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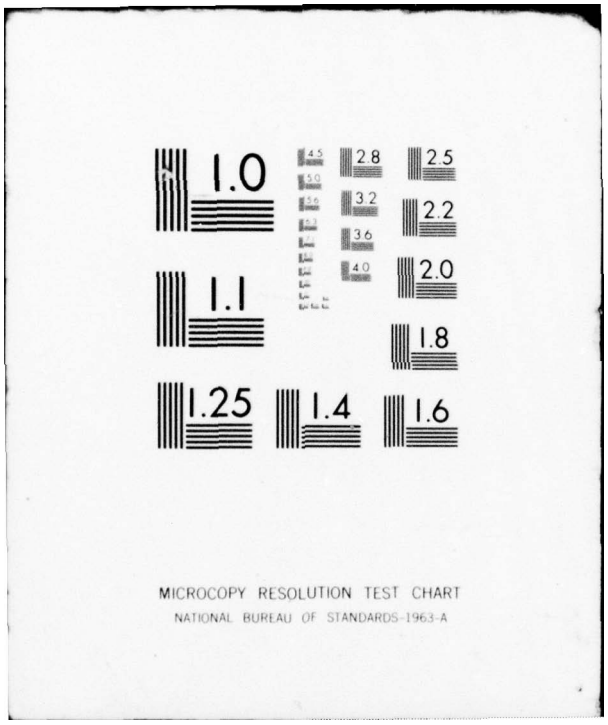
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A GENERAL APPROACH FOR KINEMATIC WAVES

by

Wilson C. Chin

Massachusetts Institute of Technology
Department of Aeronautics and Astronautics
Cambridge, Massachusetts

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Whitham's results for slowly varying wavetrains are generalized to include (i) explicitly the effect of modal dependence, (ii) the effect of low-order linear or nonlinear nonconservative terms and their role in modifying the basic competition between frequency and amplitude dispersion, (iii) the effects of moving media, rotational or irrotational, and space-time inhomogeneities, and (iv) the effect of high-order dispersive or diffusive modifications to the low-order amplitude and phase equations. A "modal law" is also derived which, when integrated over the cross-space, leads to a generalized Whitham action law modified by a source term that is introducted by nonconservative effects; the

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use of complex frequencies here is avoided, thus enabling linear and nonlinear problems to be treated on an equal basis. A self-consistent perturbation method is presented that leads to generalizations and extensions of those ideas due to Whitham, Hayes, Davey, Stewartson, Bretherton and Garrett, Taylor, Landahl, and others, and shows how each of these generalized subsets appears as a specific limit of a broader theory. A number of examples illustrating the basic ideas are presented and which lead to simple formulas that can be used in many direct applications of the general theory. Some qualitative ideas on the basic effect of the derived high-order terms are given, especially with regard to the question of kinematic shocks, and their basic relationships with other high-order theories, for example, Stewartson and Stuart's parabolic diffusion equation for wave amplitude, and a particular nonlinear Schroedinger equation that arises in studies of high-order wave dispersion, are also discussed. Based on the above results, some new ideas on wave stability are presented. One particularly important example discusses Landahl's focusing, trapping and amplification mechanism for waves riding on inhomogeneous flows in light of an analogy with the instability of transonic nozzle flows, and some speculative ideas on kinematic shock formation are given. The discussion essentially parallels the simple arguments originally put forth by Kantrowitz in gas-dynamics but in the wave-mechanical context; however, detailed mathematical analysis of the entire instability mechanism is not pursued in the present paper. Pursued in some detail here, however, is the study of wave-mechanical discontinuities (which are presumed to arise out of some wave-like instability), both with and without the effect of mean flow. Some simple calculations are given for gavity and capillary waves that illustrate the general approach undertaken, and a general theory is also presented that deals with wave-mechanical shock stability as would be influenced by low-order nonlinearities and inhomogeneities in the medium; the theory essentially sets the foundation for a general wave mechanical approach that closely parallels analogies developed from nonlinear acoustics. Other examples treated include simple generalizations of instability theories due to Whitham, Taylor, and others, and a newly discovered (and somewhat speculative) anti-cascading phenomenon arising from the idea of high-order "negative wave diffusion."

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by Wilson C. Chin*

Massachusetts Institute of Technology

Department of Aeronautics and Astronautics

Cambridge, Massachusetts

September 10, 1977

*Present Address: Boeing Commercial Airplane Company, Aerodynamics Research Group, Mail Stop 3N-29, P.O. Box 3707, Seattle, Washington, 98124.

ABSTRACT

Whitham's results for slowly varying wavetrains are generalized to include (i) explicitly the effect of modal dependence, (ii) the effect of low-order linear or nonlinear nonconservative terms and their role in modifying the basic competition between frequency and amplitude dispersion, (iii) the effects of moving media, rotational or irrotational, and space-time inhomogeneities, and (iv) the effect of high-order dispersive or diffusive modifications to the low-order amplitude and phase equations. A "modal law" is also derived which, when integrated over the cross-space, leads to a generalized Whitham action law modified by a source term that is introduced by nonconservative effects; the use of complex frequencies here is avoided, thus enabling linear and nonlinear problems to be treated on an equal basis. A self-consistent perturbation method is presented that leads to generalizations and extensions of those ideas due to Whitham, Hayes, Davey, Stewartson, Bretherton and Garrett, Taylor, Landahl, and others, and shows how each of these generalized subsets appears as a specific limit of a broader theory. A number of examples illustrating the basic ideas are presented and which lead to simple formulas that can be used in many direct applications of the general theory. Some qualitative ideas on the basic effect of the derived high-order terms are given, especially with regard to the question of kinematic shocks, and their

basic relationships with other high-order theories, for example, Stewartson and Stuart's parabolic diffusion equation for wave amplitude, and a particular nonlinear Schroedinger equation that arises in studies of high-order wave dispersion, are also discussed.

Based on the above results, some new ideas on wave stability are presented. One particularly important example discusses Landahl's focusing, trapping and amplification mechanism for waves riding on inhomogeneous flows in light of an analogy with the instability of transonic nozzle flows, and some speculative ideas on kinematic shock formation are given. The discussion essentially parallels the simple arguments originally put forth by Kantrowitz in gas-dynamics but in the wave-mechanical context; however, detailed mathematical analysis of the entire instability mechanism is not pursued in the present paper. Pursued in some detail here, however, is the study of wave-mechanical discontinuities (which are presumed to arise out of some wave-like instability), both with and without the effect of mean flow. Some simple calculations are given for gravity and capillary waves that illustrate the general approach undertaken, and a general theory is also presented that deals with wave-mechanical shock stability as would be influenced by low-order nonlinearities and inhomogeneities in the medium; the theory essentially sets the foundation for a general wave-mechanical approach that closely parallels analogies developed from nonlinear acoustics. Other examples

treated include simple generalizations of instability theories due to Whitham, Taylor, and others, and a newly discovered (and somewhat speculative) anti-cascading phenomenon arising from the idea of high-order "negative wave diffusion".

I. Introduction.

Perhaps the most exciting area in applied mathematics research today is in the field of nonlinear wave mechanics and its application to the wave stability of continuum systems. The greatest recent advances can be found in the mid-1960's, as we shall see, and many of the initial advances and applications were spearheaded by Whitham and his coworkers. The subject, as it stands today, is founded on the contributions of many investigators, among them Hayes, Stewartson, Stuart, Davey, Bretherton and Garrett, Landahl, Lighthill and others; the cumulative body of knowledge developed by these authors is capable of handling low-order frequency and amplitude dispersion, modal effects, wave damping, high-order wave diffusion and dispersion, and other simultaneous effects. However, it was often the case that various of the particular theories were devised specifically to deal with particular problems and their extension to include higher-order effects and additional physical features could not be made. The broad generality of an idea could sometimes not be readily perceived, this situation leading to many rather specialized and sometimes conflicting

theories. The basic purpose of the present paper is to reconsider anew the general problem of describing a slowly varying wavetrain by constructing a broader unifying theory that bears as few limiting assumptions as the physics permits and which reduces, in specific limits, to those theories referred to above.

Numerous applications of more or less "standard" nonlinear techniques (e.g., multiple scaling methods, averaging methods, etc.) have been made in recent years to problems involving wave motions in continuous media and an extensive survey of these studies appears in Chin (1976). These applications are mainly based on several fundamental papers and we will here summarize and discuss the basic ideas and theories. Probably the first application of WKB averaging in fluid mechanics appears in Whitham (1962), in which (for finite depth gravity waves) the fluid-dynamical conservation laws were phase-averaged so as to produce the modulation equations governing coupled large-scale changes in the wave and mean flow variables. In this averaging the Ansatz consisted of the related uniform wave solution, and one additional postulate, that of wave conservation, was introduced. This averaging method was later used in Whitham (1965a) to find slowly varying wave solutions of nonlinear equations in general and there the analogous role of the basic uniform solution as compared to that of the sinusoidal wave in linear stationary phase theory was advanced. Whitham (1965b) in a further development showed how the same modulation

equations for a system of given equations could be derived by evaluating the related Lagrangian density with the uniform wave solution (the slowly varying solution is assumed to behave locally like the planar wave), phase-averaging, and then taking appropriate variations with respect to the wave and mean flow parameters (which, during the averaging, are held constant). The power of the method is seen in its simplicity. For example, in its application to finite-depth gravity waves (Whitham, 1967) the averaging reproduced in a simple way the concepts of "radiation stress" and the results of Longuet-Higgins and Stewart (1961) obtained on the basis of detailed asymptotic analysis. Whitham also showed how, for sufficiently deep water, the competition between amplitude and frequency dispersion led to an ill-posed initial value problem which in a sense implied instability: the modulation equations were elliptic. For shallow water the governing equations were hyperbolic, and implied a "splitting" of the basic wave envelope. These ideas are quite general and apply as well to other applications of the Whitham theory. We add, in summarizing this paragraph, that Whitham's results and his "average Lagrangian method" can be derived and justified by multiple-scaling techniques, for example, as discussed in Luke (1966) and in Whitham (1970).

While Whitham's variational theory was successful in describing both frequency and amplitude dispersion, it was not able to incorporate the effects of weakly nonconservative terms, such as wave damping. This failure stemmed from the lack of a suitable variational formulation and thus precluded the

application of Whitham's ideas to many practical problems (some recent modifications of the theory (Jimenez and Whitham, 1976) now permit the treatment of such effects, but the description is confined to low-order changes and does not appear to be as general as the theory to be presented). Moreover, Whitham's method could not be generalized so as to produce the high-order dispersive terms obtained by other investigators using direct multiple-scaling methods. For example, a WKB technique applied to the study of slow modulations of a Stokes wavetrain showed that higher order terms representing modulation rates and not nonlinearity must be added to extend the validity of Whitham's equations (Chu and Mei, 1970). The same ideas were reemphasized by Davey (1972), who suggested some heuristic modifications to Whitham's theory, and by Davey and Stewartson (1974), who considered three-dimensional wave packets in water wave applications. The apparent discrepancies in the two approaches could not be found, as noted in Hayes (1973). For example, direct multiple-scaling methods led to a nonlinear Schroedinger equation for a certain complex wave amplitude, and this high-order equation embodies Whitham's low-order results (centered wavenumber expansions, however, are required in its usual derivation, so that the competing effects between amplitude and frequency dispersion are not accounted for). Furthermore we note that Whitham's method, as it presently stands, does not describe high-order diffusive effects of the kind described in Stewartson and Stuart (1971), which are certainly important in any large-time description. Thus one has in addition to low-order frequency and amplitude dispersive

terms, those terms responsible for the high-order competition between wave dispersion and wave diffusion. (Recently Yuen and Lake (1975) showed how the average Lagrangian method could be modified so as to produce the required high-order dispersive corrections, however, but certain implied assumptions are made in their analysis with regard to the relative rates of change between wavenumber and wave amplitude.) A formalism is presented in the present paper that generalizes Whitham's theory to include high-order dispersive and diffusive effects and which extends Landahl's (1972) modification of the basic action equation for linear nonconservative effects to fully nonlinear ones.

Two additional references can be cited in the context of the present discussion. Bretherton (1969) was able to show how, for linear, nondissipative problems, the phase-averaged Lagrangian density integrated over modal cross-space, satisfies Whitham's action law, thus extending its applicability to wave motions in waveguides. Under the stated restrictions Bretherton's theory was extendable to higher order, and in principle, the high-order dispersive corrections as previously discussed are reproducible. Bretherton's low-order <u>modal</u> results were generalized to include nonlinear effects by Hayes (1970b), who in addition postulated a local conservation law in the cross-space divergence operator. However, in addition to its restriction to purely conservative problems, Hayes' formalism relied on certain "well-known" variational

identities which cannot be extended to systems of arbitrary order. In the analysis to be presented, both Bretherton's and Hayes' generalizations of Whitham's ideas too are extended so that they appear as specific subsets of a broader unifying theory. The mathematical approach taken considers a general variational principle of arbitrary order in which the explicit presence of modal, propagation space and time coordinates is accounted for, in addition to that of the work function describing general nonconservative effects. Slowly varying wave solutions are sought in the form of a Luke (1966) transformation modified for modal dependence, and generalized phase and amplitude equations are derived from existence and secularity arguments. A number of specific applications are made, leading to a number of formulas useful in direct applications of the general theory, and in the case of weakly nonlinear waves, the structure of the high-order dispersive and diffusive terms is determined from a pseudo-analytic continuation method. This general theory describing the dynamics of kinematic waves completes the first part of the paper, and in the second part, some new ideas in hydrodynamic (and general wave) stability are discussed. The physical bases behind some of these ideas are discussed in detail using various analogies from gasdynamics, but because of obvious difficulties, detailed calculations are not pursued in the present paper. Some of the ideas, admittedly, are speculative, but it is the author's hope that the discussions presented

herein stimulate further investigation and study. In a sense, the second part of the present paper deals with novel ideas that hopefully would be received with the same enthusiasm as were Whitham's (1967) ideas on "elliptic instability" and wave group "splitting".

The analysis that follows essentially summarizes the main results of a doctoral thesis by the present author (Chin, 1976) presented at the Massachusetts Institute of Technology. Further papers are planned and the reader is urged to refer to that work for details of ideas and calculations briefly discussed here. Among the subjects planned for future discussion are theories detailing the wave back-interaction on inviscid shear flows, the structure of foci that result from wave trapping due to inhomogeneities, and other aspects of wave stability.

II. The General Theory. Low-Order Results.

Let us suppose that the physical system under consideration is derivable from the variational principle as given in Eq. (1), with Lagrangian density L and dissipation functional F. We consider variational principles (as opposed to specific equations) so that results common to all systems can be obtained. Also let the small parameter ε characterize an appropriate slow variation which, in addition, is proportional to the weak influence of nonconservative effects.

$$e_{\alpha,\beta,\gamma} \frac{\partial^{\alpha+\beta+\gamma}}{\partial x^{\alpha} \partial t^{\beta} \partial y^{\gamma}} \tilde{L} + L^{*} = \varepsilon F(u,\ldots) \tag{1}$$

$$L = L(\varepsilon x, y, \varepsilon t, u, u_{\alpha x, \beta t, \gamma y})$$

$$\tilde{L} = L_{u_{\alpha x, \beta t, \gamma y}} \qquad L^{*} = L_{u} \qquad e_{\alpha,\beta,\gamma} = (-1)^{\alpha+\beta+\gamma}$$

In Eq. (1) sums over various α, γ, β's are understood, x and y being the propagation and modal coordinates, and t being the time. The symbols α, β, γ indicate the respective order of partial differentiations over x, t and y. Further, it is assumed that for F = 0 the system admits propagating waves. We now attempt an asymptotic solution in the form

$$u(x,y,t) = U(\theta, X, T, y; \varepsilon) \tag{2}$$

$$X = \varepsilon x \qquad T = \varepsilon t \qquad \theta(x,t) = \varepsilon^{-1}\Theta(X,T)$$

$$\omega(X,T) = -\theta_t = -\Theta_T = -\nu(X,T) \qquad K(X,T) = \theta_x = \Theta_X$$

In the above, X and T are stretched space-time coordinates chosen to account for O(1) variations in U over large scales, θ is the phase function for periodic behaviour, and K and ω are wavenumber and frequency satisfying the condition of wave conservation. We shall neglect any exponentially small reflections and, in addition, for brevity neglect high-order scaling variables and multiple-phases (which, as will be seen, can be easily incorporated into the present approach). The key to a general analysis is the definition of the operator H in the space θ, X, T

$$H = \nu^{\beta} K^{\alpha} \frac{\partial^{\alpha+\beta}}{\partial\theta^{\alpha+\beta}} + \varepsilon\alpha\nu^{\beta} K^{\alpha-1} \frac{\partial^{\alpha+\beta}}{\partial\theta^{\alpha+\beta-1}\partial X} \tag{3}$$

$$+ \tfrac{1}{2}\varepsilon\alpha(\alpha-1)\nu^{\beta} K^{\alpha-2} K_X \frac{\partial^{\alpha+\beta-1}}{\partial\theta^{\alpha+\beta-1}} + \varepsilon\alpha\beta\nu^{\beta-1} K^{\alpha-1} K_T \frac{\partial^{\alpha+\beta-1}}{\partial\theta^{\alpha+\beta-1}}$$

$$+ \tfrac{1}{2}\varepsilon\beta(\beta-1)\nu^{\beta-2}\nu_T K^{\alpha} \frac{\partial^{\alpha+\beta-1}}{\partial\theta^{\alpha+\beta-1}} + \varepsilon\beta\nu^{\beta-1} K^{\alpha} \frac{\partial^{\alpha+\beta}}{\partial\theta^{\alpha+\beta-1}\partial T}$$

Thus, neglecting terms of $O(\varepsilon^2)$, the Euler equation specified in Eq. (1) can be rewritten as

$$e_{\alpha,\beta,\gamma} \frac{\partial^{\gamma}}{\partial y^{\gamma}} H(\tilde{L}) + L^{*} = \varepsilon F(U,\ldots) \tag{4}$$

For reasons which will later be apparent, we express Eq. (4) in the alternative form

$$e_{\alpha,\beta,\gamma} U_{\theta} \frac{\partial^{\gamma}}{\partial y^{\gamma}} H(\tilde{L}) - (\tilde{L})\left(\frac{\partial^{\gamma}}{\partial y^{\gamma}} H(U)\right)_{\theta} + \frac{\partial L}{\partial \theta} = \varepsilon U_{\theta} F(U) \tag{5}$$

where L* has been eliminated using the identity

$$\frac{\partial L}{\partial \theta} = L^{*} U_{\theta} + \tilde{L}\left(\frac{\partial^{\gamma}}{\partial y^{\gamma}} H(U)\right)_{\theta} \tag{6}$$

This form of the Euler equation leads to a convenient derivation for a "generalized action law" which, as will be seen, follows as a necessary condition for the existence of assumed asymptotic solutions in the form given in Eq. (2). Now, twice application of the differential identity

$$f^{(n)} g = \left[\sum_{K=1}^{n} (-1)^{K+1} f^{(n-K)} g^{(K-1)}\right]' + (-1)^{n} f g^{(n)} \tag{7}$$

to Eq. (5), first with primes denoting θ derivatives (where $g = U_{\theta}$ and $f = \tilde{L}_{yy}$) and second, with primes denoting y-derivatives leads directly to

$$\frac{\partial}{\partial\theta}\left(L+e_{\alpha,\beta,\gamma}\Phi\right) = \varepsilon U_\theta F(U) \tag{8}$$

$$+\varepsilon\left\{\begin{array}{l}\beta K^\alpha \nu^{\beta-1}\tilde{L}_T U_{(\alpha+\beta)\theta,\gamma y} + \beta(\beta-1)K^\alpha\nu^{\beta-2}\nu_T\tilde{L}U_{(\alpha+\beta)\theta,\gamma y}\\ +\beta K^\alpha\nu^{\beta-1}\tilde{L}U_{\gamma y,(\alpha+\beta)\theta,T} + \alpha\beta K^{\alpha-1}K_T\nu^{\beta-1}\tilde{L}U_{(\alpha+\beta)\theta,\gamma y}\end{array}\right\}$$

$$+\varepsilon\left\{\begin{array}{l}\alpha K^{\alpha-1}\nu^\beta\tilde{L}_X U_{(\alpha+\beta)\theta,\gamma y} + \alpha(\alpha-1)K^{\alpha-2}K_X\nu^\beta\tilde{L}U_{(\alpha+\beta)\theta,\gamma y}\\ +\alpha K^{\alpha-1}\nu^\beta\tilde{L}U_{\gamma y,(\alpha+\beta)\theta,X} + \alpha\beta K^{\alpha-1}\nu^{\beta-1}\nu_X\tilde{L}U_{(\alpha+\beta)\theta,\gamma y}\end{array}\right\}$$

$$+(-1)^\gamma\frac{\partial}{\partial y}\left\{\begin{array}{l}-K^\alpha\nu^\beta\{\tilde{L}_{(\gamma-1)y}U_{(\alpha+\beta+1)\theta} - \tilde{L}_{(\gamma-2)y}U_{(\alpha+\beta+1)\theta,y}+\dots\}\\ +\varepsilon\beta K^\alpha\nu^{\beta-1}\{\tilde{L}_{T,(\gamma-1)y}U_{(\alpha+\beta)\theta} - \tilde{L}_{T,(\gamma-2)y}U_{(\alpha+\beta)\theta,y}+\dots\}\\ +\varepsilon\alpha K^{\alpha-1}\nu^\beta\{\tilde{L}_{X,(\gamma-1)y}U_{(\alpha+\beta)\theta} - \tilde{L}_{X,(\gamma-2)y}U_{(\alpha+\beta)\theta,y}+\dots\}\\ +\varepsilon\{\tilde{L}_{(\gamma-1)y}U_{(\alpha+\beta)\theta} - \tilde{L}_{(\gamma-2)y}U_{(\alpha+\beta)\theta,y}+\dots\}\ X\\ \left\{\frac{\alpha(\alpha-1)}{2}K^{\alpha-2}K_X\nu^\beta + \frac{\beta(\beta-1)}{2}K^\alpha\nu^{\beta-2}\nu_T + \alpha\beta K^{\alpha-1}K_T\nu^{\beta-1}\right\}\end{array}\right\}$$

where

$$\Phi = K^\alpha\nu^\beta\{\tilde{L}_{\gamma y,(\alpha+\beta-1)\theta}U_\theta - \tilde{L}_{\gamma y,(\alpha+\beta-2)\theta}U_{\theta\theta}+\dots\} \tag{9}$$

$$+\varepsilon\alpha K^{\alpha-1}\nu^\beta\{\tilde{L}_{\gamma y,(\alpha+\beta-2)\theta,X}U_\theta - \tilde{L}_{\gamma y,(\alpha+\beta-3)\theta,X}U_{\theta\theta}+\dots\}$$

$$+\varepsilon\beta K^\alpha\nu^{\beta-1}\{\tilde{L}_{\gamma y,(\alpha+\beta-2)\theta,T}U_\theta - \tilde{L}_{\gamma y,(\alpha+\beta-3)\theta,T}U_{\theta\theta}+\dots\}$$

$$+\varepsilon\left\{\frac{\alpha(\alpha-1)}{2}K^{\alpha-2}K_X\nu^\beta + \alpha\beta K^{\alpha-1}K_T\nu^{\beta-1} + \frac{\beta(\beta-1)}{2}K^\alpha\nu^{\beta-2}\nu_T\right\}\ X$$

$$\{\tilde{L}_{\gamma y,(\alpha+\beta-2)\theta}U_\theta - \tilde{L}_{\gamma y,(\alpha+\beta-3)\theta}U_{\theta\theta}+\dots\}$$

