation	-18%	0%	57%	-3%
Statistical Power Approx.	-17%	0%	62%	-7%

total cost by 3% for the Power Approximation and by 7% for the statistical policy. These are impressive savings since we assumed that the quick delivery was <u>twice</u> as expensive as the normal delivery. Backlog protection is approximately unchanged since the expressions for P are independent of L and K. The quick-delivery system has lower holding costs and higher replenishment costs.

In Figures 4.7, 4.8, and 4.9 we present graphs of the approximations for holding cost, replenishment cost, and total cost, respectively, as functions of leadtime. Although the curves are plotted as if leadtime

Figure 4.7

Holding Cost Approximations versus Leadtime

($\sigma^2/\mu=9$, $\mu=8$, $p=9$, $K=48$, $h=1$)

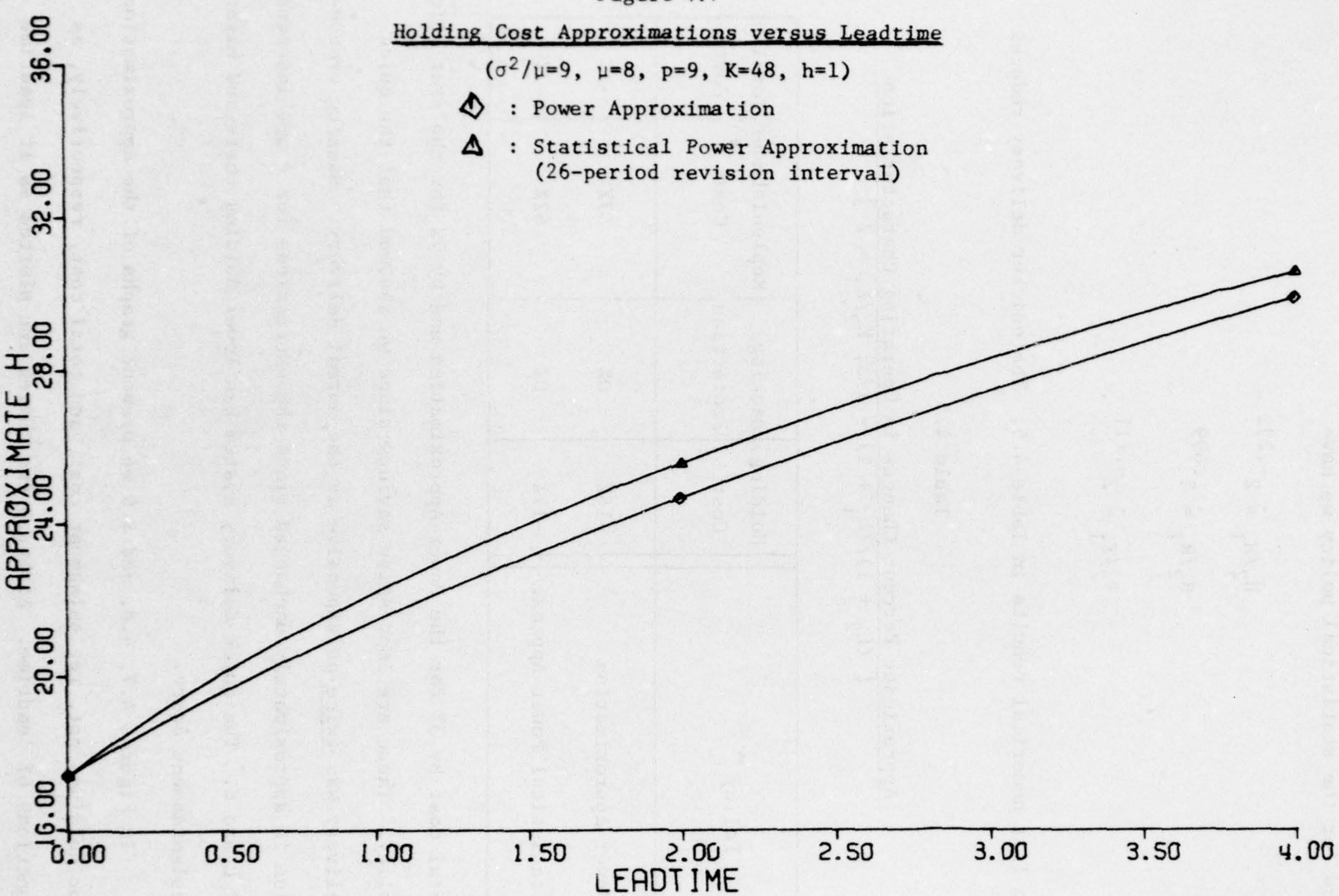


Figure 4.8

<u>Replenishment Cost Approximations versus Leadtime</u>

($\sigma^2/\mu=9$, $\mu=8$, $p=9$, $K=48$, $h=1$)

◇ : Power Approximation

△ : Statistical Power Approximation (26-period revision interval)

