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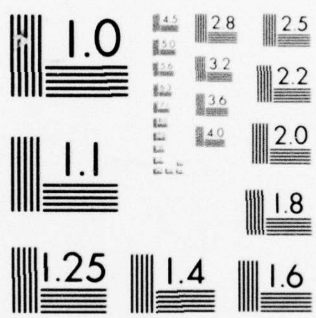

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PREFERENCE MODELING OF UTILITY SURFACES

STANFORD UNIVERSITY

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ADVANCED DECISION TECHNOLOGY PROGRAM

CYBERNETICS TECHNOLOGY OFFICE
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PREFERENCE MODELING OF UTILITY SURFACES,

by

9 Technical rept.

20 Khaled H. Nahas

DECISION ANALYSIS PROGRAM

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SUMMARY

This research proposes a new framework for constructing preference functions when the consequences of a decision are judged by multiple attributes or criteria. Methods for assessing single-attribute preference functions have been well established. The current, state-of-the-art procedure for deriving arbitrary, multiattribute preference functions, however, has required regularity assumptions to reduce the arbitrariness of the preferences. This technique, called decomposition, has been used because it usually results in a simple and appealing preference model. The difficulty with this approach is seen in the restrictiveness of the two main assumptions:

(1) the same preference regularity is imposed on all attributes, using this symmetry to achieve a simple preference function, and

(2) any n-dimensional preference function is decomposed into n one-dimensional preference functions.

In this work, no symmetry assumption is necessary, as each multiattribute preference function is tailor-fit to only those regularities that exist in a particular problem setting. Those parts of the preference function that are subject to simplifying assumptions are decomposed using a new classification scheme to

derive further independence assumptions for the standard models. Those parts of the preference function that are indecomposable are handled using a new discretization scheme along with a behaviorally motivated interpolation rule to fill the gaps. The flexibility of these methods allows an analyst to make trade-offs between the degree of accuracy desired and amount of effort needed.

This integrated framework for decomposable and indecomposable multiattribute preference functions stands as an important decision analysis aid. The usefulness of the framework is illustrated in an example of the decision to buy a new car. The relationship of the attributes can be assessed in advance, thus allowing an optimal decision to be made in the decision maker's absence.

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When Isaac Newton was asked once how he had made his discoveries, he replied, "By always thinking about them, I keep the subject constantly before me and wait 'til the first dawnings open little by little into the full light".

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Chapter 1

AN OVERVIEW

1.1 Introduction

Decision science uses multiattribute utility theory to treat problems where the consequences of a decision are judged by multiple attributes or criteria. Although the theoretical basis of the multiattribute model is relatively simple and straightforward, decision analysts face many problems when trying to apply the model. The main difficulty encountered is that of constructing the decision maker's multidimensional utility surface for a given problem. Actually, except for the case of one-attribute utilities, there are no general procedures or techniques for directly assessing arbitrary multiattribute utility surfaces. Hence, the current state of the art is to impose different regularity assumptions on the utility surface to make it less arbitrary. The main theme is to be able to derive analytically the multidimensional utility from one-dimensional utilities which are assessed directly and are completely arbitrary. Such procedures are generally called decomposition or separability techniques.

The first attempt to decompose a Von Neumann utility surface, which is what we deal with in this work, is in a 1965 article by Fishburn [8]. The article contains a system of axioms that guarantees the so-called additive utility model. Thereafter, other researchers propose other decompositional axioms that produce different simple and appealing utility forms such as Pollak's [30] multiplicative model and Keeney's [20] quasi-separable model.

Two features seem to characterize most of the work in this area. The first feature is that of the researcher's emphasis on symmetry and functional simplicity of the utility form. By symmetry, we mean that whatever preference regularity is imposed on one attribute is also imposed on all other attributes. The other feature is that of requiring full decomposition, i.e., an n-attribute utility surface is decomposed into n one-attribute utilities.

In our work, we deviate from both features. We do not insist on a prior form but, rather, we model the underlying preferences, seeking regularities (of a certain class) that exist and deriving the utility form that reflects only the acknowledged regularities. Thus, no symmetry is required, and the utility form is tailor-fit to each particular preference setting. As such, there may be utility forms where the n-attribute surface is not fully decomposed. For the indecomposable subspaces, we propose the methodology and theory of Chapter 5.

We view our work as an integrated framework for constructing multiattribute utility surfaces in two main stages.

Stage 1: Modeling the underlying preferences for a given setting. The idea here is to take advantage of whatever regularities the preference setting reflects. Many concepts of preference regularity are proposed in the literature. We have elected to work with the utility independence concept that captures a certain kind of preference regularity, which we will define later. Other concepts can be used as well.

Stage 2: If the modeling from Stage 1 does not produce a fully decomposed surface, the indecomposable subspaces are treated as such and the strategy of continuous cuts, presented in Chapter 5, is used to construct an approximation for the utilities over these subspaces.

The framework is flexible enough so that the analyst using it can continually make trade-offs between the degree of accuracy he requires of the constructed surface, and the amount of effort he and the decision maker are willing to expend on the assessment.

Even though our research is geared toward utility theory, the framework may be extended in two broad directions. First, the surface to be assessed can be any measurement theoretic surface that preserves some ordering defined on a set of objects where the order of an object is determined by multiple factors. Second, the modeling of the ordering may be conducted using any well-defined concept of consistency or regularity that is meaningful to the ordering. For instance, the problem of measuring different human senses under controlled laboratory conditions may be approached with the broad outline of this framework.

1.1.1 Basic Definitions

Since most of the terminology in utility theory is not standardized, we will define and explain some of the terms as we use them. Any decision facing the decision maker, hereafter referred to as the DM, results in a consequence to him. A consequence is judged by multiple factors or characteristics that we call attributes. An attribute x_i is defined by an attribute set, X_i, which, unless explicitly stated, can be any arbitrary set with at least two elements. The values an attribute takes need not be real numbers, and the X_i can be finite, countable, or uncountable.

Throughout our work, the number of attributes for a given problem, denoted n, is arbitrary but finite. A potential problem may have from two to three attributes, all the way up to thousands of attributes. For a problem with n attributes, the cartesian product $X = X_1 \times \ldots \times X_n$ is called the (whole) product set of attributes. In the case where decisions lead to certain consequences (no risk is involved), X is also called the choice space. For most of our work, the type of utility theory we deal with explicitly treats uncertainty as one more feature of the consequences of decisions. As such, the choice space we deal with is the space of all simple* probability distributions over X, denoted P. It should be clear to the reader that when the choice set is P, then X is a subset of P. This is so since any element in X corresponds to a degenerate probability distribution.

The DM preference structure is an ordering $\precsim$ defined on the choice space. A utility is a real valued function defined on the choice space with the main feature of preserving the ordering of the preference structure. When the choice space is X, i.e., there is no uncertainty involved, the utility over X is also called a value function. When the choice space is P and the preference structure is consistent with the system of axioms proposed by Von Neumann and Morgenstern [38], to be described later, the order preserving function is called a Von Neumann utility. We almost exclusively will deal with Von Neumann utilities. Thus, for the sake of brevity, we sometimes delete the adjective Von Neumann.

A utility surface is the part of the utility function that is defined on X. For a Von Neumann utility, once the utility surface

*A probability distribution is simple if the number of points with nonzero probability is finite.

(on X) is obtained, the utility on P is derived via the (probability) expected value operator. When assessing the DM's preferences, all the analyst has to do is to construct the DM's utility surface on X. The dimensionality of the surface is equal to the number of attributes of the problem at hand. A conditional utility is the restriction of the utility surface to the product set of some subset of the attributes. A decompositional axiom is an assumption via which the utility surface, of a certain dimension, is derived analytically from conditional utilities of lower dimensions. Since the assessment effort required to construct a utility decreases with less dimensions, a decompositional axiom, if satisfied for a given choice setting, is a welcome relief to the analyst.

1.2 Background and Related Work

In this section, we review some of the background and important previous work that relates to our research.

1.2.1 Von Neumann Utility Theory

The theoretical basis of Von Neumann and Morgenstern [38] guarantees the existence of these utilities. Since the publication of this work, several other researchers proposed other systems of axioms which are equivalent to the Von Neumann system, with minor modifications and extensions. The following system of axioms, taken from Fishburn [9], is typical and is considered standard.

Let $X = X_1 \times \ldots \times X_n$ be the product set of attributes, and let P be the space of all simple probability distributions on X. Let $\precsim$, read "not preferred to," be the preference ordering defined on P.

The relations of strict preference, $\prec$, and indifference, $\sim$, are defined from $\precsim$ as follows: Let $p_1, p_2 \in P$.

$$p_1 \prec p_2 \iff (p_1 \precsim p_2 \text{ and not } p_2 \precsim p_1)$$

$$p_1 \sim p_2 \iff (p_1 \precsim p_2 \text{ and } p_2 \precsim p_1)$$

<u>Axiom 1</u>: $\precsim$ is a weak ordering on P. That is, $\precsim$ on P is connected and transitive.

<u>Axiom 2</u>: (sure thing) If $p_1, p_2, p_3 \in P$ and if $p_1 \prec p_2$, then, for any real number $\alpha \in (0,1)$, we have

$$\alpha p_1 + (1 - \alpha)\, p_3 \prec \alpha p_2 + (1 - \alpha)\, p_3$$

<u>Axiom 3</u>: (Archimedean) If $p_1, p_2, p_3 \in P$ and if $p_1 \prec p_2$ and $p_2 \prec p_3$, then there are numbers $\alpha, \beta \in (0,1)$ such that

$$\alpha p_1 + (1 - \alpha)\, p_3 \prec p_2 \prec \beta p_1 + (1 - \beta)\, p_3$$

If these axioms are satisfied for a given preference, then we are guaranteed the existence of a real valued utility function, u, defined on P, with the following important characteristics.

(1) Order preserving: That is, if $p_1, p_2 \in P$ and $p_1 \precsim p_2$, then

$$u(p_1) \leq u(p_2)$$

where $\leq$ is the standard order, less than or equal to, defined on the real line.

(2) Expected utility property: The utility of a probability distribution $p \in P$ is

$$u(p) = \sum_{x \in X} u(x) \cdot p(x)$$

(3) u is unique up to a positive linear transformation. That is, if u and u' are two real valued functions satisfying the above two properties, then there exists two real numbers γ_1 and γ_2 with $\gamma_2 > 0$ such that

$$u = \gamma_1 + \gamma_2 u'$$

u is made unique by arbitrarily fixing its values at two points of the domain.

The first property indicates that u contains complete information about the preferences. The second property indicates that, even though u (on P) is an infinite dimensional surface, all we need to do to have u is to construct its restriction on the finite dimensional space X and then extend the domain to P analytically. The third property, which is the basis of most decomposition techniques, says that a whole family of functions are eligible candidates to represent the preferences. We should remark that there are no other properties required of u such as continuity, differentiability, or monotonicity. u can be any arbitrary surface that satisfies the above system of axioms. It is possible, as we will see later, to augment the axioms such that the evolving utility exhibits further regularity and smoothness properties.

1.2.2 Approaches for Constructing Utilities

We will review here some of the ideas for constructing utility surfaces on the product set of attributes, X. The literature (for instance, see Pratt et al [32]) contains techniques and methodologies for directly assessing utilities on a single attribute without requiring any simplifying assumptions other than the utility is of the Von Neumann variety. For the case of more than one attribute, no such methodology exists and different kinds of simplifying assumptions have to be imposed on the utility to restrict its form. This hopefully reduces the assessment effort required for construction. The most important, and most popular, procedures for introducing simplifying assumptions are the so-called decomposition or separability techniques. We will treat decomposition in detail in the next section. Here, we consider some of the other important approaches.

The Two-Step Decomposition Procedure

Here, the word decomposition is used in a different sense from that of the next section. This procedure advocates the 'decomposing' of the utility assessment effort into two stages.

(1) The assessment of the deterministic trade-offs on X. This corresponds to constructing a value function by treating X as the whole choice space (no uncertainty involved). In economics, the value function is referred to as the indifference curves.

(2) Once a value function is constructed, a Von Neumann utility is obtained by invoking a risk on a properly chosen scalar numeraire.

For more details, see Boyd [2]. This procedure is of great theoretical importance, but of limited applicability. From an application point of view, the procedure transforms the challenge of directly assessing a Von Neumann utility into the problem of assessing multidimensional indifference curves in X. The latter problem, in general, can be just as difficult or perhaps more so than the former. One instance where this procedure is used to advantage is where the indifference curves can be constructed analytically via economic modeling of the underlying deterministic trade-offs.

Dean Boyd's Work

Boyd [2], among others, has a novel stand on the construction of utilities. Recognizing that utilities are used to solve decision problems, Boyd's approach does not require the construction of the whole surface but, rather, only to assess the necessary utility information to solve the decision problem at hand. Thus, he proposes an optimization algorithm that is applied to these problems. The algorithm produces the optimal decision without requiring a priori an explicit representation of the utility. Rather, it solicits the necessary information about the utility at each iteration as needed. The main idea of the algorithm is that of approximating the utility surface at a given point by a first order Taylor approximation. Minimal knowledge about u is required to construct such local approximation. This knowledge has to be assessed directly from the DM. The theoretical fact that guides the algorithm is that, whenever the Taylor approximation maximizes the decision at the locale of the approximation, it would be sufficient for the decision to be optimal with respect to u itself. Boyd's work is an important theoretical contribution to the field even though its applicability is limited to simple decision problems.

Delta Properties Techniques

The approach here is to augment the Von Neumann system of axioms with further smoothness assumptions in such a way that the evolving utility is a member of a classical family of curves. As such, the utility assessment reduces to estimating a few parameters. The smoothness assumptions, which describe specific kinds of preference behavior, correspond to functional equations that restrict the functional form of the utility (for examples, see Keelin [18]). It is theoretically possible to develop functional equations corresponding to any of the classical functions; yet, the behavioral implication of such assumptions are hard to justify for most actual preference structures. The approach though has been useful for local approximation of preferences.

1.3 *Decompositional Techniques*

These techniques are currently the most popular approaches for dealing with multiattribute utilities. Decomposition assumptions were originally used by economists to deal with (deterministic) utilities over commodity spaces,* or what we had called value functions. As such, the assumptions are referred to as separability axioms. The first rigorous attempt at decomposing utility functions over commodity spaces is that of Samuelson [35], where he derives necessary and sufficient

*A commodity space essentially can be thought of as the product set of n attributes.

differential equation conditions corresponding to the additive utility model, i.e., the model where the utility of a commodity bundle is the sum of the utilities of each component commodity. Debreu [5] introduces a concept of preference independence and proposes an algebraic system of axioms that guarantees the additive utility model. Strotz [36] reports on empirical observations which justify the partitioning aspect of separability assumptions and proposes the notion of a utility tree to portray diagrammatically the partitioning phenomenon. Gorman [14] considers the implications between different collections of Debreu's preference independence axioms and suggests the use of utility trees for modeling preferences. Psychological research in the area of conjoint measurement (see Krantz et al [26]) proposes still other varieties of decompositional axioms that correspond to the additive utility model.

The modeling of deterministic utilities can be used to advantage for constructing Von Neumann utilities via the two-step procedure, described previously, as has been demonstrated by the work of Keelin [18].

1.3.1 The Decomposition of Von Neumann Utilities

The first attempt to directly decompose a Von Neumann utility is that of Fishburn [8]. To describe Fishburn's work, we need to introduce a few preliminaries.

Let the attribute space be $X = X_1 \times \dots \times X_n$. Let P be the space of all simple probability distributions on X. If $p \in P$, let p_{x_i} be the marginal distribution of p on X_i, $i = 1,\dots,n$.

Definition 1.1 (Fishburn). The attributes $X_1,\dots,X_n$ are mutually value independent if, for every $p, p \in P$, $p_{x_i} = p_{x_i}$ for $i = 1,\dots,n$ implies that $p \sim p'$.

The concept of value independence characterizes preference structures where the preference over lotteries depends only on the marginal distributions over the attributes and hence eliminates effects due to coupling or interactions between attributes.

Theorem 1.1 (Fishburn). Let the product set of attributes be $X = X_1 \times \dots \times X_n$. Assume the preferences satisfy the Von Neumann axioms. Then the attributes are mutually value independent if and only if, for $x = (x_1, \dots, x_n) \in X$, we can write

$$u(x_1, \dots, x_n) = \sum_{i=1}^{n} u_i(x_i) \tag{1.1}$$

where $u_i(\cdot)$ is a real valued function on the space X_i, $i = 1, \dots, n$.

Each u_i is actually a full-fledged utility defined on its respective space. Equation (1.1) is what is called the additive (Von Neumann) utility model.

1.3.2 The Work of Ralph Keeney

We will develop in detail some of Keeney's work due to the fact that it is related intimately to our research. The concept of utility independence, denoted *UI*, is central to most of this work. Before introducing this concept, however, additional notation is needed.

Let $X = X_1 \times \dots \times X_n$ be written as $Y \times Z$ where Y corresponds to an arbitrary (nonempty) subset of the attributes and Z corresponds to the complement set of attributes. For Y and Z, let us define P_y and P_z as the spaces of all marginal probability distributions on Y and Z, respectively. As such, the space of probability distributions P is $P_y \times P_z$. Hence, for $p \in P$, there exists $p_y \in P_y$ and $p_z \in P_z$ such that $p = (p_y, p_z)$. For a marginal distribution that is degenerate (i.e., only a single point in the respective space has a nonzero probability), we denote it by the certain point. As an example, for some $z_1 \in Z$, $P = (p_y, z_1)$ denotes a lottery in P with p_y marginal distribution on the Y space and a degenerate marginal distribution on Z with $p_z(z_1) = 1$. For $z_0 \in Z$, let us define the induced preference ordering, $\underset{z_0}{\lesssim}$, on P_y as follows: For p_y, $p_y' \in P_y$, we have

$$(p_y, z_0) \lesssim (p_y', z_0) \iff p_y \underset{z_0}{\lesssim} p_y'$$

<u>Definition 1.2</u> (Keeney). The (vector) attribute Y is <u>utility independent</u> of Z if, for every $z_1, z_2 \in Z$, $\underset{z_1}{\lesssim} = \underset{z_2}{\lesssim}$.

The definition characterizes preference structures where the ordering of lotteries on the Y attribute is not affected by the particular (certain) value of the Z attribute. Keeney [20] has shown that Y is UI of Z if and only if $u(y,z)$ has the following functional form:

$$u(y,z) = f_2(z) + f_1(z) \cdot u(y,z_0) \quad \text{for some } z_0 \in Z \tag{1.2}$$

where f_1 and f_2 are real valued functions with $f_1 > 0$. Equation (1.2) is sometimes used as the definition of UI.

To determine the functions f_1 and f_2, let us fix the utility surfaces' two degrees of freedom (see the third property of Von Neumann utilities, described previously) by choosing $y_0, y_1 \in Y$ such that

$$u(y_0,z_0) = 0 \; ; \qquad u(y_1,z_0) = \alpha \neq 0$$

(Here we implicitly assume that such pairs of points exist. This assumption is sometimes explicitly stated by assuming that Y is essential.)

Substituting y_0 in Eq. (1.2), we get

$$u(y_0,z) = f_2(z) + f_1(z) \cdot u(y_0,z_0) = f_2(z) \tag{1.3}$$

Also, substituting y_1 in Eq. (1.2), we get

$$\begin{aligned} u(y_1,z) &= f_2(z) + f_1(z) \cdot u(y_1,z_0) \\ \therefore f_1(z) &= \left[u(y_1,z) - u(y_0,z)\right] / \alpha \end{aligned} \tag{1.4}$$

Substituting Eqs. (1.3) *and* (1.4) back into Eq. (1.2), we get

$$u(y,z) = u(y_0,z) + 1/\alpha\left[u(y_1,z) - u(y_0,z)\right] \cdot u(y,z_0) \tag{1.5}$$

Thus, from Eq. (1.5), we are able to derive u from its restrictions on the subspaces Y and Z.

Keeney [20] uses the UI concept to develop the so-called quasi-separable model.

Definition 1.3. Let the attribute set be $X_1 \times \ldots \times X_n$. u is quasi-separable if there are real valued functions u_i on X_i, $i = 1, \ldots, n$, such that u has the following form:

$$u(x_1, \ldots, x_n) = \sum_{j_1,\ldots,j_n \in \{0,1\}} K(j_1, \ldots, j_n) \cdot R(j_1, \ldots, j_n)$$

where

$$R(j_1, \ldots, j_n) = \prod_{i=1}^{n} R_{j_i}$$

and

$$R_{j_i} = u_i(x_i) \qquad \text{if } j_i = 1$$

$$= 1 - u_i(x_i) \qquad \text{if } j_i = 0$$

The $K(j_1,\ldots,j_n)$'s are real constants.

Again, the u_i's in this definition correspond to utilities on their respective spaces.

Theorem 1.2 (Keeney). Let the product set of attributes be $X = X_1 \times \ldots \times X_n$. For $i = 1,\ldots,n$, assume the X_i is UI of its (orthogonal) complement with respect to X. Then u on X has the quasi-separable form.

In addition to the models of Theorems 1.1 and 1.2, the third most popular model is the so-called multiplicative-additive model. This model is developed, almost simultaneously, by both Keeney [20] and Pollak [30].

Theorem 1.3 (Keeney). Let the product set of attributes be $X = X_1 \times \ldots \times X_n$. Assume that $X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n$ is UI of X_i for $n-1$ of the $i = 1,\ldots,n$, and $n \geq 3$. Then either

$$u(x_1, \ldots, x_n) = \sum_{i=1}^{n} k_i u_i(x_i)$$

or

$$\left[1 + ku(x_1, \ldots, x_n)\right] = \prod_{i=1}^{n} \left[1 + kk_i u_i(x_i)\right]$$

where u and u_i are utility functions (on their respective spaces) scaled from zero to one. The k and k_i's are scaling constants with

$$-1 < k \leq \infty \;; \quad 0 < k_i < 1 \qquad \text{for } i = 1, \ldots, n$$

The value of the constant k determines whether u is additive or multiplicative. This value is determined by the k_i's whose values are to be assessed directly by the DM. The multiplicative-additive model may be compactly written as

$$u(x_1, \ldots, x_n) = \mp u_1(x_1) * \ldots * u_n(x_n) , \qquad * \in \{x,+\} \qquad (1.6)$$

Keeney [20] also derives the same model using a combination of preference independence and utility independence assumptions. For the case of only two attributes, Theorems 1.2 and 1.3 coincide and we have:

Theorem 1.4. Let $X = X_1 \times X_2$. Assume the X_1 is UI of X_2 and X_2 is UI of X_1. Then, u has the following form:

$$u(x_1,x_2) = \mp u_1(x_1) * u_2(x_2) , \qquad * \in \{x,+\}$$

Meyer [29] derives the equation (1.6) by dealing with a different set of UI assumptions.

1.3.3 The Work of Farquhar

It is clear from Definition 1.2 that the UI concept captures a certain kind of regularity by considering the induced preferences conditioned on one element, i.e., induced preferences of the form $\underset{z}{\precsim}$. Fishburn [11] introduces another regularity concept by dealing with two element conditional preferences, i.e., induced preferences of the form $\underset{z_1,z_2}{\precsim}$ where z_1 and z_2 are elements in the space Z as defined in Definition 1.2. Farquhar [6] generalizes Fishburn's work by considering n-element conditional preferences, i.e., induced preferences of the form $\underset{z_1,\ldots,z_n}{\precsim}$. The regularity assumptions that evolve from such conditional preferences are called hypercube independence assumptions (the adjective 'hypercube' is used because the n conditional elements are indexed by the vertices of n-dimensional unit hypercubes). Farquhar proves a fundamental theorem which gives the utility decomposition form corresponding to different hypercube independence assumptions. Farquhar's work is of great theoretical value since it integrates the state of the art and demonstrates the extent of varieties of preference structures. From an applications point of view, however, the consideration of n-conditional preferences is both awkward and unintuitive. By comparing some of the hypercube independence concepts to the UI concept, we believe that what makes the latter concept more appealing and amenable to introspection is the property that it always corresponds to a partition of the attribute space (we will prove this fact in Chapter 2). Most hypercube independence assumptions, however, do not have this property.

This concludes a brief review of the different results pertinent to our work.

1.4 Summary of Results and Contributions

Throughout our work, we deal with the concept of utility independence. Instead of treating specific sets of UI assumptions, or particular utility decomposition forms, we consider arbitrary sets of UI assumptions with respect to n attributes. Each arbitrary set corresponds to and defines a particular preference structure.

In Chapter 2, we characterize the utility decomposition corresponding to an arbitrary set of UI assumptions by two fundamental properties: (1) the decomposition partitions the attribute space into subspaces of lower dimensions, and (2) preferences over the subspaces reflect different levels of regularity. The concept of 'utility independence order' is introduced to capture the second property. The two properties fully characterize any decomposition corresponding to a set of UI assumptions. Next, we identify an algebraic structure (a finite semigroup or a monoid) which automates the derivation of the decomposition corresponding to a set of UI assumptions. The algebraic structure is implemented, in a natural way, using the concept of a utility tree. The tree method is a simple and visually powerful procedure for generating utility decompositions. The procedure produces utility trees that are self-contained analytical representations of the decomposition. The whole methodology may be implemented on a computer.

In Chapter 3, we start by listing the preference structures corresponding to all possible UI sets on three attributes. It is interesting to note that such a listing was one of our early research goals. Our initial attempt was to capture the different possibilities of preference interactions on three attributes by developing differential equation models, much in the same vein as that of Samuelson's [35] necessary and sufficient conditions for the additive value model. In retrospect, we believe the use of the UI concept is more meaningful and appealing. Next, we use the fundamental decomposition characterization of Chapter 2 to propose a scheme for classifying preference structures on n attributes. The scheme is most meaningful with respect to the assessment effort required for each preference structure. We use the scheme to list all 'distinct' preference structures on 2, 3, and 4 attributes. The 'distinctiveness' here, of course, is modulo the UI concept.

In Chapter 4, we deal with how UI assumptions imply each other and consider the case where different UI sets correspond to the same preference structure. We identify three instances (one of which is documented by Keeney [23]) where a given set of UI assumptions implies the satisfaction of other UI assumptions. One of the instances corresponding to a particular pattern of UI sets, which we call dichotomous chains, results in representation forms that are a generalization of the quasi-separable model of Definition 1.3.

Section 4.3 proposes a canonical form for UI sets which is helpful for visualizing the decomposition involved, along with any possible UI implications. Due to the central role of the multiplicative-additive model for UI implications, Section 4.4 presents a method for constructing a minimal number of UI assumptions corresponding to a given multiplicative-additive model. For instance, if the model involves n attributes, Theorem 1.3 (of Keeney) requires $n-1$ assumptions (Pollak [30] and Meyer [29] require n assumptions), while our construction requires a number of assumptions k where k is the smallest integer such that:

$$2^K > n$$

As an example, for a problem with a thousand attributes, Theorem 3.1 requires 999 assumptions, while our construction requires only 10 assumptions. The amount of reduction in the verification effort by the analyst is greatly reduced. The general topic of UI implications, in addition to its theoretical value, is of practical benefit to the analyst. The analyst can design the collection of UI assumptions that is most meaningful to the problem at hand. He can also reduce the number of assumptions to be verified to a minimum to eliminate any, perhaps nonobvious, duplication of effort.

Chapter 5, which relates to the second stage of the overall framework, deals with preferences over multiple attributes where no regularity assumptions are satisfied. The corresponding utility surface is called indecomposable. For such surfaces, we propose the 'Strategy of Continuous Cuts,' a flexible discretization process that collects continuous, one-dimensional, samples of information about the utility. Interpolation is used to fill the 'gaps,' due to discretization and, hence, constructs the whole surface. An appropriate rule for interpolating utility values should be behaviorally motivated. Thus, in correspondence with the continuous cuts strategy, we develop a behavior interpolation rule called 'The Risk Aversion Profile Method.' The basis of this rule is to assume a specific kind of <u>local</u> decomposition (or separability). The local decomposition is implemented using the information on the continuous cuts by assuming different behavioral assumptions about a DM's attitude toward uncertainty along each attribute. To the best of our knowledge, the rule corresponds to the first attempt where local decomposition is used to model preferences. Note that this approach is distinct from that of local fitting, using classical surfaces.

In Chapter 6, we demonstrate some of our results by an actual example. The example constructs a twelve-attribute utility function for a decision maker's preference of cars.

Finally, in the concluding chapter, we summarize the main results of our work and propose different directions for extending it.

Chapter 2

PARTIAL DECOMPOSITION OF UTILITY SURFACES

Decomposition of higher dimensional utility surfaces is attained by assuming different global regularity conditions of the surface. Several concepts of regularity exist in the literature; here, we deal exclusively with the concept of utility independence, as defined in Chapter 1. The satisfaction of a collection of UI assumptions implies the feasibility of reducing an n-dimensional utility surface to surfaces of lower dimensions. This property reduces the assessment effort required for constructing the utility.

Most of the decompositional utility models in the literature assume a symmetric set of assumptions where each attribute is treated in the same way with respect to the type of regularities imposed. In our work, we consider arbitrary sets of UI assumptions and the corresponding utility decomposition. This approach leads to a spectrum of preference structures whose ends are, on the one hand, the total decomposition of an n-dimensional utility into n 1-dimensional utilities and, on the other hand, the n-dimensional utility cannot be reduced at all. The treatment of arbitrary sets of UI assumptions is of immediate practical value since different sets of assumptions are meaningful for different problem environments.

As a preview of the chapter, Section 2.1 introduces the basic notation used throughout the text. Section 2.2 displays an algebraic spanning phenomenon that is fundamental to the kind of decomposition treated here. The concept of 'utility independence order' is introduced to capture an aspect of this phenomenon. Sections 2.3 and 2.4 characterize partial decomposition in two fundamental ways: partitioning of the attribute space, and low order regularities on the decomposed subspaces. Finally, Section 2.5 proposes a codable procedure for generating the decomposition corresponding to a set of UI assumptions. The procedure considers the algebra involved in decomposition and abstracts from it a finite algebraic structure, called a semi group, which is implemented in a natural way using the utility tree notion.

2.1 Notations

We use a powerful notation for denoting UI assumptions. Let us note (see Definition 1.2) that the UI concept is defined for any (nonempty and proper) subset of the set of attributes. Thus, for n-attributes, there are many possible UI assumptions. For a given ordering of the attributes, denote a UI assumption by an n-component vector of ones and zeros where the attributes corresponding to the ones are utility independent of the attributes corresponding to the zeros. For example, for a 4-attribute space,

$$(1,0,1,0) \overset{D}{\Longleftrightarrow} X_1 \times X_3 \quad \text{is UI of} \quad X_2 \times X_4$$

The zero vector is called the <u>null</u> assumption. It should be clear that, for n-attribute spaces, there are

$$2^n - 2$$

possible assumptions, where we discount the null assumption and the vector of all ones as being ill-defined. An arbitrary set of UI assumptions is denoted by the letter A. Let $a, b \in A$. We denote by $a \wedge b$ and $a \vee b$ the component-wise Boolean product and addition, respectively, of

a and b. Here, we are treating A as a subset of the 2^n-Boolean algebra. $\bar{a}$ denotes the Boolean complement of a, i.e., the zeros and ones of $\bar{a}$ correspond to the ones and zeros of a, respectively. As an example, let $a = (1,0,1,1)$ and $b = (1,1,0,0)$; then,

$$a \wedge b = (1,0,0,0)$$

$$a \vee b = (1,1,1,1)$$

$$\bar{a} = (0,1,0,0)$$

If u is restricted to a subspace of the attribute space, it is called a <u>conditional</u> utility and is denoted by $Cu(\cdot)$. The variables on which $Cu(\cdot)$ is defined are called <u>active</u> variables, while the variables of the complement space are called <u>parameters</u>. It is useful to denote the argument of $Cu(\cdot)$ by two different notations:

(1) The standard mathematical notation where all the variables are listed and the parameters take on constant values. For example, $Cu(x_1,\bar{x}_2,x_3,\bar{x}_4)$ is the conditional utility over (x_1,x_3) where $(\bar{x}_2,\bar{x}_4)$ is a point in the parameter space $X_2 \times X_4$.

(2) The Boolean notation is to indicate the active variables by ones and the parameter variables by zeros. This is another use for the elements of 2^n-Boolean algebra.

Both notations will be used as appropriate. Two distinct conditional utilities are <u>generically</u> different if their domain of definition is not the same. Otherwise, they are <u>parametrically</u> different.

We use the standard set theoretic notation such as $\cup$ for union, $\cap$ for intersection, and a set with an upper bar for the complement operation.

2.2 Spanning of Conditional Utilities

Before stating the main definitions, let us motivate them by an example where we point out the salient features of the decomposition technique. We will particularly articulate an algebraic phenomenon, called spanning, which is at the heart of partial decomposition.

Example 2.1.

Let the attribute space be $X \times Y \times Z$ where X, Y, and Z are scalar attributes. Let the set of satisfied UI assumptions for a particular preference structure on $X \times Y \times Z$ be $A = \{(1,0,0),(0,1,0)\}$. Let us generate the corresponding decomposition.

To apply the definition of utility independence, let (x_1,y_1,z_1), (x_2,y_1,y_1), and (x_1,y_2,z_1) be elements in $X \times Y \times Z$ with

$$u(x_1,y_1,z_1) = 0 \ ; \qquad u(x_2,y_1,z_1) = 1/\alpha \ ; \qquad u(x_1,y_2,z_1) = 1/\beta$$

where α and β are not equal to zero. We can now write

$$(1,0,0) \Rightarrow u(x,y,z) = Cu(x_1,y,z) + \alpha\Big[Cu(x_2,y,z) - Cu(x_1,y,z)\Big] \cdot Cu(x,y_1,z_1) \tag{2.1}$$

Equation (2.1) is attained similar to Eq. (1.5) of Chapter 1. Similarly, we have

$$(0,1,0) \Rightarrow u(x_1,y,z) = Cu(x,y_1,z) + \beta\Big[Cu(x,y_2,z) - Cu(x,y_1,z)\Big] \cdot Cu(x_1,y,z_1) \tag{2.2}$$

Let us decompose $Cu(x_2,y,z)$, from Eq. (2.1), via Eq. (2.2) as follows:

$$Cu(x_2,y,z) \overset{D}{=} u(x_2,y,z) = Cu(x_2,y_1,z) + \beta\left[Cu(x_2,y_2,z) - Cu(x_2,y_1,z)\right] \cdot Cu(x_1,y,z_1)$$

Decomposing $Cu(x_1,y,z)$ in the same way and substituting both expressions back in Eq. (2.1), and rearranging terms, we finally get:

$$\begin{aligned} u(x,y,z) = {} & f(x) \cdot g(y) \cdot Cu(x_2,y_2,z) \\ & + f(x)[1 - g(y)] \cdot Cu(x_2,y_1,z) \\ & + [1 - f(x)] \cdot g(y) \cdot Cu(x_1,y_2,z) \\ & + [1 - f(x)] \cdot [1 - g(y)] \cdot Cu(x_1,y_1,z) \end{aligned} \tag{2.3}$$

where $f(x) = \alpha Cu(x,y_1,z_1)$; $g(y) = \beta Cu(x_1,y,z_1)$. Let us note the following points.

(1) The set of attributes has been segmented into three mutually exclusive and collectively exhaustive subsets corresponding to the active variables of the three generic conditional utilities. As will be shown, such partitioning occurs for any set of UI assumptions.

(2) Equation (2.3) indicates that four conditional utilities on z (i.e., at four settings of the parameter space) are required to construct u. Let us identify the surface u with the set of all conditional utilities on z (the latter would be cuts along the z-direction of the former). Now, Eq. (2.3) says that this set is <u>spanned</u> by four elements in it. This observation reflects a type of low order independence on Z that is implied by the UI assumptions on X and Y. With respect to X and Y, the respective conditional utilities are

spanned by one conditional utility and a constant function. To see this point, scrutinize Eq. (2.3). These observations are the motivation for Definition 2.4 below.