We now expand the variables L, $\tilde{L}$, U and Φ in series like

$$\psi = \sum_{n=0}^{\infty} \varepsilon^n \psi^{(n)}(\theta, X, T) \tag{10}$$

and introduce the resulting expressions in Eq. (8). By equating coefficients in the varying powers of ε, the leading terms give

$$-\frac{\partial}{\partial\theta}\left(L^{(0)} + e_{\alpha,\beta,\gamma}\Phi^{(0)}\right) = \frac{\partial}{\partial y} K^{\alpha} \nu^{\beta} (-1)^{\gamma} \left\{ \tilde{L}^{(0)}_{(\gamma-1)y} U^{(0)}_{(\alpha+\beta+1)\theta} - \tilde{L}^{(0)}_{(\gamma-2)y} U^{(0)}_{(\alpha+\beta+1)\theta,y} + \ldots \right\} \tag{11}$$

Averaging over phase, we obtain the "cross-space conservation law"

$$\frac{\partial}{\partial y}\left[\frac{1}{2\pi}\int_0^{2\pi} (-1)^{\gamma} K^{\alpha} \nu^{\beta} \left\{ \tilde{L}^{(0)}_{(\gamma-1)y} U^{(0)}_{(\alpha+\beta+1)\theta} - \tilde{L}^{(0)}_{(\gamma-2)y} U^{(0)}_{(\alpha+\beta+1)\theta,y} + \ldots \right\} d\theta \right] = 0 \tag{12}$$

a generalization of Hayes' (1970) result. The bracketed quantity above can be interpreted as an energy-like density that varies slowly in time t and space x. Since $\tilde{L}$ and U both depend on x and t, Eq. (12) can be thought of as a kind of modal energy redistribution law, the exact nature of which will depend on the dynamics of the propagating wave. Expansion of Eq. (8) and retention of $O(\varepsilon)$ terms gives

$$\frac{\partial}{\partial\theta}\left(L^{(1)}+e_{\alpha,\beta,\gamma}\Phi^{(1)}\right) = U_\theta^{(0)} F(U^{(0)}) \tag{13}$$

$$+\left\{\begin{array}{l}\beta K^\alpha v^{\beta-1}\tilde{L}_\tau^{(0)} U_{(\alpha+\beta)\theta,\gamma y}^{(0)} + \beta K^\alpha v^{\beta-1}\tilde{L}^{(0)} U_{\gamma y,(\alpha+\beta)\theta,\tau}^{(0)} \\ +\beta(\beta-1)K^\alpha v^{\beta-2}v_\tau \tilde{L}^{(0)} U_{(\alpha+\beta)\theta,\gamma y}^{(0)} + \alpha\beta K^{\alpha-1}K_\tau v^{\beta-1}\tilde{L}^{(0)} U_{(\alpha+\beta)\theta,\gamma y}^{(0)}\end{array}\right\}$$

$$+\left\{\begin{array}{l}\alpha K^{\alpha-1}v^\beta \tilde{L}_X^{(0)} U_{(\alpha+\beta)\theta,\gamma y}^{(0)} + \alpha K^{\alpha-1}v^\beta \tilde{L}^{(0)} U_{\gamma y,(\alpha+\beta)\theta,X}^{(0)} \\ +\alpha(\alpha-1)K^{\alpha-2}K_X v^\beta \tilde{L}^{(0)} U_{(\alpha+\beta)\theta,\gamma y}^{(0)} + \alpha\beta K^{\alpha-1}v^{\beta-1}v_X \tilde{L}^{(0)} U_{(\alpha+\beta)\theta,\gamma y}^{(0)}\end{array}\right\}$$

$$+(-1)^\gamma \frac{\partial}{\partial y}\left\{\begin{array}{l}\beta K^\alpha v^{\beta-1}\left\{\tilde{L}_{\tau,(\gamma-1)y}^{(0)} U_{(\alpha+\beta)\theta}^{(0)} - \tilde{L}_{\tau,(\gamma-2)y}^{(0)} U_{(\alpha+\beta)\theta,y}^{(0)} + \cdots\right\} \\ +\alpha K^{\alpha-1}v^\beta\left\{\tilde{L}_{X,(\gamma-1)y}^{(0)} U_{(\alpha+\beta)\theta}^{(0)} - \tilde{L}_{X,(\gamma-2)y}^{(0)} U_{(\alpha+\beta)\theta,y}^{(0)} + \cdots\right\} \\ +\left\{\frac{\alpha(\alpha-1)}{2}K^{\alpha-2}K_X v^\beta + \frac{\beta(\beta-1)}{2}K^\alpha v^{\beta-2}v_\tau + \alpha\beta K^{\alpha-1}K_\tau v^{\beta-1}\right\} X \\ \qquad \left\{\tilde{L}_{(\gamma-1)y}^{(0)} U_{(\alpha+\beta)\theta}^{(0)} - \tilde{L}_{(\gamma-2)y}^{(0)} U_{(\alpha+\beta)\theta,y}^{(0)} + \cdots\right\} \\ -K^\alpha v^\beta\left\{\begin{array}{l}\left(\tilde{L}_{(\gamma-1)y}^{(0)} U_{(\alpha+\beta+1)\theta}^{(1)} + \tilde{L}_{(\gamma-1)y}^{(1)} U_{(\alpha+\beta+1)\theta}^{(0)}\right) \\ -\left(\tilde{L}_{(\gamma-2)y}^{(0)} U_{(\alpha+\beta+1)\theta,y}^{(1)} + \tilde{L}_{(\gamma-2)y}^{(1)} U_{(\alpha+\beta+1)\theta,y}^{(0)}\right) + \cdots\end{array}\right\}\end{array}\right\}$$

In the above the bracketed quantities are arranged so that the first contains time derivatives only and the second contains propagation space derivatives. By introducing an "average Lagrangian" with the definition

$$\bar{L} = \frac{1}{2\pi}\int_0^{2\pi} L(X,T,y,U,H(U_{\gamma y}))\, d\theta \tag{14}$$

Eq. (13) takes on a particularly simple form on phase and cross-space averaging. Assuming zero work interaction at the boundary (y_1, y_2), we obtain

$$\frac{\partial}{\partial T}\mathcal{L}_\omega^{(0)} - \frac{\partial}{\partial X}\mathcal{L}_K^{(0)} = \frac{1}{2\pi}\int_0^{2\pi}\int_{y_1}^{y_2} U_\theta^{(0)} F(U^{(0)})\, dy\, d\theta \tag{15}$$

where we have denoted

$$\mathcal{L} = \int_{y_1}^{y_2} \bar{L}\, dy \tag{16}$$

It follows that Eqs. (11) - (15) are the basic governing laws when solutions of Eq. (1) are sought in the form expressed by Eq. (2). These results are easily generalized to higher dimensions by taking $(\ )_y \to \nabla_{\vec{y}}$, $(\ )_x \to \nabla_{\vec{x}}$, and introducing the wave irrotationality requirement that $\nabla_{\vec{x}} \times \vec{K} = \vec{0}$ in addition to $\vec{K}_t + \nabla_{\vec{x}}\omega = 0$. Eq. (15) is precisely Whitham's action law when F = 0 and y-dependences are suppressed. For such "local" waves, Whitham (1970) has shown (using $L = L(u, u_t, u_x)$) that wave action conservation is "absolute", i.e., correct to all orders in ε. On this basis, Eq. (15) would be true with superscripts removed. Quite often this "absoluteness" has been taken for granted for systems of arbitrary order. But this move is not altogether clear, and

for this reason, "local" waves are reexamined here.

If in Eq. (1) y-dependences are suppressed by setting $\gamma = 0$, a tedious but straightforward extension of the above analysis shows that, in general,

$$\frac{\partial}{\partial T}\bar{L}_{\omega} + \frac{\partial}{\partial X}\bar{L}_{K} - \frac{\partial^2}{\partial T^2}\bar{L}_{\omega_T} - \frac{\partial^2}{\partial X \partial T}\bar{L}_{K_T} - \frac{\partial^2}{\partial X^2}\bar{L}_{K_X} + \cdots = -\frac{1}{2\pi}\int_0^{2\pi} U_\theta F(U)\, d\theta \qquad (17)$$

Thus, wave action conservation is not "absolute" for systems of arbitrary order. However, when the "order" of the system is two, it is, because the derivatives $\bar{L}_{\omega_T}$, $\bar{L}_{K_T}$, etc., vanish identically. In fact if we denote by A the wave action $\bar{L}_{\omega}$ in Eq. (17), one can demonstrate the existence of high-order dispersive corrections like A_{XXX} that modify the low-order action equation of Whitham's theory, and for dissipative systems, high-order Burgers' type diffusive terms that involve A_{XX} (this will be shown later in Section III). The above equation can also be obtained in a more powerful way by extending some of Whitham's ideas. Let us consider the three-variable variational principle

$$\delta \iiint_R \int_0^{2\pi} L(X,T,U,H(U))\, d\theta\, dX\, dT - \varepsilon \iiint_R \int_0^{2\pi} F(U)\, \delta U\, d\theta\, dX\, dT = 0 \qquad (18)$$

for the three-variable function $U(\theta, X, T)$. Here U and its

variations are assumed to be periodic in θ, and variations in U vanish on the boundary R. Variation of U gives, after some manipulation, the two-timed Euler equation, Eq. (4) with $\gamma = 0$. Thus, the two-timed form of the Euler equation, Eq. (1), is precisely the Euler equation obtained from the two-timed variational principle. It follows that Eq. (18) is exact because it contains the whole expansion and, in addition, it is already the averaged variational principle. We now introduce an amplitude measure "a" so that the solution for $U(\theta, X, T; \varepsilon)$ depends explicitly on the two parameters "a" and θ. Then, since

$$\delta \iint_R \bar{L}\, dX dT = \varepsilon \iint_R \left\{ \frac{1}{2\pi} \int_0^{2\pi} F(U)(U_\theta \delta\theta + U_a \delta a)\, d\theta \right\} dX dT \tag{19}$$

must hold, "a" variations lead to

$$\bar{L}_a - \frac{\partial}{\partial T}\bar{L}_{a_T} - \frac{\partial}{\partial X}\bar{L}_{a_X} + \ldots - \varepsilon \frac{1}{2\pi}\int_0^{2\pi} U_a F(U)\, d\theta = 0 \tag{20}$$

and θ variations lead to Eq. (17). Equations (17)-(20) are then the required "generalized action and phase relations". For low-order solutions the choice $U(\theta, X, T, \varepsilon) = U^{(o)}(a, \theta)$ where $U^{(o)}(a, \theta)$ is the uniform wave solution to Eq. (1) suffices, as shown by Whitham (1970). But for proper account of high-order effects, it is in general necessary to assume

$$U(\theta,X,T;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n U^{(n)}(a, a_T, a_X, \ldots, \theta, \omega, K, \ldots) \qquad (21)$$

in view of the work by Yuen and Lake (1975). This point will be discussed more fully in Section III. The proper choices for the functions $U^{(n)}$ are, in general, unclear. This difficulty is in addition compounded by the fact that variational principles are not unique - a concern that does not arise in the low-order problem because the phase and action equations are unaffected. If Eq. (21) is chosen correctly, though, the analysis carried to the appropriate level of approximation <u>does</u> produce high-order dispersive terms (as are modelled in recent studies by a certain nonlinear Schroedinger equation) and, in addition, high-order diffusion terms that can be important near kinematic shocks. Now to leading order, the evolutionary equations are

$$-\frac{\partial}{\partial T}\bar{L}_\omega^{(0)} - \frac{\partial}{\partial X}\bar{L}_K^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} U_\theta^{(0)} F(U^{(0)})\, d\theta \equiv \mathcal{F} \qquad (22)$$

$$\bar{L}_a^{(0)} = 0 \qquad U(\theta,X,T;\varepsilon) = U^{(0)}(a,\theta)$$

$$\frac{\partial K}{\partial T} + \frac{\partial \omega}{\partial X} = 0$$

and it is known (Whitham, 1967) that the physical properties of the solution vary markedly according to the type of the governing system. For example, Benjamin-Feir or Whitham-type modulational

instabilities are encountered for elliptic systems. For hyperbolic systems, the perturbation "splits" and propagates at two different speeds. And in the linear (parabolic) case the modulation equations uncouple, and the wavenumber field is determined independently of the wave amplitude. In this case, Eq. (22) shows that the concept of group velocity is still relevant to weakly nonconservative processes. It is still defined by

$$C = -\bar{L}^{(0)}_K / \bar{L}^{(0)}_\omega = \partial\omega/\partial K \tag{23}$$

since the phase relation can be written in the form $L^{(0)} = a^2 \mathcal{L}(\omega, K, X, T) = 0$. To this order, the kinematics are independent of the functional F, agreeing with the fact weakly nonconservative effects affect the phase only to $O(\varepsilon^2)$. (A more explicit formula showing the general nature of high-order dispersive and diffusive corrections to Eq. (22) will be given later in Section III.) Wave action is still defined, and again, in terms of the associated conservative system,

$$A = \bar{L}^{(0)}_\omega \tag{24}$$

but it is dissipated, in general, with strength $\overline{U_\theta F}$. The rays

are still determined by the "real group velocity", where we have avoided the inconsistencies of the complex-frequency approach. These results apply to fully nonlinear waves as well.

In practice it is sufficient to choose for $U^{(o)}$ the uniform wave solution to the associated conservative system. However the presence of the right-side integral of Eq. (22) in nonlinear systems may possibly alter the structure of the fundamental conservative solution in a significant way. This point is especially important in bifurcational and transitional studies. We may also note that by considering $\partial L/\partial T$ and $\partial L/\partial X$ instead of $\partial L/\partial\theta$ in Eq. (6), one obtains for local or modal waves the following leading order results,

$$\frac{\partial}{\partial T}\left(\omega \bar{L}_{\omega}^{(o)} - \bar{L}^{(o)}\right) - \frac{\partial}{\partial X}\left(\omega \bar{L}_{K}^{(o)}\right) = -\bar{L}_{T}^{(o)} + \omega \overline{U_{\theta}^{(o)} F(U^{(o)})} \tag{25}$$

$$-\frac{\partial}{\partial T}\left(K \bar{L}_{\omega}^{(o)}\right) - \frac{\partial}{\partial X}\left(K \bar{L}_{K}^{(o)} - \bar{L}^{(o)}\right) = \bar{L}_{X}^{(o)} + K \overline{U_{\theta}^{(o)} F(U^{(o)})} \tag{26}$$

These identify U_θ as the appropriate weighting function for F. The above equations are equivalent to Eq. (22), except that different energy-like norms are now examined along different rays. As for Eq. (22), the superscript zeroes may be removed for second order systems, but for high-order systems, additional terms must be incorporated. If we identify Eq. (1) with Hamilton's principle, we see that the above represent energy

and momentum laws respectively. This follows for, in a mathematical sense, energy is that quantity conserved when the corresponding variational principle is invariant with respect to translations in time. The conservation laws for momentum and action, for example, follow from considering translations in space and in phase. The right sides of Eqs. (25) and (26) show explicitly the role of inhomogeniety and dissipation. For example, the inhomogeniety can be prescribed (as would be the case in linear theory) or can be the result of wave-induced modifications to a given mean state. In linear theory the velocities for action, momentum, energy and wavenumber are identical, of course, since the average Lagrangian is homogeneous in a^2.

It is important to recapitulate several ideas fundamental to the above analysis. The first is that no assumptions have been made regarding the orders of the conservative and nonconservative operators in Eq. (1), the variability of their coefficients, or their linearity or nonlinearity. Secondly, the general expansions introduced in Eq. (2) include a strong modal dependence, as opposed to those used by Luke (*1966*) or Whitham (1970). One observes that when Eq. (10) is introduced in Eq. (8) with $\gamma = 0$, the left side of the resulting $O(\varepsilon)$ equation contains functions of both $U^{(0)}$ and $U^{(1)}$ while the right side contains $U^{(0)}$ exclusively. A solution uniformly valid in θ requires U, and hence each $U^{(j)}$ to be periodic in θ with

period 2π. This uniform validity is possible only if the integral of the right side with respect to θ over a period is zero. This, together with the definition given in Eq. (14), leads to the action law (Eq. (15)) directly as straightforward manipulations show. In other words, wave action conservation follows as a secular condition (in the sense of Cole (1968)) when solutions of Eq. (1) are sought in the form specified by Eq. (2). For the treatment of modal effects and high-order corrections, secularity alone is not sufficient, but phase averages can still be conveniently carried out via the two-timing introduced in Eq. (2).

It is crucial that Eq. (22) holds for equations of any order, but that the order of the system becomes important only when higher order effects are to be calculated. For second-order systems, however, Eq. (22) is exact because derivatives of frequency and wavenumber vanish identically, and superscripts may be dropped. Thus, Whitham's (1970) comments on the absoluteness of action conservation (in nondissipative media) apply only to these degenerate cases. Action conservation here turns out to be the appropriate adiabatic invariant for linear or nonlinear wave propagation in inhomogeneous dissipative systems. In nonlinear processes, this is especially important because energy cannot play too fundamental a role - it is easily transferred between component frequencies. For solutions to practical problems it is necessary to amend Whitham's (1970)

algorithm. In linear problems, for example, the uniform wave solution $U^{(o)} = a\sin\theta$ led to an average Lagrangian homogeneous in a^2 ; this is true only in the lowest approximation. Since "a" is really a function of the slow variables X and T, $O(\varepsilon^2)$ corrections to the average Lagrangian must be incorporated that will involve space-time derivatives of wavenumber and amplitude, $O(\varepsilon)$ effects having vanished by orthogonality. The existence of such high-order terms suggests a new treatment for "kinematic shocks". There thus exist high-order diffusion terms that can possibly smooth out discontinuities that appear in considering the low-order solution alone or, for example, high-order dispersive terms that may suppress shock formation. The dispersive corrections mentioned here (as will be pointed out more clearly later) resolve the inconsistencies pointed out by Chu and Mei (1970) regarding the apparent loss of certain high-order modulation terms in the Lagrangian formulation by showing that such terms <u>are</u> accounted for by the variational method. It is only that through the above inconsistency that the discrepancy arises.

Our approach also bypasses the conventional use of complex frequencies in treating dissipative problems. For example, it is <u>not</u> necessary to determine the group velocity from the real part of a complex frequency, and then, to introduce heuristically into the energy equation a damping based on the imaginary frequency. In many nonlinear problems where the use of complex

frequencies is inappropriate, Eq. (22) seems to provide the only practical alternative. Both local and modal waves, of course, were considered under the same formalism. There is still another interesting point to be noted. For nonlinear waves we can eliminate amplitude from the argument space of $\mathcal{L}_k$ and $\mathcal{L}_\omega$ using the dispersion relation. The action equation then becomes an eikonal one, from which an equation for a perturbation phase could be developed. For this system it turns out that boundary conditions cannot be satisfied if spatial growth is not assumed, for initial values applied at $x = 0$. On the other hand, initial values applied at $t = 0$ require temporal growth. This extends in a simple way Gaster's (1965) result that Orr-Sommerfeld waves (generated from a vibrating ribbon fixed in space) on a boundary layer grow spatially. It also reduces to Taylor's (1962) results for temporally growing waves.

In applying Eq. (15) to specific problems, the choice of the basic solution (or Ansatz) is crucial. From the linear results to be presented (Example 1), the uniform solution of the associated conservative system appears to suffice as a general rule because the dissipation is weak. This model can be extended to nonlinear and modal problems as well. For example, if Luke's (1966) nonlinear Klein-Gordon equation were modified to include dissipation, one would evaluate Eq. (15) by the uniform conservative solution he obtained. Finally

note how the conserved density in Eq. (12) takes on a much more complex structure than the analysis of Hayes (1970) would suggest, but reduces to his in the appropriate limit. This comment applies to the modal law Eq. (13) as well, noting that neither Bretherton's (1969) nor Hayes' theories are valid for dissipative wave flows. The evolutionary equations derived have a two-fold use. First, they can be used to describe the mathematical properties of wave-like solutions to specific equations, and this is the usual application. The second is to make certain physical assumptions about the functional form of the Lagrangian, for example, that it represents Hamilton's principle, and then, obtain alternative descriptions for physical conservation laws. This approach is pursued later in an example. One of the aims of the present paper is a systematic re-derivation and extension of both existing high and low-order theories of wave motion from one general, self-consistent approach. Before considering in detail the high-order theory, we discuss several applications of the low-order theory.

Example 1. Linear "Local" Waves.

