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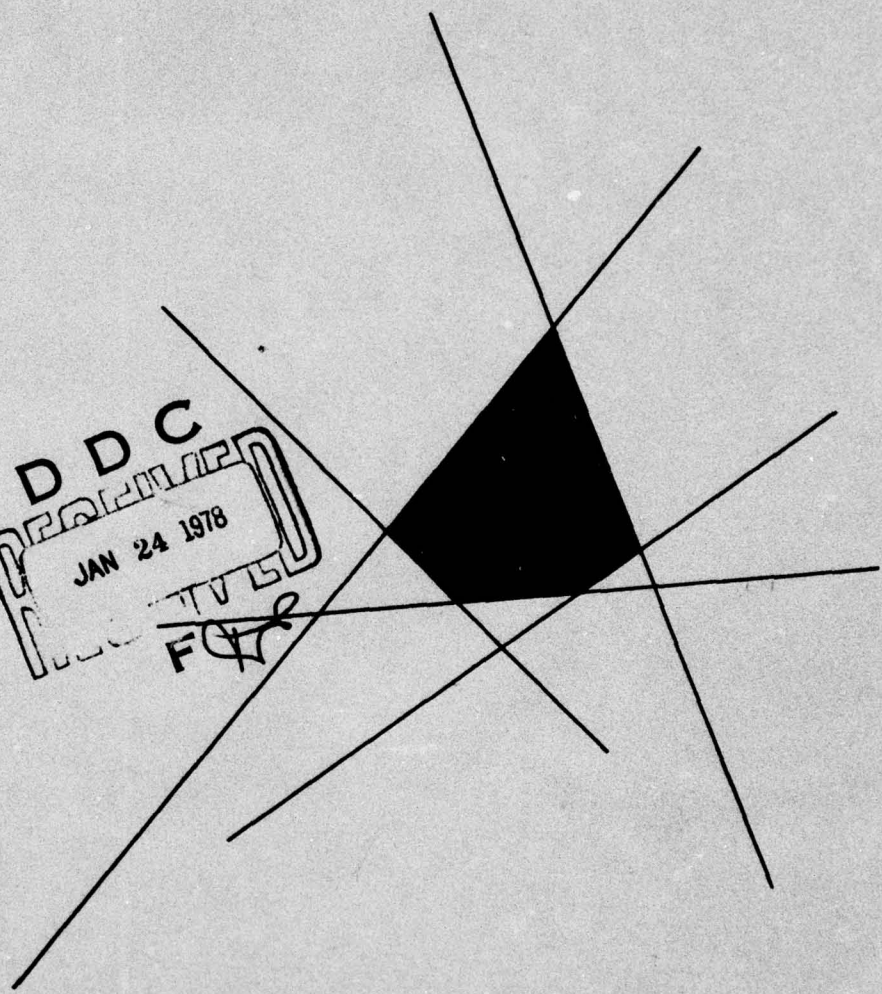
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# CONGESTION OF PRODUCTION FACTORS

by  
ROLF FÄRE  
and  
LEIF SVENSSON

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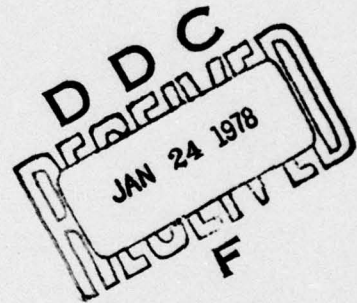
CONGESTION OF PRODUCTION FACTORS<sup>†</sup>

by

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ABSTRACT

Three different forms of congestion of production factors are defined and analyzed within an axiomatic theory of production. These forms of congestion are used to characterize a law of variable proportion.

# CONGESTION OF PRODUCTION FACTORS

by

Rolf Färe and Leif Svensson

## 1. INTRODUCTION

Examples of production technologies exhibiting congestion in the sense that if a proper subset of production factors (inputs) are kept fixed, increases in the others may obstruct output, are frequently found in agriculture, transportation and engineering. Despite the apparent dissimilarity of such production technologies a general axiomatic treatment is offered in this paper. Three forms of congestion are introduced to distinguish different strengths. These forms of congestion are related to each other under production theoretical commonly made assumptions. They are also analyzed in terms of production concepts such as essentiality, limitationality and null jointness of inputs. In the final section the three forms of congestion are used in dealing with a law of variable proportions.