(3) If Eq. (2.3) is used to generate an arbitrary conditional utility on Z, i.e., $Cu(x_i, y_i, z)$, we find that the coefficients multiplying the z-conditional utilities always add up to one. Hence, if the set $\{Cu(x_i, y_j, z): x_i \in X, y_j \in Y\}$ is embedded in an appropriate linear space, u may be characterized as a linear variety of functions, where the spanning functions are actually restrictions or cuts of u itself.

(4) The generation of any $u(x_i, y_j, z)$, for some $x_i \in X$ and $y_j \in Y$, via Eq. (2.3) can be thought of as a three-step procedure:

(i) Find $u(x_i, y_1, z)$ as a point on the function line formed by $u(x_1, y_1, z)$ and $u(x_2, y_1, z)$.

(ii) Similarly, find $u(x_i, y_2, z)$ as a point on the function line formed by $u(x_1, y_2, z)$ and $u(x_2, y_2, z)$.

(iii) Finally, derive $u(x_i, y_j, z)$ as a point on the function line formed by $u(x_i, y_1, z)$ and $u(x_i, y_2, z)$.

Note that it is just as well if we had gone the route $u(x_1, y_j, z)$, $u(x_2, y_j, z)$, and $u(x_i, y_j, z)$. See Fig. 2.1.

With the above motivation in mind, the following definitions are introduced.

<u>Definition 2.1</u>. Let the attribute space be $X \times Y$ where X and Y are vector or scalar attributes. Consider the set

$$S\left[Cu(x,y_i)\right] = \left\{Cu(x,y_i) : y_i \in Y\right\}$$

$S[Cu(\cdot)]$ is called the function space of $Cu(\cdot)$.

Definition 2.2. Let X and Y be as defined above. Consider the set

$$SP = \left\{Cu(x,y_i) : i = 1, \ldots, \ell, \; y_i \in Y\right\}$$

SP is a spanning set for $S[Cu(\cdot)]$ if, for every $Cu(\cdot) \in S[Cu(\cdot)]$,

$$Cu(\cdot) = \sum_{i=1}^{\ell} b_i(y) \; Cu(x,y_i) + b_{\ell+1}^{(y)}$$

The $b_i(\cdot)$'s are called weight functions.

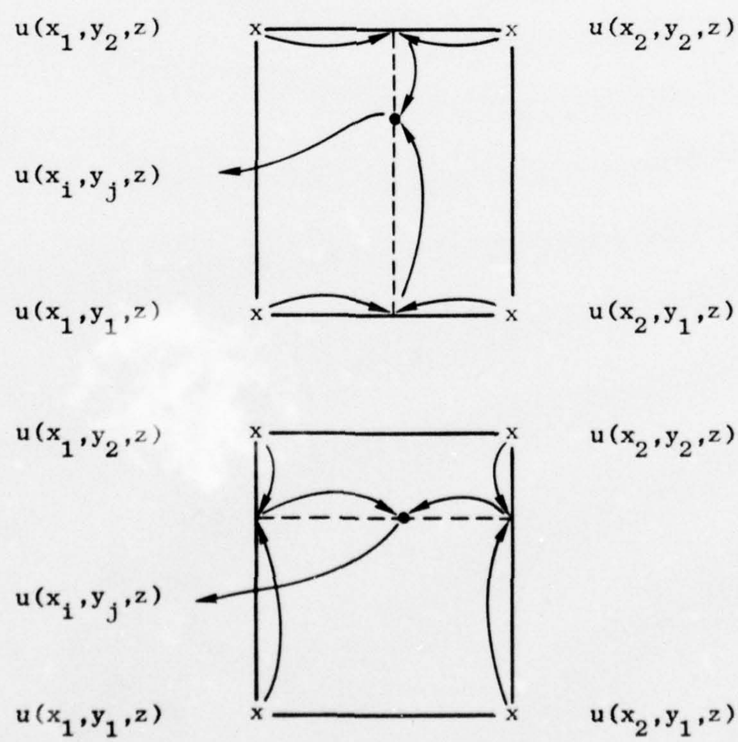


Fig. 2.1. SPANNING OF CONDITIONAL UTILITIES.

Spanning sets will generally be invoked by A, the set of UI assumptions. Hence, the following definitions.

Definition 2.3. For a particular A, The *spanning set of a given* $Cu(\cdot)$ is minimal if none of its elements can be eliminated by the assumptions of A.

Proposition 1. Let $Cu(\cdot)$ be (any conditional utility) invoked by the set of assumptions A. Then, a minimal spanning set for $Cu(\cdot)$, along with its weight functions, will construct the utility surface u.

Proof.

Let the attribute space be $X_1 \times \ldots \times X_n$. Without any loss of generality, assume we have the minimal set for $Cu(x_1,\ldots,x_r)$ where $0 < r < n$. Let $(\bar{x}_1,\ldots,\bar{x}_n)$ be a point in the attribute space. We want $u(\bar{x}_1,\ldots,\bar{x}_n)$. Two steps are involved:

(1) Interpolation step:

$$\begin{aligned} Cu(x_1, \ldots, x_r, \bar{x}_{r+1}, \ldots, \bar{x}_n) = \sum_{i=1}^{\ell} & b_i(x_{r+1}, \ldots, x_n) \\ & \cdot Cu_i(x_1, \ldots, x_r) \\ & + b_{\ell+1}(x_{r+1}, \ldots, x_n) \end{aligned}$$

where

$Cu_i(x_1, \ldots, x_r)$, $i = 1, \ldots, \ell$ minimally span $Cu(\cdot)$

(2) Substitution step: We substitute

$\bar{x}_1, \ldots, \bar{x}_r$ in $Cu(\cdot)$ to get $u(\bar{x}_1, \ldots, \bar{x}_n) = Cu(\bar{x}_1, \ldots, \bar{x}_r)$

Q.E.D.

Definition 2.4. $Cu(\cdot)$ is said to be utility independent of order k, denoted UI_k, if its minimal spanning set has cardinality k.

Let us remark that, if A has only one UI assumption, two generic conditional utilities are invoked: one is UI_1 and the other is UI_2.

Let us also remark that each assumption in A will invoke a conditional utility that is UI_1. Thus, there is a natural correspondence between $Cu(\cdot)$ that are UI_1 and utility independence assumptions.

Generating the Decomposition Corresponding to A

Each UI assumption in A will, by definition, generate a decompositional equation for u. The resulting system of equations is manipulated to produce the simplest decompositions of u. The idea is to take a conditional utility invoked by an assumption and decompose it further by one of the other assumptions. This is done recursively for each conditional utility until no more decomposition or reduction is possible. The simplest possible decomposition, which is unique, is the one that corresponds to A.

2.3 The Partitioning of the Attribute Space

The following theorem characterizes the set of all generic conditional utilities in the decomposition corresponding to A.

Theorem 2.2. Let $A = \{a_i : i = 1,\ldots,k\}$ be the set of all satisfied UI assumptions for a particular preference structure. Then,

$$\left\{Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k) : (\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k) \neq 0^* \text{ and } \tilde{a}_i \in \{a_i, \bar{a}_i\}\right\}$$

is the set of all generic conditional utilities in the decomposition corresponding to A.

Proof.

Proof by induction on k. For $k = 1$, the assertion is true by the definition of UI. Assume the assertion is true for k. Let $A' = A \cup \{a_{k+1}\}$. We will show the assertion is true for A'. We will generate the decomposition for A' by starting with the decomposition for A. Initially, a_{k+1} will be applied to each of the generic conditional utilities of A. By definition,

$$a_{k+1} \Rightarrow u(\cdot) = f\left[Cu(a_{k+1});\ Cu(\bar{a}_{k+1})\right] \tag{2.4}$$

Substituting $Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k)$ in the functional form of Eq. (2.4), we get the following functional form:

$$Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k) = f\left[Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k \wedge a_{k+1});\ Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k \wedge a_{k+1})\right]$$

Thus, we can replace all the generic utilities of A by the new ones, $Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_{k+1})$. Now, further substitutions will involve one of the UI assumptions with $Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_{k+1})$ as follows:

$$a_i = u(\cdot) = f_i\left[Cu(a_i);\ Cu(\bar{a}_i)\right], \qquad \forall\ i = 1, \ldots, k+1$$

Hence,

---

*0 here denotes the zero vector $(0,0,\ldots,0)$.

$$Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_{k+1}) = f\Big[Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_{k+1} \wedge a_i);\ Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_{k+1} \wedge \bar{a}_i)\Big] \tag{2.5}$$

But, note that

$$\begin{aligned} \tilde{a}_1 \wedge \ldots \wedge \tilde{a}_{k+1} \wedge a_i &= \tilde{a}_1 \wedge \ldots \wedge \tilde{a}_{k+1} && \text{if } \tilde{a}_i = a_i \\ &= 0 && \text{if } \tilde{a}_i = \bar{a}_i \end{aligned}$$

and

$$\begin{aligned} \tilde{a}_1 \wedge \ldots \wedge \tilde{a}_{k+1} \wedge \bar{a}_i &= 0 && \text{if } \tilde{a}_i = a_i \\ &= \tilde{a}_1 \wedge \ldots \wedge \tilde{a}_{k+1} && \text{if } \tilde{a}_i = \bar{a}_i \end{aligned}$$

Thus, Eq. (2.5) says that no new generic conditional utilities will evolve and all the current ones will be preserved. Q.E.D.

Definition 2.5. A partition of the attribute variables is a collection of mutually exclusive and collectively exhaustive subsets of the set of attributes.

Theorem 2.3. The active variables of the generic conditional utilities of a decomposition corresponding to any A form a partition.

Proof.

Any generic conditional utility is of the form

$$Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k) \quad \text{where} \quad (\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k) \neq 0$$

To show mutual exclusion, consider the following arbitrary pair of distinct Cu's:

$$Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k) \quad \text{and} \quad Cu(\tilde{b}_1 \wedge \ldots \wedge \tilde{b}_k) \; \tilde{a}_i, \tilde{b}_i \{a_i, \bar{a}_i\}$$

Since the 1's correspond to active variables, the two Cu's are mutually exclusive if

$$(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k) \wedge (\tilde{b}_1 \wedge \ldots \wedge \tilde{b}_k) = 0 \tag{2.6}$$

Equation (2.6) is true if there exists an i such that

$$\tilde{a}_i = \tilde{b}_i$$

But, at least one such i will exist or else the two Cu's are not distinct.

To show collective exhaustion, we will construct a conditional utility whose active variables contain an arbitrarily chosen attribute variable. Let the arbitrary attribute be the i^{th} one, $1 \leq i \leq n$. For each j, $1 \leq j \leq k$, choose $\tilde{a}_j \{a_j, \bar{a}_j\}$ such that the i^{th} component of $\tilde{a}_j$ is one. Now, $\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k \neq 0$ and $Cu(\tilde{a}_1 \wedge \ldots \wedge \tilde{a}_k)$ will contain the i^{th} attribute as one of its active variables.

Q.E.D.

2.4 Utility Independence Order

The previous section characterized all the generic conditional utilities of a particular decomposition. But, as Example 2.1 indicates, a conditional utility may be required at more than one setting of its parameters. Definition 2.4 refers to the required number of settings as the utility independence order of the conditional utility. The following theorem characterizes this concept.

Theorem 2.4. Let $Cu_1, \ldots, Cu_\ell$ be all the generic conditional utilities corresponding to some A. Let 0_i be the utility independence order of Cu_i, $i = 1, \ldots, \ell$. Then,

(1) $0_i \in \{2^r : r = 0,1,2, \ldots, \ell - 1\} \quad i = 1, \ldots, \ell$

(2) $\prod_{i=1}^{\ell} 0_i \leq 2^{\ell-1}$

To prove the theorem, we need the following lemma.

Lemma 2.5. For any Cu_i, the number of settings each parameter, separately, takes on is either one or two.

We will prove the theorem and lemma simultaneously.

Proof.

By induction on the number of UI assumptions in A. If $A = \{a\}$, the theorem and the lemma are true by scrutiny of Eq. (1.5) of Chapter 1. Let $A = \{a_i : i = 1, \ldots, k-1\}$, and assume that the corresponding $Cu_1, \ldots, Cu_e$ satisfy the assertions of the theorem and the lemma. Consider $A' = A \cup \{a_k\}$ with its corresponding $Cu'_1, \ldots, Cu'_e$. Without any loss of generality, denote Cu_i, $i = 1, \ldots, e$, as follows:

$$Cu_i(\cdot) = Cu_i(1_1 1_2 \ldots 1_r \, 0_{r+1} \ldots 0_k) \qquad 1 \leq r < n \qquad (2.7)$$

(This may be accomplished by permuting the attributes.)

In conformance with Eq. (2.7), a_k may be denoted as follows:

$$a_k = (1_1 \ldots 1_m 0_{m+1} \ldots 0_r \, 1_{r+1} \ldots 1_{r+\ell} 0 \ldots 0_n)$$

where $0 \leq m \leq r$; $0 \leq \ell \leq n-r$. Let

$$h_1 \equiv Cu(\underline{x}_1, \ldots, \underline{x}_m, x_{m+1}, \ldots, x_r, \underline{x}_{r+1}, \ldots, \underline{x}_{r+\ell}, x_{r+\ell+1}, \ldots, x_n)$$

$$h_2 \equiv Cu(\bar{x}_1, \ldots, \bar{x}_m, x_{m+1}, \ldots, x_r, \bar{x}_{r+1}, \ldots, \bar{x}_{r+\ell}, x_{r+\ell+1}, \ldots, x_n)$$

$$h_3 \equiv Cu(x_1, \ldots, x_m, \underline{x}_{m+1}, \ldots, \underline{x}_r, x_{r+1}, \ldots, x_{r+\ell}, \underline{x}_{r+\ell+1}, \ldots, \underline{x}_n)$$

where $\underline{x}_i$ and $\bar{x}_i \in X_i$, $i = 1, \ldots, n$.

We have, by definition,

$$a_k \Rightarrow u(\cdot) = h_1 + \alpha[h_2 - h_1] \cdot h_3 \qquad \alpha > 0, \text{ a constant} \qquad (2.8)$$

We will use this setup to argue the assertions. Let Cu_s, $1 \leq s \leq e$, be UI_{2^i} (with respect to A). Since, by the induction assumption, the number of settings of each parameter is either one or two, we can divide 2^i into 2^j and 2^{i-j}, $0 \leq j \leq i$, where 2^j is the number of settings in the parameter space $(x_{r+1}, \ldots, x_{r+\ell})$ and 2^{i-j} is the number of settings in $(x_{r+\ell+1}, \ldots, x_n)$. Let us apply a_k to Cu_s via Eq. (2.8). There are three mutually exclusive and collectively exhaustive cases to consider.

Case 1: $m = 0$.

As Eq. (2.8) indicates, Cu_s will not be decomposed further. Hence, from Theorem 2.2, it will be one of the generic conditional utilities of A'. For the number of required settings in its parameter space, note that the settings of the parameters $(x_{r+\ell+1}, \ldots, x_n)$ will be preserved at this point of the argument, while the settings of the parameters $(x_{r+1}, \ldots, x_{r+\ell})$ will be replaced by the two points $(\underline{x}_{r+1}, \ldots, \underline{x}_{r+\ell})$ and

$(\bar{x}_{r+1},\ldots,\bar{x}_{r+\ell})$. Hence, the UI order of Cu_s with respect to A is at most 2^{i-j+1}. For the case where $j = 0$, the decomposition is better off without applying a_k to Cu_s. Thus, we can conclude that the UI order of Cu_s is at most 2^i.

Case 2: $m = r$.

Again, Cu_s will not be decomposed further. The settings of the space $(x_{r+\ell+1},\ldots,x_n)$ are replaced by a single point, i.e., $(\underline{x}_{r+\ell+1},\ldots,\underline{x}_n)$. Hence, the UI order of Cu_s with respect to A is at most 2^j.

Case 3: $r > m > 0$.

By Eq. (2.8), Cu_s will be decomposed into

$Cu_{s1}(1_1 \ldots 1_m 0_{m+1} \ldots 0_n)$ and $Cu_{s2}(0_1 \ldots 0_m 1_{m+1} \ldots 1_r 0_{r+1} \ldots 0_n)$

Cu_{s1} and Cu_{s2} will be generic conditional utilities of A' (see the Proof of Theorem 2.2). Cu_{s1} will inherit all the parameter settings of the space $(x_{r+1},\ldots,x_{r+\ell})$, and only the point $(\underline{x}_{r+\ell+1},\ldots,\underline{x}_n)$ is required in its space. Thus, the UI order of Cu_{s1} (with respect to A') is at most 2^j.

Cu_{s2} will inherit all the settings of the parameters $(x_{r+\ell+1},\ldots,x_n)$, while two points will be required in the space $(x_{r+1},\ldots,x_{r+\ell})$, namely, $(\underline{x}_{r+1},\ldots,\underline{x}_{r+\ell})$ and $(\bar{x}_{r+1},\ldots,\bar{x}_{r+\ell})$. Thus, the UI order of Cu_{s2} is at most 2^{i-j+1}. So, the UI order of both Cu_{s1} and Cu_{s2} is at most 2^{i+1}. Note that, in this case, one conditional utility is replaced by two. So, $\ell' \geq \ell + 1$, and the bound in the second assertion of the theorem is made larger.

With these three cases, the second assertion of the theorem is demonstrated. To complete the proof for the first assertion and for the lemma, the possibility of further applications of any of the assumptions to available conditional utilities needs to be considered. Note that further application of the assumptions will only reduce the number of settings of the parameters. But, with an argument similar to Cases 1 or 2 above, any time the parameter settings are reduced, the reduction is a power of two and, each time a parameter takes on new settings via Eq. (2.8), the number of settings is either one or two. The decomposition corresponding to A' will be attained after a finite number of such substitution steps.

Q.E.D.

2.5 A Hypercube Referencing and Scaling Strategy

From Lemma 2.5, we note that the minimal spanning set of any conditional utility can be sufficiently referenced by parameter variables that take on at most two settings each. Thus, let us propose the following convenient referencing procedure for the different conditional utilities involved in the decomposition corresponding to any set of UI assumptions.

(1) Let the attribute space be $X_1 \times \ldots \times X_n$. Choose $(\underline{x}_1,\ldots,\underline{x}_n)$ and $(\bar{x}_1,\ldots,\bar{x}_n) \in X_1 \times \ldots \times X_n$ such that $\underline{x}_i \neq \bar{x}_i$, for every $i = 1,\ldots,n$.

(2) Consider $H = \{(h_1,\ldots,h_n): h_i \in \{\underline{x}_i,\bar{x}_i\}, i = 1,\ldots, n\}$. H is the hypercube generated by the two chosen points.

(3) Let A be the set of UI assumptions. For each $a \in A$, apply a on u by referencing the invoked

conditional utilities by vertices of faces of H. As such, all required scaling constants correspond to utilities of some of the vertices of H.

Theorem 2.4 indicates that H has enough vertices to reference any collection of conditional utilities corresponding to a set of UI assumptions. Keeney has used a restricted version of this strategy for different parts of his work (e.g., see Keeney [22]); what we did here is to formalize and generalize Keeney's idea.

An Automatum--A Finite Semigroup

When using this procedure, the process of generating the decomposition form of a utility is governed by a simple algebraic structure called a semigroup.

Definition 2.6. A set with an operation defined on it is called a *semigroup* if the operation is closed and associative. If the set has finite cardinality, then the semigroup is *finite*.

To recognize the involved semigroup, let us refer to the variables of a given conditional utility in the following way.

(1) The symbol 1 refers to an active variable.

(2) The symbol $\bar{0}$ refers to a parameter variable whose setting is in the set $\{\bar{x}_i : i = 1,\ldots,n\}$.

(3) The symbol $\underline{0}$ refers to a parameter variable whose setting is in the set $\{\underline{x}_i : i = 1,\ldots,n\}$.

This new symbolism completely specifies any conditional utility that results from the referencing strategy described above. Now, consider the set:

$$SG = \left\{(C_1, \ldots, C_n): C_i \in \{1,\bar{0},\underline{0}\},\ i = 1, \ldots, n\right\}$$

Define the operation $*$ on SG as follows: If $C, C' \in SG$, then

$$C * C' = (C_1 * C_1', \ldots, C_n * C_n')$$

where $*$ on the set $\{1,\bar{0},\underline{0}\}$ is defined by the following table:

Table 2.1

DEFINITION OF $*$

$*$		C_i': $\bar{0}$	C_i': $\underline{0}$	C_i': 1
C_i	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
	1	$\bar{0}$	$\underline{0}$	1

Note that $*$ is noncommutative. The entries in Table 2.1 are $C_i * C_i'$ (and not $C_i' * C_i$). The operation $*$ is clearly closed. To demonstrate the associativity of $*$ for SG, it is sufficient to show it for the set $\{1,\bar{0},\underline{0}\}$. Table 2.2 contains a proof (by complete enumeration) of the associativity of $*$ on $\{1,\underline{0},\bar{0}\}$.

The Tree Method

The best way to explain the use of the finite semigroup is by an example. Let the attribute space be $X \times Y \times Z$. Assume that the set of satisfied UI assumptions is $\{(1,0,0),(0,1,0)\}$. Choose $(\underline{x},\underline{y},\underline{z})$ and $(\bar{x},\bar{y},\bar{z}) \in X \times Y \times Z$ such that

$$\underline{x} \neq \bar{x}, \qquad \underline{y} \neq \bar{y}, \qquad \underline{z} \neq \bar{z}$$

Table 2.2

A PROOF OF THE ASSOCIATIVITY OF *

$1*(\bar{0}*\underline{0}) = 1*\bar{0} = \bar{0}$	$(1*\bar{0})*\underline{0} = \bar{0}*\underline{0} = \bar{0}$
$1*(\bar{0}*\bar{0}) = 1*\bar{0} = \bar{0}$	$(1*\bar{0})*\bar{0} = \bar{0}*\bar{0} = \bar{0}$
$1*(\underline{0}*\bar{0}) = 1*\underline{0} = \underline{0}$	$(1*\underline{0})*\bar{0} = \underline{0}*\bar{0} = \underline{0}$
$1*(\underline{0}*\underline{0}) = 1*\underline{0} = \underline{0}$	$(1*\underline{0})*\underline{0} = \underline{0}*\underline{0} = \underline{0}$
$\bar{0}*(1*\bar{0}) = \bar{0}*\bar{0} = \bar{0}$	$(\bar{0}*1)*\bar{0} = \bar{0}*\bar{0} = \bar{0}$
$\underline{0}*(1*\bar{0}) = \underline{0}*\bar{0} = \underline{0}$	$(\underline{0}*1)*\bar{0} = \underline{0}*\bar{0} = \underline{0}$
$\bar{0}*(1*\underline{0}) = \bar{0}*\underline{0} = \bar{0}$	$(\bar{0}*1)*\underline{0} = \bar{0}*\underline{0} = \bar{0}$
$\underline{0}*(1*\underline{0}) = \underline{0}*\underline{0} = \underline{0}$	$(\underline{0}*1)*\underline{0} = \underline{0}*\underline{0} = \underline{0}$
$\bar{0}*(\bar{0}*1) = \bar{0}*\bar{0} = \bar{0}$	$(\bar{0}*\bar{0})*1 = \bar{0}*1 = \bar{0}$
$\bar{0}*(\underline{0}*1) = \bar{0}*\underline{0} = \bar{0}$	$(\bar{0}*\underline{0})*1 = \bar{0}*1 = \bar{0}$
$\underline{0}*(\bar{0}*1) = \underline{0}*\bar{0} = \underline{0}$	$(\underline{0}*\bar{0})*1 = \underline{0}*1 = \underline{0}$
$\underline{0}*(\underline{0}*1) = \underline{0}*\underline{0} = \underline{0}$	$(\underline{0}*\underline{0})*1 = \underline{0}*1 = \underline{0}$
$1*(1*\underline{0}) = 1*\underline{0} = \underline{0}$	$(1*1)*\underline{0} = 1*\underline{0} = \underline{0}$
$1*(1*\bar{0}) = 1*\bar{0} = \bar{0}$	$(1*1)*\bar{0} = 1*\bar{0} = \bar{0}$
$1*(\bar{0}*1) = 1*\bar{0} = \bar{0}$	$(1*\bar{0})*1 = \bar{0}*1 = \bar{0}$
$1*(\underline{0}*1) = 1*\underline{0} = \underline{0}$	$(1*\underline{0})*1 = \underline{0}*1 = \underline{0}$
$\underline{0}*(1*1) = \underline{0}*1 = \underline{0}$	$(\underline{0}*1)*1 = \underline{0}*1 = \underline{0}$
$\bar{0}*(1*1) = \bar{0}*1 = \bar{0}$	$(\bar{0}*1)*1 = \bar{0}*1 = \bar{0}$
$1*(1*1) = 1*1 = 1$	$(1*1)*1 = 1*1 = 1$
$\bar{0}*(\bar{0}*\bar{0}) = \bar{0}*\bar{0} = \bar{0}$	$(\bar{0}*\bar{0})*\bar{0} = \bar{0}*\bar{0} = \bar{0}$
$\underline{0}*(\underline{0}*\underline{0}) = \underline{0}*\underline{0} = \underline{0}$	$(\underline{0}*\underline{0})*\underline{0} = \underline{0}*\underline{0} = \underline{0}$
$\bar{0}*(\bar{0}*\underline{0}) = \bar{0}*\bar{0} = \bar{0}$	$(\bar{0}*\bar{0})*\underline{0} = \bar{0}*\underline{0} = \bar{0}$
$\bar{0}*(\underline{0}*\bar{0}) = \bar{0}*\underline{0} = \bar{0}$	$(\bar{0}*\underline{0})*\bar{0} = \bar{0}*\bar{0} = \bar{0}$
$\underline{0}*(\bar{0}*\bar{0}) = \underline{0}*\bar{0} = \underline{0}$	$(\underline{0}*\bar{0})*\bar{0} = \underline{0}*\bar{0} = \underline{0}$
$\bar{0}*(\underline{0}*\underline{0}) = \bar{0}*\underline{0} = \bar{0}$	$(\bar{0}*\underline{0})*\underline{0} = \bar{0}*\underline{0} = \bar{0}$
$\underline{0}*(\bar{0}*\underline{0}) = \underline{0}*\bar{0} = \underline{0}$	$(\underline{0}*\bar{0})*\underline{0} = \underline{0}*\underline{0} = \underline{0}$
$\underline{0}*(\underline{0}*\bar{0}) = \underline{0}*\underline{0} = \underline{0}$	$(\underline{0}*\underline{0})*\bar{0} = \underline{0}*\bar{0} = \underline{0}$

Assume also that

$$u(\underline{x},\underline{y},\underline{z}) = 0$$

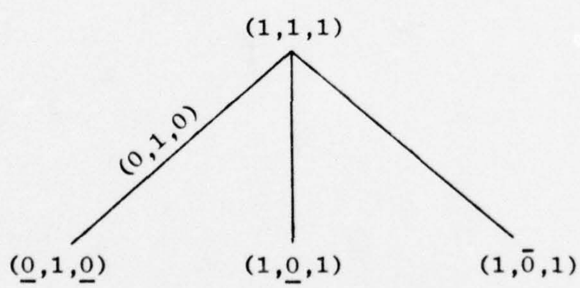
Using Eq. (1.5) of Chapter 1, we can write

$$(0,1,0) \Rightarrow u(x,y,z) = Cu(x,\underline{y},z) + \frac{[Cu(x,\bar{y},z) - Cu(x,\underline{y},z)]}{Cu(\underline{x},\bar{y},\underline{z})} \cdot Cu(\underline{x},y,\underline{z}) \tag{2.9}$$

where $Cu(\underline{x},\bar{y},\underline{z})$ is a scaling constant. Let us rewrite Eq. (2.9) with the symbolism of the semigroup:

$$u(1,1,1) = Cu(1,\underline{0},1) + \frac{[Cu(1,\bar{0},1) - Cu(1,\underline{0},1)]}{Cu(\underline{0},\bar{0},\underline{0})} \cdot Cu(0,1,0) \tag{2.10}$$

Equation (2.10) is portrayed using a three branch node as follows:



Where the vector at the node indicates the surface to be decomposed and the vectors at the end of the branches indicate the conditional utilities that are invoked due to the UI assumption. All four vectors are elements of the semigroup SG. The UI assumption used for the decomposition of the node is to be indicated along one of the branches. Let us associate the list $[(\underline{0},1,\underline{0});(1,\underline{0},1);(1,\bar{0},1)]$ with the assumption (0, 1,0). Such a list can be directly constructed for any UI assumption.

The three vectors in the list correspond to the three conditional utilities invoked by a UI assumption. The first vector of the list denotes the conditional utility that is in a one to one correspondence with the UI assumption. Since we have assumed (in the referencing strategy) that the attribute vector with a lower bar has a zero utility, all the zeros of the first vector should have a lower bar. For the other two vectors, one of them should have all zeros with a lower bar, and the other can, in general, have any collection as long as they are not all with lower bars. In this way, the second and third conditional utilities correspond to two different points in the parameter space. The description for constructing lists is made clear by a scrutiny of the derivation of Eq. (1.5) of Chapter 1.

Let us continue with the example. The list for $(1,0,0)$, constructed directly, is:

$$[(1,\underline{0},\underline{0});(\bar{0},1,1);(\underline{0},1,1)]$$

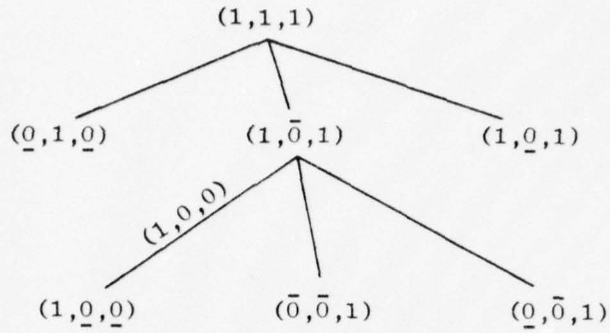
If we decompose $Cu(1,\bar{0},1)$ of Eq. (2.10) using $(1,0,0)$, we get

$$Cu(x,\bar{y},z) = Cu(x,\bar{y},z) + \frac{[Cu(\bar{x},\bar{y},z) - Cu(\underline{x},\bar{y},z)]}{Cu(\bar{x},\underline{y},\underline{z})} \cdot Cu(x,\underline{y},\underline{z}) \quad (2.11)$$

or, equivalently,

$$Cu(1,\bar{0},1) = Cu(0,\bar{0},1) + \frac{[Cu(\bar{0},\bar{0},1) - Cu(\underline{0},\bar{0},1)]}{Cu(\bar{0},\underline{0},\underline{0})} \cdot Cu(1,\underline{0},\underline{0}) \quad (2.11)$$

We can augment Eq. (2.11) on the previous tree as follows:



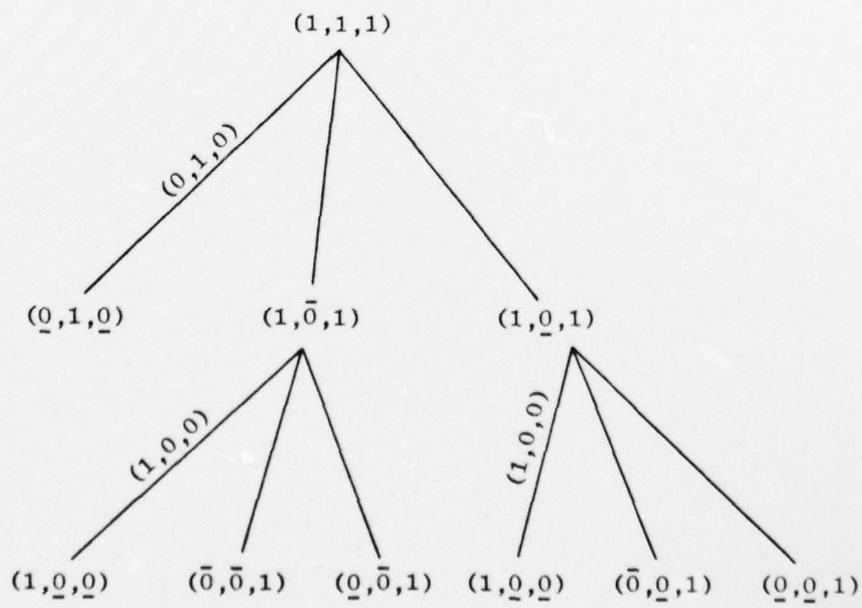
But, we can directly obtain this augmentation if we *-multiply $(1,\bar{0},1)$ by the list of the assumption $(1,0,0)$ as follows:

$$[(1,\underline{0},\underline{0});(\bar{0},1,1);(\underline{0},1,1)] * (1,\bar{0},1) = [(1,\underline{0},\underline{0});(\bar{0},\bar{0},1);(\underline{0},\bar{0},1)]$$

Decomposing $(1,\underline{0},1)$ in the same way, we get

$$[(1,\underline{0},\underline{0});(\bar{0},1,1);(\underline{0},1,1)] * (1,\underline{0},1) = [(1,\underline{0},\underline{0});(\bar{0},\underline{0},1);(\underline{0},\underline{0},1)]$$

This equation is also augmented to tree as follows:



In general, the decomposition of a conditional utility by an assumption corresponds to a three vector list. There are two ways which indicate that a conditional utility can not be decomposed any further by a UI assumption.

(1) If the first element of the evolving list is the vector of all zeros with lower bar.

(2) If the first element of the evolving list is a vector equal to the vector corresponding to the conditional utility to be decomposed.

Both of these instances correspond to cases where the evolved equation is an empty algebraic identity. A list whose first element corresponds to one of the above cases is called <u>tautological</u>. The construction of a utility tree is completed when, for every conditional utility, further decomposition by assumptions produces tautological lists. The completed tree should be such that the collection of the conditional utilities of the end branches corresponds to the collection characterized by Theorem 2.2.

The Use of the Tree

A utility tree is a self-contained analytical form of the decomposed utility surface. If the values of the conditional utilities of the end branches of the tree (the indecomposable conditional utilities) are known for a given point in the attribute space, then the overall utility of the point is obtained by working with the tree. The values of the conditional utilities are propagated upward along the tree, to the top, much in the same way as tree construction has advanced downward.

For a given node in the tree, assume that the values of its three conditional utilities are known. Then, by the use of the equation of the corresponding UI assumption, we calculate the value of the node conditional utility. This procedure is to be used, recursively, for each node until the original node of the tree is reached. The final value is the overall utility of the point.

Concluding Remarks

We have characterized the utility due to a set of UI assumptions by two fundamental properties: the partitioning of the attribute space into subspaces of lower dimensions, and the utility independence order of the evolving subspaces. These two properties, along with a scrutiny of the decomposition algebra involved leads to a simple and visually powerful procedure for generating analytical forms corresponding to a given utility decomposition.

The results of this chapter can be used to advantage by the analyst. He now can ponder questions such as: What set of assumptions to look for? For a given set of assumptions, which other assumptions further simplify the available decomposition? And, which subset of the attributes requires most of the analyst's care and attention?

The utility trees, in addition to their analytical content, may be used as a visual aid for discussion purposes with the decision maker.

Chapter 3

CLASSIFICATION OF PREFERENCES ON n-ATTRIBUTES

In this chapter, we demonstrate the extent of varieties of preference structures on n-attributes. In Section 3.1, we list the preference structures corresponding to all possible UI sets on three-attributes. Section 3.2 proposes a classification scheme that distinguishes between preferences on the basis of the two fundamental characterizations of Chapter 2, i.e., the partitioning phenomenon and the utility independence order. The scheme also reflects the assessment effort required for constructing the utility of each preference structure. We apply the scheme to list all 'distinct' preferences on two, three, and four attribute spaces. We should note that this classification is modulo the concept of utility independence. The same framework can be applied to classify preference independence for some of Farquhar's [6] hypercube independence assumptions.