First, one can obtain specialized results for rather general linear systems. Consider "local waves", that is, waves without a modal cross-structure. We examine

$$\sum_{\alpha}\sum_{\beta} a_{\alpha,\beta}(\varepsilon x,\varepsilon t)\frac{\partial^{\alpha+\beta}u(x,t)}{\partial x^{\alpha}\partial t^{\beta}} = \varepsilon\sum_{\tau}\sum_{\gamma} b_{\tau,\gamma}(\varepsilon x,\varepsilon t)\frac{\partial^{\tau+\gamma}u}{\partial x^{\tau}\partial t^{\gamma}} \tag{27}$$

where the sums are taken over combinations of $\alpha+\beta$ = even and $\tau+\gamma$ = odd. The coefficients $a_{\alpha,\beta}$ and $b_{\tau,\gamma}$ are assumed to be weakly dependent on x,t and to be sufficiently "well-behaved". The conventional approach assumes a solution proportional to $\exp i(Kx-\omega t)$, where $\omega = \omega_r + i\omega_i$ is complex and $\omega_i/\omega_r \sim O(\varepsilon)$. Neglection of $O(\varepsilon^2)$ terms in the complex dispersion relation gives

$$\sum_{\alpha}\sum_{\beta} a_{\alpha,\beta}(-1)^{\beta} i^{\alpha+\beta} K^{\alpha}\left[\omega_r^{\beta} + i\beta\omega_r^{\beta-1}\omega_i\right] = \varepsilon\sum_{\tau}\sum_{\gamma} b_{\tau,\gamma}(-1)^{\gamma} i^{\tau+\gamma} K^{\tau}\omega_r^{\gamma} \tag{28}$$

Equating real and imaginary parts we find the relations

$$\sum_{\alpha}\sum_{\beta} a_{\alpha,\beta}(-1)^{\beta} i^{\alpha+\beta} K^{\alpha}\omega_r^{\beta} = 0 \tag{29}$$

$$\frac{\omega_i}{\varepsilon} = \frac{\sum_{\tau}'\sum_{\gamma} b_{\tau,\gamma}(-1)^{\gamma} i^{\tau+\gamma} K^{\tau}\omega_r^{\gamma}}{\sum_{\alpha}\sum_{\beta} a_{\alpha,\beta}(-1)^{\beta} i^{\alpha+\beta+1}\beta K^{\alpha}\omega_r^{\beta-1}} \sim O(1) \tag{30}$$

Solutions corresponding to the neglected roots of Eq. (28) have been discarded because they are associated with rapidly varying, or alternatively, highly damped solutions. Thus, if we are looking for slowly varying wavetrains in which dissipation plays a modest role, Eq. (28) must suffice. Eq. (29) can be interpreted as a dispersion relation, consistent with that in Eq. (22) in the sense that it is not affected by weakly nonconservative effects. On the other hand, Eq. (30) provides a compact expression for the "exponential growth rate". Now, assume that the left side of Eq. (27) is derivable from a Lagrangian density, so that

$$\sum_{\alpha}\sum_{\beta} a_{\alpha,\beta} \frac{\partial^{\alpha+\beta} u}{\partial x^{\alpha} \partial t^{\beta}} \equiv L_u - \frac{\partial}{\partial x} L_{u_x} - \frac{\partial}{\partial t} L_{u_t} + \frac{\partial^2}{\partial x^2} L_{u_{xx}} + \dots \tag{31}$$

Since the term $a_{0,0}(X,T)$ does not alter the analysis we make, without loss of generalization, the normalization $a_{0,0} = 1$. Then the Lagrangian must take the form $L = \dots + \dots + \frac{1}{2}u^2$, this "fixed" term enabling us to keep track of the entire conservative operator throughout the averaging. From this it follows that

$$\bar{L}_{\nu}^{(0)}(K,\nu,a^2) = \frac{1}{4}a^2 \sum_{\alpha}\sum_{\beta} a_{\alpha,\beta}\, i^{\alpha+\beta} \beta K^{\alpha} \nu^{\beta-1} \tag{32}$$

when L is evaluated with the uniform "conservative" wave solution $U^{(0)} = a(X,T)\sin\theta$. Thus unlike conventional methods

in which dissipation is separately accounted for in a complex phase, the "a" in our approach contains both conservative and slightly nonconservative effects. Evaluating the right side of Eq. (27) with this solution gives, successively,

$$\frac{\partial}{\partial T}\bar{L}^{(o)}_{\nu}+\frac{\partial}{\partial X}\bar{L}^{(o)}_{K} = -\frac{1}{2\pi}\int_0^{2\pi}\sum_{\tau}\sum_{\gamma} b_{\tau,\gamma}\, U_{\theta}^{(o)} U^{(o)}_{(\tau+\gamma)\theta}\, K^{\tau}\nu^{\gamma}\, d\theta \qquad (33)$$

$$= -\frac{1}{2}a^2\sum_{\tau}\sum_{\gamma} b_{\tau,\gamma}\, i^{\tau+\gamma-1} K^{\tau}\nu^{\gamma}$$

$$= -2\bar{L}^{(o)}_{\nu}\,\frac{\sum_{\tau}\sum_{\gamma} b_{\tau,\gamma}\, i^{\tau+\gamma-1} K^{\tau}\nu^{\gamma}}{\sum_{\alpha}\sum_{\beta} a_{\alpha,\beta}\, i^{\alpha+\beta}(-1)^{\beta-1}\beta K^{\alpha}\omega_r^{\beta-1}} = 2\bar{L}^{(o)}_{\nu}\,\frac{\omega_i}{\varepsilon}$$

In terms of the wave action $A = \bar{L}^{(o)}_{\omega}$ and the group velocity $C_g = -\bar{L}^{(o)}_{K}/\bar{L}^{(o)}_{\omega}$ we have

$$-\frac{\partial}{\partial T}A + \frac{\partial}{\partial X}C_g A = 2\left(\frac{\omega_i}{\varepsilon}\right)A \qquad C_g = \partial\omega_r/\partial K \qquad (34)$$

which describes the long term effects of locally weak dissipation. It is essential here that the basic group velocity (which determines the kinematics) remains, to lowest order, unaffected by dissipation. The above equation has been used by Landahl (1972) and by Davey (1972). Complex frequencies, again, are unnecessary in the present approach; the purpose of Eq. (34) is to relate A to the more common "ω_i".

Example 2. Linear "Modal" Waves.

Modal waves are those that involve a cross-structure orthogonal to the propagation direction. A simple model is

$$\sum_{\alpha}\sum_{\beta}\sum_{\gamma} a_{\alpha,\beta,\gamma}(\varepsilon x,\varepsilon t,y)\frac{\partial^{\alpha+\beta+\gamma}u(x,y,t)}{\partial x^{\alpha}\partial t^{\beta}\partial y^{\gamma}} = \varepsilon\sum_{\sigma}\sum_{\tau}\sum_{\zeta} b_{\sigma,\tau,\zeta}(\varepsilon x,\varepsilon t,y)\frac{\partial^{\sigma+\tau+\zeta}u}{\partial x^{\sigma}\partial t^{\tau}\partial y^{\zeta}} \tag{35}$$

where the left side is conservative and the right side is dissipative. Now define the auxiliary system

$$\sum_{\alpha\beta\gamma} a_{\alpha,\beta,\gamma}(0,0,y)\frac{\partial^{\alpha+\beta+\gamma}u}{\partial x^{\alpha}\partial t^{\beta}\partial y^{\gamma}} = \varepsilon\sum_{\sigma\tau\zeta} b_{\sigma,\tau,\zeta}(0,0,y)\frac{\partial^{\sigma+\tau+\zeta}u}{\partial x^{\sigma}\partial t^{\tau}\partial y^{\zeta}} \tag{36}$$

having the eigensolution $\tilde{U}_s = \varphi_s(y)\exp i(Kx-\omega t)$ with mode number "s", where $\varphi_s(y)$ is determined from the solution to

$$\sum_{\alpha\beta\gamma} a_{\alpha,\beta,\gamma}(0,0,y)\varphi_{\gamma y}(y)(iK)^{\alpha}(-i\omega)^{\beta} = \varepsilon\sum_{\sigma\tau\zeta} b_{\sigma,\tau,\zeta}(0,0,y)\varphi_{\zeta y}(y)(iK)^{\sigma}(-i\omega)^{\tau} \tag{37}$$

and its appropriate homogeneous boundary conditions. The right hand sides of the above may be dropped if the resulting undamped system is free of critical layers. Then, the extension to Whitham's basic algorithm suggests an Ansatz of the form

$$U_s^{(0)}(\theta,X,T,y) = a_s(X,T)\varphi_s(y)\sin\theta \tag{38}$$

This type of assumption is reminiscent of Stuart's (1958) "shape assumption", and appears to be justifiable or physical grounds (provided that the medium is, indeed, slowly varying).

Example 3. Waves in Moving Media: Irrotational Mean Flows.

The evolutionary equations as given in Eq. (22) do not impose any restrictions on the background medium, except that it is slowly varying. We will consider fluid-dynamical problems in which waves ride on slowly varying mean currents that vary slowly in time and in the propagation direction only. Both wave and mean flow interact nonlinearly through a radiation stress that transfers energy from mean flow to wave (and vice versa). In addition, the slow variation of the medium introduces small local changes to the dispersive properties characteristic of the wave. The cumulative effects of these changes may be significant.

Let us assume that the waves under consideration propagate in a direction parallel to the mean flow $U = U(X,T)$. This base flow is irrotational because there is no dependence on the shear coordinate y. Then if superscript primes refer to a coordinate system S' moving with the mean flow U and unprimed symbols refer to ground-fixed coordinates S, the principle of Galilean invariance implies, to leading order,

$$\mathcal{L}^{(0)}(X,T,\omega,K) = \mathcal{L}'^{(0)}(X',T',\omega-UK,K) \tag{39}$$

What enters in Eq. (39) is the expected Doppler shift, as can be derived by applying Eq. (2) and its primed counterpart to any system of equations written for both S and S' (Chin, 1976).

The key point in the analysis that follows is a re-interpretation of the fictitious forces experienced in S' as equivalent Reynolds stress effects as observed from S. This principle was first enunciated by Whitham (1962) in the context of gravity waves. It may have been suggested by the fact that the fluid-dynamical equations linearized about a constant U are identical with the same equations written in moving coordinates, but the analogy is not quite complete. With the above interpretation simple results are available on defining energy densities $E' = \omega' \mathcal{L}'_{\omega'} - \mathcal{L}'$ and $E = \omega \mathcal{L}_\omega - \mathcal{L}$ (which are valid for fully nonlinear waves), as suggested by Eq. (25). Now because wave action is invariant under Galilean transformation, that is, $\mathcal{L}_\omega = \mathcal{L}'_{\omega'}$, we can write (to leading order) $(E' + \mathcal{L}')/\omega' = (E + \mathcal{L})/\omega = \mathcal{L}_\omega$. For linear systems, $\mathcal{L} = \mathcal{L}' = 0$ since the difference between phase-averaged kinetic and potential energy densities vanishes. If the resulting expression for $\mathcal{L}_\omega$ is substituted in Eq. (22) and Eq. (33) is used for the dissipation term, we are led to (on dropping primes)

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(U + \Omega_K)E = E\left[\frac{\Omega_t + U\Omega_x - K\Omega_K U_x}{\Omega} + 2w_i\right] \qquad (40)$$

where we have denoted $\Omega(K,X,T) = \omega'$ and set $\omega = U(X,T)K + \Omega$ in the wave conservation law. The group velocity here is simply $C_g = U + \Omega_K$. It is easy to see that the right-side term in Eq. (40), aside from the w_i, represents the effects

of radiation stress. For example, progressive waves on deep water satisfy $\Omega^2 = gK + T_0K^3/\rho$ where g is the acceleration due to gravity, T_0 is the surface tension, and ρ is the fluid density. Since $\Omega_t = \Omega_x = 0$ it follows that the radiation stress $R = (1+3Z)E/(2+2Z)$ where $Z = T_0K^2/\rho g$. For pure gravity waves ($Z = 0$) $R = \frac{1}{2}E$, and for pure capillary waves ($Z = \infty$) $R = 3E/2$, reproducing the results of Longuet-Higgins and Stewart (1964) in a simple way. For gravity waves in water of finite depth $h(X,T)$, the intrinsic frequency is $\Omega = (gK\tanh Kh)^{\frac{1}{2}}$. In this case Ω_x and Ω_t are nonzero. Expanding these terms out and simplifying with $h_t + (hU)_x = 0$ produces the interaction term $(\frac{1}{2} + 2Kh/\sinh 2Kh)E$ in Eq. (40), which again agrees with the results of detailed perturbation methods. These results are not altogether surprising in view of the physical ideas underlying Eq. (39). Eq. (40), of course, is valid for non-fluid-dynamical problems as well. We simply set $U = 0$ to arrive at the interaction term $E\,\Omega_t/\Omega$.

In contrast spatial, as opposed to temporal, inhomogenieties are responsible for changes in wave momentum. The mathematical basis for this is implied by Noether's theorem (Courant and Hilbert, 1953). A wave momentum law analogous to Eq. (40) can be easily derived. For linear systems, this is accomplished by noting that $E = \Omega A$ and $M = KA$ for the wave quantities. Thus $E/\Omega = M/K$ which, together with action and wave conservation, leads to

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial x}(U+\Omega_K)M = -M\left[\frac{\partial U}{\partial x} + \frac{\Omega_x}{K} - 2w_i\right] \tag{41}$$

It is also possible to extend Eq. (40) to completely nonlinear flows. This is accomplished by applying Eq. (39) to Eq. (25), with the result that

$$\frac{\partial E}{\partial T} + \frac{\partial}{\partial X}\left(U - \frac{\Omega \mathcal{L}_K}{\Omega \mathcal{L}_\Omega - \mathcal{L}}\right)E = (K\mathcal{L}_K - \mathcal{L})\frac{\partial U}{\partial X} - (\mathcal{L}_T + U\mathcal{L}_X) + \Omega \mathcal{F} \tag{42}$$

where now $E = \Omega \mathcal{L}_\Omega - \mathcal{L}$ is the wave energy density. One may have anticipated Eq. (42) in retrospect. For example, the right-side terms are for radiation stress, since $K\mathcal{L}_K - \mathcal{L}$ is a momentum flux (see Eq. (26)). The momentum analogue to Eq. (42) is just as easily derived by combining Eqs. (39) and (26). The result is simply

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial X}\left(U + \frac{\mathcal{L} - K\mathcal{L}_K}{K\mathcal{L}_\Omega}\right)M = \mathcal{L}_x - M\frac{\partial U}{\partial x} + \varepsilon K \mathcal{F} \tag{43}$$

The above laws are, of course, coupled to the dynamics of the mean flow. For further details, the reader is referred to Whitham (1967). Three propagation velocities, those for action, energy and momentum, can be defined. They are, respectively,

$$C^{(a)} = U - \mathcal{L}_K/\mathcal{L}_\Omega \qquad (44)$$

$$C^{(e)} = U - \Omega\mathcal{L}_K/(\Omega\mathcal{L}_\Omega - \mathcal{L})$$

$$C^{(m)} = U + (\mathcal{L} - K\mathcal{L}_K)/K\mathcal{L}_\Omega$$

In the linear case these reduce to $U + \Omega_K$ since the Lagrangian vanishes. Note also that the nonlinear wave kinetic energy $\tilde{K}$ satisfies <u>exactly</u> $\tilde{K} = \frac{1}{2}(E + \mathcal{L}) = \frac{1}{2}\omega A = \frac{1}{2}(\omega/K)(KA) = \frac{1}{2}C_p M$, where C_p is the phase velocity. This identity is based on E being the sum of kinetic and potential energies and $\mathcal{L}$ being the difference. It was first given by Levi-Civita (1924) for finite amplitude gravity waves in water of finite depth. But its applicability seems to extend to much more general systems. In the linear case, energy equipartition implies $E = 2\tilde{K}$ and so, $E = C_p M$, a result familiar in surface waves.

General results like Eqs. (42) and (43) were made possible by the assumed dependence of U on X,T only, that is, there is no y-dependence. Thus the dispersive properties of the wave are unmodified by the presence of the mean flow except to the extent of the expected Doppler-shift. For many problems, however, the assumption $U = U(X,T)$ is not very realistic. In deep water, for example, the nonlinear coupling between wave and mean flow disappears (Whitham, 1967) so that, effectively, the mean flow is unaffected by wave growth or decay as induced by radiation stresses. Surfaces changes in U can only be accounted for by

incorporating a y-dependence. Obviously, the above approach using Eq. (39) is no longer applicable, and a theory for the y-coupling must be separately developed. This will be outlined in the next example.

At this point we can make some remarks on wave packet growth. Usually stability theory centers on the idea of temporal growth or decay, for example, the time behaviour of a normal mode is studied in parallel flow models. Wave-like disturbances, however, propagate downstream in reality, and pass through various regions which may be stable or unstable on the basis of normal mode analysis. Not only is the space-time evolution of the wave energy density important, but its integral, the overall wave packet growth in defining stability. In general, the stability (or instability) of one does not imply the same for the other. Consider the integral $\Sigma(t)$ of $E(x,t)$ over dx between two rays $x_1(t)$ and $x_2(t)$. $\Sigma(t)$ is a net energy in some sense, with time playing a parametric role. Differentiation leads to

$$\dot{\Sigma} = \int_{x_1}^{x_2} \frac{\partial E(x,t)}{\partial t} dx + E(x_2,t)\frac{dx_2}{dt} - E(x_1,t)\frac{dx_1}{dt}$$

or,

$$\dot{\Sigma} = \int_{x_1}^{x_2'} \left\{ \frac{\partial E(x,t)}{\partial t} + \frac{\partial}{\partial x}(qE) \right\} dx$$

If q is identified with the group velocity $U + \Omega_K$, then $\dot{\Sigma} = d\Sigma/dt$ is simply the time rate of change of total energy between two rays. If we introduce Eq. (40) we find that

$$\frac{d\Sigma(t)}{dt} = \int_{X_1(t)}^{X_2(t)} E(x,t) \left[\frac{\Omega_t + U\Omega_x - K\Omega_K U_x}{\Omega} + 2w_i \right] dx \qquad (45)$$

which, in non-fluid-dynamical problems, takes on U = 0. If the inhomogeniety and the dissipation vanish, Eq. (45) shows that the energy between two rays is a constant equal to the initial energy. In general, the coefficient of E in Eq. (45) is a complicated function of K, X and T that is not known until the kinematics are completely determined. However if $\Omega_t = \Omega_x = w_i = 0$ (as in inviscid deep-water wave problems) Eq. (45) automatically implies wave energy growth whenever $U_x < 0$ since $K\Omega_K/\Omega > 0$.

Consider, for example, the quantity

$$Q \equiv \frac{\Omega_t + U\Omega_x - K\Omega_K U_x}{\Omega} + 2w_i = -\left[\frac{1}{2} + \frac{2Kh}{\sinh 2Kh}\right] U_x$$

for gravity waves. If $Q > 0$ everywhere, instability is implied, but if $Q \leq 0$ there is total stability. The energy of a wave packet thus grows or decays if, relative to the wave, U_x is negative or positive, a fact in agreement with observation. Flow inhomogenieties play a dual role in affecting both energy

density and total energy. For example, an energy singularity can develop within a wave packet while the total energy decreases. Now, let Q_{max} and Q_{min} denote the maximum and minimum values of Q in (x_1, x_2), so that $\int_{x_1(t)}^{x_2(t)} EQ\, dx \leq Q_{max} \Sigma$ and $\int_{x_1(t)}^{x_2(t)} EQ\, dx \geq Q_{min} \Sigma$. Hence, substitution in Eq. (45) shows that Q_{max} and Q_{min} can be interpreted as upper and lower bounds for total energy amplification. In linear theory these bounds are prescribed in the sense that U(X,T) is specified beforehand. Since U is slowly varying the growth rates Q are $O(\varepsilon)$, however, events occurring within the wave packet can be quite sudden.

We can also examine wave momentum. The total wave momentum $\mathcal{M} = \int_{x_1(t)}^{x_2(t)} M(x,t)\, dx$ following a wave group is governed, in linear theory, by the equation

$$\frac{d\mathcal{M}(t)}{dt} = \int_{x_1(t)}^{x_2(t)} M \left[- \frac{\partial U}{\partial x} - \frac{\Omega_x}{K} + 2\omega_i \right] dx \qquad (46)$$

where $x_{1,2}(t)$ are determined by integration of the wave conservation law. For simplicity consider waves in inviscid, deep water, where $\Omega_x = \omega_i = 0$. Then, the inequality

$$(-U_x)_{MIN} \mathcal{M}' = (-U_x)_{MIN} \int_{x_1}^{x_2} M dx \leq \int_{x_1}^{x_2} M(-U_x) dx \leq (-U_x)_{MAX} \int_{x_1}^{x_2} M dx = (-U_x)_{MAX} \mathcal{M}$$

follows, identifying the maximum and minimum of $-U_x$ as the

upper and lower bounds for the momentum amplification. For the wave action density A, one has $d/dt \int_{x_1(t)}^{x_2(t)} A(x,t)\, dx = 0$ in a conservative system. For linear or nonlinear problems, we have

$$\mathcal{A} \equiv \int_{x_1(t)}^{x_2(t)} A(x,t)\, dx = \int_{x_1(0)}^{x_2(0)} A(x,0)\, dx = \mathcal{A}_0 \tag{47}$$

where $\mathcal{A}_0$ is the total initial action. With dissipation, Eq. (47) is no longer true, but rough estimates for upper and lower bounds on the decay rates are easy to obtain. Consider the case $\Omega_x = \Omega_a = 0$. If we denote

$K_M = \text{Max } K(x,0) > K_m = \text{Min } K(x,0) > 0$ and $\gamma_M = \text{Max}(-U_x) > \gamma_m = \text{Min}(-U_x)$

it is clear, from wave conservation, that along all rays

$K_m e^{\gamma_m t} \leq K \leq K_M e^{\gamma_M t}$. Since $\omega_i = -2\nu K^2$ in deep water, where ν is the kinematic viscosity, it follows that

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} A(x,t)\, dx = 2\int_{x_1}^{x_2} \omega_i A\, dx = -4\nu \int_{x_1}^{x_2} K^2 A\, dx \tag{48}$$

and hence, for any $t = t_0$, that

$$-4\nu K_M^2 e^{2\gamma_M t_0} \leq \frac{d}{dt} \ln \mathcal{A} \leq -4\nu K_m^2 e^{2\gamma_m t_0}$$

Thus, bounds for the total action decay (or growth) can be readily formulated in terms of initial conditions and streamwise

parameters. In the linear results given above it is possible for the total energy or momentum of a wave packet to grow without bound. In this case the effect of nonlinearity must be included by, say, expanding the average Lagrangian in powers of wave action. For further details, the reader is referred to Chin (1976).

Example 4. Waves in Moving Media: Mean Shear Flows.

The previous example dealt with mean flows constant across the modal cross-space. Because the local mean speed was unique over y we invoked the principle of Galilean invariance, and this led to general results. This invariance is, however, inapplicable to the case of mean shear flows. At first, the situation appeared to be additionally complicated by the lack of a suitable variational formulation for the inviscid Rayleigh equation. This latter point was dismissed for, although a variational principle does not exist in the linearized problem, it does for the complete nonlinear one. With this settled, we opted to equivalently average over phase and modal cross-space directly the fluid-dynamical conservation laws for total mass, total momentum and total energy. The formal procedure parallels Whitham's (1962) treatment for irrotational gravity waves (of energy density E and wavenumber K) in water of finite depth h(X,T) and speed U(X,T), the latter two variables of which, on account of O(E) wave-induced effects, cannot be assumed as known a priori. E and K in addition are taken as slowly varying functions of x,t and the waves are assumed to be locally sinusoidal in the averaging. The resulting modulation equations (although E^2 terms are not explicitly present) describe the nonlinear wave back-interaction on the mean state over scales large compared to a typical period or wavelength. The consequences of the theory can be found in Whitham (1967).