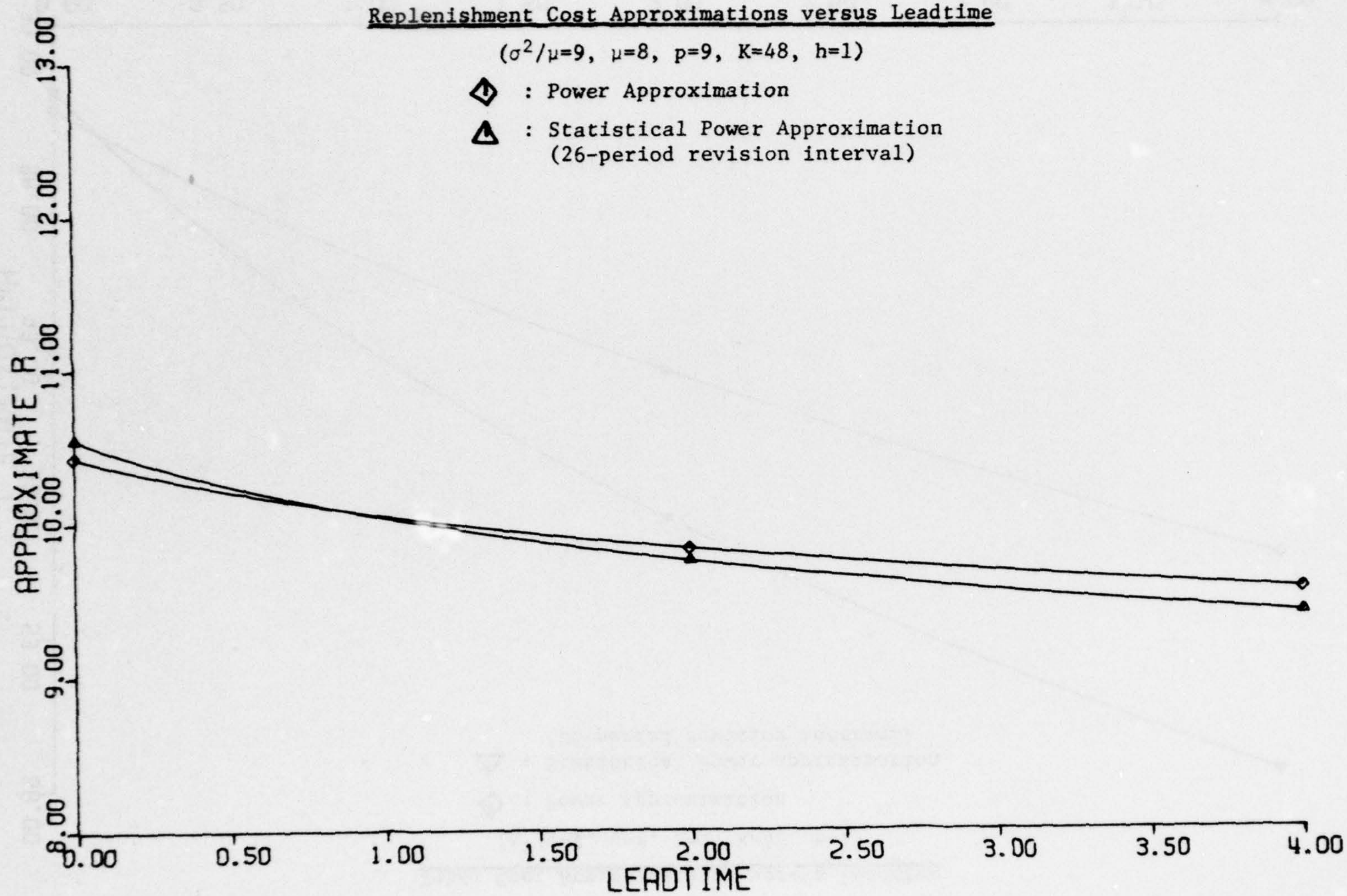


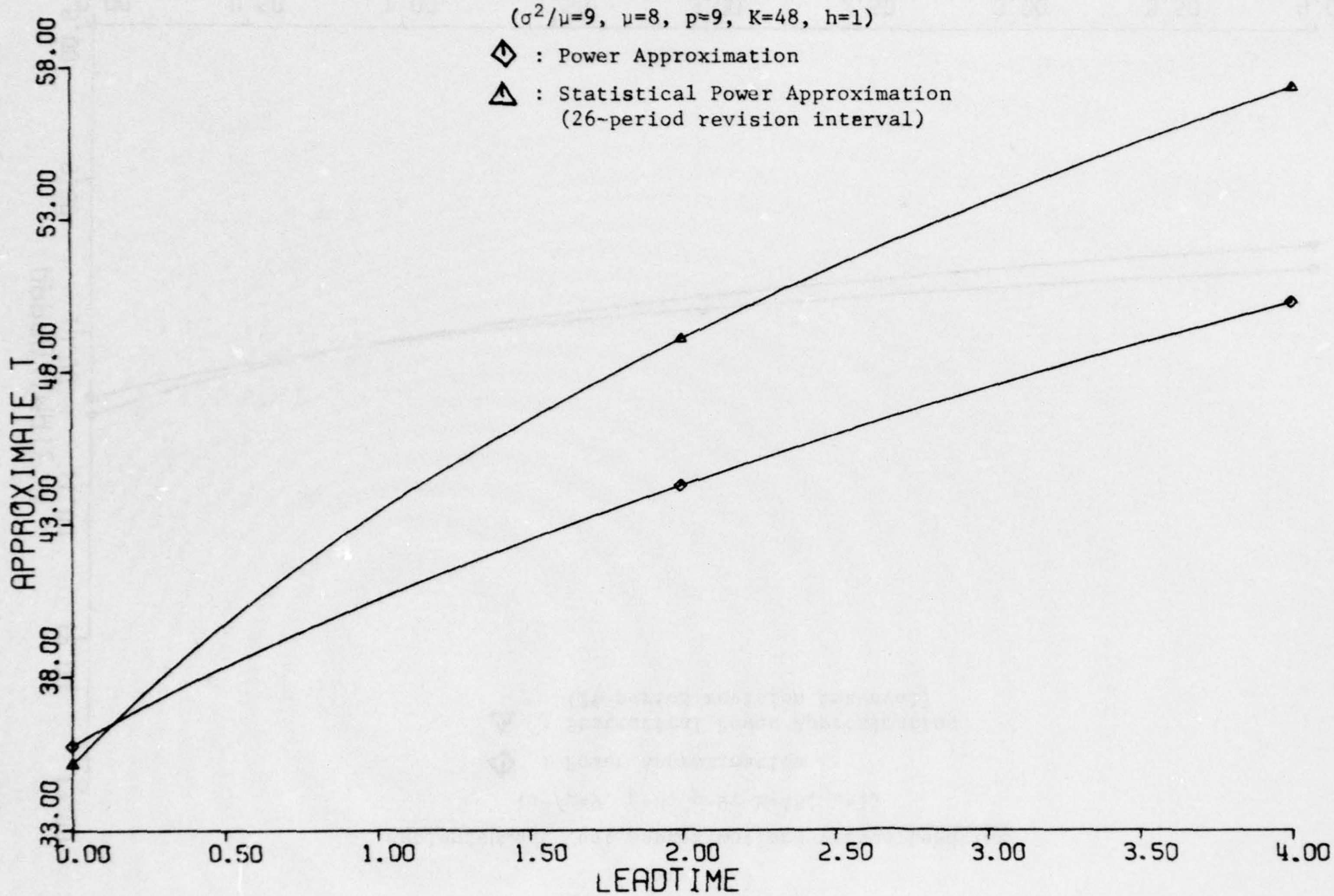
Figure 4.9

Total Cost Approximations versus Leadtime

($\sigma^2/\mu=9$, $\mu=8$, $p=9$, $K=48$, $h=1$)

◇ : Power Approximation

△ : Statistical Power Approximation
(26-period revision interval)



were a continuous variable, we note that leadtime is required to be integer-valued in our model. The curves for holding cost and replenishment cost differ only slightly for fixed- and statistical-policy approximations. Replenishment cost is especially insensitive to changes in leadtime. The curves for total cost display a greater sensitivity to leadtime for the statistical policy than for the fixed policy. The statistical policy curve also has larger values of total cost for all values of leadtime except for a small region near L equals 0. In this region the values of total cost may be considered essentially the same since the difference never exceeds 2% which is less than the average errors of approximation for total cost.

In Figures 4.10, 4.11, and 4.12 we present graphs of the approximations for holding cost, replenishment cost, and total cost, respectively, as functions of K. The replenishment cost curves are nearly identical for statistical- and fixed-policy approximations. For holding cost, the statistical-policy curve has higher values and a steeper slope than the fixed-policy curve. The total cost curves are essentially two parallel lines, with the statistical-policy curve about four units above the fixed-policy curve.