3.1 All Preference Structures on Three-Attributes

In Section 2.1, we have noted that the number of possible UI assumptions for n-attribute spaces is:

$$2^n - 2$$

A preference structure on n-attributes may be modelled and characterized by any subset of the set of all UI assumptions. As such, the number of different subsets of UI assumptions is:

$$2^{[2^n-2]} \tag{3.1}$$

Consider the attribute space $X \times Y \times Z$ where X, Y, and Z are scalar attributes. For this space, there are only $2^3 - 2$, or six, UI assumptions. They are: $\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1)\}$. Any collection of these assumptions defines a preference structure on $X \times Y \times Z$. The number of subsets of the six assumptions is

$$2^{[2^3-2]} = 2^6 = 64$$

Table 3.1 lists all of these UI sets along with the corresponding utility decomposition form.

The Construction of Table 3.1

We explain in detail the decomposition form corresponding to each entry of the table. Entries whose UI sets are the same up to a reordering of the attributes are grouped together. As such, the decomposition form of only one of them needs to be explained.

Entry 1: None of the UI assumptions is satisfied. Hence, the three dimensional utility surface is indecomposable.

Entries 2-4: The decomposition of Entry 2 is obtained using Eq. (1.5) of Chapter 1.

Entries 5-7: The decomposition of Entry 5 is obtained using Eq. (1.5) of Chapter 1.

Entries 8-10: The decomposition of Entry 8 is obtained by treating $(Y \times Z)$ as a (vector) attribute and applying Theorem 1.4 on $X \times (Y \times Z)$.

Entries 11-13: For a proof of the decomposition of Entry 11, see Example 2.1 of Chapter 2.

Entries 14-16: The decomposition of Entry 14 is obtained using Theorem 1.3 of Chapter 1.

Entries 17-22: The decomposition of Entry 17 is obtained as follows:

$$(1,1,0) \Rightarrow u(\cdot) = Cu_1(z) + \alpha\left[Cu_2(z) - Cu_1(z)\right] \cdot Cu(x,y) \tag{3.2}$$

$$(1,0,0) \Rightarrow u(\cdot) = Cu_1(y,z) + \beta\left[Cu_2(y,z) - Cu_1(y,z)\right] \cdot Cu(x) \tag{3.3}$$

$$Cu(x,y) = Cu_1(y) + \beta\left[Cu_2(y) - Cu_1(y)\right] \cdot Cu(x) \tag{3.4}$$

Substituting Eq. (3.4) back in Eq. (3.2), we get:

$$u(\cdot) = Cu_1(z) + \alpha\left[Cu_2(z) - Cu_1(z)\right] Cu_1(y)$$

$$+ \alpha\beta\left[Cu_2(z) - Cu_1(z)\right] \cdot \left[Cu_2(y) - Cu_1(y)\right] \cdot Cu(x)$$

Entry 23: Corresponds to Keeney's quasi-separable form as characterized by Definition 1.3 of Chapter 1.

Entry 24: The decomposition of Entry 24 is obtained using Theorem 1.3 of Chapter 1. The entry contains more assumptions than is necessary to generate the decomposition form. This implies that the assumptions jointly contain duplicate information about the preference structure.

Entries 25-27: See explanation for Entry 24.

Entries 28-33: See explanation for Entry 24.

Entries 34-36: The decomposition form of Entry 34 is obtained as follows:

$$(1,1,0) \Rightarrow u(\cdot) = Cu_1(z) + \alpha\left[Cu_2(z) - Cu_1(z)\right] \cdot Cu(x,y)$$

By Lemma 4.5 of Chapter 4, we have:

$$Cu(x,y) = \mp Cu(x) * Cu(y) \qquad * \in \{+,\times\}$$

Hence, we have:

$$u(\cdot) = Cu_1(z) \mp \alpha\Big[Cu_2(z) - Cu_1(z)\Big] \cdot \Big[Cu(x) * Cu(y)\Big] \qquad * \in \{+,\times\}$$

Entries 37-42: The decomposition form of Entry 37 is obtained as follows. From Entry 9, we have:

$$\{(0,1,0),(1,0,1)\} \Rightarrow u(\cdot) = \mp Cu(y) * Cu(x,z) \qquad * \in \{\times,+\} \qquad (3.5)$$

We also have:

$$(1,0,0) \Rightarrow u(\cdot) = Cu_1(y,z) + \alpha\Big[Cu_2(y,z) - Cu_1(y,z)\Big] \cdot Cu(x)$$

$$Cu(x,z) = Cu_1(z) + \alpha\Big[Cu_2(z) - Cu_1(z)\Big] Cu(x) \qquad (3.6)$$

Substituting Eq. (3.6) in Eq. (3.5), we get:

$$u(\cdot) = \mp\Big(Cu_1(z) + \alpha\Big[Cu_2(z) - Cu_1(z)\Big] \cdot Cu(x)\Big) * Cu(y) \qquad * \in \{\times,+\}$$

Entries 43-45: The UI sets for these entries correspond to what we call dichotomous chains, to be defined in Chapter 4. For an explanation of the derived form, see the proof of Theorem 4.4.

Entries 46-48: The decomposition of Entry 46 is obtained using Theorem 1.3 of Chapter 1. The remarks made for Entry 24 apply here also.

Entries 49-51: See explanation for Entry 24.

Entries 52-57: See explanation for Entry 24.

Entries 58-60: See explanation for Entry 24.

Entries 61-63: See explanation for Entry 24.

Entry 64: See explanation for Entry 24.

We observe from the construction of Table 3.1 that, even though the number of different UI sets is large, the number of different decomposition forms is relatively small. This observation is strongly demonstrated for spaces with a larger number of attributes. There are two instances where different UI sets correspond to the same decompositional forms: (1) The two UI sets are the same up to a reordering of the attributes, and (2) For a given ordering of the attributes, two unequal UI sets may contain the same information (i.e., information of the UI type) about the regularity of the preference structure. An example of the first instance is Entries 2 and 3 of Table 3.1; an example of the second instance is Entries 24 and 25. The second instance which invokes a sort of equivalence relation between different sets of UI assumptions is a source of flexibility to the analyst when modeling preferences. For a given utility decomposition, the analyst has more than one choice of UI sets guaranteeing the decomposition; he can choose the UI set most natural to the problem environment at hand. The second instance is treated in detail for n-attributes in the next chapter.

There are two more remarks particular to Table 3.1. Some entries of the table (e.g., Entry 60) correspond to more than one possible decomposition form. In such a case, the most general (or least restrictive) form is chosen. Finally, the decomposition forms of Entries 11 through 22 and Entries 34 through 45 are not explicitly treated in the literature.

Table 3.1

ALL PREFERENCES ON THREE-ATTRIBUTE SPACES

Entry	UI Assumptions						Decomposition Form
	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)	
1							$u(\cdot)$ is indecomposable
2	X						$u(\cdot) = \left(Cu(x)/\alpha\right) \cdot Cu_1(y,z)$ $+ \left[1 - Cu(x)/\alpha\right] \cdot Cu_2(y,z)$
3		X					
4			X				
5				X			$u(\cdot) = \left[Cu(x,y)/\alpha\right] \cdot Cu_1(z)$ $+ \left[1 - Cu(x,y)/\alpha\right] \cdot Cu_2(z)$
6					X		
7						X	
8	X					X	$u(\cdot) = \mp Cu(x) * Cu(y,z) \qquad * \in \{\times,+\}$
9		X			X		
10			X	X			
11	X	X					$u(\cdot) = Cu(x)/\alpha \cdot Cu(y)/\beta\, Cu_1(z)$ $+ Cu(x)/\alpha \left[1 - Cu(y)/\beta\right] \cdot Cu_2(z)$ $+ Cu(y)/\beta \left[1 - Cu(x)/\alpha\right] \cdot Cu_3(z)$ $+ \left[1 - Cu(x)/\alpha\right] \cdot \left[1 - Cu(y)/\beta\right] \cdot Cu_4(z)$
12	X		X				
13		X	X				

Table 3.1

CONTINUED

Entry	UI Assumptions						Decomposition Form
	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)	
14				X	X		
15				X		X	$u(\cdot) = \mp Cu(x) * Cu(y) * Cu(z) \quad * \in \{\times, +\}$
16					X	X	
17	X			X			
18	X				X		
19		X		X			$u(\cdot) = Cu_1(z) + \alpha[Cu_2(z) - Cu_1(z)]\ Cu_1(y)$
20		X				X	$+ \alpha\beta[Cu_2(z) - Cu_1(z)]$
21			X		X		$\cdot\ [Cu_2(y) - Cu_1(y)] \cdot Cu(x)$
22			X			X	
23	X	X	X				Keeney's quasi-separable model
24				X	X	X	
25	X			X	X		$u(\cdot) = \mp Cu(x) * Cu(y) * Cu(z) \quad * \in \{\times, +\}$
26		X		X		X	

Table 3.1

CONTINUED

Entry	UI Assumptions						Decomposition Form
	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)	
27			X		X	X	
28		X		X	X		
29			X	X	X		
30	X			X		X	$u(\cdot) = \mp Cu(x) * Cu(y) * Cu(z) \quad * \in \{\times,+\}$
31			X	X		X	
32		X			X	X	
33	X				X	X	
34	X	X		X			
35	X		X		X		$u(\cdot) = Cu_1(z) \mp \alpha\left[Cu_2(z) - Cu_1(z)\right] - \left[Cu(x) + Cu(y) \quad * \in \{\times,+\}\right.$
36		X	X			X	
37	X	X			X		
38	X	X				X	$u(\cdot) = \mp\left(Cu_1(z) + \alpha\left[Cu_2(z) - Cu_1(z)\right] \cdot Cu(x)\right) * Cu(y) \quad * \in \{\times,+\}$
39	X		X	X			

Table 3.1

CONTINUED

Entry	UI Assumptions						Decomposition Form
	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)	
40	X		X			X	$u(\cdot) = \mp\Big(Cu_1(z) + \alpha\big[Cu_2(z) - Cu_1(z)\big]$
41		X	X	X			$\cdot\, Cu(x)\Big) * Cu(y) \qquad * \in \{\times,+\}$
42		X	X		X		
43	X	X	X	X			$u(\cdot) = \mp\big[\mp Cu(x) *_1 Cu(y) + \beta\big] *_2 Cu(z)$
44	X	X	X		X		$*_1, *_2 \in \{\times,+\}$
45	X	X	X			X	
46	X			X	X	X	
47		X		X	X	X	
48			X	X	X	X	
49		X	X	X	X		$u(\cdot) = \mp Cu(x) * Cu(y) * Cu(z) \qquad * \in \{\times,+\}$
50	X		X	X		X	
51	X	X			X	X	
52	X	X		X	X		

Table 3.1

CONTINUED

Entry	UI Assumptions						Decomposition Form
	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)	
53	X		X	X	X		
54	X	X		X		X	
55		X	X	X		X	
56		X	X		X	X	
57	X		X		X	X	
58	X	X	X	X	X		
59	X	X	X	X		X	$u(\cdot) = \mp Cu(x) * Cu(y) * Cu(z) \quad * \in \{\times, +\}$
60	X	X	X		X	X	
61	X	X		X	X	X	
62		X	X	X	X	X	
63	X		X	X	X	X	
64	X	X	X	X	X	X	

3.2 Classifying Preferences on n-Attributes

If we are to construct a list similar to Table 3.1 for five attribute spaces, we would have to generate 2^{30} entries (see Eq. (3.1)). This number is extremely large and gets to be enormous for even a moderate number of attributes. The question we entertain in this section is: of the huge number of possible UI sets, how many correspond to a distinct decomposition form? Another wording of this question is: how many 'distinct' preference structures are there on n-attribute spaces? To have a well-posed question, we have to define carefully what we mean by 'distinct.'

Definition 3.1. Let A_1 and A_2 be two sets of UI assumptions that are satisfied in two preference structures on R^n. The two structures are said to be distinct if the collection of generic conditional utilities of A_1 and A_2 are not the same up to any permutation transformation of the attributes or, once they are the same, at least one conditional utility has different UI order for the two structures.

The definition distinguishes between preferences on the basis of regularities defined by UI assumptions only. The distinction is based on the two fundamental properties of a decomposition resulting from a UI set. The definition does not differentiate between two preferences that are defined by two UI sets which are the same up to a reordering of the attributes since, in this case, the two UI sets produce the same collection of generic conditional utilities and the same collection of UI order. The definition also clearly does not differentiate between two preferences defined by two UI sets that are unequal but contain the same regularity information. Thus, the definition eliminates the two types of equivalence of UI sets pointed out for Table 3.1.

Definition 3.1 is used to propose a three-stage classification scheme for preferences on n-attributes. We use the scheme to list all distinct preference structures on two-, three-, and four-attribute spaces.

The Classification Scheme

Stage 1

To distinguish among structures by only looking at their collections of generic conditional utilities. Let A be the set of satisfied UI assumptions, for a given preference, with the corresponding $Cu_1, \ldots, Cu_k$. Let d_i be the dimensionality of the domain of Cu_i, $i = 1,\ldots,k$. Let us call the numbers $[d_1,\ldots,d_k]$ the D-collection. The term "collection" is used to indicate that the indexing of the d_i's is immaterial. From the first part of Definition 3.1, two preference structures are distinct if their D-collections are not identical. Note that:

$$d_1 + \ldots + d_k = n \, , \qquad 1 \leq d_i \leq n \qquad \forall \; i = 1, \ldots, k \qquad (3.7)$$

where n is the number of attributes. Let $p(n)$ be the number of D-collections that correspond to distinct preference structures on an n-attribute space. $p(n)$ corresponds to a classical problem in combinatorics under the topic of <u>unordered partitions</u>. From Hall [15], $p(n)$ has the following generating function:

$$P(x) = \prod_{i=1}^{\infty} \left(1 - x^i\right)^{-1} , \qquad x \text{ is a real variable} \qquad (3.8)$$

Hall [15] also tabulates different values of $p(n)$.

Stage 2

To distinguish among structures by considering the collections of UI-order corresponding to a given collection of generic conditional utilities, $Cu_1, \ldots, Cu_k$. Let O_i be the UI order of Cu_i, $i = 1, \ldots, k$. The numbers $[O_1, \ldots, O_k]$ are called the O-collection. Here again, the indexing of the O_i's is immaterial. For two preferences, with k generic Cu's, to be distinct, it is sufficient that their O-collections are not identical. Theorem 2.4 gives constraints on all possible O-collections. Let $O_i' = \mathrm{Log}_2 O_i$. By Theorem 2.4,

$$\sum_{i=1}^{k} O_i' \leq k - 1; \qquad O_i' \in \{1, \ldots, k - 1\}, \quad i = 1, \ldots, K \tag{3.9}$$

Let $r(k)$ be the number of all possible O'-collections that correspond to distinct preferences, where k is the number of generic conditional utilities of the preferences. By comparing Eq. (3.7) and Eq. (3.9), we can write:

$$r(k) = 1 + \sum_{i=1}^{k-1} p(i) \qquad k = 2, \ldots, n$$

where n is the number of attributes.

Stage 3

Corresponds to a matching process between Stages 1 and 2 above. Two preferences having the same D-collection and O'-collection may still be distinct. As an example, let the number of attributes be 3. Let the

D-collection be [1,2], i.e., one 1-dimensional conditional utility and one 2-dimensional are involved. Let the O'-collection be [1,2]. There are two possible distinct preferences: (1) the 1-dimensional conditional utility is UI_1, and (2) the 1-dimensional conditional utility is UI_2.

The number of matchings corresponding to distinct preferences depends on the particularity of both collections. This problem is abstractly similar to the problem of finding all the combinatorial possibilities of matching partially distinguishable colored balls to partially distinguishable colored boxes. In general, only explicit enumeration is possible.

Enumerating All Distinct Preferences on Two, Three, and Four Attributes

In conformance with the classification scheme, Tables 3.2, 3.3, and 3.4 list all distinct preference structures on two-, three, and four-attribute spaces, respectively. The last column of each table contains the assessment effort required to construct the utility surface corresponding to each structure. The following notation is used.

> $i\ Cu^j$ means the assessment of i number of j-dimensional conditional utility is required

The following remarks pertain to Tables 3.2, 3.3, and 3.4.

(1) A similar table is constructed for a five-attribute space. We will just report that this space has 47 distinct preference structures. The number of distinct preferences should be contrasted with the number of all possible UI sets as obtained by the formula:

$$2^{[2^n-2]}$$

Table 3.2

ALL DISTINCT PREFERENCE STRUCTURES ON TWO ATTRIBUTES

Item	D-Collections	O'-Collections	Possible Matching	Required Assessment
1	[2]	[1]	[(2:1)]	1 Cu^2
2	[1,1]	[2,1]	[(1;2),(1;1)]	3 Cu^1
3		[1,1]	[(1;1),(1;1)]	2 Cu^1

Table 3.3

ALL DISTINCT PREFERENCE STRUCTURES ON THREE ATTRIBUTES

Item	D-Collections	O'-Collections	Possible Matching	Required Assessment
1	[3]	[1]	[(3;1)]	1 Cu^3
2	[2,1]	[2,1]	[(2;2),(1;1)]	2 Cu^2, 1 Cu^1
3		[2,1]	[(2;1),(1;2)]	1 Cu^2, 2 Cu^1
4		[1,1]	[(2;1),(1;1)]	1 Cu^1, 1 Cu^1
5	[1,1,1]	[4,1,1]	[(1,4),(1;1),(1;1)]	6 Cu^1
6		[2,2,1]	[(1;2),(1;2),(1;1)]	5 Cu^1
7		[2,1,1]	[(1;2),(1;1),(1;1)]	4 Cu^1
8		[1,1,1]	[(1;1),(1;1),(1;1)]	3 Cu^1

Table 3.4

ALL DISTINCT PREFERENCE STRUCTURES ON FOUR ATTRIBUTES

Item	D-Collections	O'-Collections	Possible Matching	Required Assessment
1	[4]	[1]	[(4;1)]	1 Cu^4
2	[3,1]	[2,1]	[(3;2),(1;1)]	2 Cu^3, 1 Cu^1
3		[2,1]	[(3;1),(1;2)]	1 Cu^3, 2 Cu^1
4		[1,1]	[(3;1),(1;1)]	1 Cu^3, 1 Cu^1
5	[2,2]	[2,1]	[(2;2),(2;1)]	3 Cu^2
6		[1,1]	[(2;1),(2;1)]	2 Cu^2
7	[2,1,1]	[4,1,1]	[(2;4),(1;1),(1;1)]	4 Cu^2, 2 Cu^1
8		[4,1,1]	[(2;1),(1;4),(1;1)]	1 Cu^2, 5 Cu^1
9		[2,2,1]	[(2;2),(1;2),(1;1)]	2 Cu^2, 3 Cu^1
10		[2,2,1]	[(2;1),(1;2),(1;2)]	1 Cu^2, 4 Cu^1
11		[2,1,1]	[(2;2),(1;1),(1;1)]	2 Cu^2, 2 Cu^1
12		[2,1,1]	[(2;1),(1;2),(1;1)]	1 Cu^2, 3 Cu^1
13		[1,1,1]	[(2;1),(1;1),(1;1)]	1 Cu^2, 2 Cu^1
14	[1,1,1,1]	[8,1,1,1]	[(1;8),(1;1),(1;1),(1;1)]	11 Cu^1
15		[4,2,1,1]	[(1;4),(1;2),(1;1),(1;1)]	8 Cu^1
16		[2,2,2,1]	[(1;2),(1;2),(1;2),(1;1)]	7 Cu^1
17		[4,1,1,1]	[(1;4),(1;1),(1;1),(1;1)]	7 Cu^1
18		[2,2,1,1]	[(1;2),(1;2),(1;1),(1;1)]	6 Cu^1
19		[2,1,1,1]	[(1;2),(1;1),(1;1),(1;1)]	5 Cu^1
20		[1,1,1,1]	[(1;1),(1;1),(1;1),(1;1)]	4 Cu^1

(2) The whole classification layout is modulo the concept of utility independence. Other decompositional concepts may be used as well.

(3) Two perhaps curious entries of Table 3.4 are Items 16 and 17 where, even though the two structures are distinct, *the required assessment effort is the same.*

(4) With respect to Table 3.4, the three celebrated models in the literature: the additive utility model, the quasi-separable, and the multiplicative-additive, all correspond to the last entry of the table.

The scheme may be used to generate such lists for higher dimensional utility surfaces as well.

Chapter 4

UI IMPLICATIONS

In this chapter, we will consider sets of UI assumptions which automatically imply the satisfaction of some new ones. This topic is of practical importance to the analyst since it shows him how to minimize his work and eliminate some, perhaps nonobvious, duplication of effort.

As a preview to the chapter, Section 4.1 characterizes the set of all possible implications for a given set of UI assumptions. Section 4.2 demonstrates three basic instances of UI implications, one of which is discovered and reported by Keeney [23]. Section 4.3 proposes a canonical form for sets of UI assumptions that is useful for, among other things, recognizing UI implications. Since the three sources of UI implications are intimately related to the multiplicative-additive utility model, Section 4 takes a close look at this model and constructs sets of UI assumptions for it where the number of assumptions involved is minimal.

Before we start, we propose a classification of pairs of UI assumptions that is useful and convenient for theoretical arguments.

Let a and b, $a \neq b$ be two UI assumptions.

$[a,b] \in$ class 1 if: $a \wedge b \neq 0$ and $\bar{a} \wedge b \neq 0$ and $a \wedge \bar{b} \neq 0$

and

$[a,b] \in$ class 2 if: $a \wedge b \neq 0$ and $\bar{a} \wedge b$ or $a \wedge \bar{b}$ is zero

and

$[a,b] \in$ class 3 if: $a \wedge b = 0$ and $a \neq \bar{b}$

Finally,

$[a,b] \in$ class 4 if: $a = \bar{b}$

The classification is clearly mutually exclusive and collectively exhaustive, and is unique for any reordering of the attributes.

4.1 The Set of All Possible UI Implications

Let A be a set of satisfied UI assumptions for a given preference structure.

Definition 4.1. Let b be a UI assumption. A implies b if A and $A' = A \cup \{b\}$ contain the same assumptions about the induced preference ordering of the structure.

We will restrict the set of possibilities of UI implications for a given A. If b is implied by A, then from the definition A and A' will have the same decomposition. Equivalently, b can not contain new information about the preferences not already included in A. Hence, b should, roughly speaking, be compatible with the partition boundaries of the decomposition of A. The following theorem is a careful statement of this observation.

Theorem 4.1. Let $A = \{a_i : i = 1,\ldots,k\}$ be a set of satisfied assumptions. Let A imply b, $b \notin A$. Then,

$$b = ({}_1\tilde{a}_1 \wedge \ldots \wedge {}_1\tilde{a}_k) \vee \ldots \vee ({}_\ell\tilde{a}_1 \wedge \ldots \wedge {}_\ell\tilde{a}_k)$$

where

$${}_j\tilde{a}_i \in \{a_i, \bar{a}_i\}, \qquad i = 1, \ldots, k \; ; \; j = 1, \ldots, \ell$$

$$1 \leq \ell \leq 2^k$$

and $\vee$ denotes Boolean addition.

Before proving the theorem, let us state its assertion in words. Let the partition of the attribute space (with respect to A) be X_1, ..., X_ℓ. Then the theorem says that b corresponds to a UI assumption where the union of some of the subspaces is utility independent of its complement.

<u>Proof</u>.

By contradiction. Assume that the assertion is not true. The only other possibility is for b to cut across the partition boundaries. As such, b can be used to further decompose at least one of the subspaces. Hence, b reflects new information about the induced preference orderings not already contained in A. This contradicts the definition of UI implication.

Q.E.D.

The following example is an application of Theorem 4.1.

<u>Example 4.1</u>.

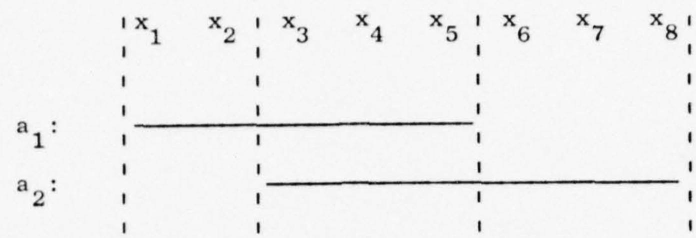
Let $A = \{(1,1,1,1,\ 1,0,0,0),(0,0,1,1,\ 1,1,1,1)\}$. The set of all possible UI implications is the following:

$$\{(1,1,0,0,\ 0,0,0,0),(0,0,1,1,\ 1,0,0,0),(0,0,0,0,\ 0,1,1,1),$$
$$(1,1,1,1,\ 1,0,0,0),(1,1,0,0,\ 0,1,1,1),(0,0,1,1,\ 1,1,1,1)\}$$

Since the number of all UI assumptions in R^8 is 2^8-2, we are able to eliminate all but the above six assumptions from consideration.

<u>Bar Diagrams</u>

A, in Example 4.1 above, may also be denoted schematically by the so-called 'bar diagram' representation as follows:



Where the variables underlined by each horizontal line are UI of their complement; the dotted vertical lines indicate the boundaries of the partition of the attribute space. As we will see in the next section, bar diagrams are very convenient for representing sets of UI assumptions.

4.2 Different Sources of Implications

In this section, we will introduce three sources of UI implications, the first two of which are basic patterns that can be discerned in arbitrary sets of UI assumptions. The third source is due to a joint occurrence of the first two ones.

4.2.1 Implications Due to Overlapping Chains

This source of implications was discovered and reported by Keeney [23]. The definitions and theorems of this subsection are, with some modifications, in Keeney [23].

Definition 4.2. Let A be a set of satisfied UI assumptions. Let $B = \{b_i: i = 1,\dots,r,\ r \geq 1\}$ be a nonempty subset of A. B is called an overlapping chain if there exists an ordering on the b_i's such that for every b_j, $j = 2,\dots,r$ there exists at least one i, $i = 1,\dots,r$ such that

$$[b_i, b_j] \in \text{Class } 1$$

Definition 4.3. Let $C = \{c = (\tilde{b} \wedge \ldots \wedge \tilde{b}_r) : (\tilde{b}_1 \wedge \ldots \wedge \tilde{b}_r) \neq 0$ and $(\tilde{b}_1 \wedge \ldots \wedge \tilde{b}_r) \neq (\bar{b}_1 \wedge \ldots \wedge \bar{b}_r)$ and $\tilde{b}_i \in \{b_i, \bar{b}_i\}$, $i = 1, \ldots, r\}$. $c \in C$ is called an overlapping chain element.

Theorem 4.2 (Keeney). Let $C = \{c_i : i = 1, \ldots, 1\}$ be as defined above. Then, every

$$\bigvee_{i \in I} c_i \qquad I \subseteq \{1, 2, \ldots, p\}$$

is a UI assumption implied by B.

The theorem says that the Boolean sum of any subset of the chain elements is a satisfied UI assumption on the preference structure corresponding to B. The reader should verify that these UI implications do conform with the assertions of Theorem 4.1.

Example 4.2.

Let $A = \{(1,1,0,0), (0,1,1,1), (0,0,0,1)\}$. Let $B = \{(1,1,0,0), (0, 1,1,1)\}$. B is clearly an overlapping chain. Also,

$$C = \{(1,0,0,0), (0,1,0,0), (0,0,1,1)\}$$

Now, by Theorem 4.2, each element in the set:

$$\{(1,0,0,0), (0,1,0,0), (0,0,1,1), (1,1,0,0), (1,0,1,1), (0,1,1,1)\}$$

is a UI implication of B. This set also happens to be the set of all possible UI implications of B as characterized by Theorem 4.1.

The next theorem relates overlapping chains to the multiplicative-additive, denoted M-A, utility model. (See Theorem 1.3 of Chapter 1.)

Theorem 4.3 (Keeney). Let B be as in Definition 4.2 above. Then, we have either

$$\left[1 + K\ Cu(b_1 \vee \ldots \vee b_r)\right] = \prod_{i=1}^{r} \left[1 + K\ Cu(b_i)\right] \quad \text{where } K \neq 0 \quad (4.1)$$

or, if $K = 0$, we have

$$Cu(b_1 \vee \ldots \vee b_r) = \sum_{i=1}^{r} Cu(b_i)$$

It is the symmetry of Eq. (4.1) that causes all the implications of Theorem 4.2.

4.2.2 Implications Via Dichotomous Chains

Let A be a set of satisfied UI assumptions.

Definition 4.4. Let $B = \{b_i : i = 1,\ldots,\ell\}$ be a nonempty subset of A. B is called a dichotomous chain if there exists an ordering on the b_i such that, for every b_j, $2 \leq j \leq \ell$, there exists a unique r, $1 \leq r \leq \ell$ such that:

(1) $b_j \wedge b_r = 0$

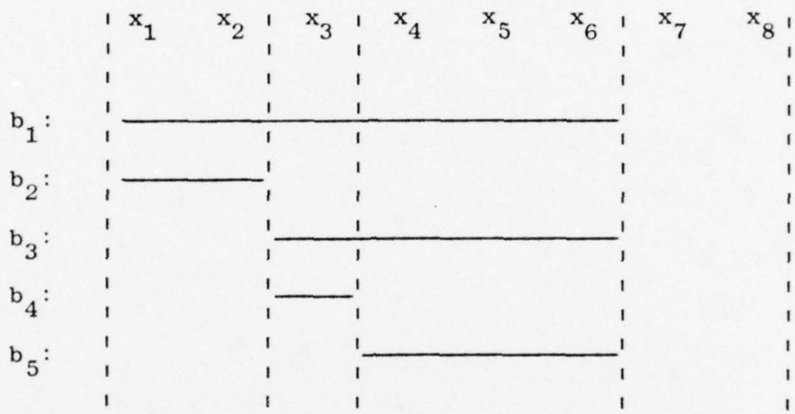
(2) $(b_j \vee b_r) \in B$ or $(b_j \vee b_r) = 1$ †

Definition 4.5. Let $C \subset B$ be the set of all c such that $c \wedge b = 0$ or $c \wedge b = c$, $\forall\, b \in B$. $c \in C$ is called a dichotomous chain element.

† 1 is the vector of all ones.

Example 4.3.

Let $B = \{(1,1,1,1,\ 1,1,0,0),(1,1,0,0,\ 0,0,0,0),(0,0,1,1,\ 1,1,0,0),$ $(0,0,0,1,\ 1,1,0,0),(0,0,1,0,\ 0,0,0,0)\}$. Choose $b_1 = (1,1,1,1,\ 1,1,0,0)$. B is a dichotomous chain. By a bar diagram, B is denoted as follows:



The following theorem is the analogue of Theorem 4.3.

Theorem 4.4. Let $B = \{b_i : i = 1,\ldots,\ell\}$ be a dichotomous chain. Order the b_i's such that $b_r,\ldots,b_\ell$, $1 < r < \ell$, are the elements of the chain. Then,

$$Cu(b_1 \vee \ldots \vee b_\ell) = \sum_{i=1}^{r} \alpha_i f_{i1}(b_1) \times \ldots \times f_{i\ell}(b_\ell)$$

where

$$f_{ij}(b_j) = Cu(b_j) \quad \text{or} \quad f_{ij}(b_j) = 1 \qquad \forall\ i,j$$

and the α_i's are real constants.

Let us call this model the generalized quasi-separable model.

Before proving this theorem, let us prove the same assertion for the two simplest cases of a dichotomous chain.

<u>Lemma 4.5</u>. Let $B = \{b_1, b_2, b_3\}$ be such that $b_2 \wedge b_3 = 0$ and $b_2 \vee b_3 = b_1$. Then, either

$$Cu(b_1) = Cu(b_2) + Cu(b_3)$$

or

$$\left[1 + K\ Cu(b_1)\right] = \left[1 + K\ Cu(b_2)\right] \cdot \left[1 + K\ Cu(b_3)\right] \qquad \text{for } K \neq 0$$

<u>Proof</u>.

B is, from the definition, a dichotomous chain. Without any loss of generality, let the 1's in b_2 correspond to vector x, the 1's in b_3 correspond to vector y, and the 1's in $\bar{b}_1$ correspond to vector z. So,

$$\begin{array}{llll} b_1 \equiv & x \text{ and } y & & \text{UI of } z \\ b_2 \equiv & x & & \text{UI of } y \text{ and } z \\ b_3 \equiv & y & & \text{UI of } x \text{ and } z \end{array}$$

Assume $u(x_1, y_1, z_1) = 0$. By definition, we have:

$$b_2 \Rightarrow u(x,y,z) = Cu(x_1,y,z) + \gamma\left[Cu(x_2,y,z) - Cu(x_1,y,z)\right] Cu(x,y_1,z_1) \tag{4.2}$$

$$b_3 \Rightarrow u(x,y,z) = Cu(x,y_1,z) + \beta\left[Cu(x,y_2,z) - Cu(x,y_1,z)\right] Cu(x_1,y,z_1) \tag{4.3}$$

where $\gamma = 1/Cu(x_2,y_1,z_1)$; $\beta = 1/Cu(x_1,y_2,z_1)$. From Eq. (4.2),

$$Cu(x,y,z_1) = Cu(x_1,y,z_1) + \gamma\left[Cu(x_2,y,z_1) - Cu(x_1,y,z_1)\right] Cu(x,y_1,z_1) \tag{4.4}$$

But, from Eq. (4.3),

$$Cu(x_2,y,z_1) = \alpha_2 + \beta(\alpha_1 - \alpha_2)\ Cu(x_1,y,z_1) \tag{4.5}$$

where $\alpha_2 = Cu(x_2,y_1,z_1) = 1/\gamma$; $\alpha_1 = Cu(x_2,y_2,z_1)$. Substituting Eq. (4.5) back into Eq. (4.4), we get:

$$Cu(x,y,z_1) = Cu(x_1,y,z_1) + \gamma\left[\alpha_2 + (\beta\alpha_1 - \beta\alpha_2 - 1)\ Cu(x_1,y,z_1)\right] \cdot Cu(x,y_1,z_1)$$

Let $\gamma(\beta\alpha_1 - \beta\alpha_2 - 1) = k$. If $k = 0$, we get the additive model. Otherwise, we can write:

$$1 + k\ Cu(x,y,z_1) = \left[1 + k\ Cu(x_1,y,z_1)\right] \cdot \left[1 + K\ Cu(x,y_1,z_1)\right]$$

Q.E.D.

The second simple case of a dichotomous chain correspond to Theorem 1.4 of Chapter 1. We restate it here for fast reference.

Lemma 4.6. Let $B = \{b_1,b_2\}$ where $b_1 = \bar{b}_2$. Then, either

$$u(b_1 \vee b_2) = Cu(b_1) + Cu(b_2)$$

or

$$\left[1 + Ku(b_1\ \ b_2)\right] = \left[1 + K\ Cu(b_1)\right] \cdot \left[1 + K\ Cu(b_2)\right] \quad \text{for } K \neq 0$$

Proof of Theorem 4.4.

Let the number of subspaces that are dichotomized by pairs of UI assumptions be p. The proof is by induction on p. For $p = 1$, the dichotomous chain corresponds to either Lemma 4.5 or Lemma 4.6; hence, the assertion of the theorem is satisfied. Assume that the assertion is satisfied for $p-1$. That is, we can write:

$$Cu(b_1 \vee \ldots \vee b_\ell) = \sum_{i=1}^{r} \alpha_i f_{i1}(b_1) \times \ldots \times f_{i\ell}(b_\ell) \qquad (4.6)$$

For some $j \in \{1,2,\ldots,\ell\}$, let the pair of UI assumptions b_{j1} and b_{j2} dichotomize the subspace corresponding to b_j. We thus can write:

$$Cu(b_j) = Cu(b_{j1}) + Cu(b_{j2}) + K\, Cu(b_{j1}) \cdot Cu(b_{j2}) \qquad (4.7)$$

where K is any real constant. (Note: Eq. (4.7) is an equivalent statement of the assertion of Lemma 4.5.)