A function  $\phi : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  between exogenous inputs ( $x$ ) and net output ( $u$ ) is called a production function if it satisfies the following axioms stated by Shephard (see [3]).

$\phi.1 \quad \phi(0) = 0$ , and  $\phi(x) > 0$  for some  $x \geq 0$ .<sup>(1)</sup>

$\phi.2 \quad \phi(x)$  is finite for  $\|x\|$  bounded.

$\phi.3 \quad \phi(\lambda \cdot x) \geq \phi(x)$  if  $\lambda \geq 1$ .

$\phi.4 \quad$ For  $\phi(\lambda \cdot x) > 0$ ,  $\phi(\lambda \cdot x) \rightarrow +\infty$  as  $\lambda \rightarrow +\infty$ .

$\phi.5 \quad \phi(x)$  is upper semi-continuous in  $x$ .

$\phi.6 \quad$ The efficient subset  $E(u) := \{x \mid \phi(x) \geq u, y \leq x \Rightarrow \phi(y) < u\}$  is bounded for  $u > 0$  and  $E(0) := 0$ .

<sup>(1)</sup>  $x \geq 0$  means  $x_i \geq 0$ ,  $i = 1, 2, \dots, n$ ,  $x \neq 0$ .

Congestion is next analyzed within the framework  $\phi.1 - \phi.6$  for a production technology.



## 2. CONGESTION

For the purpose of distinguishing between different strengths of congestion of production factors, three forms are defined and discussed in this paper, namely:

### Definition 1:

A factor combination  $\{v_{k+1}, v_{k+2}, \dots, v_n\}$ ,  $0 \leq k < n$ , is output-limitational (OL) congested at  $x^0 \in \mathbb{R}_+^n$ , if  $\phi(x^0 + y) \leq \phi(x^0)$ , for all  $y \in D(v_1, v_2, \dots, v_k)$ . (2)

### Definition 2:

A factor combination  $\{v_{n+1}, v_{k+2}, \dots, v_n\}$ ,  $0 \leq k < n$ , is monotone output-limitational (MOL) congested at  $x^0 \in \mathbb{R}_+^n$ , if  $\phi(x^0 + y'') \leq \phi(x^0 + y')$ , for all  $y'', y' \in D(v_1, v_2, \dots, v_k)$  with  $y'' \geq y'$ .

### Definition 3:

A factor combination  $\{v_{k+1}, v_{n+2}, \dots, v_n\}$ ,  $0 \leq k < n$ , is output-prohibitive (OP) congested at  $x^0 \in \mathbb{R}_+^n$ , if  $\phi(x^0 + y) = 0$ , for all  $y \in D(v_1, v_2, \dots, v_k)$ .

Clearly if a factor combination  $\{v_{k+1}, v_{n+2}, \dots, v_n\}$  is OP-congested at  $x \in \mathbb{R}_+^n$  it is MOL-congested at that input combination. Also if  $\{v_{k+1}, v_{k+2}, \dots, v_n\}$  is MOL-congested at  $x \in \mathbb{R}_+^n$  it is OL-congested there. In general, however, the converse is not true, hence the three definitions distinguish different strength of congestion. The following two examples of production functions clarify this.

$$\text{(2)} \quad D(v_1, v_2, \dots, v_k) := \left\{ x \in \mathbb{R}_+^n \mid x \geq 0, x_{v_i} = 0, i = 1, 2, \dots, k \right\}.$$

$$(1) \quad \phi(x) := \min \{x_1, |2x_1 - x_2|\} .$$

$$(2) \quad \phi(x) := \min \{x_1, x_2\} .$$

In Example 1, the production factor  $x_2$  is OL-congested at  $(x_1, x_2) = (1, 1)$ , but it is not MOL-congested at that point, since for  $y'' = (0, 2)$ ,  $y' = (0, 1)$ ,  $\phi(x + y'') = 0 < \phi(x + y') = 1$ .