Our theory, however, departs from Whitham's by recognizing that the U here, in addition to experiencing "magnitude changes" of the Whitham type, experiences "shape changes" due to the action of Reynolds stresses. This consideration does not arise in the irrotational case by virtue of Kelvin's theorem: an irrotational flow stays irrotational. What is needed to extend the above ideas is thus a suitable (phase-independent) Ansatz <u>for the mean flow</u> in the phase-modal averaging, so that within the limitations imposed by WKB theory, the simultaneous wave back-interaction process is properly described. A simple model can be constructed by observing that waves truly slowly varying in space can be described locally by the usual normal mode theory. The given laminar flow $U_\ell(y)$ corrected for wave-induced distortions of O(E) as introduced by the Rayleigh equation then can be put in the form (Chin, 1976)

$$U(y,E) = U_\ell(y) + E\,\frac{U_\ell''\,|\varphi_N|^2}{|U_\ell - c|^2} \tag{50}$$

where $\varphi_N(y,K)$ is the Rayleigh eigenfunction suitably normalized, and E includes the initially exponential temporal growth. This result agrees with that in Landahl (1973).
The key idea is to assume a slowly varying mean flow in the form

$$U(y,X,T) = \frac{\hat{U}(X,T)}{U_o}\left[U_\ell(y) + E\,\frac{U_\ell''\,|\varphi_N(y,K;X,T)|^2}{|U_\ell - c|^2}\right] \tag{51}$$

where U_o is a reference speed. Thus, in this model, the local normal mode solution for the corrected mean flow determines the form of a large-scale (wave-dependent) shape structure convected with the mean flow. This assumption is physically sound provided the waves are slowly varying. The above Ansatz reduces to Whitham's when U_ℓ = constant, since $U_\ell'' = 0$. Second-order changes in mean pressure, of course, also must be accounted for in the averaging. When the final averaged equations are combined with the kinematic requirement for wave conservation we arrive at a system of modulation equations which, in Whitham's (1967) sense, can be hyperbolic or elliptic depending on the value of Kh. These imply a splitting of the initial wave packet or a type of modulational instability. However, the new theory generalizes Whitham's by the inclusion of velocity profile curvature, and this introduces a host of new effects. The new theory can be considered as complementing the viscous one of Stewartson and Stuart (1971) by dealing with essentially inviscid flows that are not in "near-equilibrium". A detailed treatment for surface waves and channel/jet - type flows is planned in a separate paper, but a preliminary account of the main results is available in Chin (1976).

Example 5. Taylor's Waves.

The Lagrangian theory leads to simple generalizations for "Taylor waves." The effect of nonuniform currents with constant horizontal divergence on a train of gravity waves whose amplitude and length are constant in space but variable in time was first discussed by G.I. Taylor (1962). Such space-independent waves can be expected to exist for constant $U_x = \alpha$ if there are otherwise no other inhomogenieties in the medium, and Taylor has shown how these can be experimentally realized. We thus consider "deep-water" problems. If we denote by A the wave action, the nonlinear governing equations on taking $K_x = A_x = 0$ become

$$\frac{\partial}{\partial t} \ln K = -\alpha \tag{52}$$

$$\frac{\partial}{\partial t} \ln A = -\alpha \tag{53}$$

which integrate to

$$K = K_o e^{-\alpha t} \tag{54}$$

$$A = A_o e^{-\alpha t} \tag{55}$$

where K_o is the initial wavenumber and A_o is the initial wave action. Now $A = A_o K/K_o$ and we can expand the wave energy as $E = Af(K) + \frac{1}{2}A^2 g(K)$. Thus the nonlinear solution for energy becomes

$$E = \frac{A_o}{K_o} K f(K) + \frac{1}{2}\left(\frac{A_o}{K_o}\right)^2 K^2 g(K) \qquad (56)$$

For deep-water gravity waves, $f(K) = g_o^{\frac{1}{2}} K^{\frac{1}{2}}$ and $g(K) = K^3$ where g_o is the acceleration due to gravity, as can be shown from the results of Whitham (1967). Eq. (56) specializes to

$$E = \frac{A_o g_o^{1/2}}{K_o} K^{3/2} + \frac{1}{2}\left(\frac{A_o}{K_o}\right)^2 K^5 \qquad (57)$$

For linear waves $E = \frac{1}{2}\rho g_o a^2$ implies that $a \sim \lambda^{-3/4}$, reproducing Taylor's result. For capillary waves with surface tension T, $f = T^{\frac{1}{2}} K^{3/2}$ and $g = -K^3/8$ (Chin, 1976). In this case,

$$E = \frac{A_o T^{1/2}}{K_o} K^{5/2} - \frac{1}{16}\left(\frac{A_o}{K_o}\right)^2 K^5 \qquad (58)$$

which, for small amplitudes, gives $E \sim \lambda^{-5/2}$.

We can also formulate the above ideas somewhat differently. The energy equation as given in Eq. (42) is simplified by expanding the average Lagrangian in powers of A. If we assume

that $\omega = UK + f + Ag$, the equation satisfied by (the nonlinear definition of) E is, to $O(E^2)$,

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x} C^{(e)} E = \left[-E \frac{K f_K}{f} + E^2 \frac{K g f_K - g f - K f g_K}{2 f^3} \right] \frac{\partial U}{\partial x} \tag{59}$$

Assuming $E_x = K_x = 0$ as before, we can equivalently write

$$\frac{\partial E}{\partial t} = \beta_1 E - \beta_2 E^2 \tag{60}$$

where

$$\beta_1 = -\left(1 + \frac{K f_K}{f}\right) \frac{\partial U}{\partial x}. \qquad \beta_2 = -\frac{K g f_K - g f - K f g_K}{2 f^3} \frac{\partial U}{\partial x} \tag{61}$$

For pure capillaries the coefficients are $\beta_1 = -5\alpha/2$ and $\beta_2 = -5\alpha/(32T)$, while for pure gravity waves $\beta_1 = -3\alpha/2$ and $\beta_2 = 7K^2\alpha/4g_o$. In linear theory equilibrium amplitudes cannot exist for $\alpha < 0$. But in the nonlinear case, a possible equilibrium energy $E = \beta_1/\beta_2$ exists. Thus it exists for capillaries ($E = 16T$) but it does not, to this order, for gravity waves. Eq. (60) can be integrated for constant β's, giving

$$E(t) = \frac{\delta e^{\beta_1 t}}{1 + \frac{\beta_2}{\beta_1} \delta e^{\beta_1 t}} \tag{62}$$

where δ is related to the initial energy. (For more complicated problems in which the β's are time-dependent, integration in closed form is possible using Bernoulli-type integrals.)

This result is reminiscent of the amplitude equation first given by Stuart (1958) in nonlinear boundary layer stability. Essentially, if (i) $\beta_1 > 0$, $\beta_2 > 0$ an equilibrium energy exists, (ii) $\beta_1 > 0$, $\beta_2 < 0$ an equilibrium energy doesn't exist, and (iii) $\beta_1 < 0$, $\beta_2 < 0$ the disturbance takes on an equilibrium energy E_o at $t = -\infty$; however, the equilibrium is unstable. If the energy density is less than E_o, the disturbance decays to zero, whereas if it is greater than E_o, the disturbance grows. Thus, Stuart's equation describes a broader class of temporally varying problems. In other problems in which the waves are not purely steady in space and time, the modulations balance according to Eq. (22). Thus the Landau coefficients in Eq. (62) are now functions of space and time as determined from the wave kinematics. Implicit also in Eq. (22), of course, is Gaster's (1962) theorem relating temporal and spatial growth rates via the group velocity; the relationship is much more complicated in the nonlinear case.

Example 6. A Weakly Nonlinear Instability.

To examine the basic effect of nonlinearity on linear wave flows, consider homogeneous, conservative media and in Eq. (22) with $F = 0$, let us eliminate the explicit dependence of $\mathcal{L}_\omega$ and $\mathcal{L}_K$ on A by using the dispersion relation. This leads to

$$\frac{\partial}{\partial t}\mathcal{L}_\omega(\omega,K) - \frac{\partial}{\partial x}\mathcal{L}_K(\omega,K) = 0 \tag{63}$$

which can be written as the following eikonal equation for the phase function $\theta(x,t)$.

$$\mathcal{L}_{\omega\omega}(\omega,K)\frac{\partial^2\theta}{\partial t^2} - 2\mathcal{L}_{\omega K}\frac{\partial^2\theta}{\partial x\partial t} + \mathcal{L}_{KK}\frac{\partial^2\theta}{\partial x^2} = 0 \tag{64}$$

We can specialize our discussion to wavemakers. In linear theory the wave frequency is fixed on prescribing the wavenumber. But in nonlinear theory this alone is not sufficient since something about the amplitude distribution must be known. We can, of course, equivalently modulate the frequency and take $K = K_o$ and $\omega = \omega_o - \varepsilon \exp(i\alpha x)$ as initial conditions, where ε and α are prescribed constants. Then the perturbation phase function $\phi = \theta - K_o x + \omega_o t$ satisfies $\phi_x = 0$ and $\phi_t = \varepsilon \exp(i\alpha x)$ at t=0 as well as the differential equation above. Nonlinear

perturbations to the basic system thus satisfy, approximately,

$$\mathcal{L}_{\omega\omega}(\omega_o,K_o)\frac{\partial^2\phi}{\partial t^2} - 2\mathcal{L}_{\omega K,o}\frac{\partial^2\phi}{\partial x\partial t} + \mathcal{L}_{KK,o}\frac{\partial^2\phi}{\partial x^2} = 0 \tag{65}$$

The solution to this is simply the real part of

$$\theta = K_o x - \omega_o t + \varepsilon e^{i\alpha\left(x + \frac{\mathcal{L}_{\omega K,o}}{\mathcal{L}_{\omega\omega,o}}t\right)} \frac{\sinh \xi t}{\xi} \tag{66}$$

where, if $\mathcal{L}_{\omega\omega,o}\mathcal{L}_{KK,o} - \mathcal{L}^2_{\omega K,o} > 0$,

$$\xi = \frac{\alpha\left(\mathcal{L}_{\omega\omega,o}\mathcal{L}_{KK,o} - \mathcal{L}^2_{\omega K,o}\right)^{1/2}}{\mathcal{L}_{\omega\omega,o}} \tag{67}$$

The instability implied by Eq. (66) arises from the solution of an elliptic Cauchy problem. It is essentially Whitham's (1967) instability and, for example, can be realized by gravity or capillary waves on deep water. One obvious question is the effect of inhomogeniety (e.g., a slowly varying current $U = U(X,T)$) and dissipation in the medium. To explore this issue let us follow Hayes (1970) and expand the average Lagrangian $\mathcal{L}$ in the form $\mathcal{L} = A\omega - H(K,A,X,T)$. This representation is a canonical one since the only amplitude measure that appears is the wave action $A = \mathcal{L}_\omega$ itself. The modulation equations as expressed in Eq. (22) become

$$\frac{\partial A}{\partial T} + \omega_K \frac{\partial A}{\partial X} + H_{KK} \frac{\partial K}{\partial X} + H_{KX} = \mathcal{F} \tag{68}$$

$$\frac{\partial K}{\partial T} + \omega_K \frac{\partial K}{\partial X} + \omega_A \frac{\partial A}{\partial X} + \omega_X = 0 \tag{69}$$

where the frequency ω is determined from $\mathcal{L}_A = 0$, that is,

$$\omega = H_A(K, A, X, T) \tag{70}$$

Since $\omega \mathcal{L}_\omega - \mathcal{L}$ is an energy density (see Eq. (25)), the function H can be interpreted as a Hamiltonian energy function. In the present case (as in Eq. (22)) the frequency is again real; nonconservative effects appear idrectly in Eq. (68). As in the discussion leading up to Eq. (65) we consider weakly nonlinear perturbations to a uniform state (denoted by zero subscripts) and linearize Eqs. (68) and (69) about the constants K_o and A_o. If we introduce new coordinates $\tau = t$ and $\xi = x - C_o t$ where C_o is the "mean velocity" $\omega_K = H_{AK}(K_o, A_o; X, T)$, the perturbation problem for A' (where $A' = A - A_o$ and $K' = K - K_o$) can be put in the form

$$\frac{\partial^2 A'}{\partial \tau^2} - \Omega_{A,o} H_{KK,o} \frac{\partial^2 A'}{\partial \xi^2} = 2\varepsilon(\omega_i - U_x) \frac{\partial A'}{\partial \tau} \tag{71}$$

$$A'(\xi, \tau = 0) = \varphi(\xi)$$

$$\frac{\partial A'}{\partial \tau}(\xi, \tau = 0) \cong -H_{KK,o} K'_\xi(\xi, \tau = 0) \equiv \psi(\xi)$$

where $(\omega_i - U_x)$ is taken as constant locally. If ε is small we can write, approximately,

$$A'(\xi,\tau) \cong e^{\varepsilon(\omega_i - U_x)\tau} B\left(\xi \pm \sqrt{\Omega_{A,0} H_{KK,0}}\ \tau\right) \tag{72}$$

In the hyperbolic case the solution to the "conservative part" B satisfies

$$\frac{\partial^2 B}{\partial \tau^2} - \Omega_{A,0} H_{KK,0} \frac{\partial^2 B}{\partial \xi^2} = 0 \tag{73}$$

$$B(\xi,\tau=0) = \varphi(\xi) \qquad B_\tau(\xi,\tau=0) = \psi(\xi)$$

and is given by D'Alembert's formula, that is,

$$B(\xi,\tau) = \tfrac{1}{2}\left\{\varphi\left(\xi+\sqrt{\Omega_{A,0}H_{KK,0}}\ \tau\right) + \varphi\left(\xi-\sqrt{\Omega_{A,0}H_{KK,0}}\ \tau\right)\right\} + \frac{1}{2\sqrt{\Omega_{A,0}H_{KK,0}}} \int_{\xi-\sqrt{\Omega_{A,0}H_{KK,0}}\,\tau}^{\xi+\sqrt{\Omega_{A,0}H_{KK,0}}\,\tau} \psi(\eta)\, d\eta \tag{74}$$

Eq. (74) shows how the perturbation "splits"; it also illustrates the interplay between U_x, ω_i, initial conditions and nonlinearity. The latter is stabilizing in thesense that small perturbations remain small.

The elliptic initial value problem is solved by analytic continuation into the complex ξ - plane. The formulation for B as given in Eq. (73) still holds, but to indicate explicitly that $\Omega_A H_{KK} < 0$, we will introduce the barred variables

denoted by $\bar{\xi} = \xi/(-\Omega_A H_{KK})^{\frac{1}{2}}$ and $\bar{\tau} = \tau$. This leads to

$$\frac{\partial^2 B(\bar{\xi},\bar{\tau})}{\partial \bar{\tau}^2} + \frac{\partial^2 B}{\partial \bar{\xi}^2} = 0 \qquad \bar{\tau} \geq 0 \tag{75}$$

$$B(\bar{\xi},\bar{\tau}=0) = \varphi(\bar{\xi}) \qquad B_{\bar{\tau}}(\bar{\xi},\bar{\tau}=0) = \psi(\bar{\xi})$$

As suggested in Garabedian (1964), we introduce the complex variable $\bar{\xi} = \zeta + i\eta$ and <u>assume</u> that the functions $\varphi(\bar{\xi})$ and $\psi(\bar{\xi})$ are analytical functions of $\bar{\xi}$. We can then write $B(\bar{\xi},\bar{\tau}) = B(\zeta+i\eta,\bar{\tau})$, $B(\bar{\xi},\bar{\tau}=0) = \varphi(\zeta+i\eta) = \varphi_1(\eta)$ and $B_{\bar{\tau}}(\bar{\xi},\bar{\tau}=0) = \psi(\zeta+i\eta) = \psi_1(\eta)$ where, for each value of ζ, $B(\bar{\xi},\bar{\tau})$ satisfies

$$\frac{\partial^2 B}{\partial \bar{\tau}^2} - \frac{\partial^2 B}{\partial \eta^2} = 0 \tag{76}$$

Because Eq. (76) is formally hyperbolic, the solution $B(\bar{\tau},\zeta+i\eta)$ can be determined in a stable manner from initial conditions. However, this is a solution for complex $\bar{\xi}$. The physical solution is obtained in the limit $\eta \to 0$ after an explicit formula for $B(\bar{\xi},\bar{\tau})$ is obtained. Again, the D'Alembert formula applies, giving

$$B(\eta,\zeta;\bar{\tau}) = \frac{\varphi_1(\eta+\bar{\tau})+\varphi_1(\eta-\bar{\tau})}{2} + \frac{1}{2}\int_{\eta-\bar{\tau}}^{\eta+\bar{\tau}} \psi_1(\omega)\,d\omega \tag{77}$$

where the semicolon above refers to the parametric dependence on ζ (the physical solution implies the limit $\bar{\xi} = \zeta$). As a simple example consider the situation where the initial wavenumber variations are zero so that $\psi_1(\eta) = 0$. Since Eq. (76) is linear it suffices to consider a Fourier component $B(\bar{\tau}=0, \bar{\xi}) = \alpha \sin\beta\bar{\xi}$ of the initial action disturbance, that is, $\varphi_1(\eta) = \alpha \sin\beta(\zeta + i\eta)$. In this case, the physical solution is

$$B(\bar{\xi}, \bar{\tau}) = \lim_{\eta \to 0} \frac{\varphi_1(\eta+\bar{\tau}) + \varphi_1(\eta-\bar{\tau})}{2} = \alpha \sin\beta\bar{\xi} \cosh\beta\bar{\tau} \qquad (78)$$

The appearance of the $\cosh\beta\bar{\tau}$ in addition to the growth rates ω_i and U_x already in Eq. (72) enable a simple definition for an "effective ω_i".

In applying D'Alembert's formula and Garabedian's method of imaginary characteristics certain conditions on smoothness and analyticity, of course, are implicitly assumed. These are in addition to those calling for slow variation, as indicated in Eq. (2). Sufficiently non-smooth Cauchy data also possess real solutions, but solutions to these problems are not likely to be slowly varying, and hence, are not likely to be described by the present theory.

III. High-Order Effects.

The low-order formulation discussed in Section II gives results which, strictly speaking, are invalid over space/time scales greater than $O(\varepsilon^{-1})$. They are, of course, valid in the sense under which they were derived, but to deal more accurately with large-scale effects, high-order modulations as described in Eqs. (17) and (20) must be considered. Lighthill (1965a), as an example, traced the progress of gravity wave packets on deep water and showed how, on the basis of low-order Whitham theory, cusp-like energy redistributions within the packets would asymptotically result. However, the inclusion of high-order dispersive terms (Yuen and Lake, 1975) actually suppresses the Lighthill singularity; the end result is a train of solitons. Another situation where high-order dispersive and diffusive corrections are likely to be important is in the formation of so-called "kinematic shocks". These are essentially wave-mechanical discontinuities which, on the basis of low-order theory, are assumed to conserve various fluxes, for example, momentum or energy. Their probable existence was first speculated upon by Whitham (1965). However, the true nature and stability of these shocks, of course, cannot be studied without considering the structure of the high-order terms. It is entirely possible that sufficiently strong dispersive effects in the high-order correction terms can suppress their ever forming. On the other hand if the corrections are

essentially diffusive, as in gas-dynamic shocks, the low-order results would be meaningful in the sense that the solution is "properly embedded" within the high-order one. More on this will be said in Section IV.