Substituting Eq. (4.7) back in Eq. (4.6), it is clear that the representation form will be preserved.

Q.E.D.

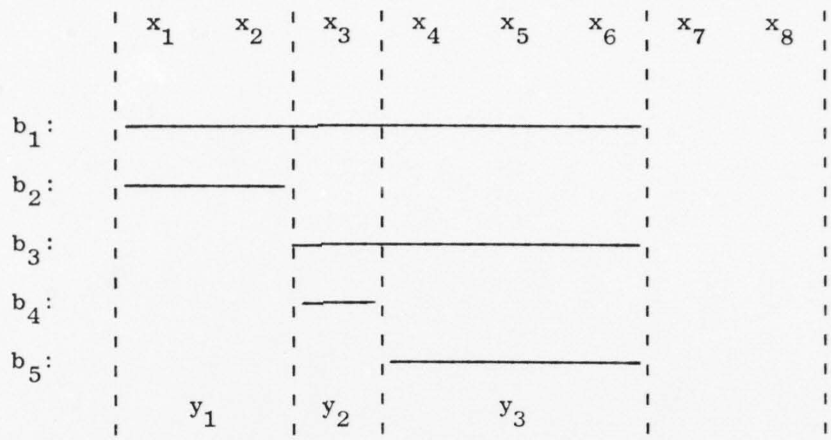
The assertion of Theorem 4.4 is too general to indicate explicitly possible UI implications of a dichotomous chain. In fact, a different kind of preference information about the structure is required before any UI implications are discerned. The following example demonstrates this point.

Example 4.4.

Let B = {(1,1,1,1 1,1,0,0),(1,1,0,0 0,0,0,0),(0,0,1,1, 1,1,0,0), (0,0,1,0, 0,0,0,0),(0,0,0,1, 1,1,0,0)}. Choose b_1 = (1,1,1,1, 1,1,0,0). B is a dichotomous chain

C = {(1,1,0,0, 0,0,0,0),(0,0,1,0, 0,0,0,0),(0,0,0,1, 1,1,0,0)}

By a bar diagram, B is denoted as follows:



Using $\{b_1, b_2, b_3\}$, we can write:

$$Cu(y_1, y_2, y_3) = Cu(y_1) + Cu(y_2, y_3) + K_1 \, Cu(y_1) \cdot Cu(y_2, y_3) \qquad (4.8)$$

where K_1 is any real constant. Also, using $\{b_3, b_4, b_5\}$, we have:

$$Cu(y_2, y_3) = Cu(y_2) + Cu(y_3) + K_2 \, Cu(y_2) \cdot Cu(y_1) \qquad (4.9)$$

where K_2 is any real constant.

We will consider three cases:

<u>Case 1</u>: $K_1 = K_2 = 0$.

As such, we can write:

$$Cu(y_1, y_2, y_3) = Cu(y_1) + Cu(y_2) + Cu(y_3) \qquad (4.10)$$

<u>Case 2</u>: $K_1 = K_2 \neq 0$.

We have:

$$\left[1 + K_1 \, Cu(y_1, y_2, y_3)\right] = \left[1 + K_1 \, Cu(y_1)\right] \cdot \left[1 + K_1 \, Cu(y_2, y_3)\right]$$

and

$$\left[1 + K_2\ Cu(y_2,y_3)\right] = \left[1 + K_2\ Cu(y_2)\right] \cdot \left[1 + K_2\ Cu(y_3)\right]$$

$$= \left[1 + K_1\ Cu(y_2,y_3)\right]$$

Hence,

$$\left[1 + K_1\ Cu(y_1,y_2,y_3)\right] = \left[1 + K_1\ Cu(y_1)\right] \cdot \left[1 + K_1\ Cu(y_2)\right]$$

$$= \left[1 + K_1\ Cu(y_3)\right] \tag{4.11}$$

These two cases correspond to the M-A model. By a scrutiny of the UI definition, we can easily discern the following implications:

$$\{(1,1,1,0,\ 0,0,0,0),(1,1,0,1,\ 1,1,0,0),(0,0,1,1,\ 1,1,0,0),$$
$$(1,1,1,1,\ 1,1,0,0),(1,1,0,0,\ 0,0,0,0),(0,0,1,0,\ 0,0,0,0),$$
$$(0,0,0,1,\ 1,1,0,0)\}$$

<u>Case 3</u>: $K_1 \neq K_2$.

As such, we can write:

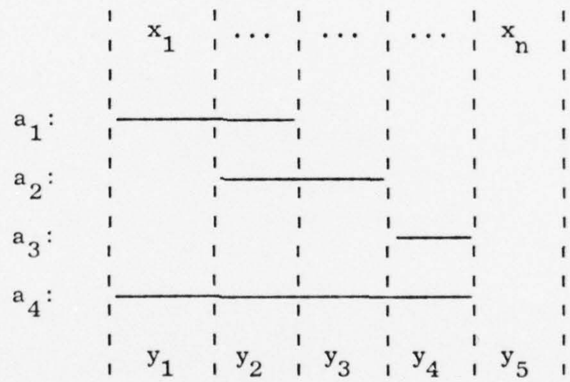
$$Cu(y_1,y_2,y_3) = Cu(y_1) + Cu(y_2) + Cu(y_3) + K_2\ Cu(y_2)\ Cu(y_3)$$
$$+ K_1\ Cu(y_1)\ Cu(y_2) + K_1\ Cu(y_1)\ Cu(y_3)$$
$$+ K_1K_2\ Cu(y_1)\ Cu(y_2)\ Cu(y_3) \tag{4.12}$$

For this case, there are no UI implication. Note that Eqs. (4.10), (4.11), and (4.12) all conform with the assertion of Theorem 4.4. The values of K_1 and K_2 may be determined by a procedure contained in Keeney [21].

4.2.3 Implications Via Mixed Chains

Here, we consider sets composed of overlapping and dichotomous chains, jointly. Since there are many combinatorial possibilities of such coexistence, we will use a series of examples to demonstrate different basic patterns. For each example, a bar diagram representation is used to denote the UI assumptions. The verification of each example should be apparent from Theorems 4.3 and 4.4.

Example 4.5.



Using $\{a_1, a_2\}$, we have either:

$$\left[1 + K_1\ Cu(y_1, y_2, y_3)\right] = \left[1 + K_1\ Cu(y_1)\right] \cdot \left[1 + K_1\ Cu(y_2)\right] \cdot \left[1 + K_1\ Cu(y_3)\right]$$

where $K_1 \neq 0$. Or, if $K_1 = 0$, we have:

$$Cu(y_1, y_2, y_3) = Cu(y_1) + Cu(y_2) + Cu(y_3)$$

Also, using $\{(a_1 \vee a_2), a_3, a_4\}$, we have:

$$Cu(y_1, y_2, y_3, y_4) = Cu(y_1, y_2, y_3) + Cu(y_4) + K_2\ Cu(y_1, y_2, y_3) \cdot Cu(y_4)$$

Let us consider the following three cases:

Case 1: $K_1 = K_2 = 0$.

As such, we can write:

$$Cu(y_1,y_2,y_3,y_4) = Cu(y_1) + Cu(y_2) + Cu(y_3) + Cu(y_4)$$

Case 2: $K_1 = K_2 \neq 0$.

As such, we have:

$$\left[1 + K_2\ Cu(y_1,y_2,y_3,y_4)\right] = \left[1 + K_2\ Cu(y_4)\right] \cdot \left[1 + K_2\ Cu(y_1,y_2,y_3)\right]$$

$$= \left[1 + K_2\ Cu(y_4)\right] \cdot \left[1 + K_1\ Cu(y_1)\right]$$

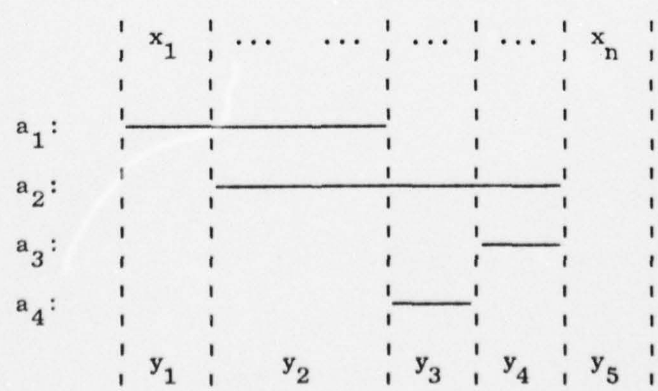
$$\cdot \left[1 + K_1\ Cu(y_2)\right] \cdot \left[1 + K_1\ Cu(y_3)\right]$$

For these two cases, the UI implications are: any subset of $\{y_1, y_2, y_3, y_4\}$ is UI.

Case 3: $K_1 \neq K_2$.

Here, by a scrutiny of the evolving functional form, the only UI implications are those associated with the overlapping chain $\{a_1, a_2\}$.

Example 4.6.



Using $\{a_1, a_2\}$, we have either:

$$\left[1 + K_1\ Cu(y_1,y_2,y_3,y_4)\right] = \left[1 + K_1\ Cu(y_1)\right] \cdot \left[1 + K_1\ Cu(y_2)\right] \cdot \left[1 + K_1\ Cu(y_3,y_4)\right]$$

where $K_1 \neq 0$. Or, if $K_1 = 0$, we have:

$$Cu(y_1,y_2,y_3,y_4) = Cu(y_1) + Cu(y_2) + Cu(y_3,y_4)$$

Using $\{(a_2 \wedge \bar{a}_1), a_3, a_4\}$, we have:

$$Cu(y_3,y_4) = Cu(y_3) + Cu(y_4) + K_2\ Cu(y_3) \cdot Cu(y_4)$$

K is any real constant.

Let us consider the following three cases:

Case 1: $K_1 = K_2 = 0$.

As such, we can write:

$$Cu(y_1,y_2,y_3,y_4) = Cu(y_1) + Cu(y_2) + Cu(y_3) + Cu(y_4)$$

Case 2: $K_1 = K_2 \neq 0$.

As such, we can write:

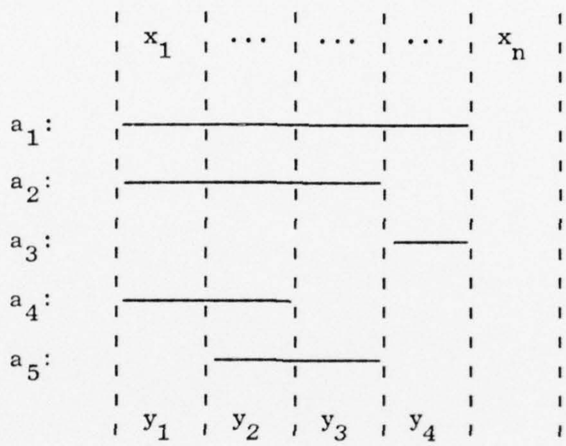
$$\left[1 + K_1\ Cu(y_1,y_2,y_3,y_4)\right] = \left[1 + K_1\ Cu(y_1)\right] \cdot \left[1 + K_1\ Cu(y_2)\right] \cdot \left[1 + K_1\ Cu(y_3)\right] \cdot \left[1 + K_1\ Cu(y_4)\right]$$

For these two cases, the UI implications are: any subset of $\{y_1, y_2, y_3, y_4\}$ is UI.

Case 3: $K_1 \neq K_2$.

For this case, the only UI implications are those associated with the overlapping chain $\{a_1, a_2\}$.

Example 4.7.



Using $\{a_4, a_5\}$, we have either:

$$\left[1 + K_1\ Cu(y_1, y_2, y_3)\right] = \left[1 + K_1\ Cu(y_1)\right] \cdot \left[1 + K_1\ Cu(y_2)\right] \cdot \left[1 + K_1\ Cu(y_3)\right]$$

where $K_1 \neq 0$. Or, if $K_1 = 0$, we have:

$$Cu(y_1, y_2, y_3) = Cu(y_1) + Cu(y_2) + Cu(y_3)$$

Using $\{a_1, a_2, a_3\}$, we have:

$$Cu(y_1, y_2, y_3, y_4) = Cu(y_1, y_2, y_3) + Cu(y_4) + K_2\ Cu(y_1, y_2, y_3) \cdot Cu(y_4)$$

Let us consider the following three cases:

Case 1: $K_1 = K_2 = 0$.

As such, we can write:

$$Cu(y_1,y_2,y_3,y_4) = Cu(y_1) + Cu(y_2) + Cu(y_3) + Cu(y_4)$$

Case 2: $K_1 = K_2 \neq 0$.

As such, we can write:

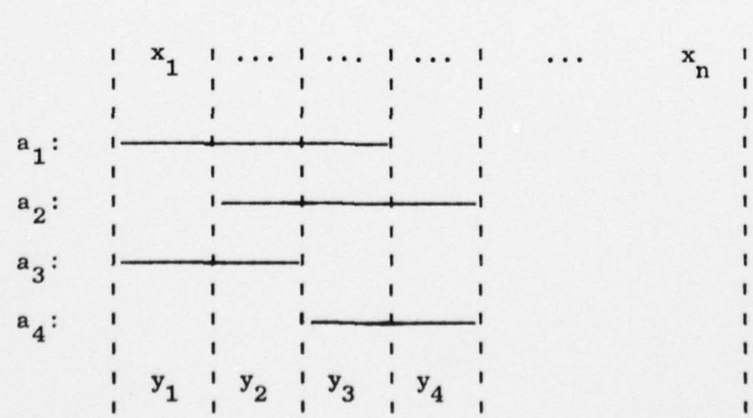
$$\left[1 + K_1 \; Cu(y_1,y_2,y_3,y_4)\right] = \left[1 + K_1 \; Cu(y_1)\right] \cdot \left[1 + K_1 \; Cu(y_2)\right] \cdot \left[1 + K_1 \; Cu(y_3)\right] \cdot \left[1 + K_1 \; Cu(y_4)\right]$$

For these two cases, the UI implications are: any subset of $\{y_1,y_2,y_3,y_4\}$ is UI.

Case 3: $K_1 \neq K_2$.

Here, by a scrutiny of the evolving functional form, the only UI implications are those associated with the overlapping chain $\{a_4,a_5\}$.

Example 4.8.



In this example,

(1) $\{a_1,a_2\}$ and $\{a_1,a_2,a_3\}$ and $\{a_1,a_2,a_4\}$ are overlapping chains.

(2) $\{(a_1 \vee a_2),a_3,a_4\}$ and $\{(a_1 \wedge a_2),(a_2 \wedge a_3),(a_1 \wedge a_4)\}$ are dichotomous chains.

(3) $\{a_1,a_2,a_3,a_4\}$ is also an overlapping chain and it subsumes all the chains above. Thus, the set of UI implications is the one associated with this chain, as characterized by Theorem 4.2.

4.3 A Canonical Form for Sets of UI Assumptions

Here, we propose a canonical form that is useful for, among other things, recognizing the kind of UI implications treated in the previous section. The form is introduced by an algorithm that constructs it for any arbitrary set of UI assumptions. The main idea of the form is to aggregate UI assumptions with respect to the space location of the regularity that they define.

The Algorithm

PART 1. Step 1: Let $a_i \in A$ be such that, if $(a_i,a_j) \in$ Class 2, then $\bar{a}_i \wedge a_j = 0$ for any $a_j \in A$.

Step 2: Let $A(1)$ be the union of all overlapping chains in A that contain a_i.

Step 3: Let $a_i \in A \setminus A(1)$ and assume it satisfies the conditions of Step 1 above. Construct $A(2)$ as in Step 2 above.

__Step 4__: Let $a_3 \in A \setminus A(1) \cup A(2)$ and assume it satisfies the conditions of Step 2 above. Construct $A(3)$ as in Step 2 above. By repeating this process, we construct $A(1),\ldots,A(j_1)$. The process stops when there does not exist $a_i \in A \setminus A(1) \cup \ldots \cup A(j_1)$ which satisfies the condition of Step 1.

Step 5: Construct $A^*(i)$, $1 \leq i \leq j_1$, as follows: $a_r \in A^*(i)$ if $a_r \in A \setminus A(1) \cup \ldots \cup A(j_1)$ and $a_r \wedge a_d = a_r$ for some $a_d \in A(i)$. Now, the $A(i)$'s and $A^*(i)$'s are a collection of mutually exclusive and collectively exhaustive subsets of A.

PART 2: For each $A^*(i) \neq \emptyset$, $i = 1,\ldots,j_1$, construct $A(i;1),\ldots,A(i;j_2)$ and $A^*(i;1),\ldots,A^*(i;j_2)$ by replacing A by $A^*(i)$ in PART 1 above.

PART 3: For each $A^*(i_1;i_2) \neq \emptyset$, $1 \leq i_1 \leq j_1$ and $1 \leq i_2 \leq j_2$, construct $A(i_1;i_2;1),\ldots,A(i_1;i_2;j_3)$ and $A^*(i_1;i_2;1),\ldots,A^*(i_1;i_2;j_3)$. This process is continued until all the $A^*(\cdot)$ are empty. The canonical form is the collection of all the $A(\cdot)$'s.

The following example demonstrates the algorithm.

__Example 4.9.__

Let $A = \{a_i: i = 1,\ldots,10\}$, where the a_i's are:

$$a_1 = (0,0,0,0,\ 0,0,0,0,\ 0,0,1,1)$$

$$a_2 = (0,1,1,1,\ 1,0,1,1,\ 1,1,0,0)$$

$$a_3 = (0,0,1,0,\ 0,1,0,0,\ 0,1,0,0)$$

$$a_4 = (1,0,0,0,\ 0,0,0,0,\ 0,0,0,0)$$

$$a_5 = (0,1,0,1,\ 1,0,1,0,\ 0,0,0,0)$$

$$a_6 = (0,1,0,1,\ 0,0,0,1,\ 1,0,0,0)$$

$$a_7 = (0,1,0,0,\ 0,0,0,0,\ 0,0,0,0)$$

$$a_8 = (0,0,0,1,\ 0,0,0,0,\ 0,0,0,0)$$

$$a_9 = (0,0,0,0,\ 0,0,0,0,\ 0,0,0,1)$$

$$a_{10} = (0,0,1,0,\ 0,0,0,0,\ 0,0,0,0)$$

Applying the algorithm, we get:

$A(1) = \{a_1\}$; $\quad A^*(1) = \{a_9\}$

$A(2) = \{a_4\}$; $\quad A^*(2) = \emptyset$

$A(3) = \{a_2,a_3\}$; $\quad A^*(3) = \{a_5,a_6,a_{10},a_7,a_8\}$

$A(1;1) = \{a_9\}$; $\quad A^*(1;1) = \emptyset$

$A(3;1) = \{a_5,a_6\}$; $\quad A^*(3;1) = \{a_7,a_8\}$

$A(3;2) = \{a_{10}\}$; $\quad A^*(3;2) = \emptyset$

$A(3;1;1) = \{a_7\}$; $\quad A^*(3;1;1) = \emptyset$

$A(3;1;2) = \{a_8\}$; $\quad A^*(3;1;2) = \emptyset$

Figure 4.1 contains a schematic representation of the canonical form for this example.

The following comments pertain to the canonical form:

(1) Each $A(\cdot)$ contains UI assumptions corresponding to a particular subspace of the attribute space. $A^*(\cdot)$ contains UI assumptions that decompose this subspace further. When $A^*(\cdot)$ is empty, this implies that the subspace corresponding to $A(\cdot)$ can not be decomposed any further.

(2) The boundaries of the attribute space partition, corresponding to A, are compatible with the partition of the attributes corresponding to the $A(\cdot)$'s whose $A^*(\cdot)$'s are empty.

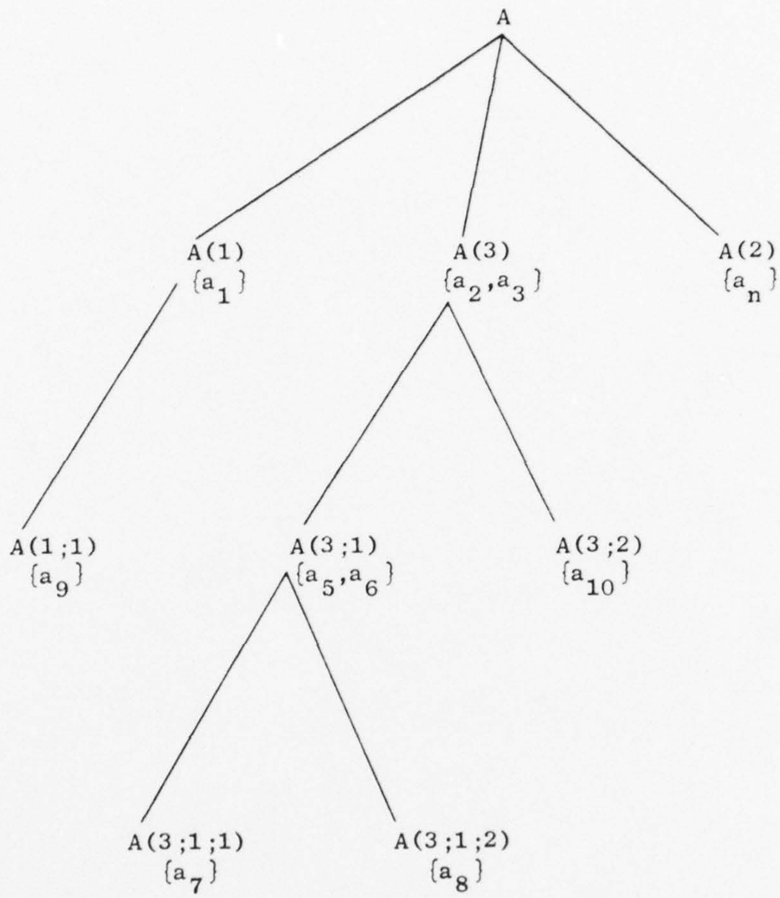


Fig. 4.1. A SCHEMATIC REPRESENTATION OF THE CANONICAL FORM OF EXAMPLE 4.9.

(3) All UI implications due to overlapping chains are discerned by scrutinizing all the $A(\cdot)$'s of A.

(4) To recognize possible UI implications due to a dichotomy, more than one $A(\cdot)$ has to be considered at a time. Many patterns are possible; the main guideline is to look for the occurrence of pairs of $A(\cdot)$'s which are singleton and have the same source, i.e., the same $A^*(\cdot)$.

4.4 The Multiplicative-Additive Model

Of the different decompositional utility models, the multiplicative-additive model is the one most used in reported applications. We have seen in Section 4.2 that there are many distinct sets of UI assumptions that produce an M-A model. In this section, we construct sets with the least number of UI assumptions that produce the M-A model:

$$u(x_1, \ldots, x_n) = \mp Cu(x_1) * \ldots * Cu(x_n) \qquad * \in \{\times, +\}$$

where n is arbitrary. We will denote such sets by A^n.

Construction of A^n

Step 1: If n is even, segment the attributes into two halves corresponding to the two sets $S(1)$ and $S(2)$. If n is odd, segment the attributes such that Card $[S(1)]$ = Card $[S(2)]$ + 1, where 'Card' stands for the cardinality of the set.

Step 2: By inductive construction, assume we have $S(i_1;i_2;\ldots;i_j)$, $i_1,\ldots,i_j \in \{1,2\}$. If Card $[S(i_1;\ldots;i_j)]$ is even, segment its attributes into two halves corresponding to $S(i_1;\ldots;i_j;1)$ and $S(i_1;\ldots;i_j;2)$. If Card $[S(i_1;\ldots;i_j)]$ is odd, segment the attributes such that Card $[S(i_1;\ldots;i_j;1)] = 1 +$ Card $[S(i_1;\ldots;i_j;2)]$.

Step 3: This construction is carried on until every $S(\cdot)$ is a singleton. Assume these last sets are indexed by $S(i_1;\ldots;i_{r-1})$.

Step 4: For every j, $1 \leq j \leq r-2$, let a_j denote that the attributes in

$$\bigcup_{\substack{i_j=1 \\ i_1,\ldots,i_{j-1}\in\{1,2\}}} S(i_1; \ldots; i_j)$$

are UI of their complement.

Step 5: This process is ended in one of two ways:

(i) If Card $[S(2_1;2_2;\ldots;2_{r-2})] = 1$, then a_{r-1} denotes that the attributes in

$$\left[\bigcup_{\substack{i_{r-1}=1 \\ i_1,\ldots,i_{r-2}\in\{1,2\}}} S(i_1;\ldots;i_{r-1})\right] \cup S(2_1;2_2;\ldots;2_{r-2})$$

are UI of their complement, and we stop.

(ii) If Card $[S(2_1;2_2;\ldots;2_{1-2})] = 2$, then a_{r-1} is defined as in Step 4, and $a_r = \bar{a}_{r-1}$.

Thus, A^n will have r or $r-1$ UI assumptions, depending on n.

Lemma 4.7. Let $A^n = \{a_i : i = 1,\ldots,k\}$ be as constructed above. Then, $k \leq \ell$, where ℓ is the smallest integer such that $2^\ell > n$.

Proof.

First, assume $n = 2^m$, m some integer. Then, $k = m+1$ because each step of the segmentation process which constructs $S(\cdot)$ reduces m by 1, and the process of constructing the a_i's ends with the case where Card $[S(2_1;\ldots;2_{m-1})] = 2$.

If $n = 2^m - 1$, for some integer m, then $k = m$ because, in this case, the process of constructing the a_i's ends by the other possibility of steps. To see this, just add a dummy variable to the attributes and construct the $S(\cdot)$'s such that $S(2_1;\ldots;2_{m-1})$ contains the dummy variable. At Step 5, the dummy is deleted.

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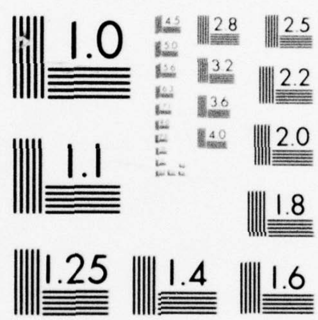

1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0

Now, for any n, there exists a unique integer ℓ such that $2^{\ell} > n \geq 2^{\ell-1}$. If $n = a^{\ell} - 1$, then Card $(A^n) = \ell$ by the argument above. If $n < 2^{\ell} - 1$, then, clearly, Card $(A^n) \leq \ell$.

Q.E.D.

Theorem 4.8. (i) A^n produces the M-A model, and (ii) of all the other sets producing the same model, A^n has the smallest cardinality.

Proof.

For the first assertion, A^n is, by construction, an overlapping chain. Thus, we know by Theorem 4.2 that we will get some M-A model. All we need to show is that each of the conditional utilities is one dimensional. Assume otherwise, i.e., there exists i and j such that x_i and x_j belong to the same $Cu(\cdot)$ in the M-A model. Let us assume, without any loss of generality, that x_i and x_j belong to $Cu(a_1 \wedge \ldots \wedge a_r \wedge \bar{a}_{r+1} \wedge \ldots \wedge \bar{a}_k)$, where $0 \leq r \leq k$. This means that x_i and x_j belong to the same set for every set used in constructing $a_1, \ldots, a_r$ and also belong to the same set of every set not used in constructing $a_{r+1}, \ldots, a_k$. But this is impossible since, sooner or later, any subset of the attributes with more than one element will be decomposed via the construction process of $S(\cdot)$'s.

For the second assertion, assume there exists a set with cardinality d, where $n > 2^d$. By Theorem 2.5 (see previous text), the set

$$\left\{Cu(\tilde{a}_1 \ldots \tilde{a}_d): \tilde{a}_i \in \{a_i, \bar{a}_i\}, i = 1, \ldots, d\right\}$$

contains all the conditional utilities of the decomposition and this set has cardinality 2^d. Hence, $2^d \geq n$--a contradiction.

Q.E.D.

Example 4.10.

Let us construct A^{15}. From Theorems 4.7 and 4.8, Card $(A^n) = 4$. Applying the algorithm, we construct the following four assumptions:

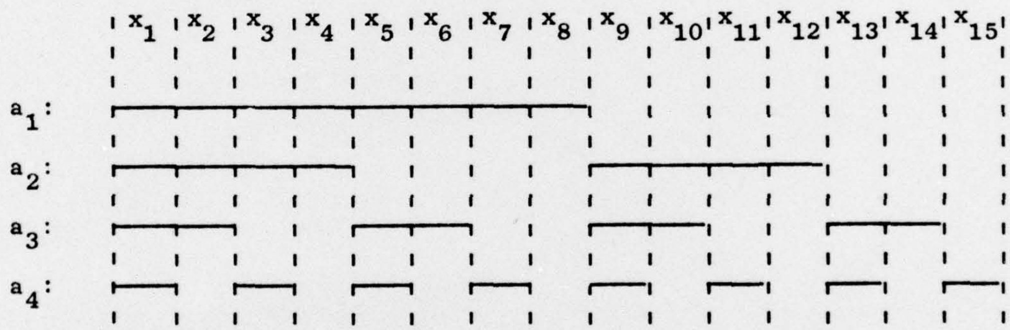
$$a_1 = (1,1,1,1,1,\ 1,1,1,0,0,\ 0,0,0,0,0)$$

$$a_2 = (1,1,1,1,0,\ 0,0,0,1,1,\ 1,1,0,0,0)$$

$$a_3 = (1,1,0,0,1,\ 1,0,0,1,1,\ 0,0,1,1,0)$$

$$a_4 = (1,0,1,0,1,\ 0,1,0,1,0,\ 1,0,1,0,1)$$

In a bar diagram representation, these assumptions are:



It should be clear that A^n are not unique, not even up to a permutation transformation of the attributes. If the M-A model to be generated is one where some of the indecomposable spaces are of higher dimensions, the same construction process is carried on by treating the vectors of each subspace as a scalar and n, of A^n, will correspond to the number of indecomposable subspaces involved.

Finally, the same type of process can be used to construct minimal sets of other decompositional concepts such as preference independence, in which case the generated model is the deterministic additive utility model.

Concluding Remarks

In this chapter, we considered how UI assumptions span each other. Since no apparent geometry is involved, the term implications is used instead of spanning. In the first section, we characterized the set of all possible UI implications for a given arbitrary set of UI assumptions. In Section 2, three sources of UI implications are introduced, and their relations to the multiplicative-additive utility model are demonstrated. Section 3 proposes a canonical form for sets of UI assumptions via which the three sources of UI implications are made transparent. Finally, due to the central role played by the multiplicative-additive model, Section 4 constructs sets with the least number of UI assumptions for generating an arbitrary M-A model.

Chapter 5

UTILITY ASSESSMENT OVER INDECOMPOSABLE SPACES

In the previous chapters, we have shown how to generate a utility surface, of a given dimension, from surfaces of lower dimensions via a modeling effort of the preference structure involved. The reason given for justifying such an effort is that it is always easier to construct utilities of lower dimensions. This standpoint is neither novel nor new. In fact, it is possible to categorize almost all proposed methodologies for constructing n-dimensional utilities into two types: one where some smoothness conditions on the surface is required, hence, limiting the utility to be a member of a classical family of surfaces, and the other is where some regularity conditions are assumed on cuts of the surface, hence, invoking decomposition. The only technology currently available for constructing a Von Neumann utility without limiting its functional form is when it is one-dimensional (see Pratt et al [32]).

The purpose of this chapter is to propose a framework for constructing utility surfaces that can not be decomposed via any of the known decompositional concepts. In other words, the preference structure is assumed to be so intertwined and intricate that a joint treatment of the attributes is required. The framework does not require any specific global smoothness conditions on the surface, and the limitation of the functional form of the constructed utility is rather minimal.

Section 5.1 introduces the so-called "strategy of continuous cuts" which is essentially a discretization process that collects continuous samples of information about the utility surface. Section 5.2 contains a general treatment of classical interpolation techniques which can be used to construct the whole utility surface by filling the "gaps" left out by the discretization process. Also contained, is a discussion of the salient features for any interpolation rule used on utilities. Section 5.3 proposes a behaviorally motivated rule of interpolation via the use of a new concept that we call "risk aversion profile."

5.1 The Strategy of Continuous Cuts

Within the framework of the previous chapters, let us assume that the set of UI assumptions, for a particular preference structure, does not fully decompose the attribute space. The challenge that faces the analyst now is how to construct the utility surface over indecomposable subspaces of higher dimensions. One possible solution is to attempt to model these utilities via other regularity concepts such as Fishburn's "Diagonal Independence" (Fishburn [11]) or Farquhar's "Hypercube Independence" (Farquhar [6]). Another possibility is to ascertain the plausibility of assuming that the utility is a member of a family of curves. As such, the utility surface is constructed by estimating a finite number of parameters.

An alternate way of approaching the problem is to start by constructing the underlying (deterministic) value function (i.e., the indifference curves for the preferences). The Von Neumann utility is then attained by invoking uncertainty on a properly chosen numeraire. This technique is called the decomposition procedure (see Boyd [2]). To construct the value function, modeling, again, is required. The basis of modeling in this case is the same as that for the utility itself, i.e., either (deterministic) separability of some sort is introduced for the attributes (e.g., see Debreu [5]), or some smoothness conditions limiting the shape of the indifference curves are assumed, see Keelin [18]. Once the value function is constructed, the Von Neumann utility is attained via the decomposition procedure, mentioned above, with not much difficulty.

When All Else Fails

The question now is--what if both the utility and its corresponding value function are not amenable to any simplifying assumptions? For this case, we propose the strategy of continuous cuts. This strategy calls for the assessment of the utility surface along continuous one-dimensional cuts. Each cut corresponds to a properly chosen numeraire. The numeraires are chosen in such a way that they spread over the entire domain of definition that is meaningful to the problem at hand. The utility surface between numeraires is approximated via some, hopefully, behaviorally motivated rules of interpolation. To propose a collection of numeraires, one attribute is chosen to be a variable, while all the others act as parameters. Thus, each numeraire corresponds to different discretized values of the parameters.

As an example, let the indecomposable subspace be $X \times Y \times Z$. We may discretize Y and Z into $y_1, \ldots, y_k$ and $z_1, \ldots, z_\ell$, respectively. Then we assess the utility over the numeraires

$$(x, y_i, z_j) \qquad \text{for every} \quad \begin{array}{l} i = 1, \ldots, k \\ j = 1, \ldots, \ell \end{array}$$

The collection $\{(y_i, z_j)\}$ are chosen to cover the part of the domain that is most pertinent and meaningful to the decision at hand. The utility over each numeraire is directly assessed via one of the different standard techniques (see Pratt et al [32]).

Each numeraire utility can initially have any reference zero and any scale (i.e., its two degrees of freedom may be arbitrarily chosen). Yet, before the assessment is over, they have to be calibrated. The procedure for calibration is easy enough. One numeraire utility is chosen as a reference. Two points on each of the other numeraire utilities are compared with points on the reference utility. In this way, the two degrees of freedom of each of the other numeraire utilities are made compatible with those of the reference numeraire utility.

With an appropriate rule of interpolation, we will construct the whole utility surface.

Applying the Strategy

When applying this strategy, there are two important points to consider: (1) How to outline a domain of definition for the utility that is most meaningful to the decision maker? And, (2) How fine should the discretization of the parameters be? The degree of fineness increases the assessment effort exponentially, but also decreases the error due to interpolation.

The assessment can be meaningful if all the deterministic and probabilistic structural dependence among the attributes are taken into account. These dependencies, which vary with different applications, act as constraining agents on the domain of the utility. Note that the structural dependencies of the attributes are different from the preference dependencies; the latter acts on the utility surface itself while the former acts on its domain.