In the second example with  $x_1 = x_2 = 2$  it is clear that  $x_2$  is MOL-congested at  $(2, 2)$  but not OP-congested.

On the other hand, if the production function exhibits strong disposability of inputs, (i.e.,  $x' \geq x \Rightarrow \phi(x') \geq \phi(x)$ ), then in definitions 1 and 2, the inequality sign may be replaced by equality and then obviously MOL- and OL-congestion coincide.

To continue, consider the following subsets of factors of production in  $\mathbb{R}_+^n$ :

$$A(OL) := \left\{ x \in \mathbb{R}_+^n \mid \{v_{k+1}, v_{k+2}, \dots, v_n\} \text{ is OL-congested at } x \right\} ,$$

$$A(MOL) := \left\{ x \in \mathbb{R}_+^n \mid \{v_{k+1}, v_{k+2}, \dots, v_n\} \text{ is MOL-congested at } x \right\}$$

and

$$A(OP) := \left\{ x \in \mathbb{R}_+^n \mid \{v_{k+1}, v_{k+2}, \dots, v_n\} \text{ is OP-congested at } x \right\} .$$

It is clear from the above discussion that  $A(OP) \subset A(MOL) \subset A(OL)$  and that  $A(OL) = A(MOL) := \{x \mid \phi(x) = \phi(x + y) \text{ for all } y \in D(v_1, v_2, \dots, v_k)\}$  whenever the production function  $\phi$  exhibits strong disposability of inputs. Also note that under this disposability condition,  $A(OP)$  may be nonempty with  $A(OL) \neq A(OP)$ . The following production function is an illustration of this case.

$$(3) \quad \phi(x) := \min \{ \max \{ 0, x_1 - a \}, x_2 \}, \quad a > 0 .$$

Clearly the production function (3) exhibits strong disposability of inputs and for  $a = 1$ , the input vector  $(x_1, x_2) = (1, 1) \in A(OP)$  with the second factor OP-congested.  $(x_1, x_2) = (3, 1)$  belongs to  $A(OL)$  with  $x_2$  OL-congested but  $(3, 1) \notin A(OP)$ , i.e.,  $A(OP)$  is nonempty with  $A(OL) \neq A(OP)$ .

For the case where only one factor is congested (i.e.,  $k = n - 1$ ) it can be shown that for a quasi-concave production function  $\phi$ ,  $A(OL) = A(MOL)$ .

Proposition 1:

Let the production function  $\phi$  be quasi-concave. If  $D(v_1, v_2, \dots, v_k)$  is one dimensional (i.e.,  $D(v_1, v_2, \dots, v_k) = D(v_1, v_2, \dots, v_{n-1})$ ), then  $A(OL) = A(MOL)$ .

Proof:

Since  $A(MOL) \subset A(OL)$  it is sufficient to assume that  $A(OL)$  is nonempty and to prove that  $A(OL) \subset A(MOL)$ . Hence let  $x^0 \in A(OL)$  and let  $y', y'' \in D(v_1, v_2, \dots, v_{n-1})$  with  $y' \leq y''$ . Then since  $D(v_1, v_2, \dots, v_{n-1})$  has dimension one and  $y' \leq y''$ ,  $x^0 + y'$  lies on the line segment between  $x^0$  and  $x^0 + y''$ . Hence by quasi-concavity,  $\phi(x^0 + y') \geq \min \{ \phi(x^0), \phi(x^0 + y'') \}$ . Thus, since  $\phi(x^0) \geq \phi(x^0 + y')$  ( $x^0 \in A(OL)$ ), it follows that  $\phi(x^0 + y') \geq \phi(x^0 + y'')$  i.e.,  $x^0 \in A(MOL)$ . Q.E.D.