Figure 4.10

Holding Cost Approximations versus Setup Cost Ratio

($\sigma^2/\mu=9$, $\mu=8$, $L=2$, $p=9$, $h=1$)

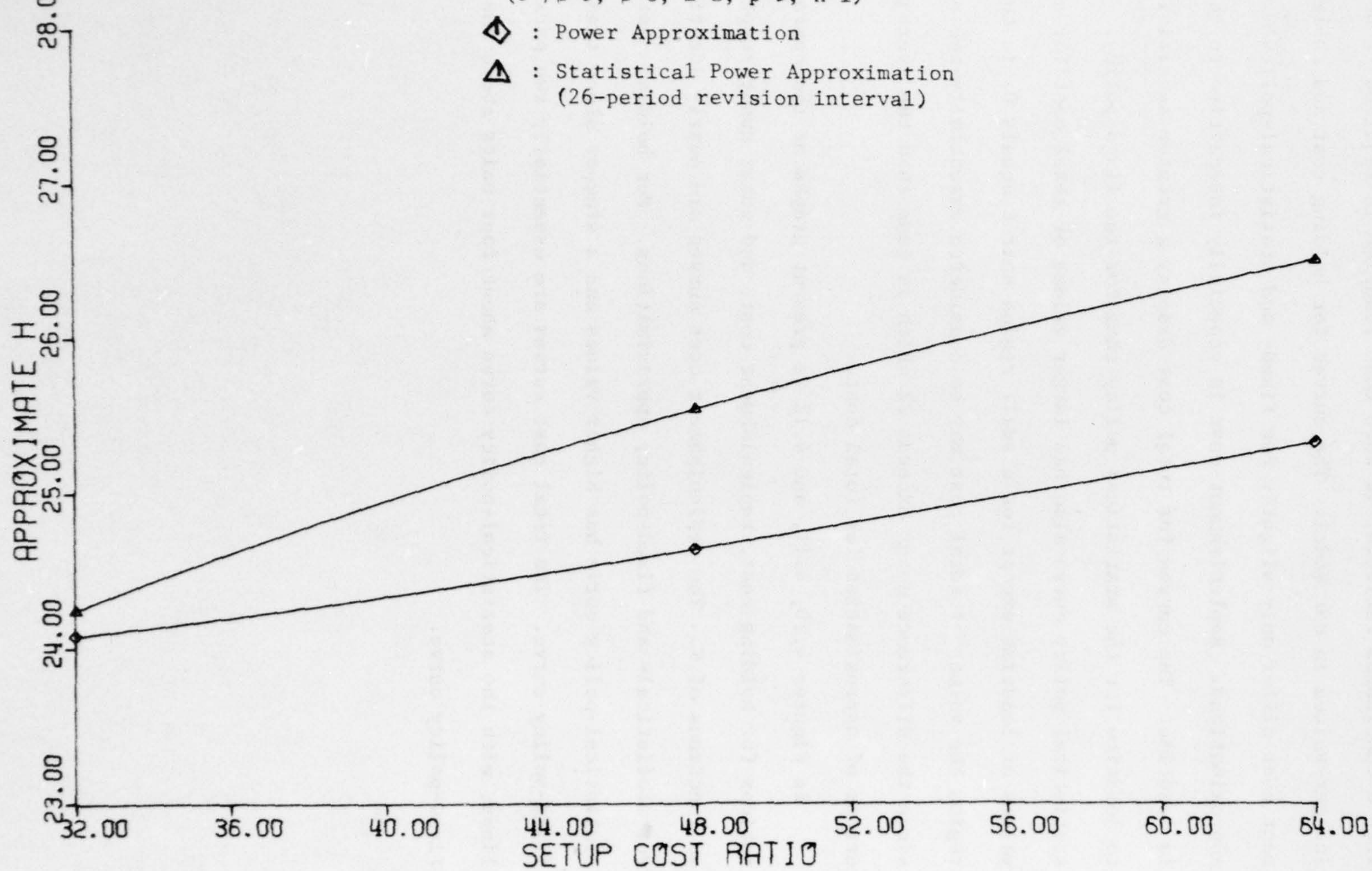


Figure 4.11

<u>Replenishment Cost Approximations versus Setup Cost Ratio</u>

($\sigma^2/\mu=9$, $\mu=8$, $L=2$, $p=9$, $h=1$)

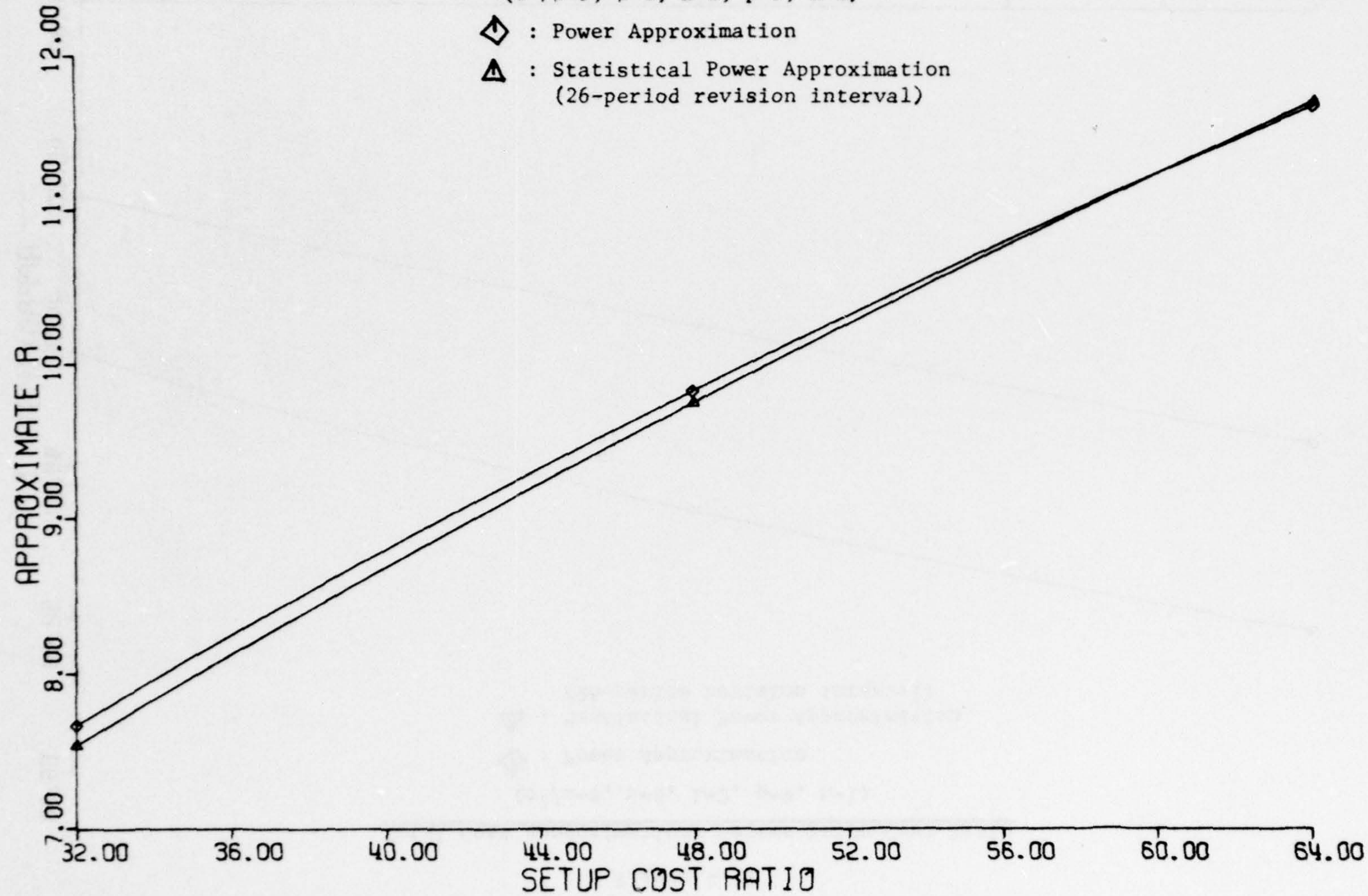
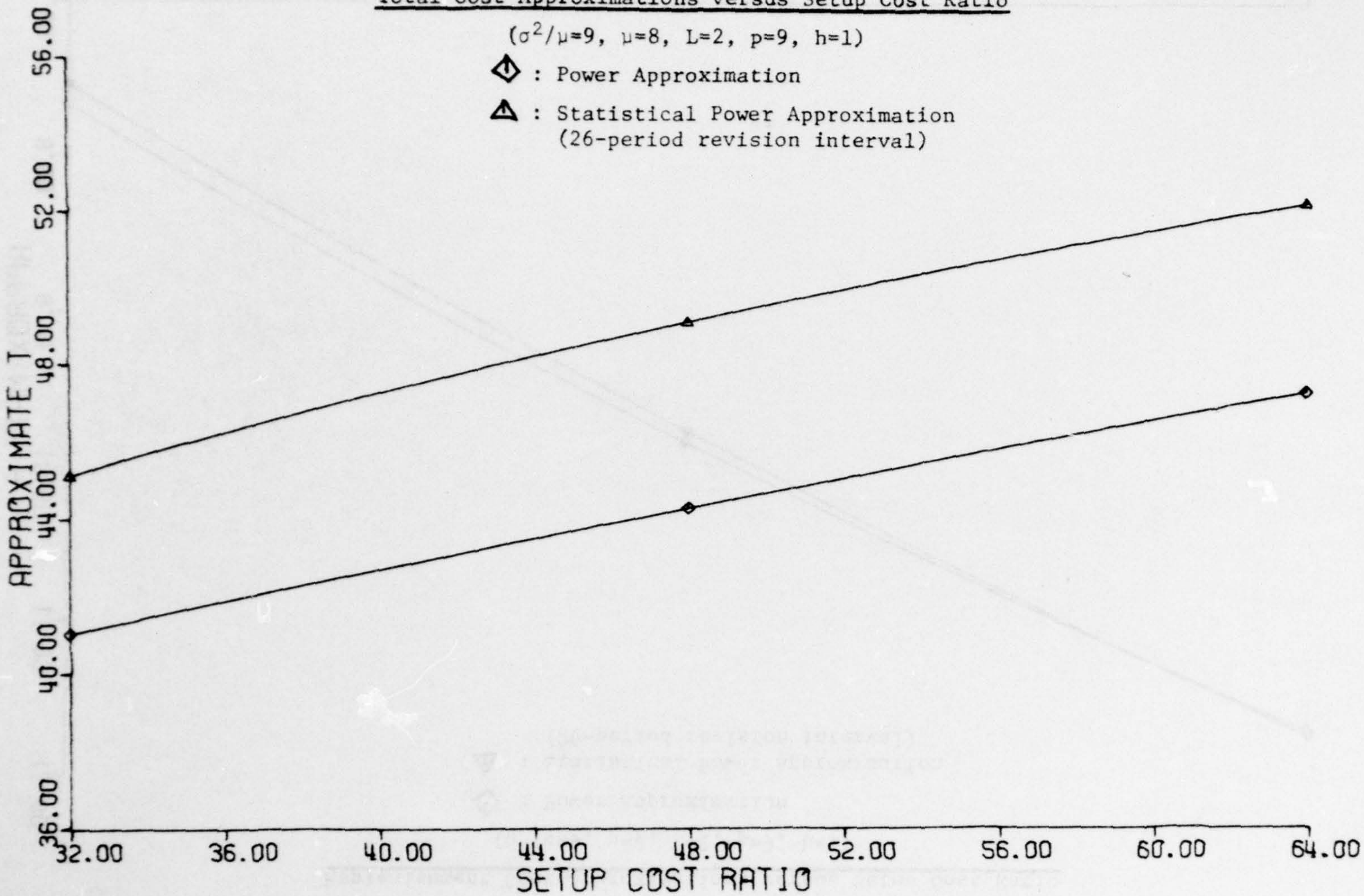