The question of minimal assessment may be answered by a joint consideration of the error bounds due to interpolation along with other sources of computational error. Different forms of sensitivity analysis can be used to resolve this issue. We propose the following types:

(1) Preassessment Sensitivity: This is done by constructing some sort of a graph that indicates what part of the domain of the utility surface is most frequently used in the calculations of the problem. The discretization process may then be designed to take full advantage of this information. This procedure will hopefully reduce the overall interpolation error.

(2) Dynamic Sensitivity: This type of sensitivity analysis is to be conducted "on line" during the assessment procedure. The idea is to attempt to sense any steep or irregular regions of the utility surface. One way to do this is to ask the decision maker some preference questions and then compare his answers with those obtained by the interpolation rule. Once a steep region of the surface is discerned, the discretization process is modified to accommodate it. This kind of sensitivity is particularly hopeful if the assessment is conducted via an interactive computer code.

(3) Post-mortem Sensitivity: This is conducted after the utility surface is built. If the optimization rule of the decision setting of the problem is sensitive to values taken from the constructed utility surface, then further assessment (on a finer discretization of the parameters) may be conducted to improve the accuracy of the utility surface.

5.2 Interpolation

For the purpose of this section, we will assume that the domain of definition of the utility is the whole product set corresponding to

$X_1 \times \ldots \times X_k$, where the ranges of the X_i's are finite real intervals. According to the strategy of continuous cuts, all but one of the attributes are discretized into a finite number of values. Let us call such variables the parameter attributes. Also, the single variable that is not discretized is called the numeraire attribute. It is, by convention, chosen to be x_1.

Now, the set

$$D = \left\{ \left(x_{1i_1}, x_{2i_2}, \ldots, x_{ki_k} \right) : x_{1i_1} \in X_1 ; x_{ji_j} \in \{\text{discretized values of } X_j\}, \text{ for every } j = 2, \ldots, k \right\}$$

is the only part of the domain that is of known utility. The utility over the complement set (with respect to the domain) is to be interpolated. Let $x_0 \in \bar{D}$. Consider the $k-1$ dimensional hyperplane in the attribute space that is orthogonal to the space X_1 and passes through the point x_0. (It should be clear that we are treating the domain here as a subset of the k-dimensional euclidean space.) Consider the lattice that is formed by the intersection of such a hyperplane with the numeraire cuts. The $k-1$ tuple $(x_{20}, \ldots, x_{k0})$ is surrounded by the points of such lattice whose utility values are known. If we just consider the part of the lattice that immediately surrounds $(x_{20}, \ldots, x_{k0})$, then almost any point in the product set (except possibly points on the boundary of the lattice) is cornered into a (smallest) hyperrectangle whose vertices are of known utilities. Figure 5.1 displays this setup for the case of three attributes.

We will investigate the classical approach of fitting a family of curves through the utilities of the lattice points and interpolating the

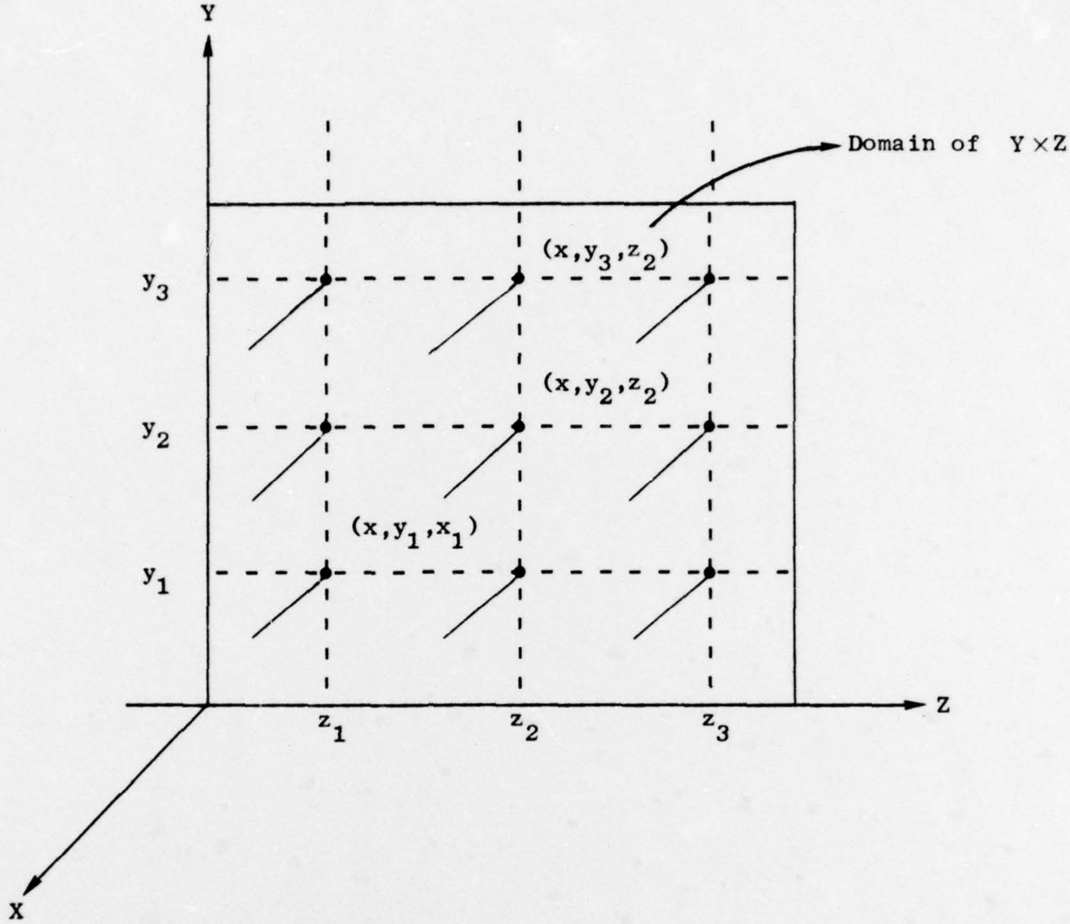


Fig. 5.1. THE CONTINUOUS CUTS STRATEGY. Note that most points of the domain are contained in rectangles whose vertices are of known utilities.

points in between or extrapolating for the points on the boundaries of the lattice. For the sake of clarity, we will treat the case of two attributes. The extension to k-attributes is straightforward. Let the attribute space be $X \times Y$. Discretize Y into y_1, y_2, and y_3. Thus, the numeraire utilities are $u(x,y_1)$, $u(x,y_2)$, and $u(x,y_3)$. Assume we want the utility at (x_0,y_0). If $y_0 \in \{y_1,y_2,y_3\}$, $u(x_0,y_0)$ is a known value corresponding to a point on one of the numeraire utilities.

Otherwise, (x_0, y_0) can be approximated by fitting a curve through the (lattice) points $u(x_0, y_1)$, $u(x_0, y_2)$, and $u(x_0, y_3)$. See Fig. 5.2. Note that the curve we fit corresponds to a conditional utility on Y. Let us demonstrate, by an example, some of the implications of this approach.

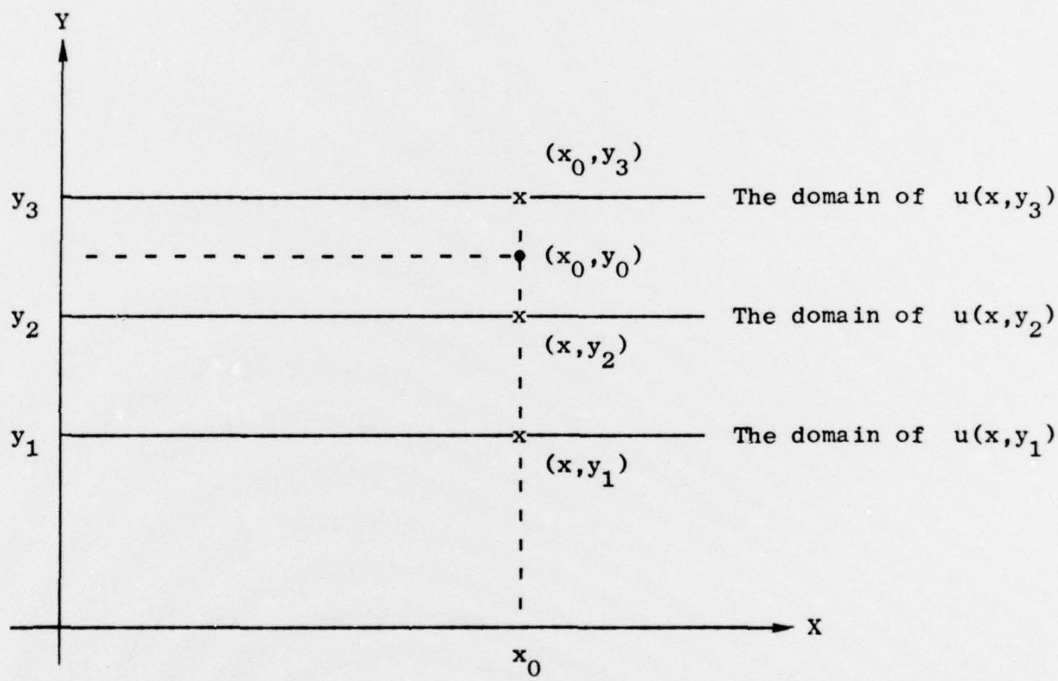


Fig. 5.2. EXAMPLE 5.1.

Example 5.1.

Let us fit a quadratic curve through the points $u(x_0, y_1)$, $u(x_0, y_2)$ and $u(x_0, y_3)$. We get

$$u(x_0, y) = \alpha_1(g) + \alpha_2(g)\, y + \alpha_3(g)\, y^2$$

where

$$g = g\left[u(x_0, y_1), u(x_0, y_2), u(x_0, y_3)\right]$$

with sufficient manipulation, $u(x_0,y)$ can also be written as

$$u(x_0,y) = f_1(y)\ u(x_0,y_1) + f_2(y)\ u(x_0,y_2) + f_3(y)\ u(x_0,y_3) \tag{5.1}$$

where $f_1(\cdot)$, $f_2(\cdot)$, $f_3(\cdot)$ are real valued functions. Since x_0 is arbitrary, we can write Eq. (5.1) as

$$u(x,y) = f_1(y)\ u(x,y_1) + f_2(y)\ u(x,y_2) + f_3(y)\ u(x,y_3) \tag{5.2}$$

So, even though we started with a procedure for point-wise interpolation, Eq. (5.2) reflects a sort of function-wise interpolation that is similar to ones induced by decompositional axioms. To see this more vividly, note that, from Eq. (5.2), three functions on X and on Y are required to (linearly) span all other conditional utilities and hence construct $u(\cdot)$. This is particularly reminiscent of some of the hypercube independence of Farquhar [6], except that here the forms of the functions on Y are explicit.

If the exponential family is used instead of the quadratic family, the functional form for $u(\cdot)$ is:

$$u(x,y) = \beta_1(g) + \beta_2(g)\ \mathrm{Exp}\left[-\beta_3(g)\ \cdot\ y\right] \tag{5.3}$$

where

$$g = g\left[u(x,y_1), u(x,y_2), u(x,y_3)\right]$$

In this case, the function-wise spanning is more subtle and nonlinear.

Thus, if we are to propose an interpolation procedure, it should produce an (approximation) surface that is not amenable to decomposition. Otherwise, we would not have had to revert to the strategy of continuous cuts. One way to avoid this difficulty is not to require that the same family of curves be used everywhere; thus different families may be used for different locales.

Desirable Properties of Utility Interpolation

In the above discussion, we reflected on one feature of using the classical approach for utility interpolation. Here, we will identify and discuss other salient features involved.

One of the first issues to be decided upon is whether the interpolating surface should be unique or a best fit, where "best" is defined in some mathematical sense. Uniqueness is, of course, a function of the number of parameters of the surface versus the amount of information about the utility to be used. Strictly speaking, the numeraire utilities contain an infinite amount of information about the surface. The analyst has to decide how much to use and from where, for a given locale. The question of uniqueness versus best fit determines whether the fidelity of the data is to be preserved or not. The theoretical basis for deciding either way depends on the kind of assumptions the analyst believes about the assessed utilities. For example, if the utility values are only accurate within an error range, then best fit may be justified. Another aspect of the question is the computational effort required for each case. Interpolation by best fit requires a lot more computations than by unique surfaces.

Another salient feature of an interpolating surface is the absence of any oscillatory behavior. This is a desirable property for a normative utility model even though, strictly speaking, Von Neumann utilities are not required to be monotonic. This feature rules out most polynomial and transcendental surfaces.

Finally, the analyst may impose different assumptions on the approximating surface that are relevant to utility theory. For instance, it may be plausible to make the behavioral assumption that the decision maker's risk aversion is smooth and displays no abrupt changes. This assumption is valid for many choice situations, and it corresponds to requiring that the utility surface be twice continuously differentiable.

5.3 Risk Aversion Profile

With the comments of the previous section in mind, we will propose a behaviorally motivated rule of interpolation. The main idea involved is to construct an interpolating surface such that risk aversion along different attributes "resembles," in a certain way, risk aversion along the numeraire utilities. For the purpose of this section, we will require the Von Neumann utility to be monotonic with respect to each of the variables. That is,

$$\frac{\partial u}{\partial x_i} \geq 0 \quad \underline{\text{or}} \quad \frac{\partial u}{\partial x_i} \leq 0 \qquad \forall\ i = 1, \ldots, k$$

Behaviorally, this means that either more of an attribute is always better (or just as good) than less, or more of an attribute is always worse (or just as good) than less, for every attribute.

We will first introduce the concept of risk aversion profile for 1-dimensional utilities. Let $u(x_1)$ be a utility over the scalar attribute x_1. Assume that $du(x_1)/d(x_1) \geq 0$. Hence, $u(x_1)$ may look like the curve in Fig. 5.3. Let x_{11} and x_{12}, where $x_{11} < x_{12}$, be two points in the domain of $u(\cdot)$. Any point $x_{1i} \in [x_{11}, x_{12}]$ can be written as

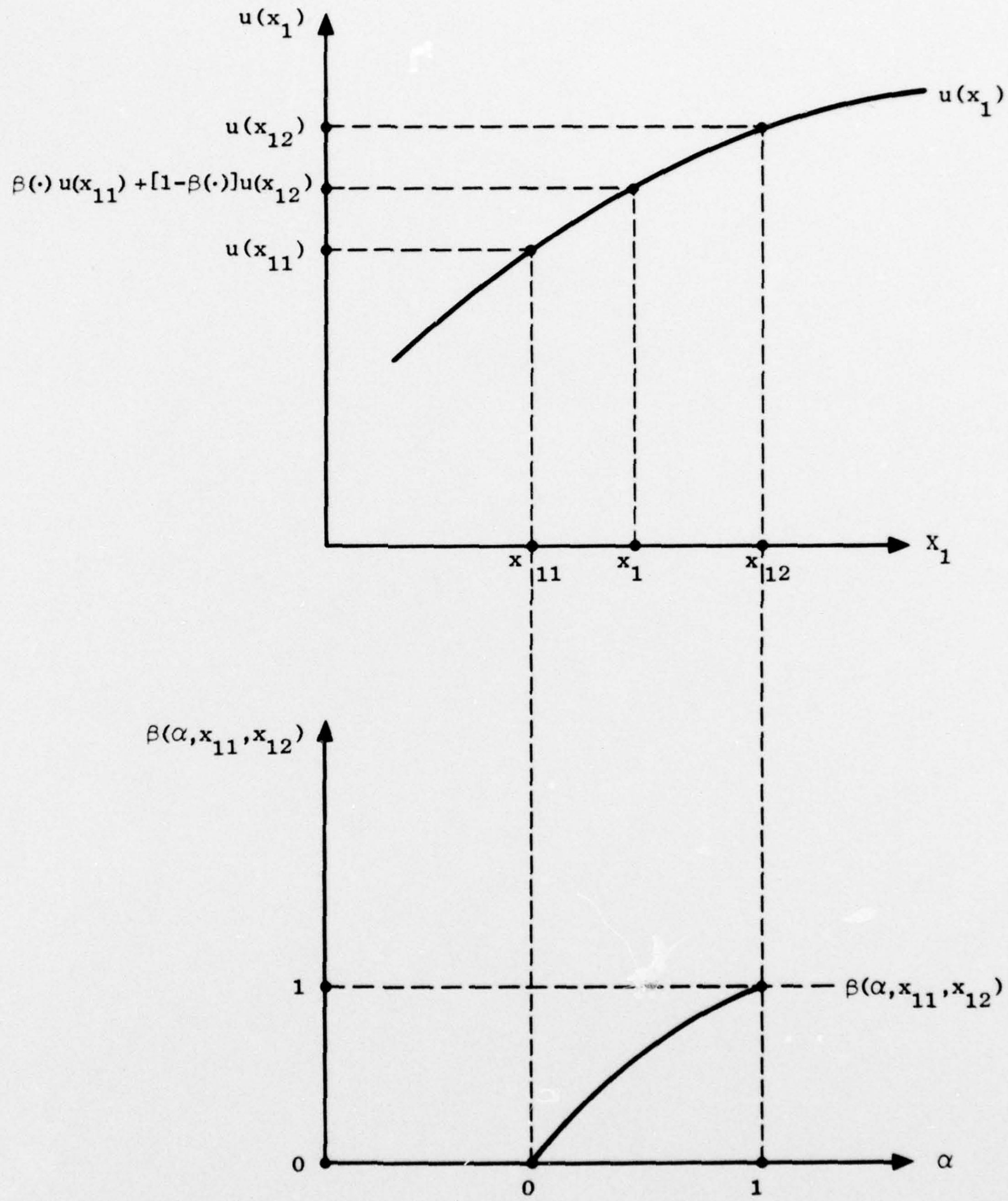


Fig. 5.3. RISK AVERSION PROFILE.

$$x_{1i} = \alpha x_{11} + (1 - \alpha)\, x_{12} \qquad \alpha \in [0,1]$$

Also, due to the monotonicity assumption, for every $\alpha \in [0,1]$ and x_{11} and x_{12}, there exists a unique $\beta(\alpha, x_{11}, x_{12}) \in [0,1]$ such that

$$u\left[\alpha x_{11} + (1 - \alpha)\, x_{12}\right] = \beta(\alpha, x_{11}, x_{12})\, u(x_{11}) + \left[1 - \beta(\alpha, x_{11}, x_{12})\right] \cdot u(x_{12}) \qquad (5.4)$$

See Fig. 5.3. $\beta(\cdot)$ is clearly a linear transformation of $u(\cdot)$ on the interval $[x_{11}, x_{12}]$. Also, $\beta(\cdot)$ is the same for every positive linear transformation of $u(\cdot)$, and hence is well-defined.

Definition 5.1. For a fixed $x_{11}, x_{12} \in X_1$, where $x_{11} < x_{12}$, $\beta(\alpha, x_{11}, x_{12})$ is the risk aversion profile of $u(\cdot)$ on the interval $[x_{11}, x_{12}]$.

5.3.1 Localized Separability

Risk aversion profiles will be used to implement a type of separability condition. Let $x_0 = (x_{10}, \ldots, x_{k0})$ be a point in $X_1 \times \ldots \times X_k$. We want to construct $u(x_0)$. Within the framework of the continuous cuts strategy, x_0 is contained in a (smallest) hyperrectangle whose vertices are of known utilities. Let $x_1 = (x_{11}, \ldots, x_{k1})$ and $x_2 = (x_{12}, \ldots, x_{k2})$ define the hyperrectangle containing x_0, where

$$x_{i1} \leq x_{i0} \leq x_{i2} \qquad \forall\ i = 2, \ldots, k$$

and

$$x_{11} = x_{10} = x_{12}$$

x_0 corresponds to some convex combination of the vertices of the hyperrectangle $[x_1;x_2]$. Also, due to the monotonicity assumption, $u(x_0)$ corresponds to some convex combination of the utility values of the vertices of the hyperrectangle. So, what we need is a k-dimensional analogue of Eq. (5.4).

Define $\alpha_2,\ldots,\alpha_k$ such that

$$x_{i0} = \alpha_i x_{i1} + (1 - \alpha_i)\, x_{i2} \qquad \forall\ i = 2,\ \ldots,\ k$$

The α_i's are well-defined and unique. Assume that for each i, $i = 2, \ldots, k$, we have $\beta_i(\alpha_i, x_{i1}, x_{i2})$. We can construct $\hat{u}(x_0)$, the approximation of $u(x_0)$ recursively, in the following manner:

$$\hat{u}\left(x_{10}, x_{20}, x_{3j_3}, x_{kj_k}\right) = \beta_2(\cdot)\ u\left(x_{10}, x_{21}, x_{3j_3},\ \ldots,\ x_{kj_k}\right) + \left[1 - \beta_2(\cdot)\right] \cdot u\left(x_{10}, x_{22}, x_{3j_3},\ \ldots,\ x_{kj_k}\right) \tag{5.5}$$

where $j_3,\ldots,j_k \in \{1,2\}$. And,

$$\hat{u}\left(x_{10}, x_{20}, x_{30}, x_{4j_4},\ \ldots,\ x_{kj_k}\right) = \beta_3(\cdot)\ \hat{u}\left(x_{10}, x_{20}, x_{31}, x_{4j_4},\ \ldots,\ x_{kj_k}\right) + \left[1 - \beta_3(\cdot)\right] \cdot \hat{u}\left(x_{10}, x_{20}, x_{32}, x_{4j_4},\ \ldots, x_{kj_k}\right) \tag{5.6}$$

where $j_4,\ldots,j_k \in \{1,2\}$, and so on. Finally, we get

$$\hat{u}(x_{10},\ \ldots,\ x_{k0}) = \beta_k(\cdot)\ \hat{u}(x_{10},\ \ldots,\ x_{k-1,0}, x_{k1}) + \left[1 - \beta_k(\cdot)\right]\ \hat{u}(x_{10},\ \ldots,\ x_{k-1,0}, x_{k2}) \tag{5.7}$$

The following proposition characterizes this recursive system in three ways.

Proposition 5.1.

(1) The recursive system of Eqs. (5.5), (5.6), and (5.7) is equivalent to the following explicit formula:

$$\hat{u}(x_0) = \sum_{j_2,\ldots,j_k \in \{1,2\}} C_{(j_2,\ldots,j_k)}\, u\left(x_{10}, x_{2j_2}, \ldots, x_{kj_k}\right) \tag{5.8}$$

where

$$C_{(j_2,\ldots,j_k)} = \prod_{i=2}^{k} C^{i}_{(j_2,\ldots,j_k)}$$

and

$$C^{i}_{(j_2,\ldots,j_k)} = \beta_i(\alpha_i, x_{i1}, x_{i2}) \qquad \text{if} \quad j_i = 1$$

$$= 1 - \beta_i(\alpha_i, x_{i1}, x_{i2}) \qquad \text{if} \quad j_i = 2$$

(2) $\hat{u}(x_0)$ is independent of the ordering of the attributes in the recursion process of Eqs. (5.5), (5.6), and (5.7).

(3) The $C_{(\cdot)}$'s of Eq. (5.8) correspond to a convex combination, i.e., each $C_{(\cdot)}$ is nonnegative and they sum up to one.

Proof.

For the first assertion, the correspondence between the recursive equations and the formula (5.8) is made apparent by starting with Eq. (5.7) and substituting for the utilities on its right side by their equivalence in the recursion. This is to be done recursively until $\hat{u}(x_0)$ involves only vertex utilities. What we will have then is Eq. (5.8).

For the second assertion, we note that any ordering of the attributes in the recursion process will produce the same $C_{(\cdot)}$'s in Eq. (5.8); hence, $\hat{u}(x_0)$ is unique and well-defined. The third assertion is demonstrated by noting that:

$$\sum_{j_2,\ldots,j_k \in \{1,2\}} C_{(j_2,\ldots,j_k)} = \prod_{i=2}^{k} \left[\beta_i(\cdot) + \left(1 - \beta_i(\cdot)\right)\right] = \prod_{i=2}^{k} 1 = 1$$

The nonnegativity of the $C_{(\cdot)}$'s is obvious from their definition.

Q.E.D.

Proposition 5.1 corresponds to a procedure where the interpolation is conducted by handling the variables separately, i.e., interpolating unidimensionally, edge by edge along the hypercube, until in a finite number of steps, converging to the point of interest. The procedure assumes a certain type of localized separability of the attributes. This fact is reflected by Eq. (5.8) by noting that, for a given locale, the $\beta_i(\cdot)$'s are conditional utilities along the i^{th} direction. In fact, a utility is represented by Eq. (5.8) if and only if, for $i = 2,\ldots,K$, X_i is UI. For a proof, see Appendix 5.A. The separability, though, is not global because the $\beta_i(\cdot)$'s are different (not the same up to a positive linear transformation) for different locales. An important feature of the procedure is that it preserves the fidelity of the assessed data. That is, if the point whose utility is to be interpolated happened to be in the domain of the numeraire utilities, Eq. (5.8) produces the exact assessed value.

All we need now is a meaningful proposition for the $\beta_i(\cdot)$'s.

5.3.2 Estimating the $\beta_i(\cdot)$'s

We will propose two behaviorally motivated methods for estimating the $\beta_i(\cdot)$'s.

Method I

The main idea here is that, for a given interval of the overall utility, each $\beta_i(\cdot)$ is equal to the average of risk aversion profiles of surrounding numeraire utilities. An equivalent way of expressing the idea is to say that the preference ordering along an attribute is the same as the (arithmetic) average of preference orderings of the surrounding numeraire utilities when both the attribute and the surrounding numeraires are restricted to correspond to a particular interval of the utility. (The arithmetic average is chosen here for the sake of simplicity only; other averages may be used as well.)

Applying Method 1

The numeraire utilities surrounding the hyperrectangle $[x_1;x_2]$ containing x_0 are:

$$u\left(x_1, x_{2j_2}, \ldots, x_{kj_k}\right) \quad \text{where} \quad j_2, \ldots, j_k \in \{1,2\}$$

For a given i, $i = 2,\ldots,k$, and a given numeraire utility $u_L(x_1) = u(x_1, x_{2j_2}, \ldots, x_{kj_k})$, $j_2,\ldots,j_k \in \{1,2\}$, assume that there exists x_{1i_L} such that

$$u\left(x_{1i_L}, x_{2j_2}, \ldots, x_{ij_i}, \ldots, x_{kj_k}\right) = u\left(x_{10}, x_{2j_2}, \ldots, x_{im_i}, \ldots, x_{kj_k}\right) \tag{5.9}$$

where $m_i \in \{1,2\}$ and $m_i \neq j_i$. Let $\beta_{i_L}(\alpha_i, x_{10}, x_{1i_L})$ be the risk aversion profile corresponding to $u_L(x_1)$, on the domain $[x_{10}, x_{1i_L}]$, evaluated at α_i. We will use the arithmetic average of all the $\beta_{i_L}(\cdot)$'s (that exist) as a value for $\beta_i(\alpha_i, x_{i1}, x_{i2})$ in Eqs. (5.5), (5.6), and (5.7).

We will demonstrate the method by an example.

Example 5.2.

Let the attribute space be $X \times Y$. Choose the numeraire utilities to be $u_1(x,y_1),\ldots,u_k(x,y_k)$ where $y_1,\ldots,y_m \in Y$. Let us try to calculate $\hat{u}(x_0,y_0)$ via the risk aversion profile using Method I. Assume that:

$$y_0 = \alpha y_i + (1 - \alpha)\, y_{i+1}\,, \qquad \alpha \in [0,1] \text{ and for some } i \in \{1,2,\ \ldots,\ m - 1\}$$

From Fig. 5.4, there exists x_i and x_{i+1} such that

$$u_i(x_i,y_i) = u_{i+1}(x_0,y_{i+1})$$

and

$$u_{i+1}(x_{i+1},y_{i+1}) = u_i(x_0,y_i)$$

Thus, using Eq. (5.4),

$$\beta_{y_i}(\alpha,x_0,x_i) = \frac{u_i(x_i,y_i) - u_i\Big[\Big(\alpha x_0 + (1 - \alpha)\, x_i\Big),y_i\Big]}{u_i(x_i,y_i) - u_i(x_0,y_i)}$$

and

$$\beta_{y_{i+1}}(\alpha,x_{i+1},x_0) = \frac{u_{i+1}(x_0,y_{i+1}) - u_{i+1}\Big[\Big(\alpha x_{i+1} + (1 - \alpha)\, x_0\Big),y_{i+1}\Big]}{u_{i+1}(x_0,y_{i+1}) - u_{i+1}(x_{i+1},y_{i+1})}$$

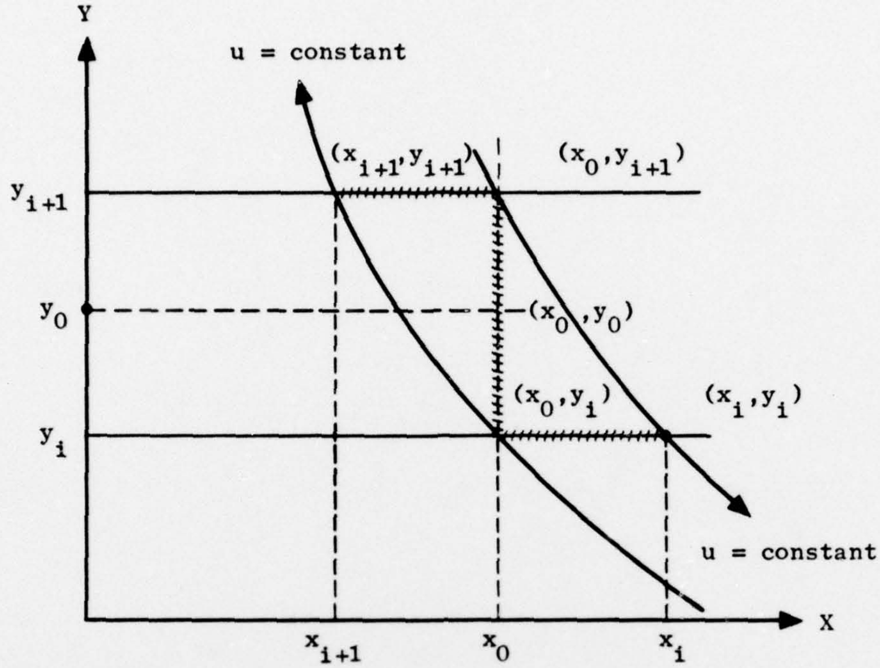


Fig. 5.4. RISK AVERSION PROFILE--AN EXAMPLE.

so

$$\beta_y(\alpha, y_i, y_{i+1}) = \left[\beta_{y_i}(\cdot) + \beta_{y_{i+1}}(\cdot)\right] \Big/ 2$$

and finally

$$\hat{u}(x_0, y_0) = \beta_y(\alpha, y_i, y_{i+1})\, u(x_0, y_i) + \left[1 - \beta_y(\alpha, y_i, y_{i+1})\right] u(x_0, y_{i+1})$$

Proposition 5.2. The interpolation via the localized separability and Method I is exact for linear utilities, i.e., utilities of the form

$$u(x_1, \ldots, x_k) = \sum_{i=1}^{k} \sigma_i x_i + \rho$$

Proof.

We will find $\hat{u}(x_0)$ where $x_0 = (x_{10}, \ldots, x_{k0}) \in X_1 \times \ldots \times X_k$ and compare it with the exact value, $\sum_{i=1}^{k} \sigma_i x_{i0} + \rho$. Let $x_1 = (x_{10}, x_{20}, \ldots, x_{k-1})$ and $x_2 = (x_{10}, x_{22}, \ldots, x_{k2})$ where $x_{i1} < x_{i2}$, $i = 2, \ldots, k$, correspond to the hyperrectangle containing x_0. Pick j, $2 \leq j \leq k$, and let α_j be such that:

$$x_{j0} = \alpha_j x_{j1} + (1 - \alpha_j)\, x_{j2}\,, \qquad \alpha_j \in [0,1] \tag{5.10}$$

Let $u_L(x_1) = u(x_1, x_{2L_2}, \ldots, x_{kL_k})$, where $L_2, \ldots, L_k \in \{1,2\}$, be one of the numeraire utilities indexed by one of the vertices of $[x_1; x_2]$. Assume that there exists $x_{1j_L} \in X_1$ such that

$$u_L\left(x_{1j_L}\right) = u\left(x_{10}, x_{2L_2}, \ldots, x_{jm_j}, x_{j+1L_{j+1}}, \ldots, x_{kL_k}\right)$$

where $m_j \in \{1,2\}$ and $m_j \neq L_j$. Using Eq. (5.4), we have

$$\beta_{j_L}\left(\alpha_j, x_{1j_L}, x_{10}\right) = \frac{u_L(x_{10}) - u_L\left[\alpha_j x_{1j_L} + (1 - \alpha_j)\, x_{10}\right]}{u_L(x_{10}) - u_L\left(x_{1j_L}\right)}$$

$$= \frac{\left[\sum_{i=2}^{k} \sigma_i x_{2L_2} + \sigma_1 x_{10} + \rho\right] - \left[\sum_{i=2}^{k} \sigma_i x_{2L_2} + \sigma_1\left[\alpha_j x_{1j_L} + (1 - \alpha_j)\, x_{10}\right] + \rho\right]}{\left[\sum_{i=2}^{k} \sigma_i x_{2L_2} + \sigma_1 x_{10} + \rho\right] - \left[\sum_{i=2}^{k} \sigma_i x_{2L_2} + \sigma_1 x_{1j_L} + \rho\right]}$$

$$= \frac{-\alpha_j \sigma_1\left(x_{1j_L} - x_{10}\right)}{\sigma_1\left(x_{10} - x_{1j_L}\right)} = \alpha_j$$

Since the choice of $u_L(x_1)$ is arbitrary (with respect to the ones surrounding the hyperrectangle), all $\beta_{j_L}(\cdot)$ that exist will be equal to α_j. So, their average is equal to α_j, and hence the estimate for $\beta_j(\alpha_j, x_{jL}, x_{j2})$ is α_j. Since j is arbitrarily chosen, we can write

$$\beta_j(\alpha_j, x_{j1}, x_{j2}) = \alpha_j , \qquad \forall\ j = 2, \ldots, k \tag{5.11}$$

We will use Eq. (5.11) along with the system of equations (5.5)-(5.7) to estimate the value $\hat{u}(x_0)$.

$$\begin{aligned}
\hat{u}\left(x_{10}, x_{20}, x_{3j_3}, \ldots, x_{kj_k}\right) &= \alpha_2 u\left(x_{10}, x_{21}, x_{3j_3}, \ldots, x_{kj_k}\right) \\
&\quad + (1 - \alpha_2)\, u\left(x_{10}, x_{22}, x_{3j_3}, \ldots, x_{kj_k}\right) \\
&= \alpha_2 \left(\sum_{i=3}^{k} \sigma_i x_{ij_i} + \sigma_1 x_{10} + \sigma_2 x_{21} + \rho\right) \\
&\quad + (1 - \alpha_2) \left(\sum_{i=3}^{k} \sigma_i x_{ij_i} + \sigma_1 x_{10} + \sigma_2 x_{22} + \rho\right) \\
&= \left(\sum_{i=3}^{k} \sigma_i x_{ij_i}\right) + \sigma_1 x_{10} \\
&\quad + \sigma_2 \left[\alpha_2 x_{21} + (1 - \alpha_2)\, x_{22}\right] + \rho
\end{aligned}$$

But, Eq. (5.10), we get

$$\begin{aligned}
&= \sum_{i=3}^{k} \sigma_i x_{ij_i} + \sigma_1 x_{10} + \sigma_2 x_{20} + \rho \\
&= u\left(x_{10}, x_{20}, x_{3j_3}, \ldots, x_{kj_k}\right) \\
&\qquad \forall\ j_3, \ldots, j_k \in \{1,2\}
\end{aligned} \tag{5.12}$$

Equation (5.12) indicates that the first step of the recursion is exact. Let us make the inductive assumption that the $(k-1)^{st}$ step is exact, i.e., $\hat{u}(x_{10},\ldots,x_{k-1,0},x_{k1})$ and $\hat{u}(x_{10},\ldots,x_{k-1,0},x_{k2})$ corresponds to the exact values. Hence, by Eq. (4.7),

$$\begin{aligned}
\hat{u}(x_{10}, \ldots, x_{k0}) &= \alpha_k \hat{u}(x_{10}, \ldots, x_{k-1,0}, x_{k1}) \\
&\quad + (1-\alpha_k)\, \hat{u}(x_{10}, \ldots, x_{k-1,0}, x_{k2}) \\
&= \alpha_k \left(\sum_{i=2}^{k-1} \sigma_i x_{i0} + \sigma_1 x_{10} + \sigma_k x_{k1} + \rho \right) \\
&\quad + (1-\alpha_k) \left(\sum_{i=2}^{k-1} \sigma_i x_{i0} + \sigma_1 x_{10} + \sigma_k x_{k2} + \rho \right) \\
&= \sum_{i=2}^{k-1} \sigma_i x_{i0} + \sigma_1 x_{10} + \sigma_k \left[\alpha_k x_{k1} + (1-\alpha_k)\, x_{k2} \right] + \rho \\
&= u(x_{10}, \ldots, x_{k0})
\end{aligned}$$

Q.E.D.