For many applications in continuum physics the basic wave solution is weakly nonlinear, and hence, expandable via Fourier series in powers of amplitude squared. For these cases the structural form of the high-order modulation terms depends only on the primary harmonic, that is, the basic linear wave solution. In this physical limit one can obtain a more transparent but completely equivalent re-expression of Eqs. (17) and (20), in which the corrections to low-order theory depend only on the real and imaginary frequencies $\Omega^R(K)$ and $\Omega^i(K)$ corresponding to the uniform wave solution. To facilitate the discussion, let us first consider purely linear problems without flow inhomogenieties, so that Fourier superposition is valid. The mathematical approach is a simple one. Expansions about a center wavenumber K_o and a real frequency ω_o for the modulations are introduced as, for example, in Davey (1972), but unlike Davey the series solution is not truncated to a finite number of terms. The full series solution is then resummed in a type of analytic continuation to produce solutions valid for all wavenumbers that are consistent with the Lagrangian approach. The technique used is analogous to, for example, identifying the series $\sum_{n=0}^{\infty} x^n$ with $(1-x)^{-1}$, which has a wider range of convergence. We now consider the linear

superposition of monochromatic wave components exp i(Kx-ωt), each satisfying the complex dispersion relation

$$\omega = \Omega(K) = \Omega^R + i\Omega^i \tag{79}$$

Here K is a real wavenumber with Ω^R and Ω^i being respectively the real and imaginary parts of $\Omega(K)$. In the neighborhood of the center wavenumber K_o, Taylor expansion gives the series $\omega = \sum_{n=0}^{\infty} \frac{1}{n!} \Omega_{nK}(K_o)(K-K_o)^n$. If $|\Omega^i/\Omega^R| \sim O(\varepsilon) << 1$ we can, as in Lighthill (1965b), Fourier superpose elementary solutions and arrive at the more general solution

$$\tilde{F}(x,t) = \int_{-\infty}^{\infty} B(K)\, e^{i(Kx-\omega t)}\, dK \tag{80}$$

The function B(K) is proportional to the Fourier transform of the initial condition and it exists when the initial disturbances are localized. Without loss of generality let us re-express Eq. (80) in the form

$$\tilde{F}(x,t) = \psi(x,t)\, e^{i[K_o x - \Omega_o^R t]} \tag{81}$$

$$\psi = \int_{-\infty}^{\infty} B(K)\, e^{i[(K-K_o)x - (\Omega-\Omega_o^R)t]}\, dK \equiv \int_{-\infty}^{\infty} G\, dK \tag{82}$$

so that F(x,t) consists of a purely periodic part and an amplitude function $\psi(x,t)$ that includes dissipative effects. This follows the approach introduced in Section II. Applying the operator $\int_{-\infty}^{\infty} \ldots B(K)\, e^{i[(K-K_o)x-(\omega-\Omega_o^R)t]}\, dK$ to the Taylor-expanded dispersion relation (which holds for any component wave) and noting that $i\frac{\partial\psi}{\partial t} = \int_{-\infty}^{\infty}(\omega-\Omega_o^R)\,G\,dK$ and $(-i)^n\frac{\partial^n\psi}{\partial x^n} = \int_{-\infty}^{\infty}(K-K_o)^n\,G\,dK$ leads to the following identity for ψ.

$$i\frac{\partial\psi}{\partial t} = i\Omega_o^i\psi + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\,\Omega_{nK,o}\,\frac{\partial^n\psi}{\partial x^n} \tag{83}$$

Next, as in Eq. (2), we seek WKB solutions of the form

$$\psi = a(X,T)\,e^{i\tilde{\theta}(x,t)} \tag{84}$$

where $\tilde{\theta}$ is the perturbation phase function, that is, $\tilde{\theta}_x = K-K_o = \mathcal{K}(X,T)$ and $-\tilde{\theta}_t = \omega-\Omega_o^R = \varpi(X,T)$. Now the multiple-scaling introduced in Eq. (2) implies that, for <u>any</u> function $u(x,t) = U(\tilde{\theta},X,T)$,

$$u_{nx}(x,t) = K^n U_{n\tilde{\theta}} \tag{85}$$

$$+ \varepsilon\left[nK^{n-1} U_{(n-1)\tilde{\theta},X} + \frac{n(n-1)}{2} K^{n-2} K_X U_{(n-1)\tilde{\theta}} \right]$$

$$+\varepsilon^2 \left[\begin{array}{l} \frac{n(n-1)}{2} K^{n-2} U_{(n-2)\tilde{\theta},XX} + \frac{n(n-1)(n-2)}{2} K^{n-3} K_X U_{(n-2)\tilde{\theta},X} \\ + \frac{n(n-1)(n-2)(n-3)}{8} K^{n-4} K_X^2 U_{(n-2)\tilde{\theta}} \\ + \frac{n(n-1)(n-2)}{6} K^{n-3} K_{XX} U_{(n-2)\tilde{\theta}} \end{array} \right]$$

$$+\varepsilon^3 \left[\begin{array}{l} \frac{n(n-1)(n-2)(n-3)(n-4)}{12} K^{n-5} K_X K_{XX} U_{(n-3)\tilde{\theta}} \\ + \frac{n(n-1)(n-2)(n-3)}{6} K^{n-4} K_{XX} U_{(n-3)\tilde{\theta},X} \\ + \frac{n(n-1)(n-2)(n-3)}{4} K^{n-4} K_X U_{(n-3)\tilde{\theta},XX} \\ + \frac{n(n-1)(n-2)}{6} K^{n-3} U_{(n-3)\tilde{\theta},XXX} \\ + \frac{n(n-1)(n-2)(n-3)}{24} K^{n-4} K_{XXX} U_{(n-3)\tilde{\theta}} \\ + \frac{n(n-1)(n-2)(n-3)(n-4)}{8} K^{n-5} K_X^2 U_{(n-3)\tilde{\theta},X} \\ + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{48} K^{n-6} K_X^3 U_{(n-3)\tilde{\theta}} \end{array} \right]$$

$$+ \ \ldots$$

$$+ \varepsilon^n U_{nX}$$

as can be readily shown by induction. Let us expand the function $\psi(\tilde{\theta},X,T)$ in Eq. (84) following Eq. (85) and substitute the results into Eq. (83), remembering that $\Omega_{nK,o}=\Omega^{R}_{nK,o}+i\Omega^{i}_{nK,o}$ satisfies $|\Omega^{i}_{nK,o}/\Omega^{R}_{nK,o}| \ll 1$. One sees that to obtain $O(\varepsilon^2)$ corrections to the linear phase relation only $O(\varepsilon^2)$ terms in Eq. (85) need to be retained. But to obtain $O(\varepsilon^2)$ corrections to the low-order amplitude equation (modified to include the effects of dissipation) we must retain the $O(\varepsilon^3)$ terms in Eq. (85). Equating real and imaginary parts in the equation just obtained gives, respectively,

$$a\varpi = \varepsilon \sum_{n=1}^{\infty} \frac{\Omega^{i}_{nK,o}}{n!}\left[nK^{n-1}a_X + \frac{n(n-1)}{2}K^{n-2}K_X a \right] \tag{86}$$

$$+\sum_{n=1}^{\infty}\frac{\Omega^{R}_{nK,o}}{n!}\left[aK^{n} - \varepsilon^2\left\{ \begin{array}{l} \frac{n(n-1)}{2}K^{n-2}a_{XX} + \frac{n(n-1)(n-2)}{2}K^{n-3}K_X a_X \\ + \frac{n(n-1)(n-2)(n-3)}{8}K^{n-4}K_X^2 a \\ + \frac{n(n-1)(n-2)}{6}K^{n-3}K_{XX}a \end{array} \right\} \right]$$

$$\varepsilon a_T = \Omega_o^i a - \varepsilon \sum_{n=1}^{\infty} \frac{\Omega_{nK,o}^R}{n!} \left[nK^{n-1} a_X + \frac{n(n-1)}{2} K^{n-2} K_X a \right] \tag{87}$$

$$+ \sum_{n=1}^{\infty} \frac{\Omega_{nK,o}^i}{n!} \left[\begin{array}{l} aK^n \\ -\varepsilon^2 \left\{ \begin{array}{l} \frac{n(n-1)}{2} K^{n-2} a_{XX} + \frac{n(n-1)(n-2)}{2} K^{n-3} K_X a_X \\ + \frac{n(n-1)(n-2)(n-3)}{8} K^{n-4} K_X^2 a + \frac{n(n-1)(n-2)}{6} K^{n-3} K_{XX} a \end{array} \right\} \end{array} \right]$$

$$+ \varepsilon^3 \sum_{n=1}^{\infty} \frac{\Omega_{nK,o}^R}{n!} \left[\begin{array}{l} \frac{n(n-1)(n-2)(n-3)(n-4)}{12} K^{n-5} K_X K_{XX} a \\ + \frac{n(n-1)(n-2)(n-3)}{6} K^{n-4} K_{XX} a_X + \frac{n(n-1)(n-2)}{6} K^{n-3} a_{XXX} \\ + \frac{n(n-1)(n-2)(n-3)}{4} K^{n-4} K_X a_{XX} \\ + \frac{n(n-1)(n-2)(n-3)}{24} K^{n-4} K_{XXX} a \\ + \frac{n(n-1)(n-2)(n-3)(n-4)}{8} K^{n-5} K_X^2 a_X \\ + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{48} K^{n-6} K_X^3 a \end{array} \right]$$

The first equation describes the perturbation phase corresponding to the slowly varying wavetrain and the second one describes amplitude. However, since we expect our results to extend to all wavenumbers, we sum the above series. This leads to a phase relation of the form

$$\omega = \Omega^R + \varepsilon \Omega^i_K \frac{a_X}{a} + \frac{1}{2}\varepsilon K_X \Omega^i_{KK} - \frac{1}{2}\varepsilon^2 \Omega^R_{KK} \frac{a_{XX}}{a} \tag{88}$$

$$- \frac{1}{2}\varepsilon^2 \Omega^R_{3K} \frac{a_X K_X}{a} - \frac{1}{8}\varepsilon^2 \Omega^R_{4K} K_X^2 - \frac{1}{6}\varepsilon^2 \Omega^R_{3K} K_{XX}$$

After some manipulation the amplitude equation, in X-T variables, becomes

$$\frac{\partial}{\partial T} a^2 + \frac{\partial}{\partial X} \Omega^R_K a^2 = 2 \frac{\Omega^i}{\varepsilon} a^2 \tag{89}$$

$$- \varepsilon^2 \left[a a_{XX} \frac{\Omega^i_{KK}}{\varepsilon} + a a_X K_X \frac{\Omega^i_{3K}}{\varepsilon} + \frac{1}{4} a^2 K_X^2 \frac{\Omega^i_{4K}}{\varepsilon} + \frac{1}{3} a^2 K_{XX} \frac{\Omega^i_{3K}}{\varepsilon} \right]$$

$$+ \varepsilon^2 \left[\begin{array}{l} \frac{1}{3} a a_{XXX} \Omega^R_{3K} + \frac{1}{2} K_X a a_{XX} \Omega^R_{4K} \\ + \frac{1}{3} a a_X K_{XX} \Omega^R_{4K} + \frac{1}{12} a^2 K_{XXX} \Omega^R_{4K} \\ + \frac{1}{6} a^2 K_X K_{XX} \Omega^R_{5K} + \frac{1}{4} K_X^2 a a_X \Omega^R_{5K} \\ + \frac{1}{24} a^2 K_X^3 \Omega^R_{6K} \end{array} \right]$$

Because $\Omega^i / \Omega^R \sim O(\varepsilon)$ <u>all</u> high-order corrections to the basic low-order phase are $O(\varepsilon^2)$, as would be expected for weakly nonconservative waves that are truly slowly varying. (This is also consistent with the approximate Fourier superposition

used.) In the Lagrangian approach, the vanishing of $O(\varepsilon)$ terms would be attributable to the orthogonality of the trigonometric functions, and identical results (Chin, 1976) can be obtained by extending the method of Example 1 in the manner of Yuen and Lake (1975), the latter of which will be later discussed. Similarly, all high-order corrections to the low-order amplitude equation with damping are $O(\varepsilon^2)$. In general, the indicated corrections involve coefficients that depend only on high-order wavenumber derivatives of Ω^R and Ω^i, which are in turn defined by the planar monochromatic wave solution for a particular problem.

The above results show how, in a simple way, the complex dispersion relation of the monochromatic wave completely determines the dynamics of the slowly varying one, as would be expected from the physics of the problem. The high-order terms in Eqs. (88) and (89) based on $\Omega^R(K)$ and its wavenumber derivatives are, of course, directly related to the high-order Lagrangian terms in Eqs. (17) and (20); those based on Ω^i appear on account of the dissipation functional F. But the Fourier integral approach given here provides a much more transparent indication of the structure of the high-order dispersive and diffusive effects than does the equivalent variational one; the results derived here are much simpler to use and do not bear the limiting assumptions central to other approaches employing centered wavenumber expansions. Let us

now consider the effect of weak nonlinearity. As previously discussed the high-order modulation terms for a weakly nonlinear system are determined from the primary harmonic, that is, they are as given in Eqs. (88) and (89). Thus, nonlinear Stokes-type corrections can be incorporated in the above results in a simple "additive" manner. For simplicity let us first deal with conservative problems. The low-order amplitude equation is just

$$\frac{\partial}{\partial t} a^2 + \frac{\partial}{\partial x} \Omega_K^R(K) a^2 = 0 \tag{90}$$

where $\omega = \Omega^R(K)$ is the linear dispersion relation. For moderately small amplitudes a Stokes expansion implies that

$$\omega = \Omega^R(K) + f_2(K)\, a^2 \tag{91}$$

and the wave conservation condition $\partial K/\partial t + \partial \omega/\partial x = 0$ can be expanded out to give

$$\frac{\partial K}{\partial t} + \left\{ \Omega_K^R(K) + f_{2_K}(K)\, a^2 \right\} \frac{\partial K}{\partial x} + f_2(K) \frac{\partial a^2}{\partial x} = 0 \tag{92}$$

The important coupling arises in $f_2(K)\,\partial a^2/\partial x$ since it implies an O(a) correction in the characteristic velocities. The other new term merely corrects the coefficient of the existing term in $\partial K/\partial x$ and contributes only at the $O(a^2)$ level. Similarly, the nonlinear corrections to Eq. (90) would be various terms of $O(a^4)$ providing relative corrections of $O(a^2)$ to the coefficients of the existing terms in $\partial a^2/\partial x$ and $\partial K/\partial x$. Thus, in the first assessment of nonlinear effects, we can retain Eq. (90) and use the new dispersion relation as given in Eq. (91), dropping the nonlinear correction to Ω^R_K in Eq. (92). This is, in essence, Whitham's (1967) approximation. But since high-order modulation effects arising from linear terms must eventually interact with the nonlinearity, Whitham's approximation should be modified to describe this effect.
As is clear from the above discussion, the leading high-order corrections will modify only the phase. Bearing in mind that for near-linear problems the corrections are "additive" in the sense previously discussed we have, on assuming that wavenumber variations are less rapid than those in amplitude (or, if Ω^R_{3K}, Ω^R_{4K}, ... can be neglected) the following approximation

$$\omega = \Omega^R(K) + f_2(K)\,a^2 - \frac{1}{2}\varepsilon^2 \Omega^R_{KK}\,\frac{a_{XX}}{a} \tag{93}$$

which must be solved together with Eq. (90). These equations, again, hold only for homogeneous, conservative media.

To understand the qualitative nature of the above approximation, let us re-expand Eqs. (90) and (93) about $K = K_o$, say, consistently with our assumption of slow wavenumber variations. Again considering the conservative case we have, with $C_o = \Omega^R_{K,o}$,

$$(a^2)_t + ((C_o + \Omega^R_{KK,o}\tilde{\theta}_x)a^2)_x = 0 \tag{94}$$

$$\omega - \omega_o = C_o(K-K_o) + \tfrac{1}{2}\Omega^R_{KK,o}(K-K_o)^2 + f_2(K_o)a^2 - \tfrac{1}{2}\frac{\Omega^R_{KK,o}a_{xx}}{a} \tag{95}$$

Using the notation $\psi = ae^{i\tilde{\theta}}$, these equations combine to give

$$i(\psi_t + C_o\psi_x) + \tfrac{1}{2}\Omega^R_{KK,o}\psi_{xx} = f_2(K_o)|\psi|^2\psi \tag{96}$$

This "nonlinear Schroedinger equation" has been derived in a number of contexts by various authors using different techniques. But our discussion shows that it is really nothing more than a specialized form of Eqs. (17) and (20). This, in fact, will be clearly illustrated in Example 7.

Now in "self-focusing" problems, Zakharov and Shabat (1972) have shown that for initial conditions which approach zero sufficiently rapidly as $|x| \to \infty$ (corresponding to "pulses"), Eq. (96) can be solved <u>exactly</u> by the inverse scattering

technique. They discovered that an initial wave envelope pulse of arbitrary shape will eventually disintegrate into a number of solitons and an unsteady oscillatory "tail" interpreted as noise. The number and structure of these solitons and the structure of the tail are completely determined by the initial conditions. But the tail is relatively small and unimportant for pulse-type initial conditions; it disperses linearly, resulting in a $t^{-\frac{1}{2}}$ amplitude decay. Each of the resulting solitons is a permanent progressive wave characterized by (four) parameters in amplitude, speed, position and phase, but unlike the solitary wave solutions of the KdV equation for shallow water, the amplitude and speed parameters bear no direct relation to each other except that they are the imaginary and real parts of an associated scattering problem. It is important, though, that these solitons are stable in the sense that they can survive interactions with each other without permanent change, except a possible shift in (the parameters for) position and phase. The time scale for the soliton formation was found to be in direct proportion to the length of the pulse and in inverse proportion to the amplitude of the pulse. The above result shows that the end product of unstable "elliptic" modulations, unlike Lighthill's (1965a) low-order cusp solution, is a train of solitons. The exact behaviour depends on the relative magnitudes of $\Omega^{R}_{KK,o}$ and $f_2(K_o)$. But the same qualitative features can be expected for unstable

gravity and capillary waves, or for that matter, all waves that are "elliptically unstable" in Whitham's (1967) sense. To show that the approximation underlying Eqs. (90) and (93), in fact, can be obtained by Lagrangian techniques, it suffices for our purposes to present one illustration.

Example 7. Gravity Waves in Deep Water.

We quote the example considered by Yuen and Lake (1975), who chose for the basic Ansatz (corresponding to Eq. (21)) the wave solution, correct to $O(\varepsilon, Ka)$,

$$\eta = a(\varepsilon X, \varepsilon t)\cos\theta + \tfrac{1}{2}Ka^2\cos 2\theta + \dots \tag{97}$$

$$\varphi = \frac{\omega a}{K}e^{Ky}\sin\theta + \left[\frac{a_t}{K}\cos\theta + \frac{\omega a_x}{K^2}(1-Ky)\cos\theta\right]e^{Ky} + \tfrac{1}{2}\omega a^2 e^{2Ky}\sin 2\theta + \dots \tag{98}$$

The Lagrangian density is just the pressure, that is,

$$L = \int_{-\infty}^{\eta}\left[\varphi_t + \tfrac{1}{2}(\varphi_x^2 + \varphi_y^2) + gy\right]dy \tag{99}$$

where $\eta(x,t)$ is the free-surface position, y is the modal coordinate, $\varphi(x,y,t)$ is the velocity potential, g is the acceleration due to gravity, with the fluid density being unity. Then, the average Lagrangian, correct to $O(\varepsilon^2, K^2a^2)$, is

$$\mathcal{L} = -\frac{\omega^2 a^2}{4K} + \frac{g a^2}{4} + \frac{a_t^2}{4K} + \frac{\omega a_x a_t}{4K^2} + \frac{\omega^2 a_x^2}{8K^3} + \frac{a a_{tt}}{2K} + \frac{3}{4}\frac{\omega a a_{xt}}{K^2} + \frac{3}{8}\frac{\omega^2 a a_{xx}}{K^3} + \frac{g K^2 a^4}{8} \tag{100}$$

and we may note that, in this approximation, space-time derivatives of frequency and wavenumber do <u>not</u> appear. After some algebra, they found that θ variations gave

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x} C_g a^2 = 0 \qquad C_g = \tfrac{1}{2}\left(\frac{g}{K}\right)^{1/2} \tag{101}$$

while "a" variations gave, on simplifying, the phase relation

$$\omega = \sqrt{gK}\left[1 + \tfrac{1}{2}K^2 a^2 + \frac{a_{xx}}{8aK^2}\right] \tag{102}$$

Now this is precisely the approximation deduced in Eqs. (90) and (93). To see this, simply note that the Stokes solution gives $f_2(K) = \frac{1}{2} g^{\frac{1}{2}} K^{5/2}$ while $\mathcal{L}^R_{KK} = -\frac{1}{4} g^{\frac{1}{2}} K^{-3/2}$ from the linear part of the dispersion relation. The above results were also derived by Chu and Mei (1970) using multiple-scaling methods directly from the governing equations. All these results, of course, are really special cases of Eqs. (88) and (89) modified for weak nonlinearity, which do not bear the limitations imposed by "centered wavenumber" expansions.

This is especially important for, in analyses of the latter kind, one implicitly assumes (in some nondimensional sense) that amplitude variations dominate those in wavenumber - otherwise the exparsions are meaningless. In another sense, the generalized modulation equations derived are useful because it is not even clear that a "centered wavenumber" stays constant on the scale over which the high-order modulation terms become important. It is crucial to note the equivalence between Eqs. (20) and (17) and Eqs. (88) and (89). They remain applicable for large values of $K_x(X,T)$.

To the above results for conservative and homogeneous media we now add the effects of dissipation. The same line of reasoning that led to Eq. (93) in the nondissipative case now requires that we retain in Eq. (88) the $\varepsilon\Omega_K^i \frac{a_X}{a}$ term, which balances the dispersion term, both being $O(\varepsilon^2)$. At the same time the low-order damping term $2\frac{\Omega^i}{\varepsilon}a^2$ in Eq. (89), first suggested in Landahl (1972), must be kept (see Example 1). The first of these corrections seems to have appeared in a number of studies in hydromagnetic waves, according to Malkus (1976), though in a less general form. In the next approximation, high-order corrections to the "simply-damped" amplitude equation involve the diffusion terms $-\varepsilon^2\frac{\Omega^i_{KK}}{\varepsilon}a\,a_{XX}$, etc., and the dispersion terms $\frac{1}{3}\varepsilon^2\Omega^R_{3K}a\,a_{XXX}$, etc., in Eq. (89), and the remaining terms in Eq. (88). The etc.'s here generally refer to terms involving higher differentiations of Ω^R and Ω^i with respect to wavenumber and the appearance of the terms

K_X , K_{XX} and K_{XXX} (which would have been omitted in the centered wavenumber approach). Thus the appearance of these terms depends, in a sense, on "how dispersive" or "how dissipative" the original uniform wavetrain is. For example, the vanishing of Ω^R_{3K} would leave only diffusive corrections to Eq. (89); the phase relation, though, would still bear the effects of high-order dispersion. In this kind of limit, the most obvious effect of diffusion would appear to be a gradual decay of the solitons as are produced in the Zakharov and Shabat (1972) solution. No general conclusions are available, at this point, for Eqs. (88) and (89) with (or without) their nonlinear corrections. Some future studies are planned, however, to test the separate effects of strong dispersion and strong diffusion for various kinds of waves. However some speculation is possible on what types of "strange" behaviour might be expected. For example the "diffusion coefficient" $-\varepsilon\Omega^i_{KK}$ in Eq. (89) in the case of deep-water surface waves is just the kinematic viscosity (Lamb, 1932) which is positive, as expected. This positivity is also apparent from existing numerical studies in boundary layer and vortex instability; to the author's knowledge, the plot of $\Omega^i(K)$ against K has always appeared concave down. A surprising result, however, is found in warm electron plasma waves. Classically the waves are Landau-damped, that is, $\Omega^i(K) < 0$, so that according to the low-order form of Eq. (89), the waves always decay. However, direct calculation

of $-\Omega^i_{KK}$ shows that there exists a range of wavenumbers bearing negative diffusion. Thus, on a large time scale, there may occur a transfer of energy back into certain of the formerly decayed waves. The possibility of sign variations in $-\Omega^i_{KK}$ is of fundamental importance, especially from the viewpoint of developing deterministic wave models of turbulent shear flow (Chin, 1976); it allows, for example, energy transfer by classical cascading and by reverse-cascading.

We also mention, in context, the "nonlinear instability burst" found by Stewartson and Stuart (1971), who studied the stability of wave systems in plane Poiseuille flow. They considered an infinitesimal centered disturbance imposed on a fully developed plane Poiseuille flow at a Reynolds number R slightly greater than the critical value R_c for instability. The disturbance was assumed as a wave modulated in space and time, whose amplitude "a" satisfied a nonlinear parabolic equation. For finite values of $(R-R_c)t$, Hocking, Stewartson and Stuart (1972) showed that the amplitude develops an infinite peak at the center of the wave group. This result also appears as a special limit of the present theory, although the number of assumptions required seems somewhat restrictive. In Stewartson's approach, center wavenumber expansions for $K = K_o$ are taken about a point of "maximum growth", that is, $\Omega^i_K(K_o) = 0$. If we further assume that the waves are weakly dispersive ($\Omega^R_{KK}(K_o) \cong 0$) the phase relation becomes $\omega \cong \Omega^R(K)$ and the amplitude equation reduces to

$$\frac{\partial a}{\partial T} + \Omega_K^{(R)}(K_0)\frac{\partial a}{\partial X} = a\,\frac{\Omega^i(K_0)}{\varepsilon} - \frac{1}{2}\varepsilon^2\,\frac{\Omega^i_{KK}(K_0)}{\varepsilon}\,\frac{\partial^2 a}{\partial X^2} \qquad (103)$$

Stewartson then determined the coefficients of "a" and its derivatives on the basis of linear stability theory and added $O(a^3)$ nonlinear corrections to the growth rate $\Omega^i(K)$ by expanding the production term in powers of amplitude. This led to a nonlinear parabolic equation, whose solution was obtained by matching to the stationary phase solution of linear theory (assuming zero disturbances far from the wave center). The present author's own view is that Stewartson's theory (as generalized here) is overly restrictive in the sense that the kinematic coupling between wavenumber and wave amplitude is not considered. This may be correct in the initial stage, but it is certainly not so in the latter stages during which the amplitude singularity appears. We mention in passing that the structure of Landahl's (1972) focus (based on the effects of ray focusing due to inhomogenieties using Whitham's equations modified for dissipation) was also obtained in Chin (1976). Matched asymptotic expansions were used to connect the "inner" focal structure (based on a modified form of Eq. (89) allowing for inhomogeniety) where diffusive effects would be important to an "outer" solution based on Whitham's equations modified to account for nonconservative effects. The waves were assumed (1) to be only weakly dispersive, (2) to have $-\Omega^i_{KK}$ approximately constant, and (3) to satisfy $(dA/A)/(dK/K) \gg 1$ near the focus.