Figure 4.12

Total Cost Approximations versus Setup Cost Ratio

($\sigma^2/\mu=9$, $\mu=8$, $L=2$, $p=9$, $h=1$)



4.4 Sensitivity to Unit Holding Cost

Suppose the unit holding cost changes from h_1 to h_2. What happens to the operating characteristics? We assume that the designer chooses to maintain the same level of backlog protection, so p/h will remain unchanged. From (39) and (40) we see that all the other characteristics are of the form

$$\text{Cost} \doteq c\ h\ (K/h)^{\alpha},$$

where c and α are independent of h. Let Cost_1 and Cost_2 be the values of a cost component when h equals h_1 and h_2, respectively. We then have

$$\text{Cost}_2/\text{Cost}_1 \doteq (h_2/h_1)^{1-\alpha}.$$

Specifically, for the Power Approximation we have

$$H_2/H_1 \doteq (h_2/h_1)^{.931}$$

$$R_2/R_1 \doteq (h_2/h_1)^{.403}$$

$$T_2/T_1 \doteq (h_2/h_1)^{.796},$$

and for the statistical policies we have

$$H_2/H_1 \doteq (h_2/h_1)^{.900}$$

$$R_2/R_1 \doteq (h_2/h_1)^{.373}$$

$$T_2/T_1 \doteq (h_2/h_1)^{.808}.$$

We present numerical results in Table 4.6 for the case of h_2/h_1 equal 1.1.

Table 4.6

<u>Approximate Percentage Increases in Costs</u>
($h_2/h_1 = 1.1$; p/h held constant)

Policy	Holding Cost	Replenishment Cost	Total Cost
Power Approximation	9%	4%	8%
Statistical Power Approximation	9%	4%	8%

In Figures 4.13, 4.14, and 4.15 we present graphs of the approximations for holding cost, replenishment cost, and total cost, respectively, as functions of h. As in the discussion above, we have held p/h constant at 9. The curves for holding cost and replenishment cost are nearly the same for fixed and statistical policies. The statistical-policy curve for total cost is about 10% higher than the fixed-policy curve. Note that the curves for holding cost and total cost are nearly straight lines.

Figure 4.13

Holding Cost Approximations versus Unit Holding Cost

($\sigma^2/\mu=9$, $\mu=8$, $L=2$, $p/h=9$, $K=48$)