Unfortunately, as illustrated below, this proposition does not hold when there are more than one congested factor of production.

$$(4) \quad \phi(x) := \min \{x_1, \max \{0, x_2 - x_3 - x_1\}\} .$$

The production function (4) is quasi-concave and it is next seen that the factor combination  $\{x_2, x_3\}$  is OL-congested, at  $(x_1, x_2, x_3) = (1, 3, 1)$  but not MOL-congested there. From (4) it follows that,

$$\phi(1, 3 + y_2, 1 + y_3) = \min \{1, \max \{0, 1 + y_2 - y_3\}\} \leq \phi(1, 3, 1) = 1$$

for all  $y_i \geq 0$ ,  $i = 2, 3$ . Hence  $\{x_2, x_3\}$  is OL-congested at the input vector  $(1, 3, 1)$ . However to see that  $\{x_2, x_3\}$  is not MOL-congested there, choose  $(1, 3, 2)$  and  $(1, 4, 2)$  giving  $\phi(1, 3, 2) = 0$  and  $\phi(1, 4, 2) = 1$ .

### 3. PROPERTIES OF OL-, MOL- AND OP-CONGESTION

In this section OL-, MOL- and OP-congestion are further investigated and these notions are related to other production theoretical concepts.

A factor combination  $\{v_1, v_2, \dots, v_k\}$ ,  $1 \leq k < n$ , is termed essential if  $\phi(y) = 0$  for all  $y \in D(v_1, v_2, \dots, v_k)$ . Essentiality is related to the set  $A(OL)$  by:

Proposition 2:

A factor combination  $\{v_1, v_2, \dots, v_k\}$ ,  $1 \leq k < n$ , is essential if and only if  $0 \in A(OL)$ .

Proof:

Assume  $0 \in A(OL)$  then  $\phi(0 + y) \leq \phi(0)$  for all  $y \in D(v_1, v_2, \dots, v_k)$  and by axiom  $\phi.1$  (i.e.,  $\phi(0) = 0$ ) the factor combination  $\{v_1, v_2, \dots, v_k\}$  is essential. Conversely, assume  $\{v_1, v_2, \dots, v_k\}$  essential, then by axiom  $\phi.1$ ,  $\phi(0 + y) \leq \phi(0) = 0$  for all  $y \in D(v_1, v_2, \dots, v_k)$  and  $0 \in A(OL)$ . Q.E.D.

The notion of weak limitationality, stating that a factor combination  $\{v_1, v_2, \dots, v_k\}$ ,  $1 \leq k < n$ , is weakly limitational if there exists a positive bound  $(x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0)$  on  $\{v_1, v_2, \dots, v_k\}$  such that  $\phi(x)$  is bounded for all  $x \in \left\{ x \in \mathbb{R}_+^n \mid (x_{v_1}, x_{v_2}, \dots, x_{v_k}) \leq (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \right\}$ , was introduced into economic theory of production by R. W. Shephard (see [2]). He proved that a factor combination is weakly limitational if and only if it is essential. In this light, Proposition 2 gives an alternative characterization of weak limitationality.

A further characterization of the condition " $0 \in A(OL)$ " and hence for weak limitationality is next given for quasi-concave production functions, namely:

Proposition 3:

If the production function  $\phi$  is quasi-concave,  $0 \in A(OL)$  if and only if  $A(OL)$  is nonempty.

Proof:

Assume  $x^0 \in A(OL)$ , if  $x^0 = 0$  then there is nothing to prove so let  $x^0 \geq 0$ .  $x^0 \in A(OL)$  implies  $\phi(x^0) \geq \phi(x^0 + y)$  for all  $y \in D(v_1, v_2, \dots, v_k)$  and by property  $\phi.3$  of the production function,  $\phi(x^0) \geq \phi(\lambda \cdot x^0 + \lambda \cdot y)$ ,  $\lambda \in [0, 1]$ . Define  $y^0 = \lambda \cdot y / (1 - \lambda)$  for  $\lambda \in (0, 1)$ , then  $y^0 \in D(v_1, v_2, \dots, v_k)$  and by quasi-concavity,

$$\phi(x^0) \geq \phi(\lambda \cdot x^0 + \lambda \cdot y) = \phi(\lambda \cdot x^0 + (1 - \lambda) \cdot y^0) \geq \min [\phi(x^0), \phi(y^0)] .$$

Thus  $\phi(x^0) \geq \phi(y^0)$ . Since  $y$  was arbitrarily chosen,  $\phi(x^0) \geq \phi(y^0)$  for all  $y^0 \in D(v_1, v_2, \dots, v_k)$  and by property  $\phi.4$ ,  $\phi(y^0) = 0$  for all  $y^0 \in D(v_1, v_2, \dots, v_k)$ , implying that  $\phi(0 + y^0) \geq \phi(y^0)$  i.e.,  $0 \in A(OL)$ . Q.E.D.