As was expected, the effect of diffusion was a smoothing of the amplitude peak. The thickness of the region over which diffusive effects are likely to be importnat was found to be of $O(\varepsilon^{1/3})$ in unscaled x-space. This thickness decreases with decreasing viscosity ε, as would be expected on physical grounds. In contrast, Stewartson's "nonlinear instability burst" for channel flows occupies a length of $O(|\log \varepsilon|^{1/2})$ which increases with decreasing viscosity. The details of the above study will be presented in a separate paper.

IV. Discontinuous Solutions.

Perhaps the most fascinating area of nonlinear wave mechanics is the study of shocks and their formation. The possible existence of these wave mechanical discontinuities was first suggested by Whitham (1965) and a good discussion on the subject appears in Chapter 15 of Whitham (1974). However there is little experimental evidence as to their actual existence. The observability of the shocks suggested by Whitham's low-order theory may, in practice, be somewhat obscured by the effect of high-order dispersive and diffusive terms as were previously derived in Section III and the effects of low-order inhomogeniety, which are certain to affect their stability. The first type of shock considered by Whitham deals with the question of "breaking" and arises when the low-order modulation equations are hyperbolic. The dependence of the characteristic velocities on the modulation variables introduces the usual hyperbolic distortion, and "compressive" modulations in a simple wave solution will develop multivalued regions. It is possible, Whitham notes, that these solutions actually represent superpositions of two or more wavetrains with different ranges of K and "a". The solutions, though, would not be correctly described by Eq. (2); some type of multiphase generalization would be needed. Whitham compares this situation to that in linear theory where the group velocity $C_g(K)$ decreases toward the front. Then, since values of K propagate with

velocity $C_g(K)$, some kind of overlapping would occur. In the linear case the complete process is described by Fourier integrals; presumably, in the nonlinear case, some type of nonlinear superposition would hold. There does, however, exist the possibility that higher order terms become important near breaking and prevent the development of multivalued solutions. This is certainly the case if the only $O(\varepsilon^2)$ corrections kept are the dispersive ones, that is, those dependent on $\Omega^R(K)$ as shown in Eqs. (88) and (89). The resulting equations then become similar in form to the Boussinesq and Korteweg-deVries equations, and by analogy, breaking would be suppressed. This behaviour would most likely apply to small symmetric modulations, as noted by Whitham (1974), and eventual development into series of solitary waves would occur, as would be suggested by the solution of Zakharov and Shabat (1972). On the other hand, strong unsymmetrical modulations may break in some sense. If the high-order diffusion terms indicated in Eqs. (88) and (89) are more dominant than the dispersive ones, the resulting amplitude equation resembles a modified Burgers' equation. In the narrow "breaking region" where diffusion balances inertial effects, the "$-\Omega^i_{KK}a_{xx}$" term can be expected to produce a continuous shock structure. It is interesting to note that relative effects between high-order diffusion and dispersion will in general be different for different wavenumbers. Thus, the situation will be much more complicated than the

analogous case of gasdynamic shocks. There is still another important difference. Provided $-\Omega^{i}_{KK}$ is nonzero, it is always possible to define a diffusion coefficient —— even for a perfectly inviscid system. For surface waves on deep water there is no difficulty; the diffusion coefficient so defined is just the kinematic viscosity. But for waves on inviscid shear flows, wave diffusion will be present in general. It is even possible for this diffusion to be negative in a certain range of wavenumbers (as was the case for waves on warm electron plasmas). Then the gasdynamic analogy calling for a smooth shock structure would not hold. It is not clear, though, what would happen; but the high-order equations derived in Section III (suitably modified for nonlinearity) can, in principle, be considered numerically as initial value problems.

The second class of shocks posed by Whitham arises in the search for weak solutions to the modulation equations. For problems in which a mean background state does not enter (e.g., gravity waves in deep water), the only dependent variables are those for the wavenumber K and the wave amplitude "a". The values of "a" and K upstream and downstream of the discontinuity would be connected by postulated jump conditions, for example, those conserving energy flux, momentum flux, frequency, etc. The actual choice of jump conditions is open to question, however. One can argue that energy and momentum are conserved in the detailed description for the variable u (see Eqs. (1) and

(2)) and should therefore be retained in the slowly varying approximation. These shocks would therefore represent a source of oscillations and involve jumps in the adiabatic invariants. "If, on the other hand," quoting Whitham (1974), "the discontinuities are supposed to represent phenomena not covered by the original equation, but covered by some even more detailed description involving dissipation of some kind, then the choice would be different. Although momentum would probably be conserved, energy presumably would not." Action flux is not likely to be conserved, "but one could make a case for frequency. With dissipation smooth oscillatory changes between different constant states may be constructed in dispersive models." The constant end states referred to above would be the result of dissipation dampening out the oscillations on the two sides of the transition region. The qualitative effects of diffusion were considered in Whitham (1974, p. 482) using a simple model. There he added a Burgers type term to the KdV equation to produce a simple model for the structure of bores. The actual terms, of course, are given in Eqs. (88) and (89) if the system is weakly nonlinear. The dissipation term is really $-\Omega^{i}_{KK} a_{xx}$, suggesting that the unsteady formation and resulting shock structure (not attempted here) is much more complex that Whitham's model would suggest, although his results are qualitatively correct. The actual results, of course, would be highly wavenumber dependent. The existence of the above shocks, at this point, is speculative.

But they would correspond to the low-order weak solutions of the full equations (in the appropriate conservation form) with the dissipation functional F set identically to ze at the outset. The high-order terms in Eqs. (88) and (89) do provide the answers as to the structure of the transition region if the above shocks indeed exist. They also lay the groundwork for a first-cut phase plane analysis.

A strong reason, perhaps, for studying discontinuous solutions of the above kind is suggested by an analogy with gasdynamics. Consider the propagation of one-dimensional unsteady disturbances in a transonic decelerating channel flow. It is possible to trace the evolution of expansion and compression pulses originating downstream of a linearly decelerating sonic region by examining the equations for one-dimensional unsteady isentropic flow. Kantrowitz (1947) was first to show that such disturbances are deformed as they propagate upstream, and that the disturbances tended to collect in the sonic region. The deformation of the pulse shape actually causes weak shock waves to form, either at the head front or the rear of the pulse; expansion pulses, which terminate in shock waves at their rear, are subsequently "consumed" by their own shock waves, while compression pulses, which have shock fronts, grow in strength as their shock fronts progress into the supersonic region. Ultimately, in a real channel, they would destroy the supersonic flow, converting it into subsonic flow. This simple example suggests the existence of flows which cannot support arbitrary infinitesimal perturbations, but which must break down locally into finite amplitude initially small-scale oscillations. Instabilities of the above kind are "catastrophic" in nature, but they are not restricted to transonic flows. One can speculate that similar highly unsteady, nonlinear physical mechanisms might be responsible for other "strong" fluid-dynamical instabilities, for example, as in hydraulic jump formation, vortex breakdown, and perhaps the sudden

transitions observed in boundary layer instability. The transonic analogy leads us to envision an initially (essentially linear) phase during which selected self-excited secondary waves focus on a larger scale inhomogeneous primary wavetrain. Subsequent trapping and continued wave growth in the absence of dissipative effects would then precipitate a rapid, almost discontinuous, and certainly a strongly nonlinear readjustment of the mean flow in order to satisfy certain global constraints (for example, conservation of total mass, momentum or energy). This type of breakdown should be a distinct possibility for any inhomogeneous continuum system which can support propagating waves of scale small relative to the scale of the inhomogeneities The "initial phase" (as described above) is amenable to general linear analysis using recently developed ideas in kinematic wave theory, and first results pursued along these lines have been given by Landahl (1972). (Kinematic wave theory is developed using phase averages, but identical results can be derived using equivalent averages over phase shift; in this case the averaging is one over a random superposition of linear waves and is, perhaps, more relevant to the physical situation). In his linear analysis, Landahl used the modified action equation accounting for weakly nonconservative effects, as given in Eq. (34). The actual kinematics are therefore describable (to this order) by the "real" dispersion relation $\Omega^{(R)} = \Omega^{(R)}(K, \varepsilon x, \varepsilon t)$, and the equation for wave conservation can then, within the context of linear theory, be used to predict when space/time singularities in the wave trajectories can be expected to occur.

In fact, some manipulation shows that (Landahl, 1972) focusing occurs whenever

$$\Omega_K^{(R)} - C_o = \frac{\Omega_{Kt}^{(R)} + \Omega_K^{(R)}\Omega_{KX}^{(R)} - \Omega_{KK}^{(R)}\Omega_X^{(R)}}{\Omega_{KX}^{(R)} + \Omega_{KK}^{(R)}\frac{\partial K}{\partial X}} \to 0$$

is satisfied, where C_o is the critical velocity $-(\partial\Omega_K^{(R)}/\partial t)/(\partial\Omega_K^{(R)}/\partial x)$. In the steady case, the critical condition reduces to the approach toward a zero in the linear group velocity (Landahl's linear theory does not account for the nonlinear response of the supporting medium). This condition in itself implies increased wave amplitudes on account of "wave tube" convergence. If this slowing down is additionally accompanied by trapping, self-excited waves (as determined from a local application of conventional normal mode theory) would then surely amplify forever, at least on the basis of linear theory (details are given in Landahl, 1972). One would then surmise that the sudden onset of this kind of strong instability would precipitate a highly nonlinear, unsteady readjustment of the mean flow, in much the same way that wave trapping and amplification in transonic nozzles lead to shock formation and choking. Of course, Landahl's analysis is a linear one and therefore does not account for the nonlinear response of the mean flow. But the physics of the present study is in the same spirit of Kantrowitz's classic analysis, in which the small-perturbation isentropic equations are used to describe the <u>onset</u> of the instability mechanism,

only because the general unsteady, nonlinear problem is intractable. Then, as in Kantrowitz (1947), if the model in fact describes the true hypothesized instability process, one can therefore proceed to study the properties of the shocks that are formed in the latter stages. (Of course, one can study shocks for their own sake, but it is often helpful to envision some physical generating mechanism. Some of the ideas proposed here might be tested by numerical integration of the complete system of unsteady, nonlinear, high-order modulation equations subject to appropriate initial conditions. Extreme care in the analysis must be taken, however, since the results depend sensitively on the delicate balance between high-order wave dispersion and diffusion, not to mention the artificial dispersive and diffusive effects of truncation errors.)

The idea of a hypothetical instability mechanism possibly leading to a final configuration with discontinuous end states is not new, and some inferences can at least be drawn from hydraulic jump formation and, in vortex breakdown, they are at least suggested by the "finite transition" or "conjugate flow" theories of Benjamin (1962). This latter analysis was first to demonstrate the possibility of an <u>infinitesimal</u> transition from a supercritical state which cannot support standing waves, to a subcritical state which can (but the theory does not offer an explanation as to the origin of the discontinuity); the theory essentially postulates in an ad hoc manner the physical necessity for the existence of waves on the subcritical side, so

as to maintain the continuity of total momentum flux. We can improve on Benjamin's theory by considering final end states that are essentially inviscid and essentially "organized" in the sense that a wave description is still applicable; the end state of the "downstream" portion would consist of a highly enhanced wave riding on a mean flow of reduced energy, formed from an "upstream" portion whose wave is initially infinitesimal in amplitude and which rides on a more energetic mean flow. We can modify Benjamin's conjugate flow approach (which, in effect, considers only the mean flow in developing the discontinuous solutions) by connecting the end states through the introduction of two more free parameters, a wavenumber K and a wave energy density E of a superimposed wave system (however small), in addition to the man velocity and the mean cross-sectional distance. In the context of hydraulic jumps (which are here treated for simplicity), the proposed model would connect E, K, the mean height h and the mean speed U by conserving across the discontinuity total energy, total momentum, total mass and waves (i.e., we fix the frequency, for a total of four jump conditions). This model thus augments Benjamin's by systematically accounting for the presence of a wave however small, and through the nonlinear coupling, establishes whether or not conjugate solutions do, in fact, exist for all parameter ranges. Let $\hat{U}$ be the depth-independent "mass transport velocity", as given in Phillips (1969), which already accounts for the mass flow induced by the wave. For simplicity, we

examine the shallow water case, so that the wavenumber K does not explicitly appear in the conservation laws for total mass, momentum or energy. The results we are to derive, of course, are valid in quasi-steady problems in which the local coordinates move with the shock. In these steady coordinates, the governing equations for total mass, momentum and energy integrate to, for a flat bottom,

$$\rho h u = m$$

$$\rho h u^2 + \tfrac{1}{2}\rho g h^2 + \tfrac{3}{2} E = \tilde{P}$$

$$\tfrac{1}{2}\rho h u^3 + \rho g u h^2 + \sqrt{gh}\, E + \tfrac{5}{2} E u = \tilde{Q}$$

where m, $\tilde{P}$ and $\tilde{Q}$ are the prescribed (total) mass, momentum and energy fluxes, ρ is the fluid density, h is the mean depth and u is the relative velocity $\hat{U}-U_s$ (U_s being the shock speed). Let us introduce the nondimensional variables

$$\bar{h} = h/\left(\frac{2m^2}{g\rho^2}\right)^{1/3} = h/h_o\,, \quad \bar{u} = u/\left(\frac{gm}{2\rho}\right)^{1/3} = u/u_o$$

$$\bar{E} = E/\left(\frac{4}{27}\,\frac{m^4 g}{\rho}\right)^{1/3} = E/E_o$$

so that the above conservation laws become, respectively,

$$hu = 1 \tag{104}$$

$$hu^2 + h^2 + E = \frac{\tilde{P}}{\rho h_o u_o^2} \equiv P > 0 \qquad (105)$$

$$hu^3 + 4uh^2 + \frac{4}{3}\sqrt{2}\,h^{1/2}E + \frac{10}{3}Eu = \frac{2\tilde{Q}}{\rho h_o u_o^3} \equiv Q > 0 \qquad (106)$$

If E = 0 (bars have been dropped for convenience) and Eq. (106) is deleted, one obtains the classical hydraulic jump formulation in which mean flow mass and momentum fluxes are fixed (Lamb, 1932). However, let us eliminate u from Eq. (105) using Eq. (104) to produce an equation for E (in terms of h and P) which, in turn, is used in Eq. (106) to give an equation for h. Setting $H = h^{\frac{1}{2}}$, a ninth order equation is obtained as

$$H^9 - \frac{1}{4}\sqrt{2}\,H^6 - PH^5 + \frac{3}{8}\sqrt{2}\,QH^4 + H^3 - \frac{5}{4}\sqrt{2}\,PH^2 + \frac{7}{8}\sqrt{2} = 0 \qquad (107)$$

where P, Q and H are positive and all roots obtained must be such that $E \geq 0$ to qualify as physically realistic conjugate solutions. A rough idea for the type of jumps typical of Eqs. (104-106) is obtained by comparison with the results of the classical formulation. We denote by "1" and "2" the upstream and downstream states respectively, and require $h_2 > h_1$ in the usual way. For the case P = 2.12 (corresponding to a Froude number of $3^{\frac{1}{2}}$) we have $h_2/h_1 = 2$, $h_1 = 0.55$ and $h_2 = 1.10$. The mean flow energy flux (in the classical case)

$Q_m = hu^3 + 4uh^2 = 4h + h^{-2}$ is 5.50 upstream and 5.23 downstream, the deficit ratio being 5.23/5.50 = 95.0%. The wave theory we propose gives similar results, but here the loss in mean flow energy is made up for by the presence of radiating waves. For comparison's sake, set P = 2.12 and Q = 5.50. The conjugate depths are now obtained as $h_1 = 0.55$ and $h_2 = 1.03$, and calculations show $E_1 = 0.0$ while $E_2 = 0.084$. In other words, the waves are practically nonexistent on Side "1", but on Side "2" they acquire a sudden visibility. The ratio of downstream to upstream mean energy flux is 92.1% (as opposed to 95.0% before), and the rest goes into waves. The phase velocity can be shown to decrease across the jump so that, since the frequency is fixed, the wavenumber increases. The sudden visibility of the wave is enhanced by both increased wave amplitude and wavenumber; the classical mean flow results as described in Lamb (1932) are, in fact, qualitatively reproducible by the above wave model without special appeal to turbulent dissipation.

More general results are obtained by examining positive E solutions of Eq. (107). There is a simple way to construct the entire family of solutions, and this is seen by rewriting Eq. (107) in the form

$$P = \frac{\frac{3}{8}\sqrt{2}\,H^4}{H^5 + \frac{5}{4}\sqrt{2}\,H^2}\,Q + \frac{H^9 - \frac{1}{4}\sqrt{2}\,H^6 + H^3 + \frac{7}{8}\sqrt{2}}{H^5 + \frac{5}{4}\sqrt{2}\,H^2} \qquad (108)$$

Thus, every value of the positive parameter H defines a

straight line in the P-Q plane. The intersection of two lines (or more) at a point (P_o, Q_o) shows that there exists two (or more) conjugate values of H at that point. This interpretation requires us only to study the slope and P-intercept functions of Eq. (108); not every point on a straight line in the P-Q plane is physically significant, though, since it is necessary for the corresponding wave energy to be positive. The simplest criterion for this is found from Eq. (105), that is,

$$E = P - H^4 - \frac{1}{H^2} \geq 0$$

Thus, for $E \geq 0$, it is necessary that $P \geq P^* = H^4 + H^{-2}$. A useful presentation of some results is shown in Figure 1. Various sets of H_1, H_2 are considered such that $h_2/h_1 = 2.25$, so that we in fact assume (H_1, H_2) to be a conjugate pair. Then, each H determines a straight line in the P-Q plane, with the intersection point P_o, Q_o determined from the solution of two simultaneous linear equations derived from Eq. (108). Repeating the process for various h's generates different solutions, as shown in Figure 1 (different values of h_2/h_1 were also considered, and the same qualitative results were obtained). The results show that every point along the solid portion of the line determines <u>two</u> conjugate solutions; the jump in h from h_1 to h_2 is always accompanied by an increase in the wave energy density and a decrease in the mean flow energy flux Q_m previously defined, again agreeing qualitatively with

classical theory, but without appeal to turbulent dissipation. Each point along the solid curve here possesses two roots H with positive E's, while the dashed portion indicates the existence of one negative root; in this latter case the solution is unique and conjugate solutions are not possible. It follows that conjugate solutions, at least in the case of shallow water gravity waves, are possible only for sufficiently large P's or Q's.

A second kind of discontinuous solution is typified by the flow of gravity waves over infinitely deep constant speed currents. Because the current in this overly simplified model does not vary with depth, jumps in the mean speed U_m must be disallowed, for they would otherwise imply infinite changes in mean energy. The wave in this sense decouples from the mean flow and the only discontinuities that form are those in the wave parameters themselves (jumps in U_m for y-dependent mean flows can conceivably be analyzed in terms of jumps in $\hat{U}$ as given in Eq. (51), but it is unclear as to whether the corresponding shape function remains invariant across the discontinuity). By varying the parameter U_m a family of related conjugate solutions can be obtained and these solutions will form the present focus of attention. It will be helpful here to envision surface or plasma waves in steady flow and, for reasons previously indicated, discontinuities across which frequency (ω_o) and momentum flux (m_o) are fixed (the modifications needed to consider wave

breaking are not considered here). Some analysis shows (this will be verified later) that when the (real) frequency is expanded in the wave action density A, that is,

$$U_m K + F(K) + AG(K) = \omega_o$$

the corresponding expression for momentum flux becomes

$$U_m KA + AKF_K + \tfrac{1}{2}A^2(G + KG_K) = m_o$$

Solutions can be readily obtained by eliminating U_m between these two equations, tabulating A as the wavenumber K is varied, then solving for U_m from either of the above equations, and finally solving for the momentum velocity C_{mom} defined by

$$C_{MOM} = U_m + F_K + \tfrac{1}{2}A\left(\frac{G + KG_K}{K}\right)$$

The resulting equations are valid for both waves advancing and receding with respect to the mean flow, of course, provided appropriate sign changes in F(K) and G(K) are made. Only real solutions of A with $A > 0$ are admissible, and hence, multivalued solutions may or may not exist for all values of the parameter U_m. For illustrative purposes let us consider two weakly nonlinear irrotational examples, first gravity waves and second, capillary waves. In the case of gravity waves, the appropriate dispersion functions $F = g^{\frac{1}{2}}K^{\frac{1}{2}}$ and $G = K^3$ where, again, g is the acceleration due to gravity. The barred nondimensional variables

$$\bar{K}^{\frac{1}{2}} = \frac{g^{\frac{1}{2}}}{2|\omega_o|} K^{\frac{1}{2}} \qquad \bar{A} = \frac{128}{g^3} |\omega_o^5| A$$

$$\bar{U} = \frac{4}{g}|\omega_o| U_m \qquad \bar{C} = \frac{4}{g}|\omega_o| C_{mom} \qquad p = 256 \frac{m_o \omega_o^4}{g^3}$$

are introduced, and we shall assume without loss of generality, that $\omega_o > 0$ throughout. Figure 2 shows the complete family of $p, \bar{C} > 0$ "positive moving" waves consisting of both waves advancing and receding with respect to the mean flow, while Figure 3 shows the complete solution for "negative moving" waves with $p, \bar{C} < 0$. For instance, from Figure 2, only one wave solution exists when $\bar{U} \ll -1$ but three solutions exist for $\bar{U} \gg 0$, say, corresponding to a given flux p. Figure 3 shows how for positive $\bar{U}$, two wave solutions can be found for a given negative flux $p < 0$ (corresponding to a "strong" high-amplitude and a "weak" low-amplitude solution across which energy flux decreases), but solutions for sufficiently negative $\bar{U}$'s cannot be found (this may point to the importance of unsteady effects). The "singular point" occurring at $\bar{U} \cong -1$ approximately corresponds to the position determined from linear theory as that producing a zero in the linear group velocity. For capillary waves (where $F = T_o^{\frac{1}{2}} K^{3/2}$ and $G = -K^3/8$) the (barred) nondimensional variables

$$\bar{K} = \frac{T_o^{1/3}}{|\omega_o|^{2/3}} K \qquad \bar{A} = \frac{|\omega_o|}{4 T_o} A \qquad \bar{U} = \frac{U_m}{|\omega_o T_o|^{1/3}} \qquad \bar{C} = \frac{C_{mom}}{|\omega_o T_o|^{1/3}}$$

with surface tension T_o and nondimensional momentum flux $s = m_o/T_o$ can be defined. Let us here assume $\omega_o < 0$ throughout. For various specified values of positive s, the complete solutions (consisting of both waves advancing and receding with respect to $\bar{U}$) are plotted in Figure 4; negative moving solutions with $s < 0$ are shown in Figure 5. From Figure 4 for positive moving waves, multivalued solutions corresponding to various fixed values of s exist for sufficiently negative $\bar{U}$'s, but for positive $\bar{U}$'s the solutions are single-valued; similar considerations hold for negative moving waves, as is evident from Figure 5. Again, the $\bar{U}$ position about which the multi-valued/single-valued character of the problem pivots is determined approximately by the value of $\bar{U}$ for which the linear group velocity is zero. On Figures 2-5 the nondimensional wavenumber distributions are also shown.