Method II

This method is motivated by the observation that a risk aversion profile over an interval of an attribute (of an indecomposable space) changes for different levels of the overall utility and also changes as to where the attribute interval is located with respect to the overall range of the attribute. (For a vivid example of this observation, see Fig. 6.5 of Chapter 6.) As such, the method estimates $\beta_i(\cdot)$ for attribute i by a numeraire risk aversion profile corresponding to the same utility interval and also where the numeraire interval approximately has the same position with respect to the overall numeraire range as that of the attribute interval involved.

As an example of Method II, consider Fig. 5.5. Assume we need to interpolate the utility of the point (x_0, y_0). To implement the localized separability, we need an estimate of $\beta_y(\cdot)$ around the point (x_0, y_0). $\beta_y(\cdot)$ is defined on the attribute interval (y_3, y_4) that corresponds to higher values of Y. Thus, if we are to use for it a numeraire risk aversion profile, it should correspond to higher values of the numeraire attribute. We also want both the numeraire and attribute profiles to correspond to the same interval of the overall utility. Thus,

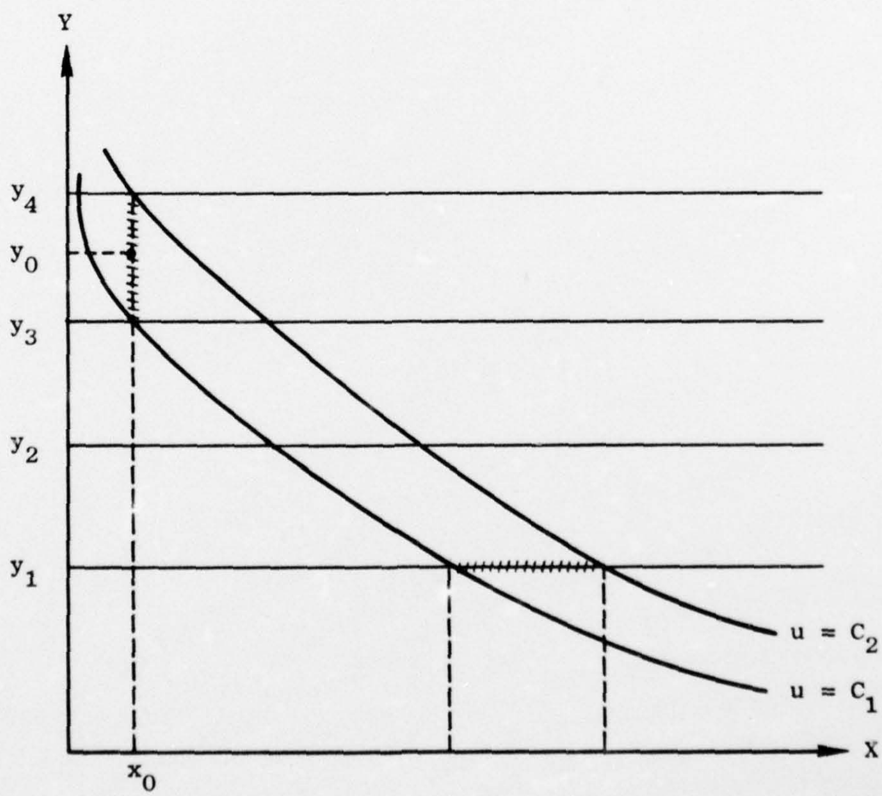


Fig. 5.5. RISK AVERSION PROFILE--METHOD II.

the appropriate choice is that part of the numeraire utility $u(x,y_1)$ that is shaded in the figure.

It should be clear that Methods I and II will essentially coincide for a large part of the domain. Even though Method I is easier to apply, Method II is an improvement over Method I for a large class of problems.

A final note concerning the case where the behavioral assumptions of both methods are not satisfied. For such problems, localized separability may still be used. What is needed is an alternative way of proposing conditional utilities along each of the parameter attributes. One way to proceed is via the idea of canonical utilities. A canonical utility is the analyst's conception of a "representative" behavior of the decision maker's preferences along each attribute. Such a conception usually evolves when the analyst develops a good grasp of the problem and the preference structure. A canonical utility will reflect any discerned peculiar preference behavior along an attribute. The appropriate interval of a canonical utility is used, after proper rescaling, along the corresponding edge of the hypercube surrounding the point of interest.

Concluding Remarks

We have dealt with spaces that are not amenable to decomposition. We introduced the strategy of continuous cuts which discretize the domain of the utility into continuous one-dimensional cuts. Utility is to be directly assessed over these cuts via existing standard techniques. For the rest of the domain, we proposed the risk aversion profile method as a rule of interpolation. The core of the method is a particular variety of local decomposition. The method produces approximation utility surfaces where risk aversion along the different attributes resembles that of the assessed utility cuts. The method is exact for linear utilities.

Appendix 5.A

<u>Theorem</u>. Let the attribute space be $X_1 \times \ldots \times X_k$. Let $x_1, x_2 \in X_1 \times \ldots \times X_k$ be such that $x_{i1} \neq x_{i2}$ for all $i = 1,\ldots,k$. The utility surface may be represented as

$$u(x_1, \ldots, x_k) = \sum_{j_2,\ldots,j_k \in \{1,2\}} f_{(i_2,\ldots,j_k)}(x_2, \ldots, x_k)$$

$$\cdot\, Cu\left(x_1, x_{2j_2}, \ldots, x_{kj_k}\right) \qquad (5.A.1)$$

where

$$f^{\ell}_{(j_2,\ldots,j_k)}(\cdot) = \prod_{\ell=2}^{\ell=k} f^{\ell}_{(j_2,\ldots,j_k)}(\cdot)$$

and

$$f^{\ell}_{(j_2,\ldots,j_k)}(\cdot) = Cu(x_i) \qquad \text{if} \quad j_i = 1$$

$$= 1 - Cu(x_i) \qquad \text{if} \quad j_i = 2$$

If and only if, for $i = 2,\ldots,k$, X_i is UI.

<u>Proof</u>.

Assume that Eq. (5.A.1) represents the utility surface. For $i = 2, \ldots, k$, the terms of Eq. (5.A.1) may be rearranged in the following manner:

$$u(x_1, \ldots, x_k) = g_{1i}(x_1, \ldots, x_{i-1}, \ldots, x_k)\, Cu(x_i)$$

$$+ g_{2_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \qquad \forall\, i = 2, \ldots, k$$

$$(5.A.2)$$

But, by definition, Eq. (5.A.2) implies that X_i is UI for $i = 2,\ldots,k$.

Now, assume that, for $i = 2,\ldots,k$, X_i is UI. We will use induction (on k) to prove the representation of Eq. (5.A.1). For $k = 2$,

$$(0,1) \overset{D}{\Rightarrow} u(x_1,x_2) = Cu(x_{12},x_2) + \alpha\left[Cu(x_{11},x_2) - Cu(x_{12},x_2)\right] \cdot Cu(x_1,x_{22})$$

where α is some real constant.

Equivalently,

$$u(x_1,x_2) = \left[\alpha\, Cu(x_1,x_{22})\right] \cdot Cu(x_{11},x_2) + \left[1 - \alpha\, Cu(x_1,x_{22})\right] \cdot Cu(x_{12},x_2) \tag{5.A.3}$$

By letting $Cu(x_1,x_{22}) = \alpha\, Cu(x_1,x_{22})$, Eq. (5.A.3) corresponds to Eq. (5.A.1).

Assume that the assertion is true for $k-1$. Let the attribute space be $X_1 \times \ldots \times X_k$. Treat $X_1 \times X_2$ as a single (vector) attribute. By the inductive assumption, we can write

$$u(x_2,x_3, \ldots, x_k) = \sum_{j_3,\ldots,j_k \in \{1,2\}} f_{(j_3,\ldots,j_k)}(x_3, \ldots, x_k) \cdot Cu\left(x_1,x_{3j_3}, \ldots, x_{kj_k}\right) \tag{5.A.4}$$

where x_2 corresponds to (x_1,x_2).

Let us decompose each $Cu(x_1,x_{3j_3},\ldots,x_{kj_k})$ via the assumption that X_2 is UI:

$$\begin{aligned}\mathrm{Cu}\left(x_1,x_{3j_3},\ \ldots,\ x_{kj_k}\right) = {} & \mathrm{Cu}\left(x_1,x_{22},x_{3j_3},\ \ldots,\ x_{kj_k}\right) \\ & + \beta\left[\mathrm{Cu}\left(x_1,x_{21},x_{3j_3},\ \ldots,\ x_{kj_k}\right)\right. \\ & \left. - \mathrm{Cu}\left(x_1,x_{22},x_{3j_3},\ \ldots,\ x_{kj_k}\right)\right] \\ & \cdot \mathrm{Cu}(x_{12},x_2,x_{32},\ \ldots,\ x_{k2})\end{aligned}$$

where β is some real constant. Equivalently,

$$\begin{aligned}\mathrm{Cu}\left(x_1,x_{3j_3},\ \ldots,\ x_{kj_k}\right) = {} & \beta\ \mathrm{Cu}(x_{12},x_2,x_{32},\ \ldots,\ x_{k2}) \\ & \cdot \mathrm{Cu}\left(x_1,x_{21},x_{3j_3},\ \ldots,\ x_{kj_k}\right) \\ & + \left[1 - \beta \cdot \mathrm{Cu}(x_{12},x_2,x_{32},\ \ldots,\ x_{k2})\right] \\ & \cdot \mathrm{Cu}\left(x_1,x_{22},x_{3j_3},\ \ldots,\ x_{kj_k}\right) \qquad (5.A.5)\end{aligned}$$

Substituting Eq. (5.A.5) into Eq. (5.A.4) produces the required result.

Q.E.D.

Chapter 6

AN APPLICATION: BUYING A NEW CAR

In this chapter, we will demonstrate part of the theory of previous chapters by an actual application. The application involves Mr. Manna as a decision maker (denoted DM). Mr. Manna is an international student at Stanford University. He is about to finish his graduate program of study and return home. Mr. Manna, who henceforth will be refferred to as the DM, would like to obtain a 1978 model, four-door sedan for himself. Since this model will only be available two months after his departure, the DM has engaged a friend to choose one of the new cars and ship it to him. The friend was a bit reluctant to accept such delegation since he was not sure what would be an appropriate choice for the DM. At this point, the author, as an analyst, steps in to help communicate the DM's preferences of cars to the friend. The strategy is to construct the DM's multiattribute Von Neumann utility over the different possible choices of cars.

6.1 The Model

We distinguish between car features which the DM insists upon as requirements and those about which he is willing to trade off and compromise. We consider the former kind of features as constraints, and the latter as attributes. Any chosen car will have all the required features along with different levels of the attributes. The best choice will represent the best balance among the attributes.

For the required features, the DM wants a four-door sedan with automatic transmission, power steering, air-conditioning, and an AM-FM radio.

For the compromising features, we identify the following attributes, along with their measurements and ranges.

(1) Total Purchase Price: Denoted x_1, is the overall cost of the transaction in U. S. dollars. The DM considers the range $4,000 - $6,000 as being appropriate for the class of car he wants.

(2) City Mileage: Denoted x_2, is the mileage per gallon in the city, as reported by EPA (Environmental Protection Agency) testing.

(3) Highway Mileage: Denoted x_3, is the mileage per gallon on the highway, as reported by EPA testing.

(4) Resale Value: Denoted x_4, is the approximate worth of the car after three years of usage, as a percentage of the initial price. x_4, for a given car, is to be estimated from historical data of similar cars. Such data is easily available. An appropriate range for x_4 is 35% - 70%.

When talking about the 'cost' of the car, we mean the vector (x_1, x_2, x_3, x_4). Variations among cars due to maintenance cost are minor and hence are dismissed.

(5) Motor Performance: The overall performance of the car is influenced by many design features. To compare performance, the following two attributes are considered sufficient.

(a) Starting Acceleration: Denoted x_5, is the time, in seconds, it takes a car to accelerate on level road from 0 - 30 miles per hour, as measured and reported by <u>Consumer Report</u> magazine. An appropriate range for x_5 is 4.5 - 7.0 seconds.

(b) Passing Acceleration: Denoted x_6, is the time, in seconds, it takes a car to accelerate on level road from 45 to 60 miles per hour, as reported by Consumer Reports. An appropriate range for x_6 is 7.0 - 15.0 seconds. When talking about 'motor performance,' we mean the doubleton (x_5, x_6).

(6) The Brake System: As a measure of the brake system performance, we define the attribute x_7, the distance in feet, for level braking from 60 miles per hour to a stop (with no wheels locked), as measured and reported by Consumer Reports. An appropriate range for x_7 is 170 - 220 feet.

(7) Maintainability: Here, the DM is concerned about the availability of maintenance and spare parts in his home country. It is decided to measure maintainability according to the popularity at home of the make of car. Thus, we define the pervasiveness of a make of car, x_8, as the percentage of the imports of such a make with respect to total imports. Such data may be obtained from the Import Statistics Year Book of the DM's home country. An appropriate range for x_8 is 1 - 10%.

(8) Trunk Size: Denoted x_9, is the volume in cubic feet of the car trunk, as reported in the car manual. An appropriate range for x_9 is 10 - 30 feet^3.

(9) Front Seat Size: An important feature of the front seat is leg room, x_{10}, as measured and reported by Consumer Reports. It is implicitly assumed that the interior design is proportioned in such a way that more leg room coincides with a large--hence, more comfortable--seat size. The size of the back seat is not of crucial importance to the DM. An appropriate range for x_{10} is 39 - 45 inches.

(10) Seat Upholstery: Denoted x_{11}, seat upholstery has only three possibilities: vinyl, leather, or cloth. The preference of upholstery concerns ease of cleaning and retainment of a fresh long time look. The DM's preference is for vinyl, then leather, and, lastly, cloth.

(11) Measurement Gauges: In addition to the standard ones such as a speedometer or an odometer, DM would like to have additional gauges such as an ampmeter and a voltmeter for the car's electrical system, a temperature gauge, a RPM gauge, and the like. We will let x_{12} denote the number of extra (nonstandard) gauges in a car. The range for x_{12} is $\{0,1,2,3,4,5\}$.

Table 6.1 summarizes all the attributes along with their ranges.

6.2 Preference Modeling

(i) The Cost Variables: It becomes apparent that there is a deterministic relationship relating the preferences over the cost variables, x_1 through x_4. Hence, it is decided to construct an economic model that will relate these variables. The present value model seems appropriate enough. A Von Neumann utility over (x_1, x_2, x_3, x_4) is obtained by invoking risk over the present value as a numeraire via the two step decomposition approach reported in Chapter 1.

For the present value model, the following assumptions are agreed upon with the DM.

(a) The DM will use his car over a three-year period before selling it.

(b) The DM will drive, on the average, about 10,000 miles per year, 60% of which is highway driving and the balance is for city driving.

Table 6.1

A LIST OF CAR ATTRIBUTES, ALONG WITH THEIR APPROPRIATE RANGES

Attributes	Symbol	Range
COST:		
Purchase Price	x_1	4,000 - 6,000 Dollars
City Mileage	x_2	8 - 20 MPG
Highway Mileage	x_3	15 - 25 MPG
Resale Value	x_4	30 - 70% (Unitless)
MOTOR PERFORMANCE:		
Starting Acceleration	x_5	4.5 - 7.0 Seconds
Passing Acceleration	x_6	7 - 15 Seconds
BRAKES	x_7	170 - 220 Feet
MAINTAINABILITY	x_8	1 - 10% (Unitless)
TRUNK SIZE	x_9	10 - 30 Feet^3
FRONT SEAT SIZE	x_{10}	38 - 45 Inches
SEAT UPHOLSTERY	x_{11}	{Vinyl, Leather, Cloth}
MEASUREMENT GAUGES	x_{12}	{0,1,2,3,4,5}

(c) A 13% discount rate is the DM's time-value of money. The DM will actually finance the car at about that rate.

(d) The cost of gasoline is $0.30 per gallon. With these assumptions, the present cost, denoted PC, of a car is:

$$\left[PC = x_1 + 0.3\ \frac{4000}{x_2} + \frac{6000}{x_3} \cdot (2.42) - (0.683)\ x_4 \cdot x_1\right] \qquad (6.1)$$

In Eq. (6.1), the cash flows are discounted yearly. With this model at hand, for the rest of the chapter we will suppress the attributes x_1 through x_4 and deal only with the present cost, PC, as an adequate measure of the overall cost of having a car.

(ii) Motor Performance: It is believed, from the DM's responses to questions, that the starting and passing accelerations, x_5 and x_6, are intricately linked, preference-wise, and thus have to be treated jointly. In accordance with our theory, the space $X_5 \times X_6$ is indecomposable. The justification for taking this stand may be deduced from the data collected and assorted for the space $X_5 \times X_6$.

(iii) A Multiplicative-Additive Model: We will use the following vector to describe fully a car:

$$(PC, p, x_7, x_8, x_9, x_{10}, x_{11}, x_{12})$$

where PC stands for present value as defined by Eq. (6.1), $p = (x_5, x_6)$ stands for motor performance, and x_7 through x_{12} are as defined in Table 6.1. From a scrutiny of the DM responses, we suspect the plausibility of the following multiplicative-additive model:

$$u(PC, p, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) = \mp Cu(PC, p, x_7, x_8)$$

$$* \; Cu(x_9) * Cu(x_{10}) * Cu(x_{11}) * Cu(x_{12}) \qquad * \in \{\times, +\} \qquad (6.2)$$

To test the validity of this model, we used the theory of Section 4.4 to construct the set A_1, a minimal set of UI assumption:

$$A_1 = \{(1,1,1,1,1,1,0,0),(1,1,1,1,1,0,1,0),(1,1,1,1,0,0,0,1)\}$$

The satisfaction of the assumptions in A_1 will guarantee the representation of Eq. (6.2).

(iv) A Quasi-Separable Submodel: In addition to the assumptions of A_1, the following set also seems worthy of testing and verification:

$$A_2 = \{(1,0,0,0,0,0,0,0),(0,1,0,0,0,0,0,0),$$
$$(0,0,1,0,0,0,0,0),(0,0,0,1,0,0,0,0)\}$$

If the assumptions of A_2 are satisfied, then $Cu(PC, p,x_7,x_8)$ of Eq. (6.2) is modeled using Keeney's [20] quasi-separable model. This means that $Cu(PC,p,x_7,x_8)$ is analytically derived from conditional utilities over PC, p, x_7, and x_8, separately.

Let $A = A_1 \cup A_2$. If the assumptions in A are verified, then, except for the attribute P, all required utility assessments are over scalar attributes.

Figure 6.1 sketches the total preference modeling effort for the problem.

Verification of UI Assumptions

For verifying the UI assumptions, two techniques are used. The first technique corresponds to confronting the DM with a two-branch lottery whereby he will get either a favorable package with probability α or an unfavorable package with probability $(1-\alpha)$. He is to compare such a lottery with choosing a particular package, for certain. All three packages involve different levels of the active attributes (the attributes corresponding to the 1's in a UI assumption). The parameter attributes are completely suppressed unless the DM has asked about them.

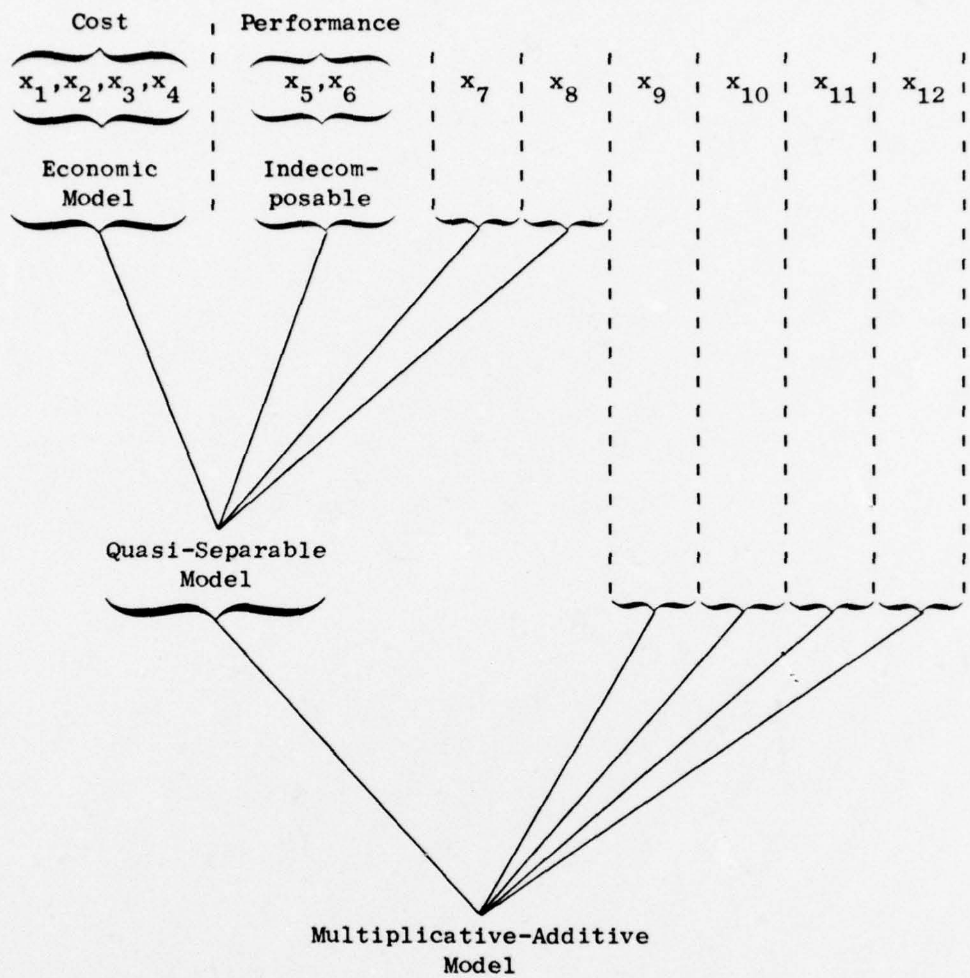


Fig. 6.1. PREFERENCE MODELING.

He is told only that their level is the same for all three packages. The three packages are varied throughout the active attributes' space. Figure 6.2 demonstrates this procedure. It is felt that, should the DM request to know the actual level of the parameter attributes, this would indicate the failure of the UI assumption. This technique indicates to us that the two types of acceleration are indecomposable.

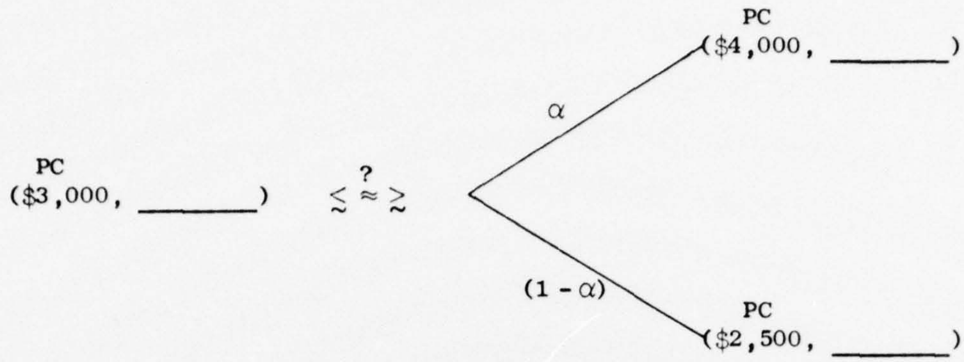


Fig. 6.2. VERIFYING THE UI ASSUMPTION (1,0,0,0,0,0,0,0).

The other procedure is suggested by Keeney [21]. It involves the same three packages as above, but with the parameter levels specified. The UI assumption is satisfied if different parameter levels do not influence the DM's preferences throughout the space of active attributes. Both techniques are used to verify the UI assumptions in A.

6.3 The Assessment of Utilities

Before discussing the assessment effort for the different submodels, we introduce two special car packages: the superior car and the inferior car. The superior car, denoted (B,B,B,B,B,B,B,B), corresponds to the car where each attribute is at its best level (B stands for best). The inferior car, denoted (w,w,w,w,w,w,w,w), corresponds to the car where each attribute is at its worst level (w stands for worst). The levels best and worst are determined by the range of attributes in Table 6.1. To fix the two degrees of freedom of the 12-dimensional utility surface, we will assume:

$$u(\text{superior}) = 1 \; ; \qquad u(\text{inferior}) = 0 \tag{6.3}$$

All assessed utilities and constants eventually will be referenced and scaled using Eq. (6.3).

Required Assessments

(i) The Multiplicative-Additive Model: Using Theorem 1.3 of Chapter 1, we need to assess four one-dimensional utilities over x_9, x_{10}, x_{11}, and x_{12}, and one 5-dimensional utility over PC $\times$ performance $\times X_7 \times X_8$. The latter utility will be derived analytically from utilities of lower dimensions using the quasi-separable model below. Also needed are five scaling constants which reflect the relative merit of each of the attributes in the M-A model. They correspond to C_3 through C_7 in Table 6.2.

(ii) The Quasi-Separable Model: This model is defined over the space PC $\times$ performance $\times X_7 \times X_8$. The model is diagrammatically depicted as a utility tree in Fig. 6.3. The required assessments are: three one-dimensional utilities for PC, X_7, and X_8, and one 2-dimensional utility over performance. Also required is the assessment of 2^4 constants corresponding to the utility values of the vertices of the hypercube defined by (B,B,B,B,w,w,w,w) and (w,w,w,w,w,w,w,w). These constants are listed as C_2, C_3, and C_8 through C_{21} in Table 6.2.

(iii) The Performance Space: Using the continuous cuts strategy of Chapter 5, the starting acceleration, X_5, is discretized into three levels: worst, average, and best. The utility over passing acceleration, X_6, is assessed at these three levels, where all other attributes are at a fixed, unspecified level. Thus, for this space, three one-dimensional utilities are required,

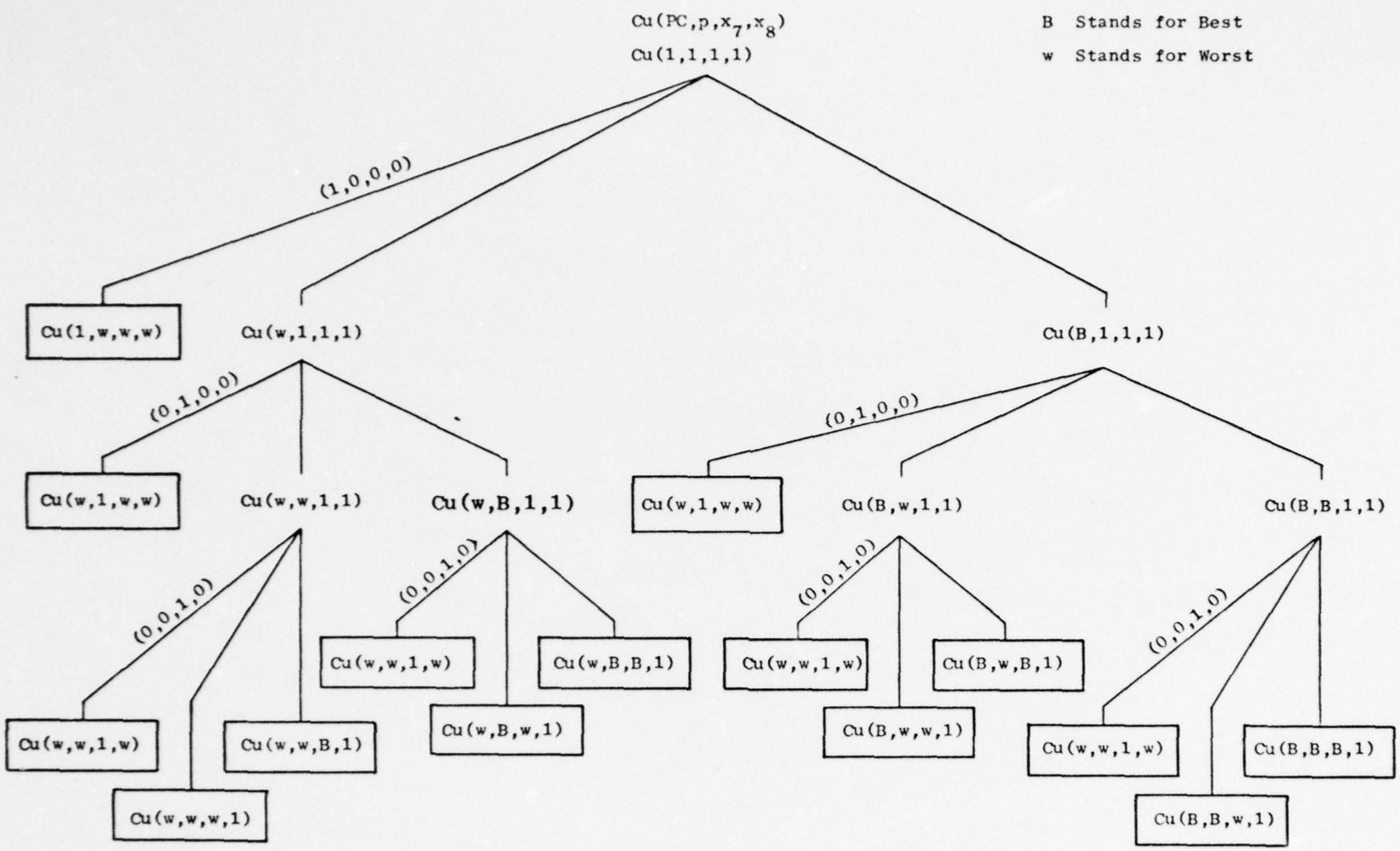


Fig. 6.3. A UTILITY TREE FOR THE QUASI-SEPARABLE MODEL. The parameter attributes, x_9 through x_{12}, are implicitly set at their worst level.

along with four constants (see Table 6.3) to calibrate the three cuts.

In summation, the overall assessment effort requires ten one-dimensional utilities and twenty-five constants.

Assessing the Utilities

All the ten one-dimensional utilities are assessed using standard techniques (see Pratt et al [32]) on a zero to one scale, i.e., the best level of the active attribute corresponds to a value of one, and the worst level corresponds to a value of zero. Figures 6.4 through 6.11 contain these utilities. During the assessment, the parameters of each utility are set at an average value chosen by the DM. This makes the assessment effort easy and fast for the DM. For each utility, about 6 to 8 points are assessed. Check lotteries are also used to verify the (relative) reproducibility of the assessed utility points. A French curve is used to connect the points.

Assessing the Constants

The first two constants of Table 6.2 are arbitrarily chosen by Eq. (6.3) to fix the two degrees of freedom of the 12-dimensional utility surface. Each of the other constants is estimated by comparing its corresponding car to a lottery involving two cars with known utilities. For example, C_3 is estimated by comparing choosing the car (B,B,B,B, w,w,w,w) to the lottery:

$$\begin{array}{l} \nearrow^{\alpha} (B,B,B,B,B,B,B,B) \equiv \text{Superior Car} \\ \searrow_{1-\alpha} (w,w,w,w,w,w,w,w) \equiv \text{Inferior Car} \end{array}$$

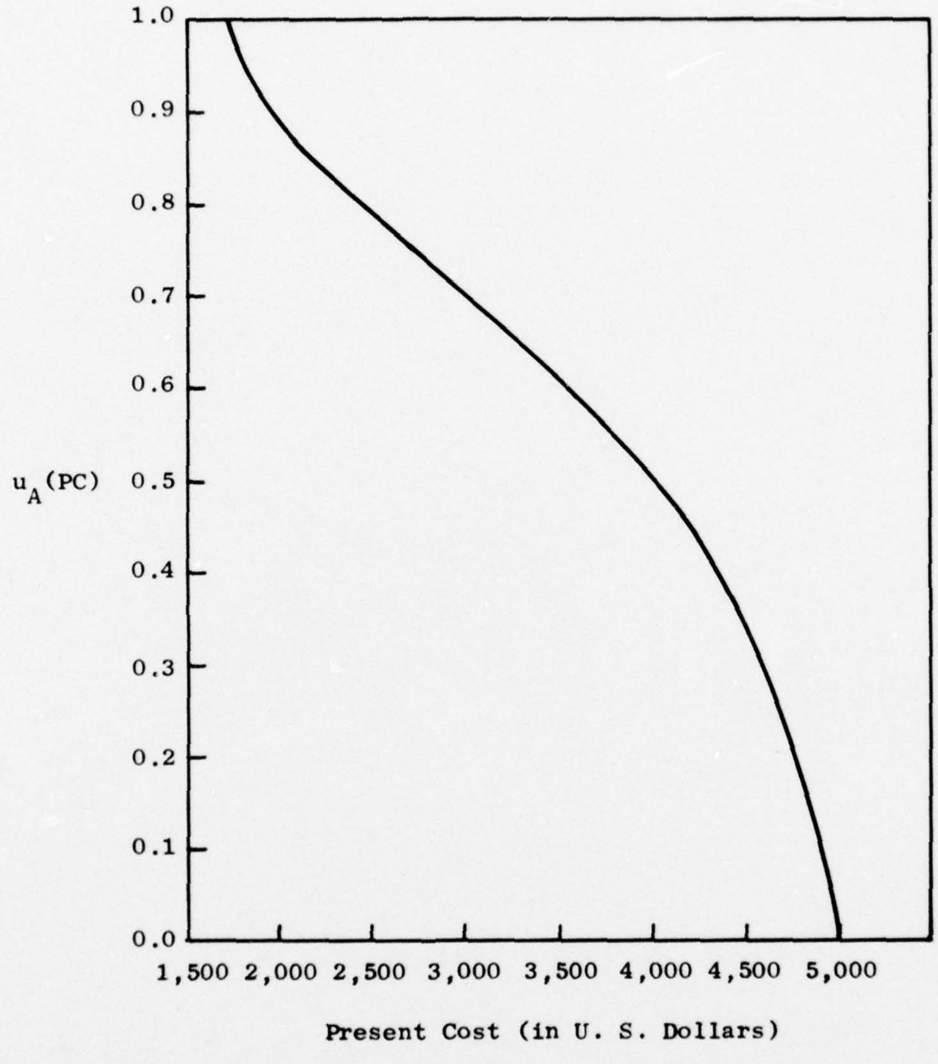


Fig. 6.4. THE ASSESSED UTILITY OVER PRESENT COST. (The subscript A in $u_A(\cdot)$ stands for assessed.)