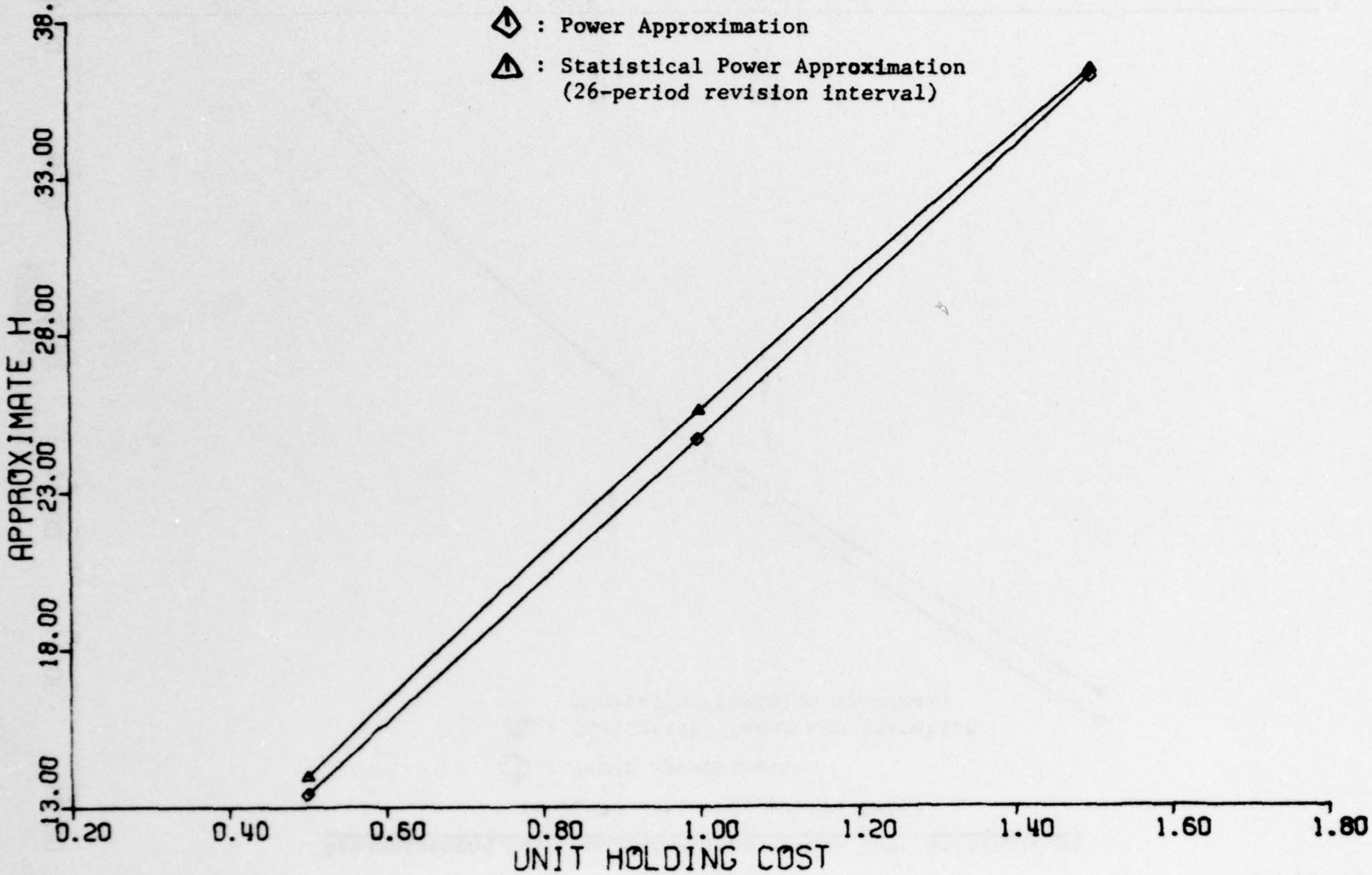


Fig. 4.14

Replenishment Cost Approximations versus Unit Holding Cost

($\sigma^2/\mu=9$, $\mu=8$, $L=2$, $p/h=9$, $K=48$)

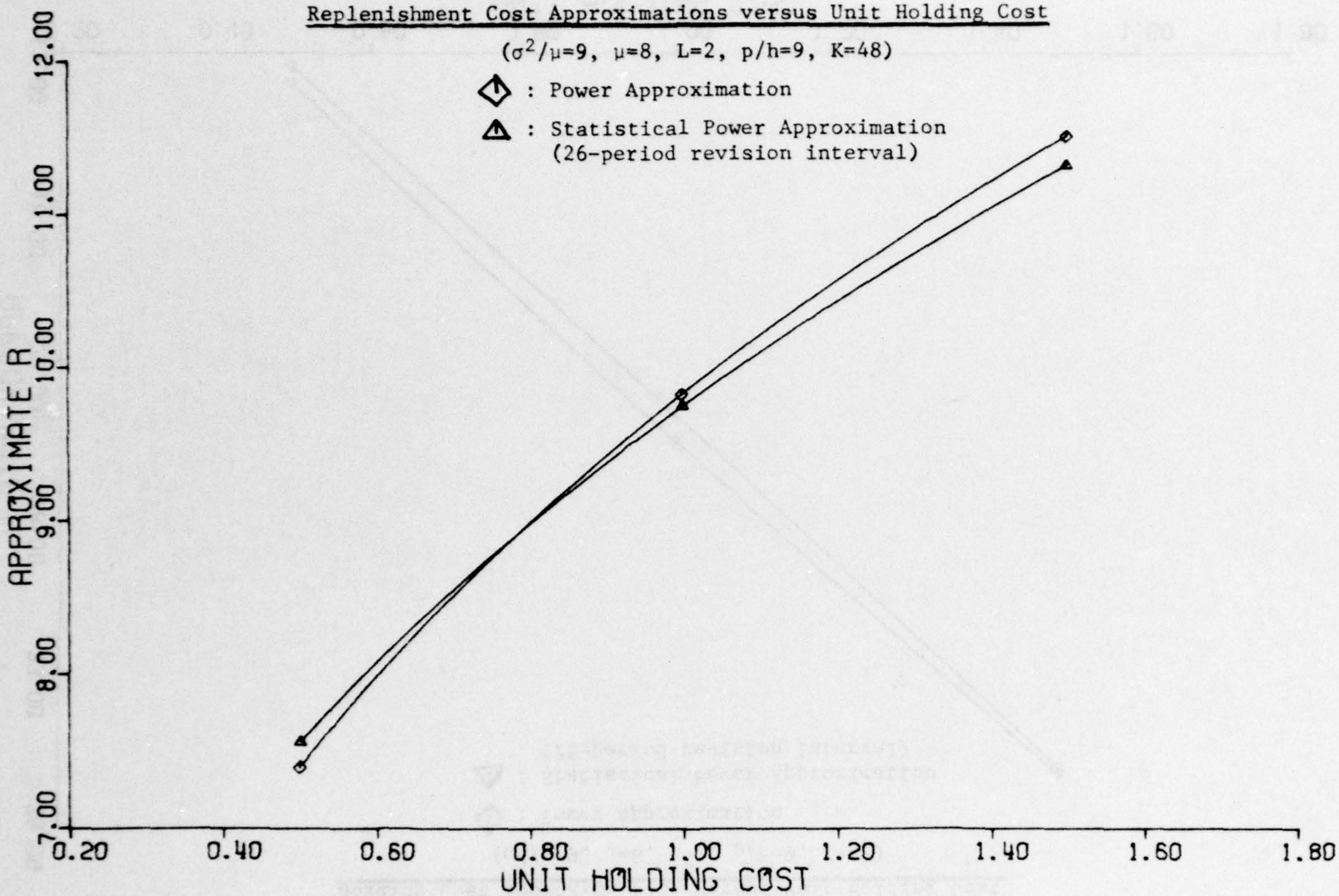
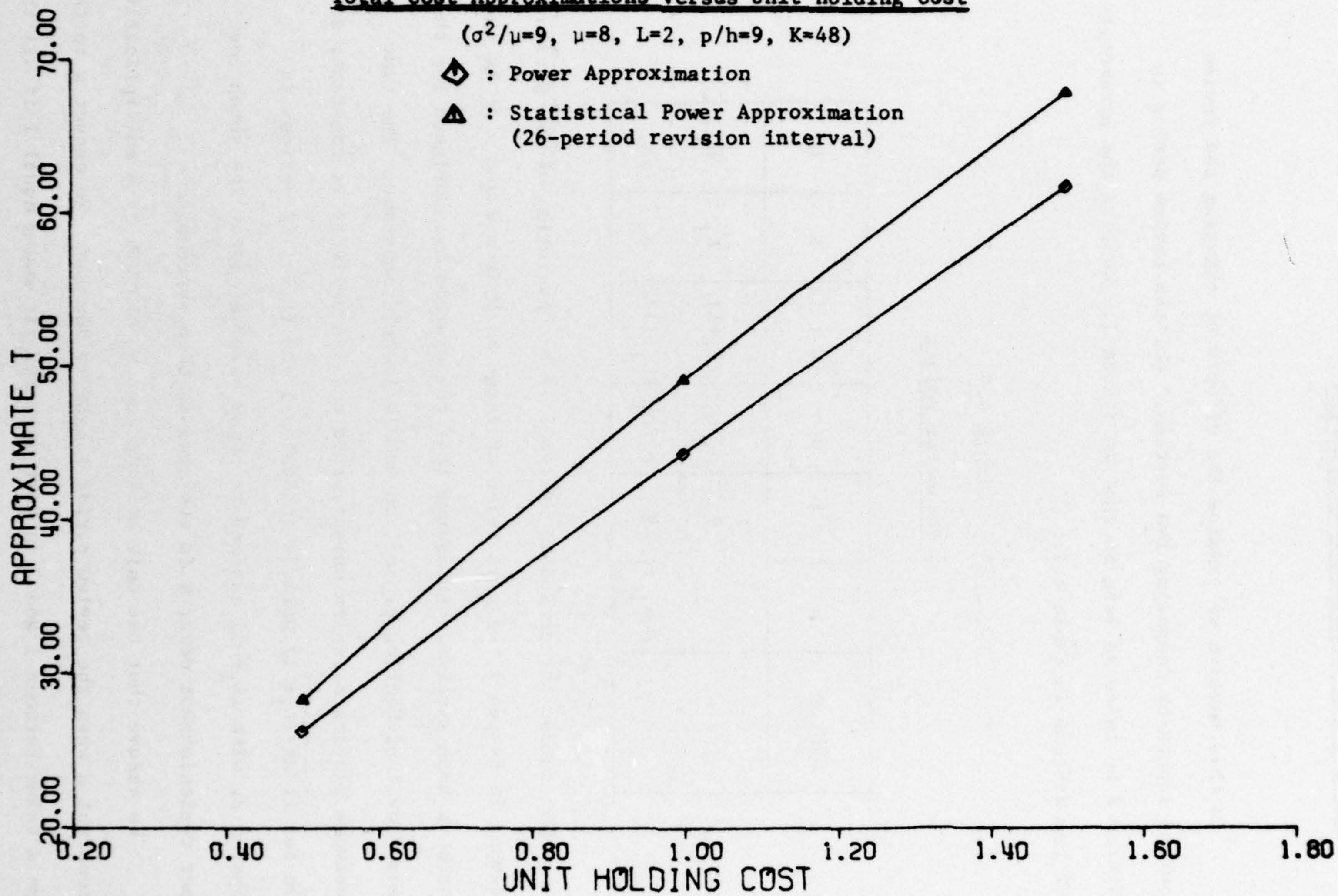