In practice the jumps discussed above are formed on currents that vary slowly in space and time, and it is important here to note that the above results are applicable as well in quasi-stationary coordinates moving with the shock. One question of fundamental importance concerns the observability of the hypothesized shocks, and this can be answered in part by examining their stability with respect to self-induced disturbances. In the simple analysis that follows, high-order modulation terms (that in reality allow or disallow the shock formation) will be ignored; we in effect assume the existence of a shock and study its stability as would be affected by low-order inhomogeneities in the medium. Two types of shocks can be afforded

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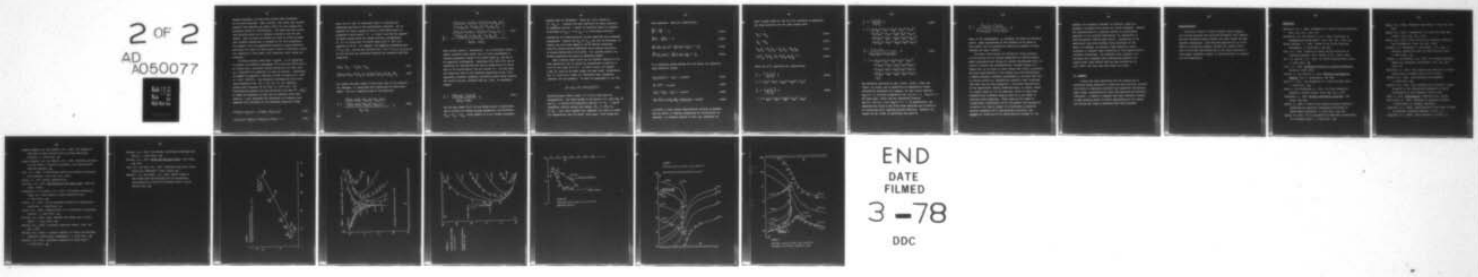

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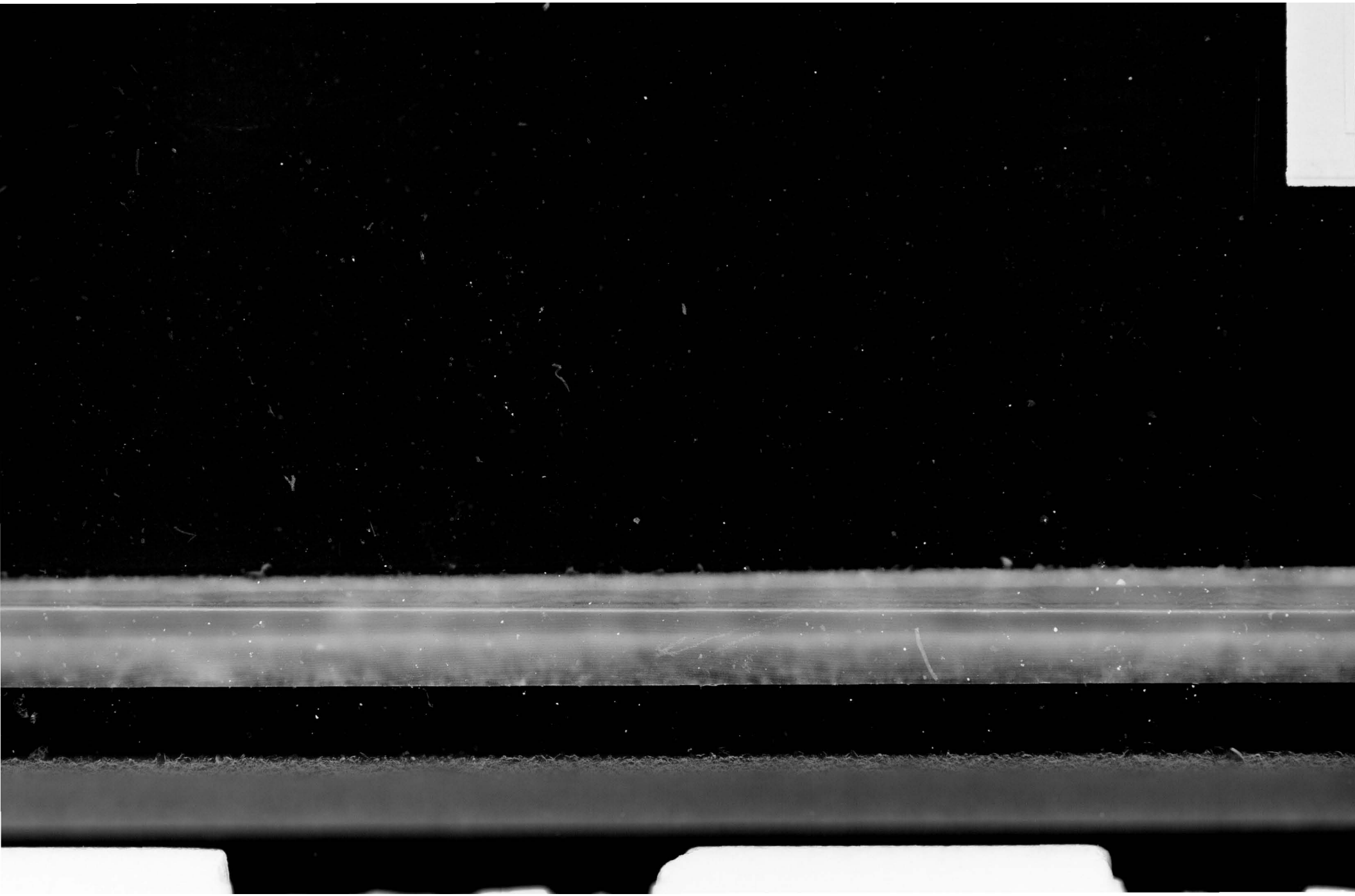

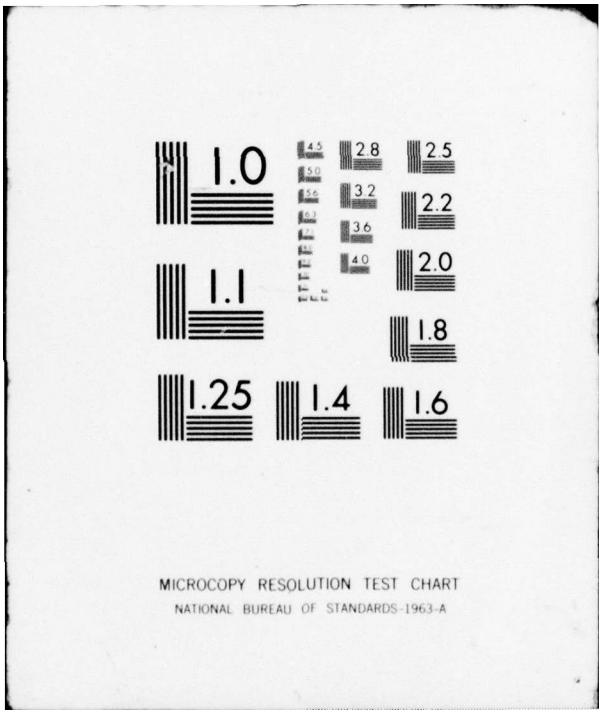
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general treatment, and they have already been introduced in the above analyses. Type A shocks, like those just treated, conserve wave momentum and waves, while the wave energy flux decreases across the discontinuity. The mean flow here enters only parametrically and it formally uncouples from the wave system. Type B shocks are defined as those in which the mean flow responds dynamically to changes in the wave parameters, for example, as in the generalized hydraulic jumps considered previously for water of finite depth; these by definition invoke the constancy of total mass, total momentum, total energy and frequency.

We shall consider first Type A shocks. It is convenient to assume a perturbation Lagrangian $\mathcal{L}'$ in the form $\mathcal{L}'=A\Omega-H(K,A)$ as viewed from a coordinate system moving with the mean flow U_m ; henceforth all subscript m's will be deleted. (Previously we assumed an expanded H function written in powers of A, with $H = AF(K) + \frac{1}{2}A^2G(K)$. The frequency following the flow U was thus determined from $\mathcal{L}'_A = 0$ as $\Omega = F + AG$, implying a ground-fixed frequency of the form $\omega = UK + F + AG$. The corresponding momentum flux was determined from $UKA + \mathcal{L}' - K\mathcal{L}'_K$.) To begin the analysis, we note that an unsteady shock with speed $U_s = U_s(T)$ satisfies the conditions for frequency and momentum flux continuity in the following respective forms

$$(U-U_s)K_1 + H_A(K_1,A_1) = (U-U_s)K_2 + H_A(K_2,A_2) \tag{109}$$

$$(U-U_s)A_1K_1 + A_1H_A(K_1,A_1) + K_1H_K(K_1,A_1) - H(K_1,A_1) = \ldots \tag{110}$$

where the "1" and "2" subscripts refer to upstream and downstream portions of the discontinuous solution. Let us consider the basic problem in which a stationary shock is displaced a small amount ξ. A, K and U can thus be expanded by Taylor series about the steady solution corresponding to $\xi = 0$, noting that the shock speed $U_s(T) \cong \dot{\xi}(T)$ is a quantity of $O(\xi)$. For example, the suggested expansions give $A \cong A_o + A_o' \xi$ and for any function S(A), $S(A) \cong S(A_o) + S_A(A_o)A_o' \xi$. To O(1) we have the following frequency and momentum flux jump conditions

$$U_{1,o}K_{1,o} + H_{A_{1,o}} = U_{2,o}K_{2,o} + H_{A_{2,o}} \tag{111}$$

$$U_{1,o}A_{1,o}K_{1,o} + A_{1,o}H_{A_{1,o}} + K_{1,o}H_{K_{1,o}} - H_{1,o} = U_{2,o}A_{2,o}K_{2,o} + A_{2,o}H_{A_{2,o}} + K_{2,o}H_{K_{2,o}} - H_{2,o} \tag{112}$$

the weakly nonlinear forms of which were used in the previous two examples, "o" subscripts here indicating the basic mean state. The $O(\xi)$ equations lead to the equations

$$\dot{\xi} = \frac{\left(U_{2,o}'K_{2,o} + U_{2,o}K_{2,o}' + H_{AA_{2,o}}A_{2,o}' + H_{AK_{2,o}}K_{2,o}'\right) - \left(U_{1,o}'K_{1,o} + U_{1,o}K_{1,o}' + H_{AA_{1,o}}A_{1,o}' + H_{AK_{1,o}}K_{1,o}'\right)}{K_{2,o} - K_{1,o}} \xi \tag{113}$$

and

$$\frac{\dot{\xi}}{\xi} = \frac{\left\{\begin{array}{l} U_{2,0}A_{2,0}K'_{2,0} + U_{2,0}A'_{2,0}K_{2,0} + U'_{2,0}A_{2,0}K_{2,0} + A_{2,0}H_{AA_{2,0}}A'_{2,0} \\ + A_{2,0}H_{AK_{2,0}}K'_{2,0} + K_{2,0}H_{KA_{2,0}}A'_{2,0} + K_{2,0}H_{KK_{2,0}}K_{2,0}' \end{array}\right\} - \left\{\begin{array}{l} U_{1,0}A_{1,0}K'_{1,0} + U_{1,0}A'_{1,0}K_{1,0} + U'_{1,0}A_{1,0}K_{1,0} + A_{1,0}H_{AA_{1,0}}A'_{1,0} \\ + A_{1,0}H_{AK_{1,0}}K'_{1,0} + K_{1,0}H_{KA_{1,0}}A'_{1,0} + K_{1,0}H_{KK_{1,0}}K_{1,0}' \end{array}\right\}}{A_{2,0}K_{2,0} - A_{1,0}K_{1,0}} \qquad (114)$$

where primes denote ξ derivatives. As in Kantrowitz (1947), a number of growth rates exist, each of which corresponds to various permissible values of the shock speed U_s, and are to be separately investigated. Equations (113) and (114) can be simplified by invoking wave and action conservation on either side of the discontinuity (but not across it). On this basis one finds that each of the bracketed quantities in Eq. (113) separately vanishes, implying a neutrally stable shock solution. The second solution, obtained from Eq. (114), is remarkably simple:

$$\dot{\xi} = \frac{-A_{2,0}K_{2,0}U'_{2,0} + A_{1,0}K_{1,0}U'_{1,0}}{A_{2,0}K_{2,0} - A_{1,0}K_{1,0}}\,\xi \qquad (115)$$

Now the mean speed U(X,T) in the present model is prescribed, and to fulfill the slowly varying assumption, the derivative $U'_{2,o} = U'_{1,o} = U'_{(0)}$ (with respect to X in a steady coordinate

system) must be continuous. Thus, Eq. (115) reduces to $\dot{\xi} = -U'_{(0)}\,\xi$, exactly the same condition for shock stability in transonic nozzles! A shock is therefore stable or unstable accordingly as $U'_{(0)} > 0$ or $U'_{(0)} < 0$, this simple criterion showing how the shock stability is wave amplitude and wavenumber independent. Results for $U'_{2,o} \neq U'_{1,o}$ can also be discussed, of course, but this would appear to be of limited usefulness since the slowly varying assumption would require continuity of second derivatives of U as well. In a uniform medium, the shocks considered here would be neutrally stable.

Type B shocks (which allow for the dynamic response of the mean parameters) can be treated by considering in addition to (A,ω,K) the triad (P,γ,β). In water waves, for example, P and β would be the mean height and mean speed, respectively, and $\gamma = \gamma(\beta,P,A,K)$ would be a Bernoulli-type "dispersion relation" for the pressure. We take the Lagrangian in the form

$$\mathcal{L} = A\omega + P\gamma - H(K,A,\beta,P,\lambda) \qquad (116)$$

following Hayes (1973), where $\lambda = \lambda(X,T)$ describes the inhomogeneity. The total energy of the system is $H = \omega\mathcal{L}_\omega + \gamma\mathcal{L}_\gamma - \mathcal{L}$ and $\mathcal{L}_A = 0$ defines the dispersion relation $\omega = H_A$. A second dispersion relation arises from taking $\mathcal{L}_P = 0$, that is, $\gamma = H_P$. Thus there remains four unknowns, for which we invoke the conservation laws for waves, total mass, total energy and

total momentum. They are, respectively,

$$\frac{\partial K}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \tag{117a}$$

$$\frac{\partial}{\partial t}\mathcal{L}_\gamma - \frac{\partial}{\partial x}\mathcal{L}_\beta = 0 \tag{118a}$$

$$\frac{\partial}{\partial t}(\omega\mathcal{L}_\omega + \gamma\mathcal{L}_\gamma - \mathcal{L}) - \frac{\partial}{\partial x}(\omega\mathcal{L}_K + \gamma\mathcal{L}_\beta) = -\mathcal{L}_t \tag{119a}$$

$$\frac{\partial}{\partial t}(K\mathcal{L}_\omega + \beta\mathcal{L}_\gamma) + \frac{\partial}{\partial x}(\mathcal{L} - K\mathcal{L}_K - \beta\mathcal{L}_\beta) = \mathcal{L}_x \tag{120a}$$

In a coordinate system moving with the shock, the respective jump conditions become

$$H_A(K, A, \beta, P, \lambda) - U_s K = \text{constant} \tag{117b}$$

$$H_\beta - U_s P = \text{constant} \tag{118b}$$

$$H_A H_K + H_P H_\beta - U_s H = \text{constant} \tag{119b}$$

$$AH_A + PH_P - H + KH_K + \beta H_\beta - U_s(KA + \beta P) = \text{constant} \tag{120b}$$

As before, a basic steady discontinuous solution is assumed, and the effect of slightly displacing the discontinuity is examined. An analysis similar to that just completed for

Type A shocks leads to, for the O(1) heirarchy of equations, the jump conditions for the basic steady flow:

$$H_{A_{1,0}} = H_{A_{2,0}} \tag{117c}$$

$$H_{\beta_{1,0}} = H_{\beta_{2,0}} \tag{118c}$$

$$H_{A_{1,0}} H_{K_{1,0}} + H_{P_{1,0}} H_{\beta_{1,0}} = H_{A_{2,0}} H_{K_{2,0}} + H_{P_{2,0}} H_{\beta_{2,0}} \tag{119c}$$

$$A_{1,0} H_{A_{1,0}} + P_{1,0} H_{P_{1,0}} - H_{1,0} + K_{1,0} H_{K_{1,0}} + \beta_{1,0} H_{\beta_{1,0}} = \dots \tag{120c}$$

while the O(ξ) equations are, respectively,

$$\dot{\xi} = \frac{(\ )_2 - (\ )_1}{K_{2,0} - K_{1,0}}\ \xi \tag{117d}$$

$$(\ ) = H_{AK} K' + H_{AA} A' + H_{A\beta} \beta' + H_{AP} P' + H_{A\lambda} \lambda'$$

$$\dot{\xi} = \frac{(\ )_2 - (\ )_1}{P_{2,0} - P_{1,0}}\ \xi \tag{118d}$$

$$(\ ) = H_{\beta K} K' + H_{\beta A} A' + H_{\beta\beta} \beta' + H_{\beta P} P' + H_{\beta\lambda} \lambda'$$

$$\dot{\xi} = \frac{(\ )_2 - (\ )_1}{H_{2,0} - H_{1,0}}\ \xi \tag{119d}$$

$$\begin{aligned}(\ ) = {} & H_A(H_{KK}K' + H_{KA}A' + H_{K\beta}\beta' + H_{KP}P' + H_{K\lambda}\lambda') \\ & + H_K(H_{AK}K' + H_{AA}A' + H_{A\beta}\beta' + H_{AP}P' + H_{A\lambda}\lambda') \\ & + H_P(H_{\beta K}K' + H_{\beta A}A' + H_{\beta\beta}\beta' + H_{\beta P}P' + H_{\beta\lambda}\lambda') \\ & + H_\beta(H_{PK}K' + H_{PA}A' + H_{P\beta}\beta' + H_{PP}P' + H_{P\lambda}\lambda')\end{aligned}$$

$$\dot{\xi} = \frac{(\ )_2 - (\ )_1}{(K_{2,0}A_{2,0} + \beta_{2,0}P_{2,0}) - (K_{1,0}A_{1,0} + \beta_{1,0}P_{1,0})}\ \xi \tag{120d}$$

$$\begin{aligned}(\ ) = {} & A(H_{AK}K' + H_{AA}A' + H_{A\beta}\beta' + H_{AP}P' + H_{A\lambda}\lambda') \\ & + P(H_{PK}K' + H_{PA}A' + H_{P\beta}\beta' + H_{PP}P' + H_{P\lambda}\lambda') \\ & + K(H_{KK}K' + H_{KA}A' + H_{K\beta}\beta' + H_{KP}P' + H_{K\lambda}\lambda') \\ & + \beta(H_{\beta K}K' + H_{\beta A}A' + H_{\beta\beta}\beta' + H_{\beta P}P' + H_{\beta\lambda}\lambda') \\ & - H_\lambda\lambda'\end{aligned}$$

The bracketed quantities in Eqs. (117d), (118d), (119d) and (120d), of course, can be simplified by applying the steady differential equations (for example, the wave action equation $\partial H_K / \partial x = 0$, the wave conservation law $\partial H_A / \partial x = 0$, $\partial H_\beta / \partial x = 0$ for mass conservation, and the consistency equation $\partial\beta / \partial t + \partial\gamma / \partial x = 0$ or $\partial H_P / \partial x = 0$). On substitution, the amplification rates in the first three equations can be shown to be identically zero, implying neutral stability. However, the result for Eq. (120d) is nontrivial and leads to

$$\dot{\xi} = \frac{-H_{\lambda_{2,0}}\lambda'_{2,0} + H_{\lambda_{1,0}}\lambda'_{1,0}}{(K_{2,0}A_{2,0}+\beta_{2,0}P_{2,0}) - (K_{1,0}A_{1,0}+\beta_{1,0}P_{1,0})}\,\xi \qquad (121)$$

Thus, if the inhomogeneity λ' vanishes, the shock is neutrally stable. Some applications of this formula can be found in Chin (1976), but the qualitative results are similar to those obtained for Type A shocks.

The basic ideas behind the instability theory advanced in this section can be briefly summarized: an infinitesimally small disturbance wave riding on a slowly varying mean flow, on account of space-time inhomogeneities, becomes trapped in a region that on the basis of local normal mode theory is unstable. The rapid amplification of the trapped wave then precipitates a sudden nonlinear, unsteady readjustment of the mean flow that is locally viewed as a "generalized hydraulic jump" involving both mean flow and wave parameters. The actual observability of the hypothesized "shock" mechanism would, of course, depend to a great extent on the all-important effect of high-order dispersive and diffusive terms as well as on the effect of low-order shock stability. These ideas on the origin of "catastrophic transitions" are at the present time speculative in nature, but they do appear to be motivated by plausible physical arguments and by gas-dynamical analogies; they explain the formation of the discontinuities assumed in, for

example, the classical treatment of hydraulic jumps and Benjamin's conjugate flow theory for vortex breakdown. Because the nonlinearities of a physical system are particularly important in its detailed description, the possibility of deducing general results seems remote. Application of the generalized high-order Whitham theory developed in this paper, for example, to hydraulic jump formation, vortex breakdown or boundary layer transition in this regard would be extremely interesting. One immediate need, perhaps, is a nonlinear wave trapping theory paralleling Landahl's for linear flows; some initial work has been carried out in Chin (1976), but much more work needs to be done.