As the following counterexample shows, Proposition 3 does not hold if the production function is not assumed quasi-concave. Let

$$(5) \quad \phi(x) : = \begin{cases} x_1 & \text{for } x \in \{(x_1, x_2) \mid x_1 \geq 0, x_2 = 0\} \\ x_2 & \text{for } x \in \{(x_1, x_2) \mid x_1 = 0, x_2 \geq 0\} \\ 0 & \text{otherwise} \end{cases}$$

then (5) is a production function but it is not quasi-concave. The input  $x_2$  is clearly OL-congested at  $(x_1, x_2) = (1, 1)$ , but not at  $(0, 0)$ , showing that Proposition 3 does not hold in general without the assumption of quasi-concavity.

As for MOL- and OP-congestion, note that

$$A(OP) = A(MOL) \cap \phi^{-1}(0) = A(OL) \cap \phi^{-1}(0)$$

where  $\phi^{-1}(0) := \{x \in \mathbb{R}_+^n \mid \phi(x) = 0\}$ . Consequently,

$$0 \in A(OP) \iff 0 \in A(MOL) \iff 0 \in A(OL).$$

From the last of the above two expressions it is clear that in Propositions 2 and 3,  $A(OL)$  can be replaced by  $A(MOL)$  or  $A(OP)$ .

For the special case of a homothetic production function it is next shown that the set of OL-congested input vectors is a cone, if there are such input vectors. Thus assume  $x^0 \in A(OL)$ , then  $F(G(x^0 + y)) \leq F(G(x^0))$  for all  $y \in D(v_1, v_2, \dots, v_k)$ , where  $\phi(x) := F(G(x))$  is a homothetic production function with  $G(\lambda \cdot x) = \lambda \cdot G(x)$ ,  $\lambda > 0$ . From the homogeneity of the case function  $G$ , it is clear that when  $x^0 \in A(OL)$ ,  $G(\lambda \cdot x^0 + \lambda \cdot y) \leq G(\lambda \cdot x^0)$ , and since  $D(v_1, v_2, \dots, v_k)$  is a cone,  $F(G(\lambda \cdot x^0 + \tilde{y})) \leq F(G(\lambda \cdot x^0))$ ,  $\lambda > 0$  and  $\tilde{y} \in D(v_1, v_2, \dots, v_k)$  implying that  $\lambda \cdot x^0 \in A(OL)$  or that  $A(OL)$  is a cone.

Similar proofs apply for the sets  $A(MOL)$  and  $A(OP)$ . Thus one has:

Proposition 4:

If the production function  $\phi$  is homothetic and  $A(OL) [A(MOL), A(OP)]$  is not empty,  $A(OL) [A(MOL), A(OP)]$  is a cone.



#### 4. A PARTICULAR RESULT FOR OP-CONGESTION

The concept of jointness between production factors was introduced in [1] to express the idea that for a positive output rate, certain requirements may have to be placed on the input mixes. In the same paper jointness was related to a form of congestion. Here a characterization of jointness in terms of OP-congestion is discussed and the result is compared to some of those found in [1]. For this reason define:

##### Definition 4:

A factor combination  $\{v_{k+1}, v_{k+2}, \dots, v_n\}$ ,  $1 \leq k < n$ , is null joint with  $\{v_1, v_2, \dots, v_k\}$  if and only if  $x \in \overline{C(1)}^{(3)}$ ,  $(x_{v_1}, x_{v_2}, \dots, x_{v_k}) = 0$  implies  $(x_{v_{k+1}}, x_{v_{k+2}}, \dots, x_{v_n}) = 0$ .