Fig. 4.15

Total Cost Approximations versus Unit Holding Cost

($\sigma^2/\mu=9$, $\mu=8$, $L=2$, $p/h=9$, $K=48$)



4.5 Sensitivity to Review Period Length

In this section we examine the effects of changing the review period length by comparing two systems. Let the review period in System 2 be twice as long as that in System 1. We list the parameters of the systems in Table 4.7.

Table 4.7

Parameter Values

System	μ	σ^2/μ	p/h	(L + 1)	K	h
1	μ_1	9	p_1/h_1	(L_1+1)	K_1	h_1
2	$2\mu_1$	9	p_1/h_1	$(L_1+1)/2$	K_1	ah_1

The demand per period in System 2 is a convolution of two periods' demand in System 1, so μ is twice as large in System 2 and σ^2/μ is the same in both systems. We assume that the systems are designed for the same level of backlog protection so p/h is held constant. The time between placing a replenishment order and its delivery is constant, so we set it at (L_1+1) periods in System 1 and $(L_1+1)/2$ periods in System 2, with (L_1+1) assumed to be an even integer. The setup cost per replenishment order K is the same in both systems.

We assume that the unit holding cost h_1 changes by a multiplicative constant a when the review period length is doubled. We expect a to take on a value between 1 and 2 in most systems. It would equal 1 if all the holding cost were attributable to review functions that occur once each period. On the other hand, a would equal 2 if all the holding cost were proportional to time-in-storage.

We now examine the effects of review period length on the cost components of the Power Approximation. Since σ^2/μ and p/h are the same in both systems, we express the cost approximations (39), (40) for Systems 1 and 2 in the form

$$\text{Cost}_1 \doteq c\, h_1 \mu_1^{\alpha} (L_1 + 1)^{\beta} (K_1/h_1)^{\delta}$$

$$\text{Cost}_2 \doteq c(ah_1)\ (2\mu_1)^{\alpha} [(L_1 + 1)/2]^{\beta}\ (K_1/ah_1)^{\delta}$$

where c, α, β, and δ are independent of review period length. We take the ratio of <u>costs per unit time</u> by dividing the costs in System 2 by twice the costs in System 1, yielding

$$\text{Cost}_2/2\ \text{Cost}_1 \doteq 2^{\alpha-\beta-1} a^{1-\delta} .$$

Specifically, we have

$$H_2/2H_1 \doteq .540\ a^{.931}$$

$$R_2/2R_1 \doteq .771\ a^{.403}$$

$$T_2/2T_1 \doteq .573\ a^{.796} .$$

In Table 4.8 we list approximate values for the percentage by which costs in System 2 differ from those in System 1, for several values of a. Substantial cost reductions occur for the longer revision interval when a is close to 1, and very small changes occur when a equals 2.

For the statistical policies we assume that the policy revision interval is the same length of time in both systems. That is, revision

Table 4.8

Approximate Percentage Changes in Costs per Unit Time
(Power Approximation Policies)

a	Holding Cost	Replenishment Cost	Total Cost
1	-46%	-23%	-43%
1.5	-21%	- 9%	-21%
2.0	3.0%	1.9%	-.5%

occurs every T_1 periods in System 1, and every $T_1/2$ periods in System 2, with T_1 assumed to be an even integer. We use the approximations obtained in Chapter 9 that are explicit functions of revision interval length. We set p/h equal to 9 and use the approximations

$$H'_{(9,T)} = 4.000\, h\, \mu^{.4587 + .00042T} (L+1)^{.2994 + 1.250/T} (K/h)^{.1280}$$

$$(43)\quad R'_{(9,T)} = .2872\, h\, \mu^{.5540} (L+1)^{-.0568 - .3930/T} (K/h)^{.6333}$$

$$T'_{(9,T)} = 6.002\, h\, \mu^{.4680} (L+1)^{.2659 + .9762/T} (K/h)^{.2008 + .1115/T} .$$