V. Summary.

A theory has been constructed for the description of wavetrains slowly modulated in space and time and which includes the effects of high-order diffusive and dispersive corrections, modal terms, nonconservative terms, and amplitude and frequency dispersion. A number of examples have been considered leading to many formulas useful in direct applications of the theory, and various new ideas in stability have been discussed.

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FIGURE 1

Gravity Waves in Shallow Water:
Conjugate Solutions for the Case $h_2/h_1 = 2.25$

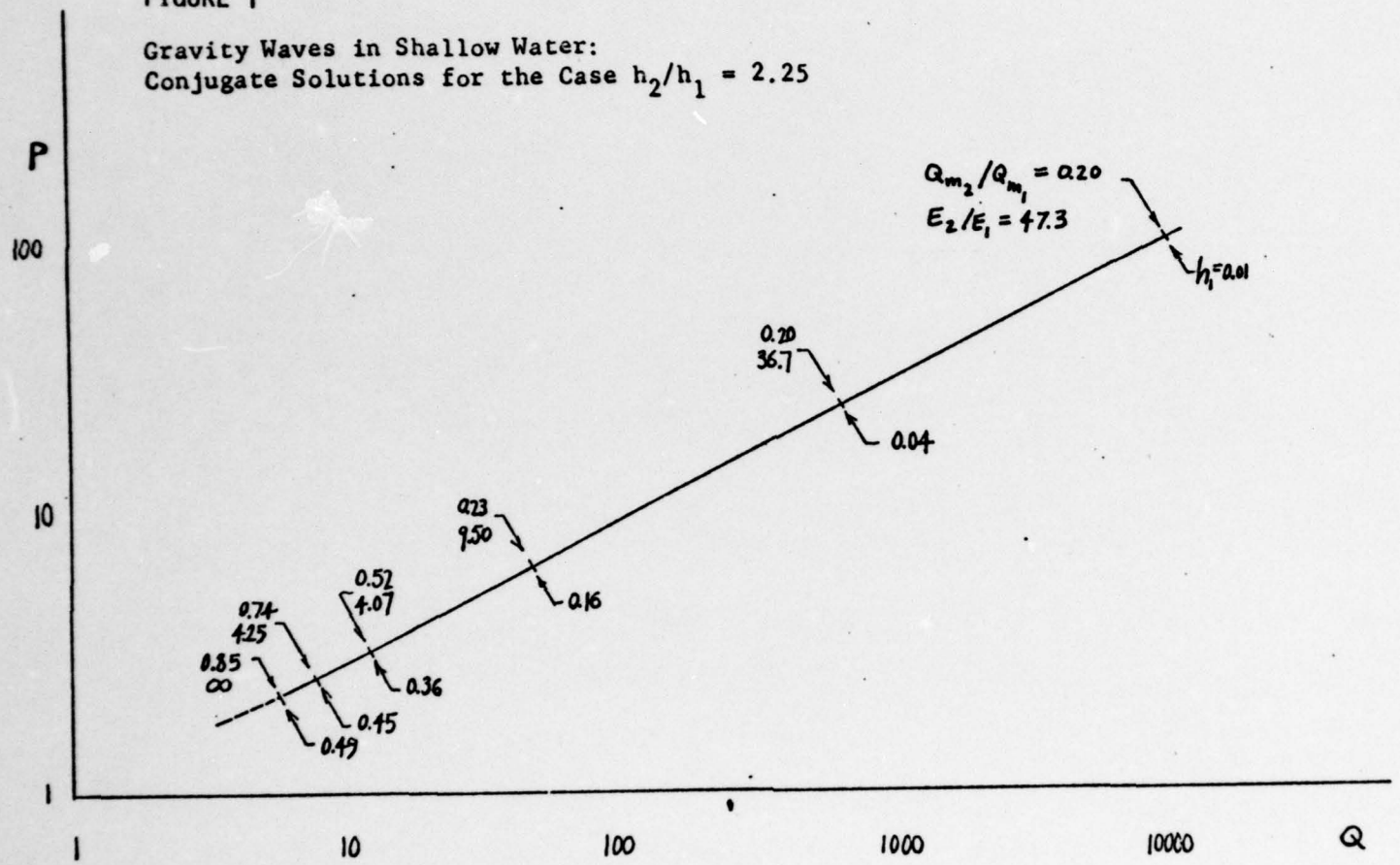
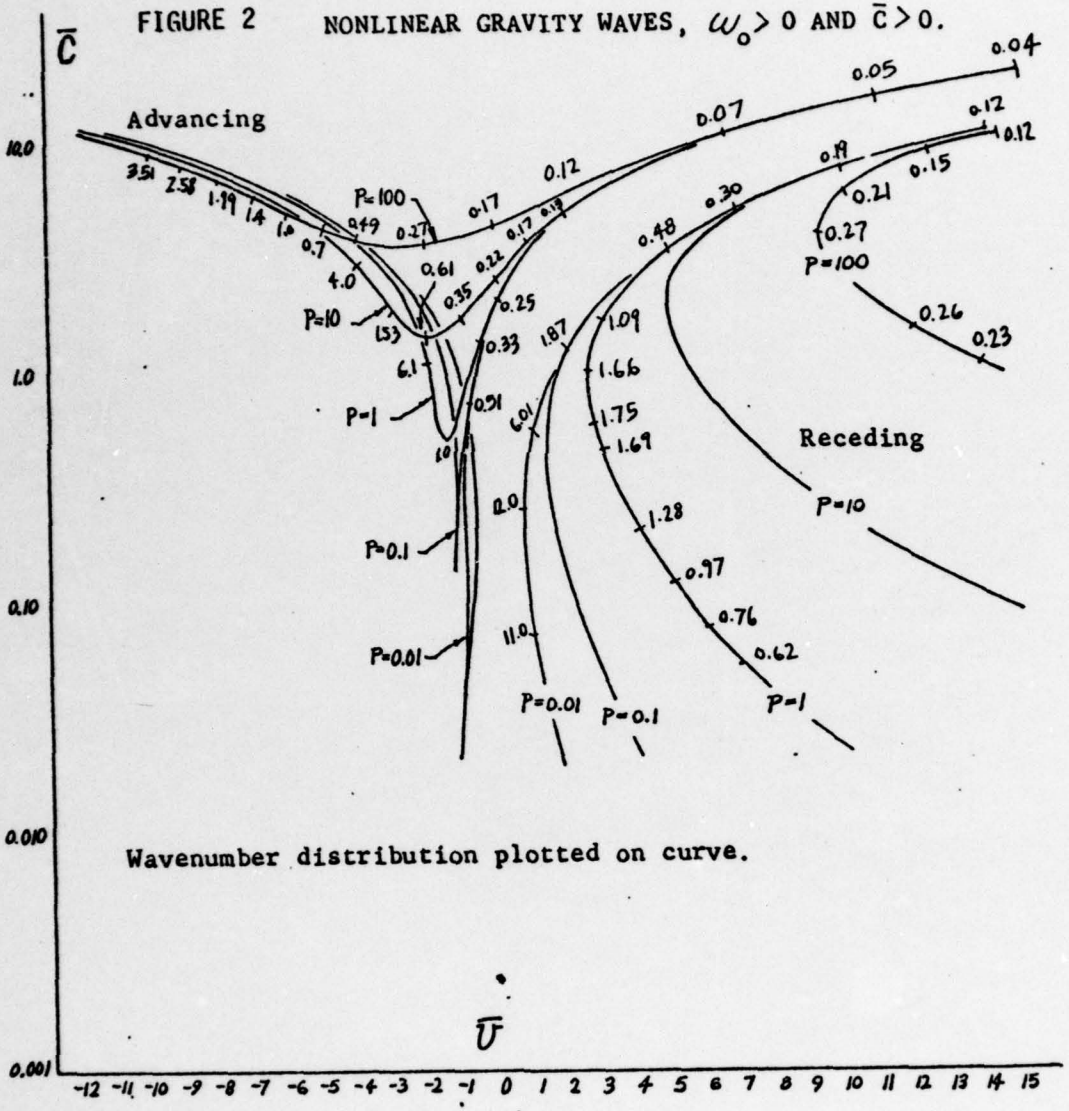


FIGURE 2 NONLINEAR GRAVITY WAVES, $\omega_o > 0$ AND $\bar{C} > 0$.



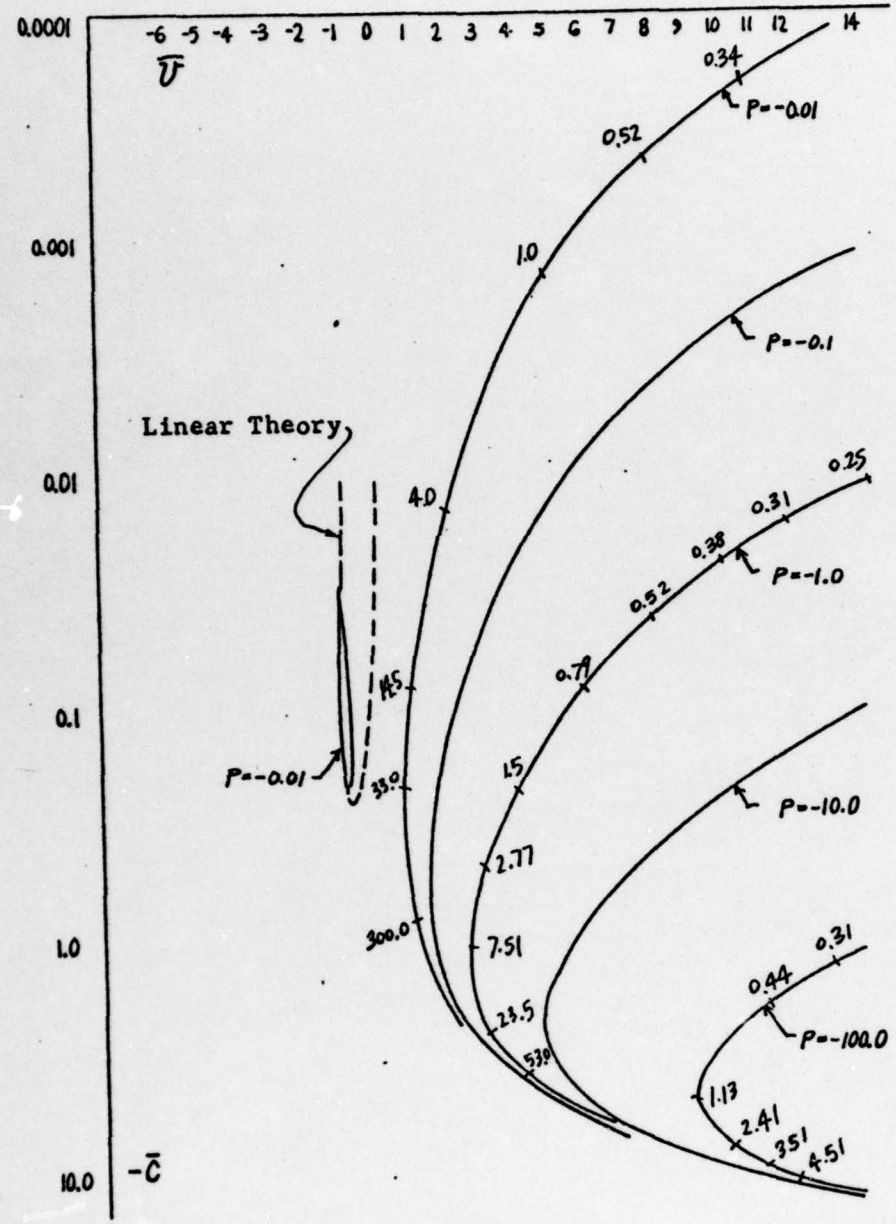
Wavenumber distribution plotted on curve.

FIGURE 3A

NONLINEAR GRAVITY WAVES

$\omega_o > 0$ and $\bar{C} < 0$.

Wavenumber distribution plotted on curve.



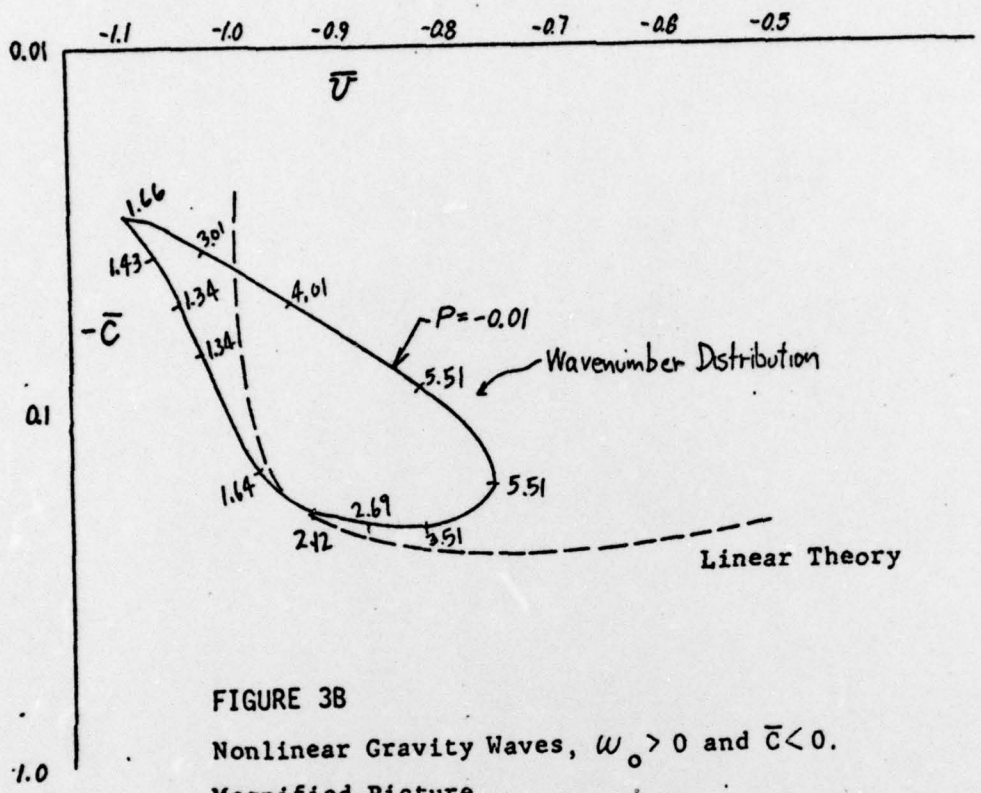


FIGURE 3B

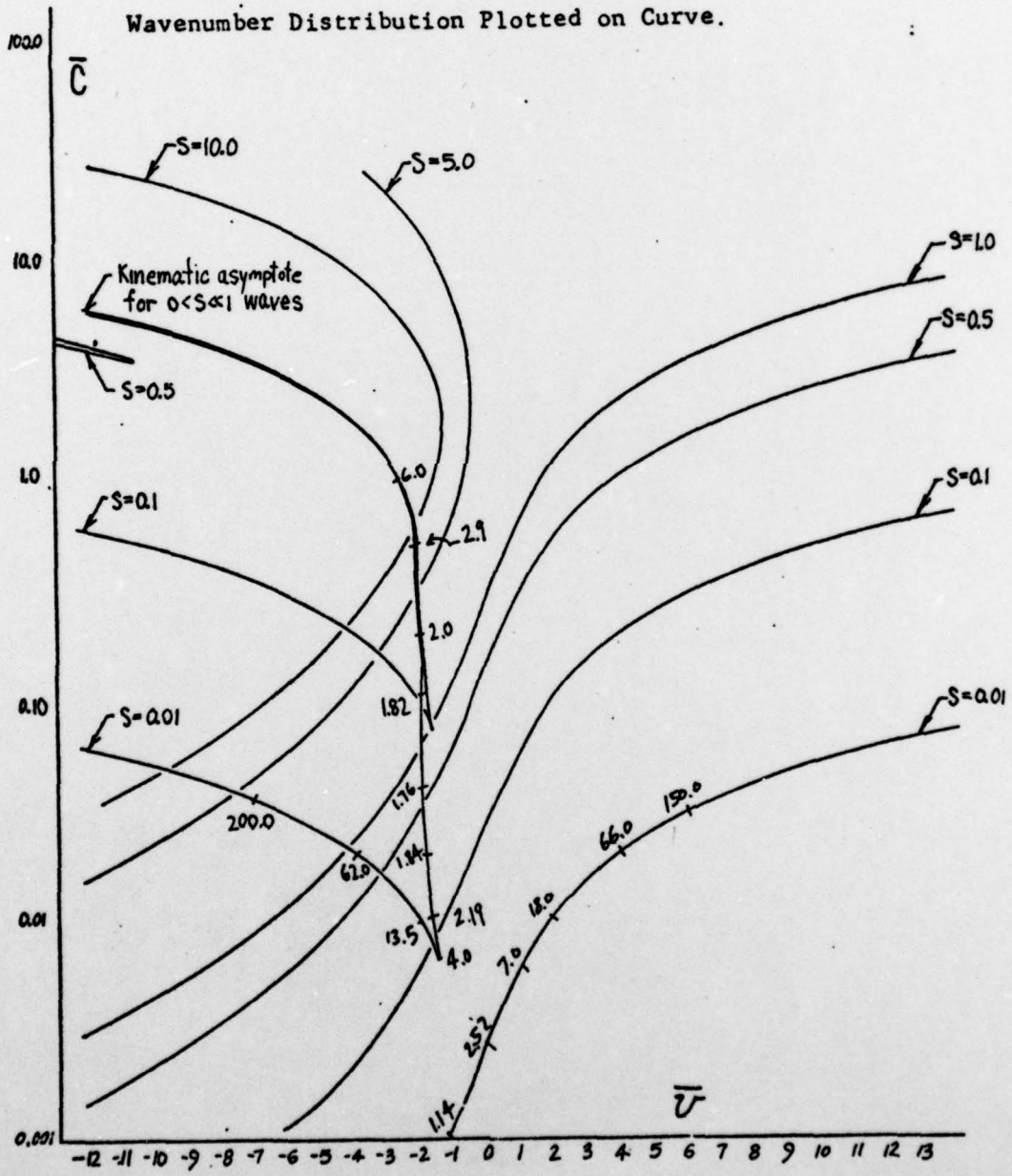
Nonlinear Gravity Waves, $\omega_o > 0$ and $\bar{C} < 0$.

Magnified Picture.

FIGURE 4

Nonlinear Capillary Waves, $\omega_o < 0$ and $\bar{c} > 0$.

Wavenumber Distribution Plotted on Curve.



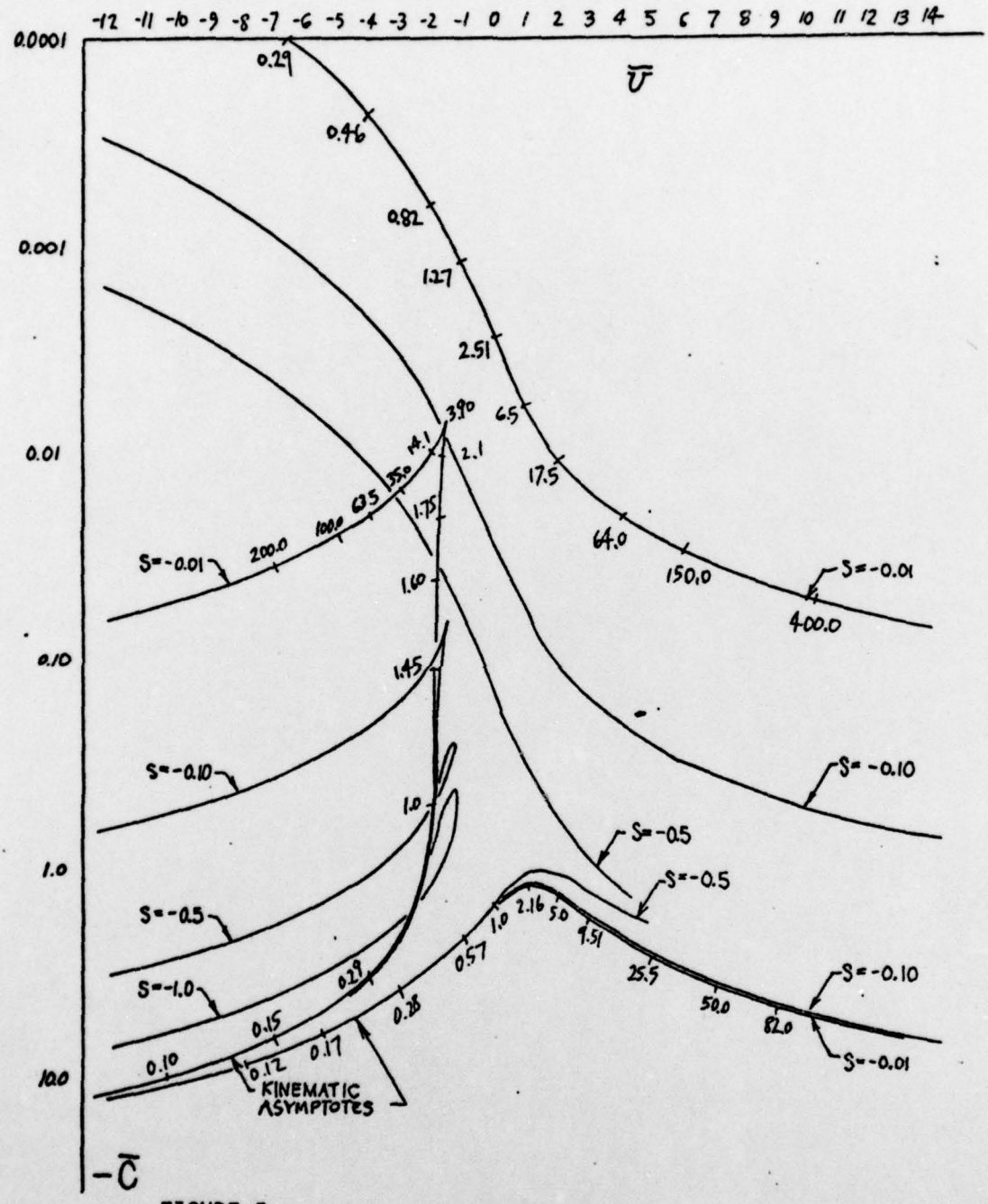


FIGURE 5

Nonlinear Capillary Waves, $W_o < 0$ and $\bar{C} < 0$.
Wavenumber distribution plotted on curve.