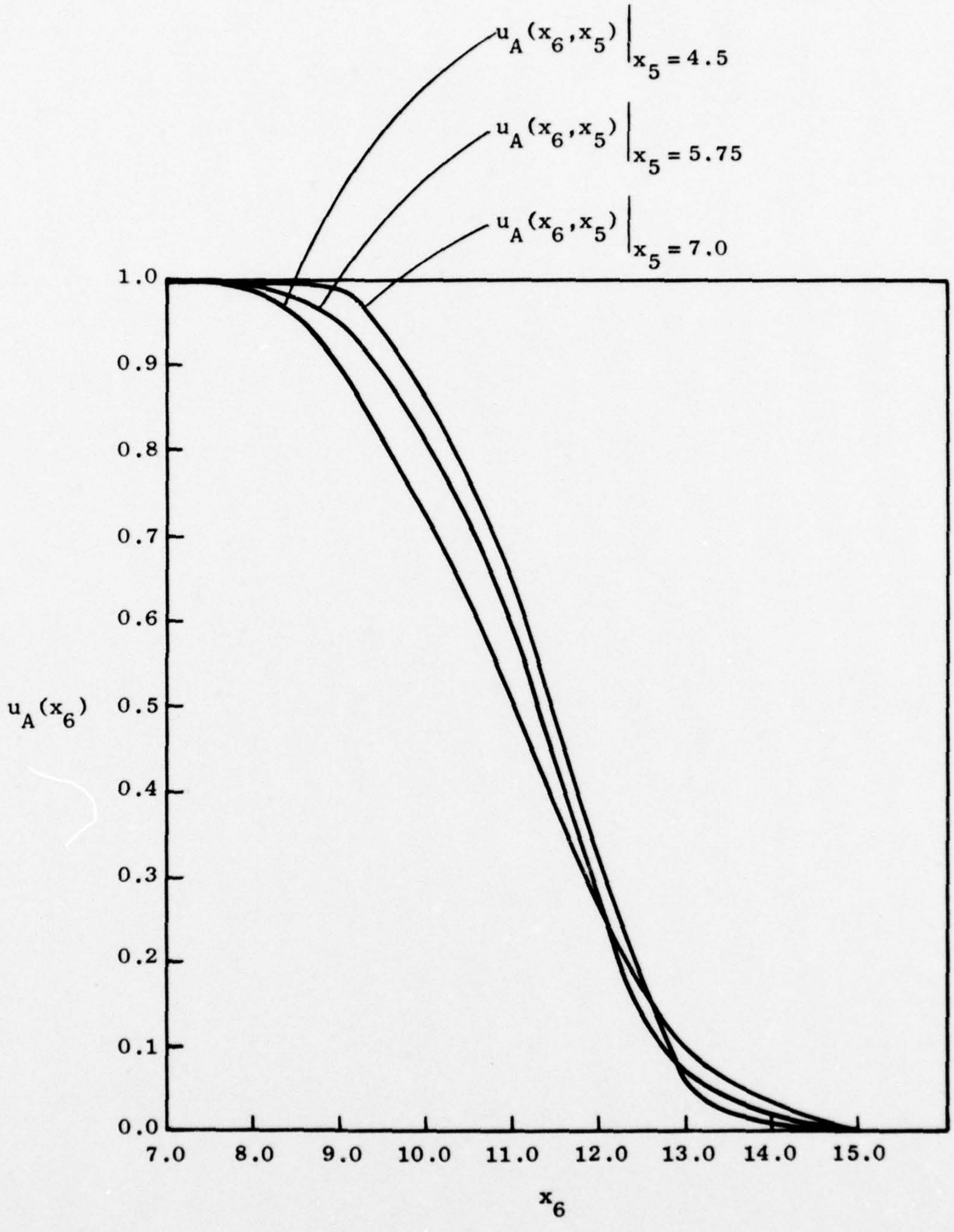


Fig. 6.5. THE ASSESSMENT ON THE THREE NUMERAIRES IN THE PERFORMANCE SPACE.

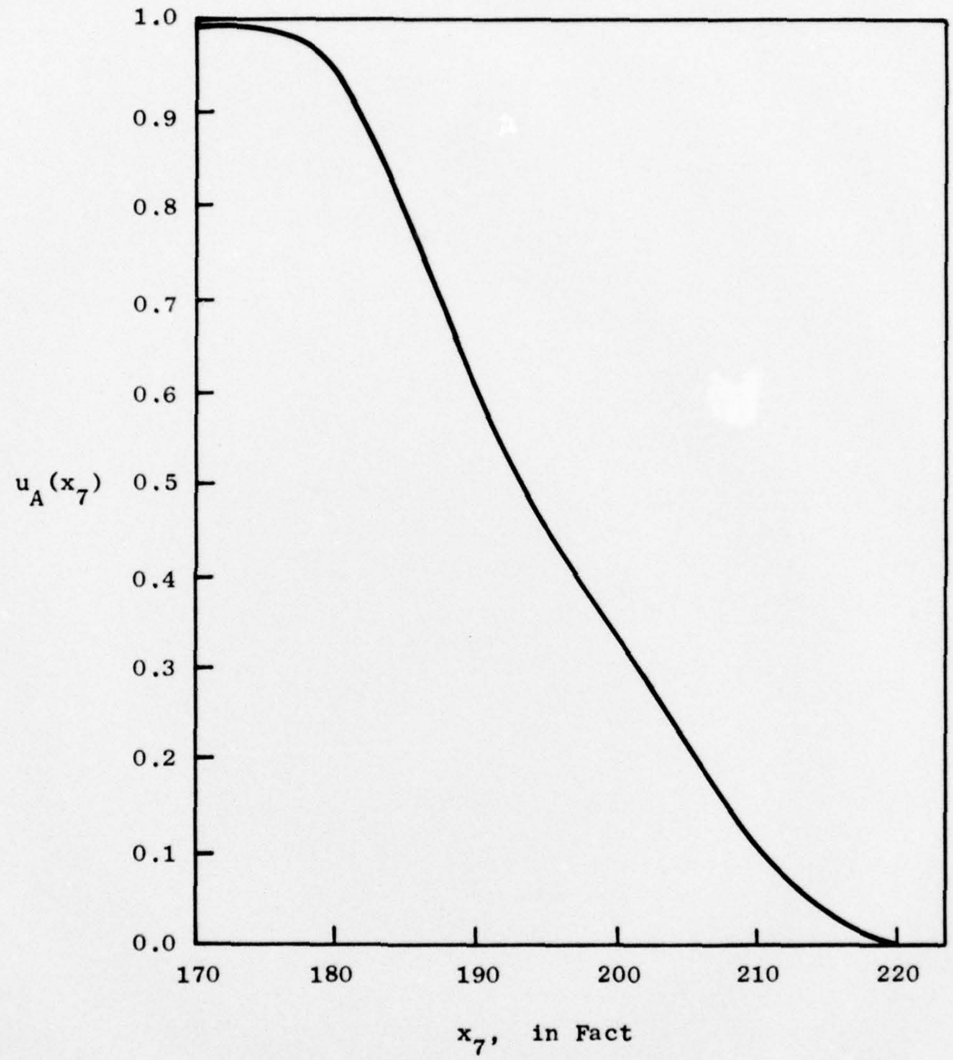


Fig. 6.6. THE ASSESSMENT OF UTILITY OVER x_7 (BRAKES).

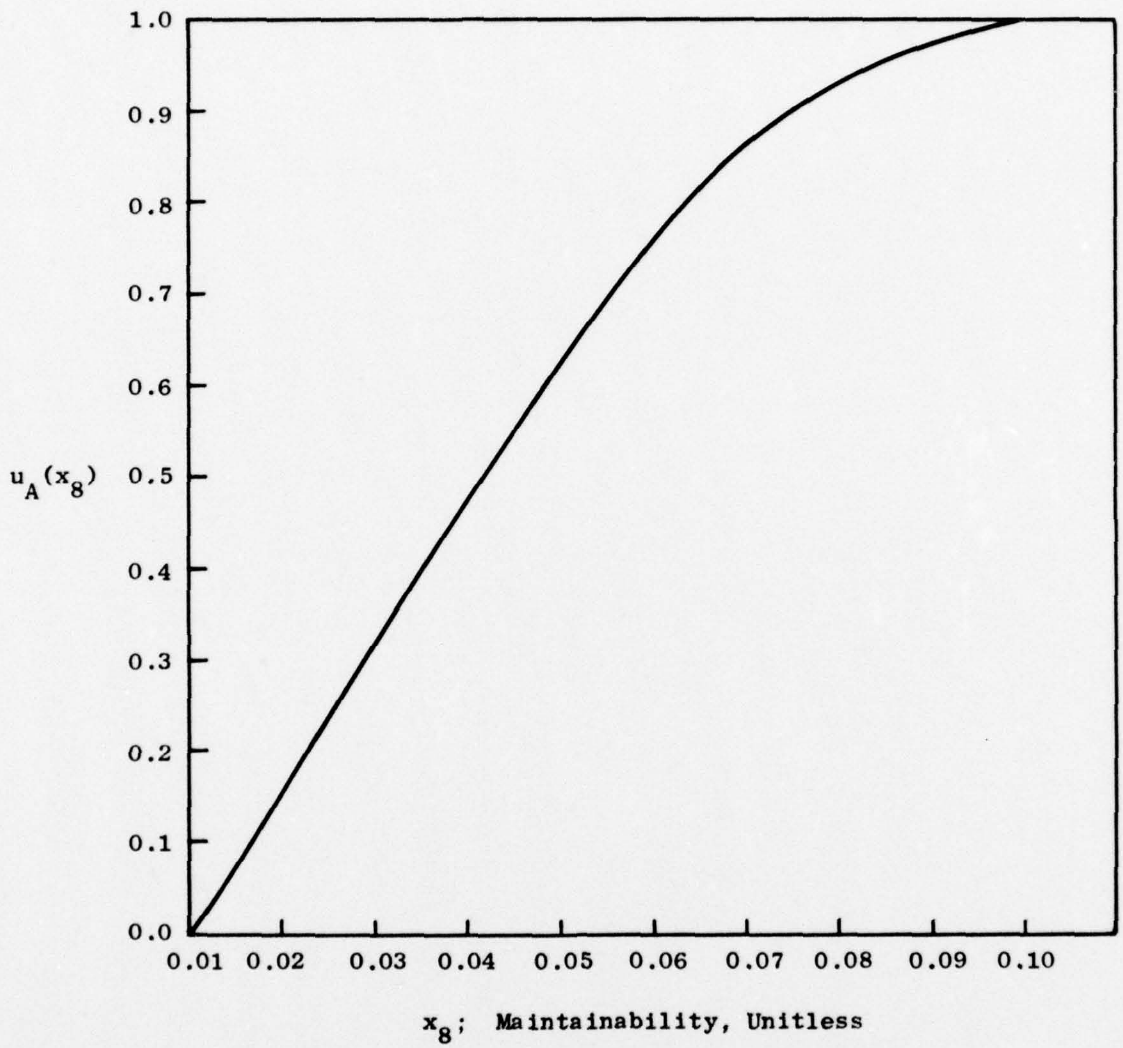


Fig. 6.7. THE ASSESSMENT OF UTILITY OVER x_8 (MAINTAINABILITY).

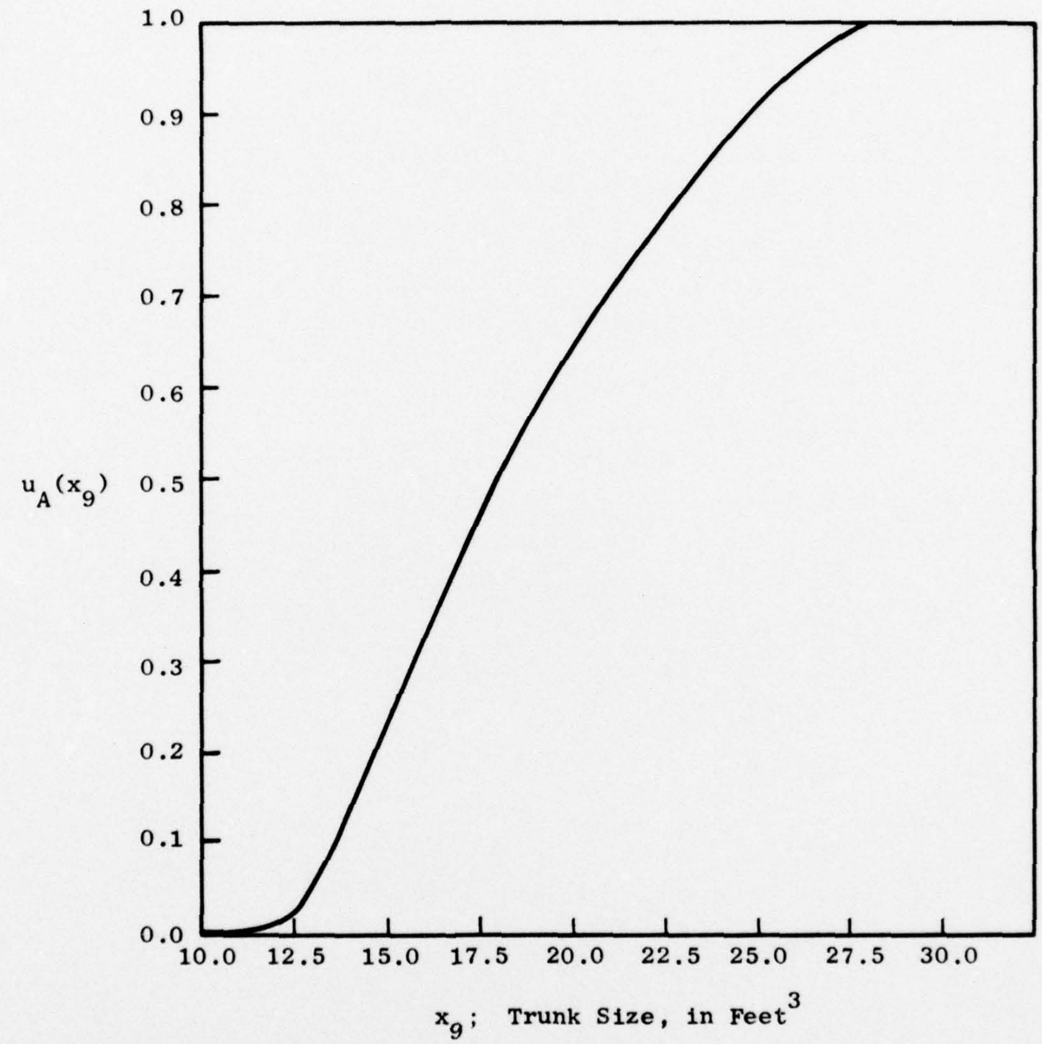


Fig. 6.8. THE ASSESSMENT OF UTILITY OVER TRUNK SIZE.

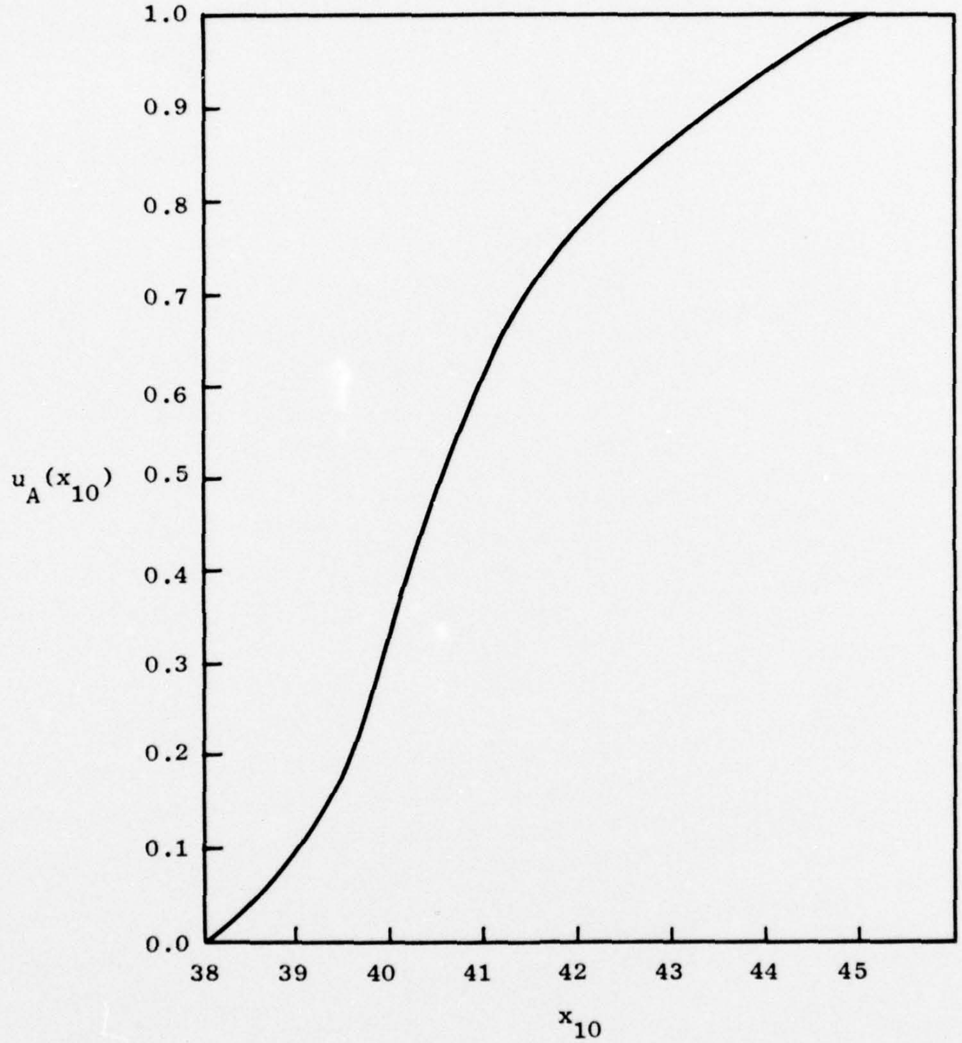


Fig. 6.9. THE ASSESSMENT OF UTILITY OVER FRONT SEAT SIZE, x_{10}.

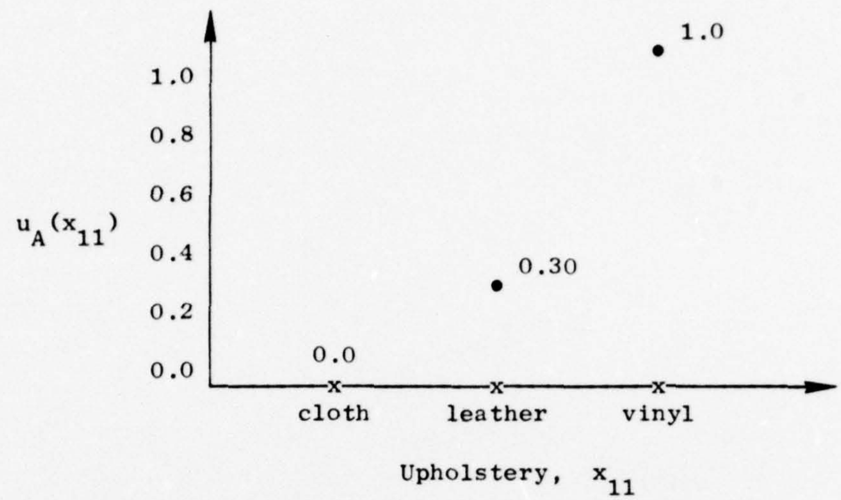


Fig. 6.10. UTILITY ASSESSMENT OVER UPHOLSTERY.

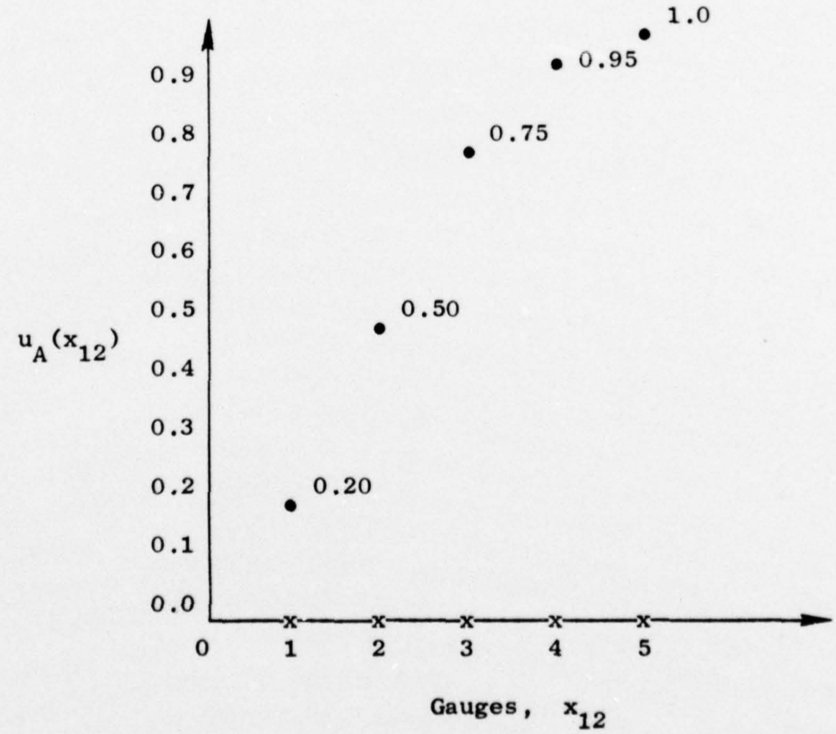


Fig. 6.11. UTILITY ASSESSMENT OVER x_{12}.

An $\alpha \in [0,1]$ where choosing (B,B,B,B,w,w,w,w) is equivalent to choosing the lottery is sought. For such an α, we have:

$$u(B,B,B,B,w,w,w,w) = \alpha\, u(\text{Superior Car}) + (1 - \alpha)\, u(\text{Inferior Car})$$
$$= \alpha\, C_1 + (1 - \alpha)\, C_2 = C_3$$

To get consistent results and reduce the assessment difficulties for the DM, different lotteries are used to estimate different constants. The last column of Table 6.2 contains the utility values of the pair of cars used in the lottery to estimate a particular constant. For example,

$$C_{16} = \alpha C_{13} + (1 - \alpha)\, C_8$$

where α is such that:

$$(B,B,\ w,\ w,\ w,w,w,w) \quad \begin{cases} \xrightarrow{\alpha} (B,B,\ w,\ B,\ w,w,w,w) \\ \xrightarrow{(1-\alpha)} (B,w,\ w,\ w,\ w,w,w,w) \end{cases}$$

It is clear that careful preplanning had to be undergone before the actual assessment is conducted.

For the sake of uniformity, since all the assessed utilities range from zero to one, we will also require that the utility over performance have the range 0 - 1. That is, we will assume:

$$Cu(\text{Best Performance}) \equiv Cu(\text{Best Starting Acceleration, Best Passing Acceleration}) = 1$$

and

$$Cu(\text{Worst Performance}) \equiv Cu(\text{Worst Starting Acceleration, Worst Passing Acceleration}) = 0$$

Table 6.2

THE ASSESSED UTILITY CONSTANTS

Utility		Attribute Levels								Reference Points
Assessed Value	Symbol	PC	P	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	
1.0	c_1	B	B	B	B	B	B	B	B	Arbitrary
0.0	c_2	w	w	w	w	w	w	w	w	
0.78	c_3	B	B	B	B	w	w	w	w	(c_1,c_2)
0.05	c_4	w	w	w	w	B	w	w	w	(c_1,c_2)
0.02	c_5	w	w	w	w	w	B	w	w	(c_1,c_2)
0.05	c_6	w	w	w	w	w	w	B	w	(c_1,c_2)
0.001	c_7	w	w	w	w	w	w	w	B	(c_1,c_2)
0.6	c_8	B	w	w	w	w	w	w	w	(c_1,c_2)
0.5	c_9	w	B	w	w	w	w	w	w	(c_1,c_2)
0.1	c_{10}	w	w	B	w	w	w	w	w	(c_1,c_2)
0.2	c_{11}	w	w	w	B	w	w	w	w	(c_1,c_2)
0.74	c_{12}	B	B	B	w	w	w	w	w	(c_2,c_3)
0.70	c_{13}	B	B	w	B	w	w	w	w	(c_2,c_3)
0.66	c_{14}	B	w	B	B	w	w	w	w	(c_2,c_3)
0.62	c_{15}	w	B	B	B	w	w	w	w	(c_2,c_3)
0.68	c_{16}	B	B	w	w	w	w	w	w	(c_8,c_{13})
0.26	c_{17}	w	w	B	B	w	w	w	w	(c_{11},c_{15})
0.64	c_{18}	B	w	w	B	w	w	w	w	(c_8,c_{14})
0.63	c_{19}	B	w	B	w	w	w	w	w	(c_8,c_{14})
0.56	c_{20}	w	B	B	w	w	w	w	w	(c_9,c_{15})
0.58	c_{21}	w	B	w	B	w	w	w	w	(c_9,c_{15})

Thus, the three numeraire utilities on the performance space will be calibrated with respect to these two points. Four calibration points (see Fig. 6.12) are assessed as such and reported in Table 6.3.

Now, using an interpolation rule, such as the one suggested in Section 5.3, we have a two-dimensional utility surface over the entire performance space whose range is zero to one.

The Final Analytical Model of the Overall Utility

We will derive the final overall utility model via the following steps.

Step 1: The first four attributes, x_1, x_2, x_3, and x_4 are mapped into the present value numeraire via Eq. (6.1).

Step 2: Using the risk aversion profile method, the utility over any pair of starting and passing acceleration is interpolated from the utilities over the three continuous cuts.

Step 3: The utility over the space PC $\times$ performance $\times X_7 \times X_8$ is derived, via the tree method of Chapter 2, using the utility tree depicted in Fig. 6.3. From the utility tree, all we need are the utilities outlined by squares. Each one of these utilities is the same, up to a positive linear transformation, as one of the assessed utilities of Figs. 6.4 through 6.11. Also, the domain of each squared utility contains two points whose utility values correspond to two constants in Table 6.3. Since the constants in Table 6.2 are assessed with respect to the reference and scale points of Eq. (6.1), each squared utility, with reference and

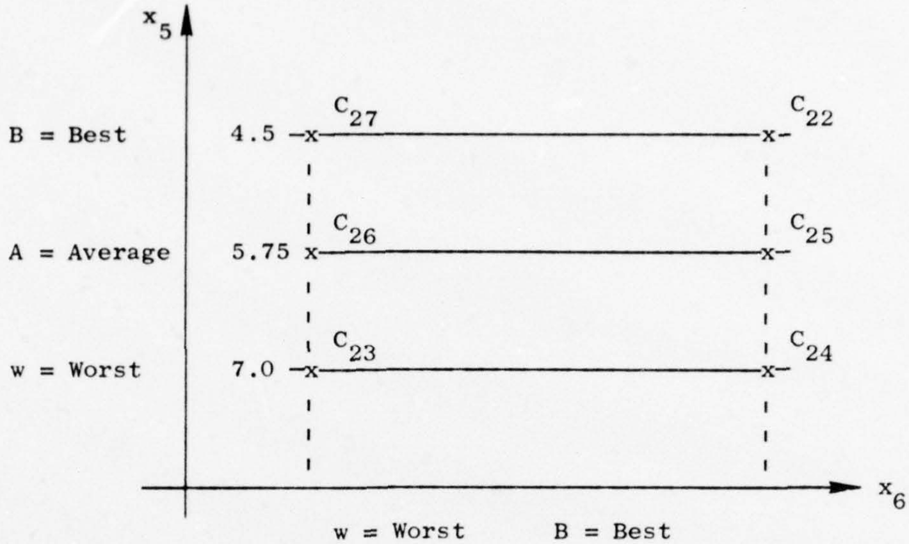


Fig. 6.12. DISCRETIZING THE PERFORMANCE SPACE.

Table 6.3

CALIBRATION CONSTANTS FOR THE NUMERAIRE UTILITIES OF THE PERFORMANCE SPACE

	Utility		Performance	
	Symbol	Assessed Value	x_5	x_6
Arbitrarily Chosen	C_{22}	1.0	B	B
Arbitrarily Chosen	C_{23}	0.00	w	w
	C_{24}	0.75	w	B
	C_{25}	0.85	A	B
	C_{26}	0.45	A	w
	C_{27}	0.70	B	w

scale compatible with Eq. (6.1) may be obtained using the assessed utilities and the scaling constants of Table 6.2. As an example:

$$Cu(w,B,w,1) = \beta + \gamma u_A(x_8) \equiv Cu(w,B,w,1,w,w,w,w)$$

where $u_A(x_8)$ is the assessed utility depicted in Fig. 6.7 and β and γ are chosen such that:

$$Cu(w,B,w,w,w,w,w,w) = C_9$$

$$Cu(w,B,w,B,w,w,w,w) = C_{21}$$

We thus have all the data necessary to construct a representation for $Cu(PC,p,x_7,x_8)$ that is compatible with the two degrees of freedom, defined by Eq. (6.1), of the overall utility.

Step 4: We now have all the pieces to construct the overall utility via the multiplicative-additive model. We will rewrite Eq. (6.2) in the following equivalent, but more convenient, representation (see Keeney [22]):

$$\begin{aligned} 1 &+ ku(PC,p,x_7,x_8,x_9,x_{10},x_{11},x_{12}) \\ &= \left[1 + kk_1\ Cu(PC,p,x_7,x_8)\right] * \left[1 + kk_2\ Cu(x_9)\right] \\ &\quad * \left[1 + kk_3\ Cu(x_{10})\right] + \left[1 + kk_4\ Cu(x_{11})\right] \\ &\quad * \left[1 + kk_5\ Cu(x_{12})\right] \qquad * \in \{\times,+\} \end{aligned} \tag{6.4}$$

where

(i) The parameter attributes of each utility are set at their worst level.

(ii) $k_1 = u(B,B,B,B,w,w,w,w) = C_3$

$k_2 = u(w,w,w,w,B,w,w,w) = C_4$

$k_3 = u(w,w,w,w,w,B,w,w) = C_5$

$k_4 = u(w,w,w,w,w,w,B,w) = C_6$

$k_5 = u(w,w,w,w,w,w,w,B) = C_7$

(iii) Each utility in Eq. (6.4) is to be assessed on a zero to one scale.

(iv) k is to be estimated from the k_i's. If $\Sigma_{i=1}^{5} k_i = 1$, $k = 0$ and we have the additive model. If $\Sigma_{i=1}^{5} k_i \neq 1$, we have the multiplicative models, and k is estimated by the equation:

$$1 + k = \prod_{i=1}^{5} (1 + kk_i) \tag{6.5}$$

where

$$-1 < k < \infty$$

For our model, since

$$\sum_{i=1}^{5} k_i = C_3 + C_4 + C_5 + C_6 + C_7 = 0.901 \neq 1$$

we have the multiplicative model. Using Eq. (6.5), along with the constraint $0 < k < \infty$, we estimate $k = 1.01$.

To implement Eq. (6.4), we use directly the assessed conditional utilities for x_9, x_{10}, x_{11}, and x_{12}. To have $Cu(PC,p,x_7,x_8)$ range from zero to one, the derived values from Step 3 above have to be multiplied by $1/Cu(B,B,B,B,w,w,w,w)$. But, by Eq. (6.4), $Cu(PC,p,x_7,x_8)$ is to be multiplied by $k_1 = C_3 = Cu(B,B,B,B,w,w,w,w)$. So, it is just as well if we use the values for $Cu(PC,p,x_7,x_8)$ as calculated by Step 3 and eliminate the constant k_1 from Eq. (6.4).

This concludes the construction of the overall utility surface.

6.4 Using the Constructed Utility

We have decided to choose a number of actual cars and estimate their utilities by using the constructed utility surface, and also by constructing the direct assessment of the DM. If we have been successful in capturing the essence of the DM's preferences, the two sets of figures should be compatible.

Table 6.4 contains a description of four 1977 model cars that satisfy the requirements of the decision maker. Most of the data is taken from Consumer Report [4]. The data on upholstery, gauges, and trunk size are obtained from local dealers. The DM assessed the data for maintainability, as he perceives it. The DM initially is asked to rank the four cars ordinarily. Then cardinal ordering is obtained via lottery questions by assuming the first choice to have unit utility value and the last choice to have a zero utility. The first column of Table 6.5 contains the cardinal utility ordering for the four cars. Using the constructed utility surface, the utility of each car is calculated and reported in the second column of Table 6.5. To compare the calculated utilities with the assessed ones, we rescaled the calculated utilities so that the highest and lowest values are one and zero, respectively. Column three of Table 6.5 contains the calculated utilities after rescaling. (Appendix 6.A contains a sample of the calculations involved; Appendix 6.B contains different sensitivity analysis calculations.)

Table 6.4

CONTAINS THE FEATURES OF FOUR, 1977 MODEL, ACTUAL CARS
Most of the data are taken from Consumer Report (February 1977)

Attribute	Symbol	Car 1	Car 2	Car 3	Car 4
Purchase price	x_1	\$5237	\$4734	\$4217	\$5230
City mileage	x_2	10.5 MPG	10.5 MPG	9.5 MPG	9.5 MPG
Highway mileage	x_3	20.0 MPG	19.5 MPG	17.5 MPG	18.5 MPG
Resale value	x_4	0.6	0.58	0.50	0.52
Starting Acceleration	x_5	5.0 sec	6.2 sec	5.1 sec	5.6 sec
Passing Acceleration	x_6	8.2 sec	13.0 sec	8.8 sec	9.5 sec
Brakes	x_7	180 ft	185 ft	205 ft	205 ft
Maintainability	x_8	0.07	0.05	0.04	0.04
Trunk size	x_9	25 ft^3	20 ft^3	22.5 ft^3	20 ft^3
Front seat size	x_{10}	41.5"	42"	41"	43"
Seat upholstery	x_{11}	Vinyl	Vinyl	Vinyl	Vinyl
Gauges	x_{12}	3	2	2	3

Table 6.5

COMPARING DIRECTLY ASSESSED VS CALCULATED
UTILITY VALUES FOR FOUR ACTUAL CARS

Car	Directly Assessed Utility	Calculated Utility	Calculated Utility (Scaled)
Car 1	1.0	0.88884	1.0
Car 2	0.4	0.69450	0.0
Car 3	0.6	0.80613	0.57
Car 4	0.0	0.74965	0.28

From Table 6.5, the calculated utilities correctly indicate the first and second choices of the DM. The third and fourth cars, though, interchange positions in the calculated ranking. If we consider the table from the point of view of comparing pairs of cars, the calculated utilities have correctly chosen the better car for five out of the six possible pairs. We do believe that the constructed utility captures the bulk of the DM's preferences.

6.5 Summary

A utility surface is constructed for an actual decision maker. The utility is defined over the space of all cars of a particular class. Twelve attributes are identified as the most important features the DM seeks in a car. Four of the attributes which reflect different aspects of the cost (price, mileage, resale value) of a car are related via an economic model. Two of the attributes reflecting motor performance are considered preference inseparable and are treated as such via the theory of Chapter 5. The economic model along with motor performance and two more attributes are preference-modeled using Keeney's [20] quasi-separable utility model. The tree method of Chapter 2 is used to depict the utility tree of this model. The quasi-separable model along with the rest of the attributes are joined via the multiplicative-additive model to form the overall utility of the DM. To justify the multiplicative-additive model, the construction of Section 4.4 is used to generate a minimal set of UI assumptions.

To test the constructed utility, the utility of four actual cars are directly assessed by the DM. The assessed values are then compared with the values estimated by the constructed utility. It is believed that the constructed surface is a fair representation of the DM's preferences of cars.

Appendix 6.A

We will show, step by step, all the calculations necessary to obtain the utility value for car 1 of Table 6.4. Such value is, of course, referenced by the two points defined by Eq. (6.3).