We express these cost components in the form

$$\text{Cost} \doteq c\, h\, \mu^{\alpha + rT} (L+1)^{\beta + m/T} (K/h)^{\delta + n/T}$$

where $c, \alpha, \beta, \delta, r$, m, and n are independent of revision interval length. As before, we take the ratio of cost in System 2 to twice the cost in System 1, yielding

$$\text{Cost}_2/2\ \text{Cost}_1 \doteq \left[2^{\alpha-\beta-1} a^{1-\delta}\right] \cdot \left[2^{rT_1/2 - 2m/T_1} a^{-2n/T_1}\right]$$
$$\cdot \left[\mu_1^{-rT_1/2} (L_1 + 1)^{m/T_1} (K_1/h_1)^{n/T_1}\right] .$$

Specifically, we have

$$H_2/2H_1 \doteq .5584\, a^{.872}\left[2^{.00021T_1 - 2.50/T_1}\, \mu_1^{-.00021T_1}\, (L_1 + 1)^{1.250/T}\right]$$

$$R_2/2R_1 \doteq .7636\, a^{.367}\left[2^{.786/T_1}\, (L_1 + 1)^{-.393/T_1}\right]$$

$$T_2/2T_1 \doteq .5752\, a^{.799}\left[a^{-.223/T_1}\, 2^{-1.95/T_1}\, (L_1 + 1)^{.976/T_1}\, (K_1/h_1)^{.112/T_1}\right] .$$

We evaluate a numerical example for the case of μ_1 equals 4, L_1 equals 3, K_1/h_1 equals 64, and T_1 equals 26. The results are listed in Table 4.9 for three values of a. Each entry in the table is the percentage by

Table 4.9

Approximate Percentage Changes in Cost per Unit Time
(Statistical policy with $\mu_1 = 4$, $L_1 = 3$, $K_1/h_1 = 64$, $T_1 = 26$)

a	Holding Cost	Replenishment Cost	Total Cost
1	-44%	-24%	-41%
1.5	-21%	-11%	-19%
2.0	1.8%	-1.5%	1.3%

which a cost in System 2 differs from a cost in System 1. The effects of the change in revision interval length are nearly the same as for the Power Approximation. Substantial cost reductions occur for the longer revision interval when a is near 1, and very small changes occur when a equals 2.

4.6 Sensitivity to Variance-to-mean Ratio

In Figures 4.16, 4.17, and 4.18 we present graphs of the approximations for holding cost, replenishment cost, and total cost, respectively, as functions of σ^2/μ. We have curves only for Power Approximation policies because the statistical-policy approximations do not have an explicit dependence on σ^2/μ. We do not have a curve for backlog protection since the approximation for P is a function of only p/h.

The curve for holding cost is nearly linear with an increase of approximately 70% when σ^2/μ is increased from one to nine. The replenishment cost curve is convex in shape, decreasing by about 20% when σ^2/μ increases from one to nine. For total cost we see a nearly linear relationship and an increase of about 50% when σ^2/μ is increased from one to nine.

Figure 4.16

Holding Cost Approximation versus Variance-to-mean Ratio

(Power Approximation policy with $\mu=8$, $L=2$, $p=9$, $K=48$, $h=1$)

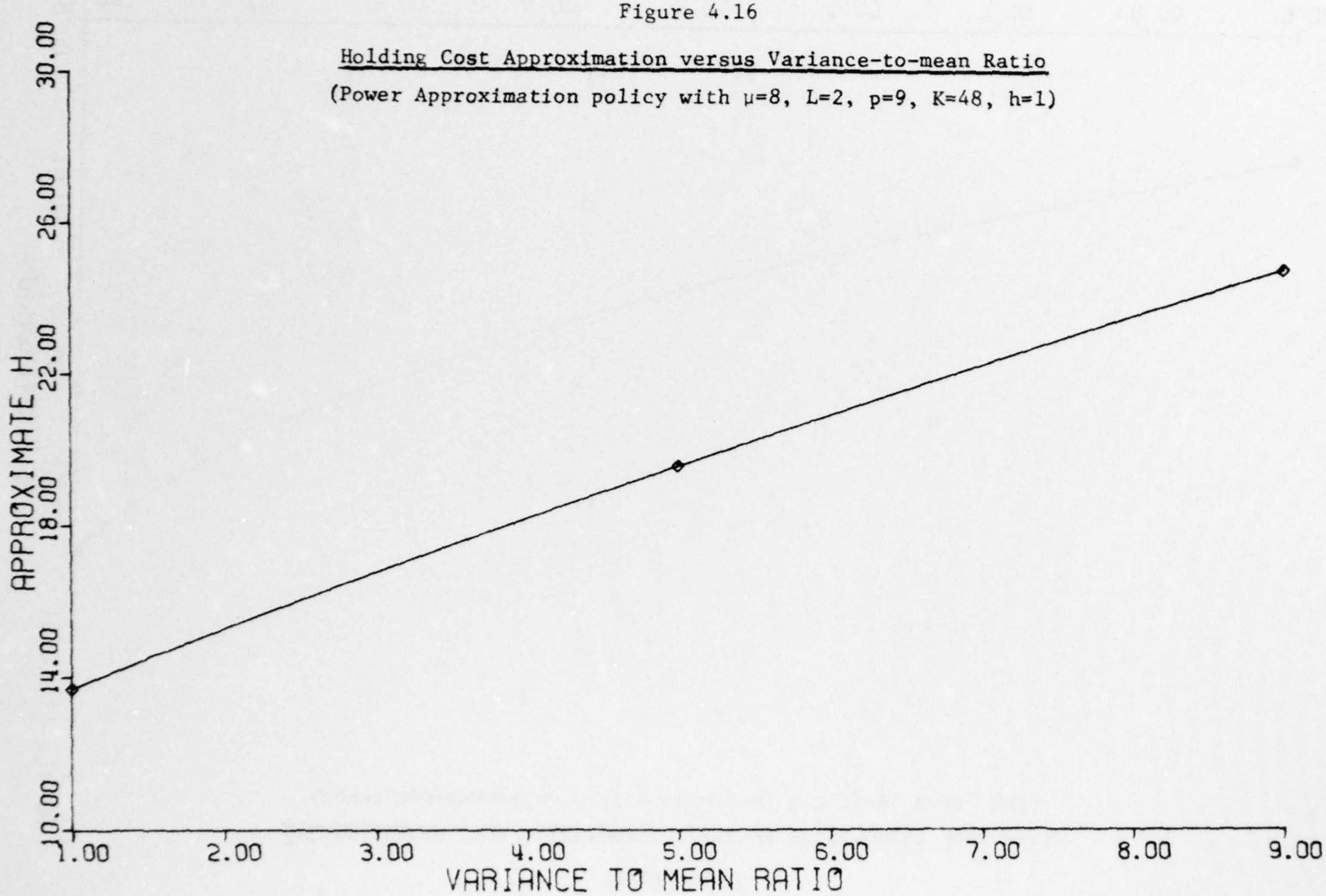


Figure 4.17

<u>Replenishment Cost Approximation versus Variance-to mean Ratio</u>

(Power Approximation policy with μ=8, L=2, p=9, K=48, h=1)

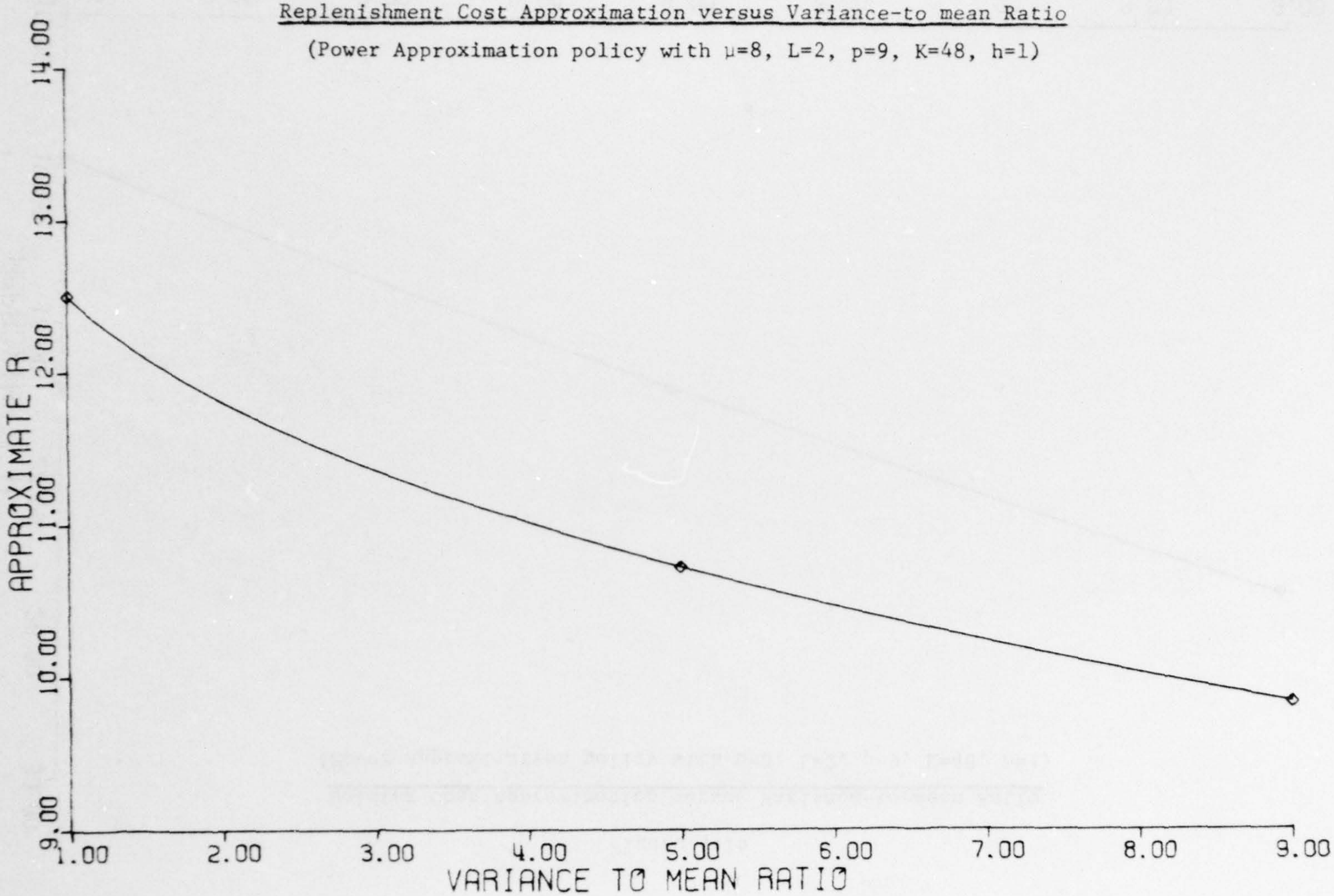
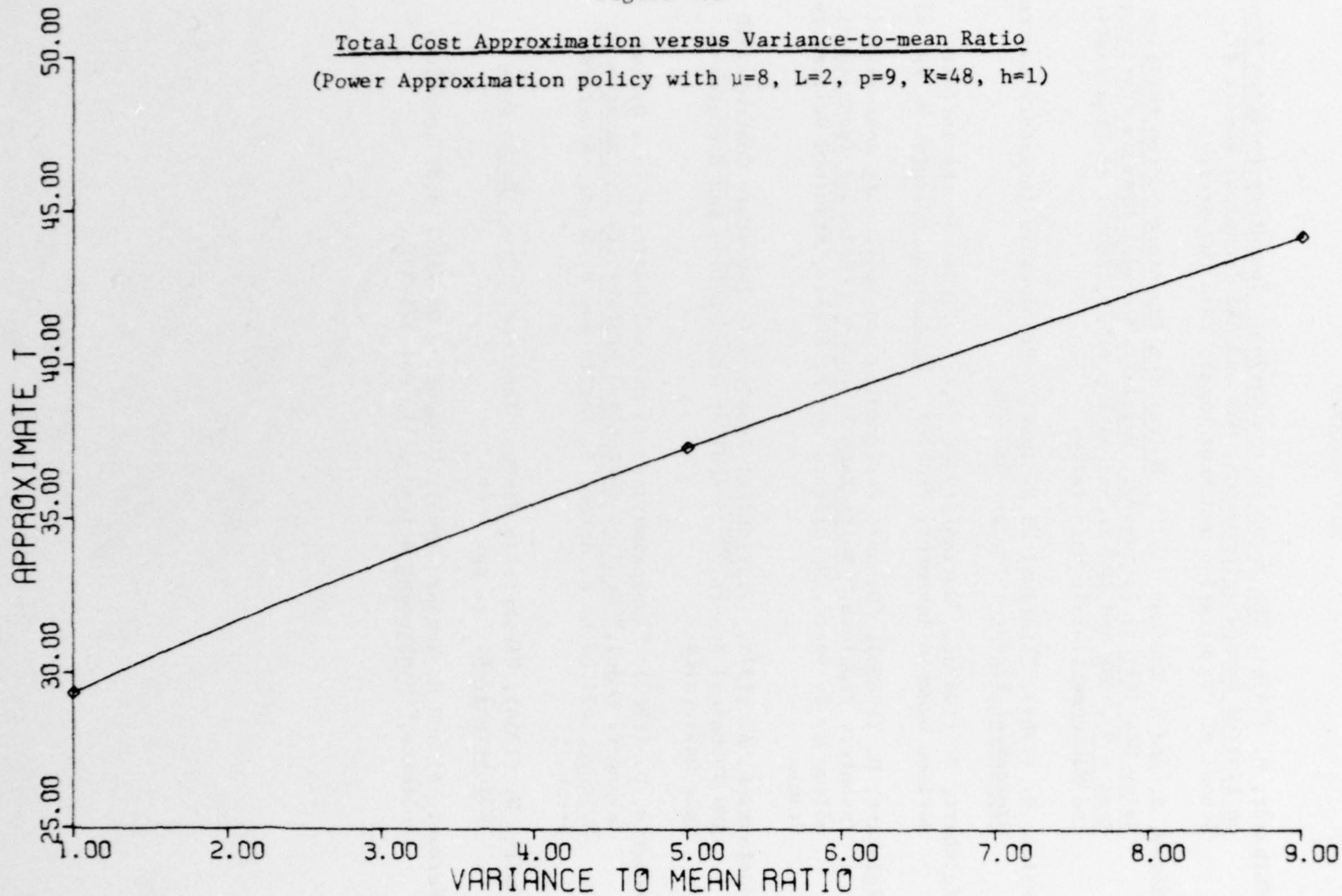


Figure 4.18

Total Cost Approximation versus Variance-to-mean Ratio

(Power Approximation policy with $\mu=8$, $L=2$, $p=9$, $K=48$, $h=1$)



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