The calculations:

Step 1:

(1) $$\text{Present Cost (Car 1)} = x_1 + 0.7255\left(\frac{4000}{x_2} + \frac{6000}{x_3}\right) - 0.687\, x_4 x_1$$

$$= \$3585 \qquad (6.A.1)$$

(2) $u_A(PC) = u_A(3585) = 0.59$ (from Fig. 6.4)

(3) $$Cu(PC,w,w,w,w,w,w,w) = C_8 \cdot u_A(PC)$$

$$= 0.6 \times 0.59 = \boxed{0.354} \qquad (6.A.2)$$

Step 2: Calculating the Utility of Performance

(1) $u_A(8.2,4.5) = 0.97$ (from Fig. 6.5)

(2) $u_A(8.2,5.75) = 0.98$ (from Fig. 6.5)

(3) $$u(8.2,4.5) = (1 - C_{27}) \cdot u_A(8.2,4.5) + C_{27}$$

$$= 0.3 \times 0.97 + 0.7 = 0.991 \text{ (scaled)} \qquad (6.A.3)$$

(4) $$u(8.2,5.75) = C_{26} + (C_{25} - C_{26})\, u_A(8.2,5.75)$$

$$= 0.45 + 0.40 \times 0.98 = 0.84 \text{ (scaled)} \qquad (6.A.4)$$

Equations (6.A.3) and (6.A.4) are used so that the utility of (8.2,5.75) and (8.2,4.5) are in conformance with the reference points C_{22} and C_{23} of Table 6.3.

(5) We will use the risk aversion profile (Method I) of Chapter 5 to estimate $u(8.2,5.0)$:

(5.1) $5.0 = \alpha 4.5 + (1 - \alpha)\ 5.75 \Rightarrow \alpha = 0.6$

(5.2) Let $u(x_6,4.5) = 0.84$

(5.3) Hence, $u_A(x_6,4.5) = \dfrac{u(x_6,4.5) - C_{27}}{(1 - C_{27})}$

$$u_A(x_6,4.5) = \frac{0.84 - 0.7}{0.3} = 0.467$$

$$\Rightarrow x_6 = 11.1 \text{ sec} \quad \text{(from Fig. 6.5)}$$

(5.4) $u_A[8.2\alpha + (1 - \alpha)\ 11.1,4.5] = u_A(9.36,4.5)$

$$= 0.952 \quad \text{(scaled)}$$

(5.5) $u(9.36,4.5) = C_{27} + [1 - C_{27}]\ u_A(9.36,4.5)$

$$= 0.952 \quad \text{(scaled)}$$

(5.6) Estimating $\beta(0.6)$:

$$u(9.36,4.5) = \beta(0.6)\ u(8.2,5.75) + [1 - \beta(0.6)] \cdot u(8.2,4.5)$$

or

$$0.952 = \beta(0.6) \times 0.84 + [1 - \beta(0.6)]\ 0.991$$

$$\Rightarrow \beta(0.6) = 0.258 \qquad (6.A.5)$$

For car 1, Eq. (6.A.5) is the only estimate of $\beta(\cdot)$ that exists.

(5.7) The interpolated utility of $(8.2,5.0) = \hat{u}(8.2,5.0)$

$= \beta(0.6)\ u(8.2,4.5) + [1 - \beta(0.6)] \cdot u(8.2,5.75)$

$= 0.258 \times 0.84 + 0.742 \times 0.991 = 0.95$

(6) $Cu(w,p,w,w,w,w,w,w) = C_9 \cdot \hat{u}(x_6,x_5)$

$\therefore\ Cu(w,(8.2,5.0),w,w,w,w,w,w) = 0.5 \times 0.95 = \boxed{0.476}$ (6.A.6)

Step 3: Calculating Cu(3585,(8.2,5.0),180,0.07) via the tree method:

(1) $u_A(x_7) = u_A(180) = 0.95$ (from Fig. 6.6)

$Cu(w,w,x_7,w,w,w,w,w) = C_{10} \cdot u_A(x_7)$

$\Rightarrow Cu(w,w,180,w,w,w,w,w) = 0.1 \times 0.95 = \boxed{0.095}$ (6.A.7)

(2) $u_A(x_8) = u_A(0.07) = 0.86$ (from Fig. 6.7)

$Cu(w,w,w,x_8,w,w,w,w) = C_{11} \cdot u_A(x_8)$

$\Rightarrow Cu(w,w,w,0.07,w,w,w,w) = 0.2 \times 0.86 = \boxed{0.172}$ (6.A.7)

(3) $Cu(w,w,B,x_8,w,w,w,w) = C_{10} + (C_{17} - C_{10})\ u_A(x_8)$

$\Rightarrow Cu(w,w,B,0.07,w,w,w,w) = 0.1 + 0.16 \times 0.86 = 0.240$ (6.A.8)

(4) $Cu(w,B,w,x_3,w,w,w,w) = C_9 + (C_{21} - C_9)\ u_A(x_8)$

$\Rightarrow Cu(w,B,w,0.07,w,w,w,w) = 0.5 + 0.08 \times 0.86 = 0.57$ (6.A.9)

(5) $Cu(w,B,B,x_8,w,w,w,w) = C_{20} + (C_{15} - C_{20})\ u_A(x_8)$

$\Rightarrow Cu(w,B,B,0.07,w,w,w,w) = 0.56 + 0.06 \times 0.86 = 0.61$ (6.A.10)

(6) $Cu(B,w,w,x_8,w,w,w,w) = C_8 + (C_{18} - C_8)\ u_A(x_8)$

$\Rightarrow Cu(B,w,w,0.07,w,w,w,w) = 0.6 + 0.4 \times 0.86 = 0.63$ (6.A.11)

(7) $Cu(B,w,B,x_8,w,w,w,w) = C_{19} + (C_{14} - C_{19})\ u_A(x_8)$

$Cu(B,w,B,0.07,w,w,w,w) = 0.63 + 0.03 \times 0.86 = 0.656$ (6.A.12)

(8) $Cu(B,B,w,x_8,w,w,w,w) = C_{16} + (C_{13} - C_{16})\, u_A(x_8)$

$$Cu(B,B,w,0.07,w,w,w,w) = 0.68 + 0.02 \times 0.86 = 0.70 \qquad (A.A.13)$$

(9) $Cu(B,B,B,x_8,w,w,w,w) = C_{12} + (C_3 - C_{12})\, u_A(x_8)$

$$Cu(B,B,B,0.07,w,w,w,w) = 0.74 + 0.04 \times 0.86 = 0.774 \qquad (6.A.14)$$

(10) $Cu(w,w,x_7,x_8,w,w,w,w) = Cu(w,w,w,x_8,w,w,w,w)$

$$+ [Cu(w,w,B,x_8,w,w,w,w) - Cu(w,w,w,x_8,w,w,w,w)] \cdot \frac{Cu(w,w,x_7,w,w,w,w,w)}{C_{10}}$$

$$\Rightarrow Cu(w,w,180,0.07,w,w,w,w) = 0.2366 \qquad (6.A.15)$$

(11) $Cu(w,B,x_7,x_8,w,w,w,w) = Cu(w,B,w,x_8,w,w,w,w)$

$$+ [Cu(w,B,B,x_8,w,w,w,w) - Cu(w,B,w,x_8,w,w,w,w)] \cdot \frac{Cu(w,w,x_7,w,w,w,w,w)}{C_{10}}$$

$$\Rightarrow Cu(w,B,180,0.07,w,w,w,w) = 0.61 \qquad (6.A.16)$$

(12) $Cu(B,w,x_7,x_8,w,w,w,w) = Cu(B,w,w,x_8,w,w,w,w)$

$$+ [Cu(B,w,B,x_8,w,w,w,w) - Cu(B,w,w,x_8,w,w,w,w)] \cdot \frac{Cu(w,w,x_7,w,w,w,w,w)}{C_{10}}$$

$$\Rightarrow Cu(B,w,180,0.07,w,w,w,w) = 0.655 \qquad (6.A.17)$$

(13) $Cu(B,B,x_7,x_8,w,w,w,w) = Cu(B,B,w,x_8,w,w,w,w)$

$$+ [Cu(B,B,B,x_8,w,w,w,w) - Cu(B,B,w,x_8,w,w,w,w)] \cdot \frac{Cu(w,w,x_7,w,w,w,w)}{C_{10}}$$

$$\Rightarrow Cu(B,B,180,0.07,w,w,w,w) = 0.77 \qquad (6.A.18)$$

(14) $Cu(w,(x_5,x_6),x_7,x_8,w,w,w,w) = Cu(w,w,x_7,x_8,w,w,w,w)$

$$+ [Cu(w,B,x_7,x_8,w,w,w,w) - Cu(w,w,x_7,x_8,w,w,w,w)] \cdot \frac{Cu(w,(x_5,x_6),w,w,w,w,w,w)}{C_9}$$

$$\Rightarrow Cu(w,(5.0,8.2),180,0.07,w,w,w,w) = 0.59 \qquad (6.A.19)$$

(15) $Cu(B,(x_5,x_6),x_7,x_8,w,w,w,w) = Cu(B,w,x_7,x_8,w,w,w,w)$

$$+ [Cu(B,B,x_7,x_8,w,w,w,w) - Cu(B,w,x_7,x_8,w,w,w,w)] \cdot \frac{Cu(w,(x_5,x_6),w,w,w,w,w,w)}{C_9}$$

$$\Rightarrow Cu(B,(5.0,8.2),180,0.07,w,w,w,w) = 0.764 \qquad (6.A.20)$$

(16) $Cu(PC,(x_5,x_6),x_7,x_8,w,w,w,w) = Cu(PC,(x_5,x_6),x_7,x_8)$

$$= Cu(w,(x_5,x_6),x_7,x_8,w,w,w,w) + [Cu(B,(x_5,x_6),x_7,x_8,w,w,w,w)$$

$$- Cu(w,(x_5,x_6),x_7,x_8,w,w,w,w)] \cdot \frac{Cu(PC,w,w,w,w,w,w,w)}{C_8}$$

$$\Rightarrow Cu(3585,(5.0,8.2),180,0.07,w,w,w,w) = \boxed{0.6927} \qquad (6.A.21)$$

Step 4: **Calculating the Overall Utility**

(1) $u_A(x_9) = \boxed{0.92}$ **(from Fig. 6.8)**

(2) $u_A(x_{10}) = \boxed{0.70}$ **(from Fig. 6.9)**

(3) $u_A(x_{11}) = \boxed{1.0}$ **(from Fig. 6.10)**

(4) $u_A(x_{12}) = \boxed{0.75}$ **(from Fig. 6.11)**

(5) $1 + 1.01 \times u(\text{car } 1) = [1 + 1.01 \times Cu(3585,(5.0,8.2),180,0.07,$

$$w,w,w,w)] \cdot [1 + 1.01 \times C_4 u_A(x_9)] \cdot [1 + 1.01 \times C_5 u_A(x_{10})] \cdot [1 + 1.01 \times C_6 u_A(x_{11})] \cdot [1 + 1.01 \times C_7 u_A(x_{12})]$$

$$\Rightarrow [1 + 1.01 \times u(\text{car } 1)] = 1.70094 \times 1.04646 \times 1.01414 \times 1.0505 \times 1.0007575 \qquad \therefore u(\text{car } 1) = \boxed{0.889}$$

Appendix 6.B

SENSITIVITY ANALYSIS

Here, we conduct different sensitivity analyses on the preference model to discern its robustness and the sensitivity of trade offs between pairs of attributes.

6.B.1 Sensitivity Analysis of the Preference Model

The preference model we used for the application involves seven UI assumptions which are depicted in Fig. 6.B.1 as assumptions a_1 through a_7. This model is referred to in this appendix as Model I. We propose Models II and III which are modifications of Model I. Model II, which is defined by assumptions a_1 through a_{12} of Fig. 6.B.1, replaces the quasi-separable component of Model I by a multiplicative-additive component. This corresponds to assuming further regularities of the preference structure on the subspace x_1 through x_8. Model III, which is defined by assumptions a_1 through a_7 and a_{13} and a_{14}, replaces the quasi-separable component by a dichotomous chain component. For this model, we dichotomize x_1 through x_8 into $\{x_1, x_2, x_3, x_4, x_7\}$ and $\{x_5, x_6, x_8\}$ where the first group of attributes are all cost related and the second group are performance related (motor and brake performance). This dichotomy seems quite natural for the problem and the corresponding UI assumptions, even though unverified, seem quite plausible.

With respect to the concept of utility independence, Model II corresponds to the most regular preference structure, then, Model III and, lastly, Model I.

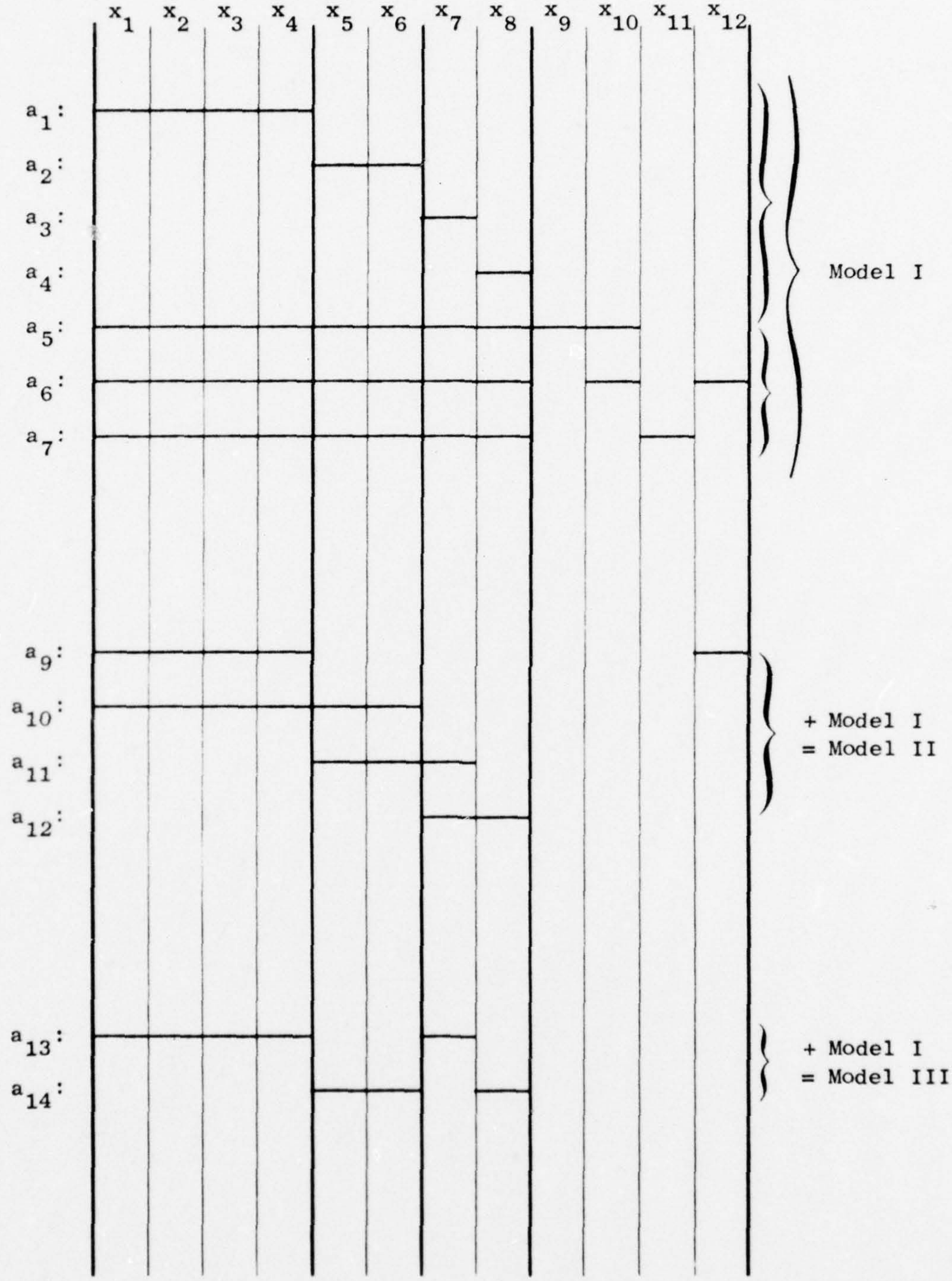


Fig. 6.B.1. PREFERENCE MODELING USING A DIFFERENT SET OF UI ASSUMPTIONS.

The analytical form of Models II and III are derived. For Model II, we get

$$[1 + Ku] = [1 + 0.6Ku_A(PC)] \cdot [1 + 0.5K\hat{u}_A(P)] \cdot [1 + 0.1Ku_A(x_7)]$$

$$\cdot [1 + 0.2Ku_A(x_8)] \cdot [1 + 0.05Ku_A(x_9)] \cdot [1 + 0.02Ku_A(x_{10})]$$

$$\cdot [1 + 0.05Ku_A(x_{11})] \cdot [1 + 0.001Ku_A(x_{12})]$$

where u is the overall utility of a car and $\hat{u}(P)$ is the utility of performance as obtained by the interpolation method. All other utilities are obtained from their respective graphs of Chapter 6. K for this model is equal to -0.76.

The analytical form of Model III is:

$$[1 + Ku] = [1 + 0.6K\ Cu(PC,p,x_7,x_8)] \cdot [1 + 0.05Ku_A(x_9)]$$

$$\cdot [1 + 0.02Ku_A(x_{10})] \cdot [1 + 0.05Ku_A(x_{11})] \cdot [1 + 0.001Ku_A(x_{12})]$$

where u is the overall utility of a car, K for this model is equal to +1.01, and

$$Cu(PC,p,x_7,x_8) = 0.714 + 0.714\ Cu'(P) \cdot Cu'(x_8) \cdot Cu'(PC) \cdot Cu'(x_8)$$

$$- 1.55\ Cu'(P) \cdot Cu'(x_8) - 1.57\ Cu'(PC) \cdot Cu'(x_7)$$

where

$$Cu'(PC) = 0.70\ u_A(PC) - 1$$

$$Cu'(x_7) = 0.12\ u_A(x_7) - 1$$

$$Cu'(P) = 0.6\ \hat{u}(P) - 1$$

$$Cu'(x_8) = 0.24\ u_A(x_8) - 1$$

All $u_A(\cdot)$ are obtained from their respective graphs in Chapter 6.

The analytical models are used to calculate the utilities of each car of Table 6.4. Table 6.B.1 contains the calculated values. It is clear from the table that Models I and III produce similar utility values while Model II is slightly different. All models give the same ordinal ranking of the four cars.

From an assessment point of view, Models II and III require less effort than Model I. In particular, Model II requires the assessment of only twelve constants as opposed to the twenty-five constants required of Model I. This translates into a reduction of about two hours worth of assessment time. Model III requires only eighteen constants. In retrospect, we would have been better off using Model III instead of Model I due to the resemblance of their utilities.

Table 6.B.1

SENSITIVITY OF PREFERENCE MODELING

Item	Utility Values			
	Car 1	Car 2	Car 3	Car 4
Direct assessment	1.0	0.4	0.60	0.0
Model I	0.88884	0.69450	0.80613	0.74965
Model I (scaled)	1.0	0.0	0.57	0.28
Model II	0.86132	0.67020	0.80736	0.73394
Model II (scaled)	1.0	0.0	0.72	0.33
Model III	0.88881	0.69445	0.80609	0.74961
Model III (scaled)	1.0	0.0	0.57	0.28

6.B.2 Trade Off Analysis

Here, we address questions of the following type: How much is some percentage improvement of an attribute for a given car worth in terms of the present value of the car or its initial purchase price?

Table 6.B.2 contains such calculations. In the table, we calculate changes in the present value of a car due to a 15% increase (or decrease) in the performance of the car, where all other attributes are fixed at their given values. These calculations correspond to moving along the same indifference curve of the attributes PC and P. To properly interpret the figures of the table, let us consider the first column corresponding to car 1. For this car, the decision maker should be indifferent between the original car 1 as defined by Table 6.4 (of Chapter 6) and the same car where performance improves by 15% and present cost increases by 315 dollars. Likewise, the decision maker should be indifferent between the original car 1 and car 1 where performance is decreased by 15% and present cost is decreased by 310 dollars.

The same sort of calculations are conducted for the attributes x_7, x_9, and x_{10}. They are reported in Tables 6.B.3 through 6.B.5. To translate the changes in present cost to changes in the initial purchase price, x_1, we use the multipliers of the last row of the table.

Finally, all such calculations are repeated using preference Model II instead of preference Model I. (We have ignored Model III since it is similar to Model I.)

Table 6.B.2

SENSITIVITY OF PRESENT COST TO PERFORMANCE CHANGES

Item	Car 1	Car 2	Car 3	Car 4
Performance, (x_5,x_6)-- old value	(5,8.2)	(6.2,13)	(5.1,8.8)	(5.6,9.5)
Utility, $\hat{u}(x_5,x_6)$	0.952	0.308	0.919	0.826
Performance, (x_5,x_6)-- new value	(4.63,7.0)	(5.83,11.8)	(4.73,7.6)	(5.23,8.3)
Utility, $\hat{u}(x_5,x_6)$	1.0	0.563	0.991	0.895
% change in performance	+ 15	+ 15	+ 15	+ 15
% change in $\hat{u}(x_5,x_6)$	+ 5	+ 83	+ 8	+ 8
Worth in present cost	+ 315	+ 903	+ 527	+ 400
Performance, (x_5,x_6)-- new value	(5.38,9.4)	(0.575,14.2)	(5.5,10)	(5.98,10.7)
Utility, $\hat{u}(x_5,x_6)$	0.862	0.03	0.802	0.69
% change in performance	- 15	- 15	- 15	- 15
% change in $\hat{u}(x_5,x_6)$	- 9	- 99	- 13	- 16
Worth in present cost	- 310	- 672	- 723	- 550
Worth with respect to x_1: multiply ΔPC by	1.69	1.66	1.52	1.56

Table 6.B.3

SENSITIVITY OF PRESENT COST TO CHANGES IN BRAKE VALUES

Item	Car 1	Car 2	Car 3	Car 4
Brakes, x_7--old value	180	185	205	205
Utility, $u_A(x_7)$	0.95	0.78	0.22	0.22
Brakes, x_7--new value	172.5	177.5	197.5	197.5
Utility, $u_A(x_7)$	1	0.97	0.38	0.38
% change in x_7	+ 15	+ 15	+ 15	+ 15
% change in $u_A(x_7)$	+ 5	+ 24	+ 73	+ 73
Worth in present cost	+ 115	+ 203	+ 227	+ 250
Brakes, x_7--new value	187.5	192.5	212.5	212.5
Utility, $u_A(x_7)$	0.67	0.52	0.06	0.06
% change in x_7	- 15	- 15	- 15	- 15
% change in $u_A(x_7)$	- 30	- 33	- 73	- 73
Worth in present cost	- 560	- 97	- 323	- 150
Worth with respect to x_1: multiply ΔPC by	1.69	1.66	1.52	1.56

Table 6.B.4

PRESENT COST SENSITIVITY TO CHANGES IN TRUNK SIZE

Item	Car 1	Car 2	Car 3	Car 4
Trunk size, x_9--old values	25	20	22.5	20
Utility, $u_A(x_9)$	0.92	0.65	0.79	0.65
Trunk size, x_9--new values	28	23	25.5	23
Utility, $u_A(x_9)$	1.0	0.82	0.93	0.82
% change in x_9	+ 15	+ 15	+ 15	+ 15
% change in $u_A(x_9)$	+ 9	+ 26	+ 18	+ 26
Worth in present cost	+ 170	+ 93	+ 157	+ 250
Trunk size, x_9--new values	22	14.5	19.5	14.5
Utility, $u_A(x_9)$	0.77	0.41	0.61	0.41
% change in x_9	- 15	- 15	- 15	- 15
% change in $u_A(x_9)$	- 16	- 37	- 23	- 37
Worth in present cost	- 290	- 297	- 393	- 394
Worth with respect to x_1: multiply ΔPC by	1.69	1.66	1.52	1.56

Table 6.B.5

SENSITIVITY OF PRESENT COST TO CHANGES IN SEAT SIZE VALUES

Item	Car 1	Car 2	Car 3	Car 4
Seat size, x_{10}--old values	41.5"	42"	41"	43"
Utility, $u_A(x_{10})$	0.7	0.77	0.6	0.87
Seat size, x_{10}--new value	42.55"	43.05"	42.05"	44.05"
Utility, $u_A(x_{10})$	0.83	0.87	0.77	0.94
% change in x_{10}	+ 15	+ 15	+ 15	+ 15
% change in $u_A(x_{10})$	+ 19	+ 13	+ 28	+ 8
Worth in present cost	+ 85	+ 75	+ 97	+ 75
Seat size, x_{10}--new value	40.45"	40.95"	39.95"	41.95"
Utility, $u_A(x_{10})$	0.47	0.6	0.33	0.77
% change in x_{10}	- 15	- 15	- 15	- 15
% change in $u_A(x_{10})$	- 33	- 22	- 45	11
Worth in present cost	- 105	- 75	- 273	- 75
Worth with respect to x_1: multiply ΔPC by	1.69	1.66	1.52	1.56

Table 6.B.6

SENSITIVITY OF PRESENT COST TO PERFORMANCE CHANGES (MODEL II)

Item	Car 1	Car 2	Car 3	Car 4
Performance, (x_5, x_6)-- old value	(5,8.2)	(6.2,13)	(5.1,8.8)	(5.6,9.5)
Utility $\hat{u}(x_5, x_6)$	0.952	0.308	0.919	0.826
Performance, (x_5, x_6)-- new value	(4.63,7.0)	(5.83,11.8)	(4.73,7.6)	(5.23,8.3)
Utility, $\hat{u}(x_5, x_6)$	1.0	0.563	0.991	0.895
% change in performance	+ 15	+ 15	+ 15	+ 15
% change in $\hat{u}(x_5, x_6)$	+ 5	+ 83	+ 8	+ 8
Worth in present cost	+ 215	+ 800	+ 625	+ 300
Performance, (x_5, x_6)-- new value	(5.38,9.4)	(0.575,14.2)	(5.5,10)	(5.98,10.7)
Utility, $\hat{u}(x_5, x_6)$	0.862	0.03	0.802	0.69
% change in performance	- 15	- 15	- 15	- 15
% change in $\hat{u}(x_5, x_6)$	- 9	- 99	- 13	- 16
Worth in present cost	- 435	- 950	- 323	- 550
Worth with respect to x_1: multiply ΔPC by	1.69	1.66	1.52	1.56

Table 6.B.7

SENSITIVITY OF PRESENT COST TO CHANGES IN BRAKE VALUES (MODEL II)

Item	Car 1	Car 2	Car 3	Car 4
Brakes, x_7--old value	180	185	205	205
Utility, $u_A(x_7)$	0.95	0.78	0.22	0.22
Brakes, x_7--new value	172.5	177.5	197.5	197.5
Utility, $u_A(x_7)$	1	0.97	0.38	0.38
% change in x_7	+ 15	+ 15	+ 15	+ 15
% change in $u_A(x_7)$	+ 5	+ 24	+ 73	+ 73
Worth in present cost	+ 8	+ 100	+ 100	+ 100
Brakes, x_7--new value	187.5	192.5	212.5	212.5
Utility, $u_A(x_7)$	0.67	0.52	0.06	0.06
% change in x_7	- 15	- 15	- 15	- 15
% change in $u_A(x_7)$	- 30	- 33	- 73	- 73
Worth in present cost	- 185	- 75	- 150	- 90
Worth with respect to x_1: multiply ΔPC by	1.69	1.66	1.52	1.56

Table 6.B.8

PRESENT COST SENSITIVITY TO CHANGES IN TRUNK SIZE (MODEL II)

Item	Car 1	Car 2	Car 3	Car 4
Trunk size, x_9--old values	25	20	22.5	20
Utility, $u_A(x_9)$	0.92	0.65	0.79	0.65
Trunk size, x_9--new values	28	23	25.5	23
Utility, $u_A(x_9)$	1.0	0.82	0.93	0.82
% change in x_9	+ 15	+ 15	+ 15	+ 15
% change in $u_A(x_9)$	+ 9	+ 26	+ 18	+ 26
Worth in present cost	+ 15	+ 50	+ 50	+ 100
Trunk size, x_9--new values	22	14.5	19.5	14.5
Utility, $u_A(x_9)$	0.77	0.41	0.61	0.41
% change in x_9	- 15	- 15	- 15	- 15
% change in $u_A(x_9)$	- 16	- 37	- 23	- 37
Worth in present cost	- 60	- 100	- 75	- 50
Worth with respect to x_1: multiply ΔPC by	1.69	1.66	1.52	1.56

Table 6.B.9

SENSITIVITY OF PRESENT COST TO CHANGES IN SEAT SIZE VALUES (MODEL II)

Item	Car 1	Car 2	Car 3	Car 4
Seat Size, x_{10}--old values	41.5"	42"	41"	43"
Utility, $u_A(x_{10})$	0.7	0.77	0.6	0.87
Seat size, x_{10}--new value	42.55"	43.05"	42.05"	44.05"
Utility, $u_A(x_{10})$	0.83	0.87	0.77	0.94
% change in x_{10}	+ 15	+ 15	+ 15	+ 15
% change in $u_A(x_{10})$	+ 19	+ 13	+ 28	+ 8
Worth in present cost	+ 15	+ 10	+ 10	+ 20
Seat size, x_{10}--new value	40.45"	40.95"	39.95"	41.95"
Utility, $u_A(x_{10})$	0.47	0.6	0.33	0.77
% change in x_{10}	- 15	- 15	- 15	- 15
% change in $u_A(x_{10})$	- 33	- 22	- 45	11
Worth in present cost	- 35	- 50	- 73	- 15
Worth with respect to x_1: multiply ΔPC by	1.69	1.66	1.52	1.56

Chapter 7

SUMMARY OF RESULTS AND FURTHER RESEARCH

In this concluding chapter, we highlight the salient features of our contribution and suggest different directions for extending this research.

7.1 Summary of Results

We propose the modeling of preference structures via sets of UI assumptions. As such, different preferences correspond to different UI sets, and different UI sets result in varying decompositional forms of the utility surface, preserving the preference ordering. Utility decomposition implies that an n-dimensional surface can be analytically derived from surfaces (or utilities) of lower dimensions. In this case, the assessment effort required for constructing the utility may be greatly reduced.

We consider arbitrary sets of UI assumptions. For a given UI set, two fundamental properties fully characterize its corresponding decomposition: the partitioning of the attribute space into subspaces of lower dimensions, and a type of low order regularity operating on each of the subspaces. The concept of "utility independence order" is introduced to capture such low order regularities. These two properties underly a proposed codable procedure called the tree method for generating the utility decomposition form corresponding to any UI set. The procedure rests on an automatum, or a finite semigroup, which is an abstraction of the decomposition algebra involved. The procedure produces tree-like structures that are both a self-contained analytical representation of the utility decomposition and a visually powerful aid for demonstration and discussion purposes. We believe that this part of our contribution gives the analyst much facility and insight with regard to modeling preferences via UI sets and the whole decomposition technique in general.

Next, we use the two characterizations of utility decomposition to propose a natural scheme for classifying preferences on n-attributes. We use the scheme to list all distinct (modulo the UI concept) preference structures on four-attribute spaces.

The treatment of arbitrary UI sets leads naturally to questions of implications and equivalence between such sets. We thus consider the case where the satisfaction of a set of UI assumptions automatically implies the satisfaction of further assumptions. We discuss three instances of UI implications, one of which has been discovered by Keeney. Of the other two instances, the implication due to dichotomous chains leads to a generalized version of Keeney's quasi-separable utility model. Dichotomous chains also are believed to remedy a conceptually weak point of the UI primitive with respect to the fact that it is one directional, i.e., if X UI Y, it does not mean that Y UI X. For a dichotomous chain which involves a collection of nested subspaces, the UI assumption is satisfied in both directions, with respect to each subspace.

To discern any occurrence of UI implications, a canonical form is proposed for UI sets. The canonical form also indicates, visually, other properties of the induced decomposition.

The multiplicative-additive model which plays a central role in UI implications is considered on its own merits. We propose a construction which produces a minimal number of UI assumptions corresponding to a given multiplicative-additive model. Such a construction reduces the verification effort required by the analyst.

A UI set may not lead to a full decomposition of the utility surface. Thus, an analyst may have to construct conditional utilities of more than one dimension that are indecomposable. For such surfaces, we propose the strategy of continuous cuts, a discretization process which requires the assessment of one-dimensional utilities on one of the attributes where the other attributes are parameterized into a finite number of points. An interpolation rule is to be used to approximate the utility throughout the domain. For a rule, we consider a procedure that assumes a local decomposition due to a particular choice of a set of UI assumptions. The implementation of the local decomposition relies on simplifying behavioral assumptions about the decision maker's attitude toward uncertainty along each of the attributes. All the information required for this procedure is extracted from the assessed one-dimensional utilities. When comparing this procedure with the current state of the art, of fitting classical families of curves locally, we find that our procedure requires a relatively modest amount of calculations. This is so because the utility of a point is calculated step by step, undimensionally, with respect to each attribute. Hence, no systems of equations need to be solved, as is the case with other methods.

7.2 Future Research

The following is a list of suggestions for future research that extends and complements our contribution.

(1) In Example 2.1 of Chapter 2, we observed a spanning phenomenon that characterizes the utility surface as a linear

variety of functions that are part of the surface itself. Such a characterization may be established as a general mathematical property of decomposition due to any UI set.

(2) It may be useful to identify and document actual examples of preferences corresponding to the distinct classes of some of the tables of Chapter 3.

(3) In the chapter on UI implications, we stopped short of identifying <u>all</u> possible sources of implications. We have a strong feeling that the three instances we discussed are all the sources there are. This statement, of course, has to be proved or disproved.

(4) On the treatment of indecomposable utilities, our proposed methodology may require a great deal of assessment effort. Specific schemes of sensitivity analysis need to be proposed to calibrate this effort with other parts of the overall assessment for the purpose of enhancing the total accuracy of the overall utility.

(5) There seems to be a great deal of variety of regularity and smoothness assumptions proposed for modeling preferences. A theory is needed to structure the state of the art and compare preferences corresponding to different generic assumptions as to their restrictiveness or equivalence. The development of a hierarchy of preference interdependencies between the attribute seems like a fruitful way to proceed.

(6) More on the practical side, the utility concept seems to be illusive and slippery at times. Some of the questions that require answers in this area are: How does the utility over a set of objects develop in the decision maker's mind? If it changes over time, why should it change? And, what about other concepts that refine, specialize, or extend the utility concept?

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This research proposes a new framework for constructing preference functions when the consequences of a decision are judged by multiple attributes or criteria. Methods for assessing single-attribute preference functions have been well established. The current, state-of-the-art procedure for deriving arbitrary, multiattribute preference functions, however, has required regularity assumptions to reduce the arbitrariness of the preferences. This technique, called decomposition, has been used

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) because it usually results in a simple and appealing preference model. The difficulty with this approach is seen in the restrictiveness of the two main assumptions:

(1) the same preference regularity is imposed on all attributes, using this symmetry to achieve a simple preference function, and

(2) any n-dimensional preference function is decomposed into n one-dimensional preference functions.

In this work, no symmetry assumption is neccessary, as each multiattribute preference function is tailor-fit to only those regularities that exist in a particular problem setting. Those parts of the preference function that are subject to simplifying assumptions are decomposed using a new classification scheme to derive further independence assumptions for the standard models. Those parts of the preference function that are indecomposable are handled using a new discretization scheme along with a behaviorally motivated interpolation rule to fill the gaps. The flexibility of these methods allows an analyst to make trade-offs between the degree of accuracy desired and amount of effort needed.

This integrated framework for decomposable and indecomposable multiattribute preference functions stands as an important decision analysis aid. The usefulness of the framework is illustrated in an example of the decision to buy a new car. The relationship of the attributes can be assessed in advance, thus allowing an optimal decision to be made in the decision maker's absence.