The restrictions enforced on  $A(OP)$  by null jointness for a quasi-concave production function is clear from:

##### Proposition 5:

For a quasi-concave production function  $\phi$ , a factor combination  $\{v_{k+1}, v_{k+2}, \dots, v_n\}$ ,  $1 \leq k < n$ , is null joint with  $\{v_1, v_2, \dots, v_k\}$  if and only if there is an input vector  $x^0 \in A(OP)$  such that  $\phi(z^0) > 0$  for some  $z^0 := x^0 - y$ ,  $y \in D(v_1, v_2, \dots, v_k)$  and

$$\left\{ \{x \mid \|x\| > \|x^0\|\} \cap \{x \mid x \geq 0, (x_{v_1}, x_{v_2}, \dots, x_{v_k}) = (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0)\} \right\} \subset A(OP).$$

<sup>(3)</sup> $\overline{C(1)}$  is the closed cone spanned by the input vectors yielding the output rate  $u = 1$ , formally,  $\overline{C(1)} := \{x \mid x = \lambda \cdot y, \lambda > 0, \phi(y) = 1\}$ . Note that by property  $\phi.4$  of the production function,  $C(1) = C(u)$ , for  $u > 0$ .

Proposition 3 shows that if  $A(OP)$  nonempty,  $0 \in A(OP)$  and thus,  $D(v_1, v_2, \dots, v_k) \subset A(OP)$ . This together with property  $\phi.3$  of the production function, implies that in Proposition 5,  $\phi(x) = 0$  for all  $x \in \left\{ \left\{ x \mid \|x\| > \|x^0\| \right\} \cap \left\{ x \mid x \geq 0, \left( x_{v_1}, x_{v_2}, \dots, x_{v_k} \right) \leq \left( x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \right) \right\} \right\}$ . Thus the relationship between Proposition 5 and the characterization of null jointness found in [1], namely:

Proposition 6:

For a quasi-concave production function  $\phi$ , a factor combination  $\{v_{k+1}, v_{k+2}, \dots, v_n\}$ ,  $1 \leq k < n$ , is null joint with  $\{v_1, v_2, \dots, v_k\}$  if and only if for each positive bound  $\left( x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \right)$  on  $\{v_1, v_2, \dots, v_k\}$  there is a  $\delta > 0$  such that for all  $x \in \left\{ \left\{ x \mid \|x\| > \delta \right\} \cap \left\{ x \mid x \geq 0, \left( x_{v_1}, x_{v_2}, \dots, x_{v_k} \right) \leq \left( x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \right) \right\} \right\}$ ,  $\phi(x) = 0$ , is clear.

In proving Proposition 5, the following lemma is of use:

Lemma:

Let  $K \subset \mathbb{R}_+^n$ ,  $0 \in K$  be a closed cone such that the intersection  $K \cap D(v_1, v_2, \dots, v_k)$ ,  $1 \leq k < n$ , is empty. Then for each positive bound  $\left( x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \right)$  on the subvector  $(v_1, v_2, \dots, v_k)$ , the set  $K \cap \left\{ x \mid x \geq 0, \left( x_{v_1}, x_{v_2}, \dots, x_{v_k} \right) \leq \left( x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \right) \right\}$  is compact.

Proof:

Let  $\left( x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \right)$  be any positive bound on the subvector  $(v_1, v_2, \dots, v_k)$ ,  $1 \leq k < n$ , and define  $S^0 := K \cap \left\{ x \mid x \geq 0, \left( x_{v_1}, x_{v_2}, \dots, x_{v_k} \right) \leq \left( x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \right) \right\}$ . The set  $S^0$  is closed as

an intersection of two closed sets. Thus we only have to show that it is bounded. Assume conversely that there is an infinite sequence  $\{x^\ell\} \subset S^0$  such that  $\|x^\ell\| \rightarrow +\infty$  as  $\ell \rightarrow +\infty$ . Since by assumption the subvector  $(x_{v_1}^\ell, x_{v_2}^\ell, \dots, x_{v_k}^\ell)$  is bounded,  $\|x_{v_{k+1}}^\ell, x_{v_{k+2}}^\ell, \dots, x_{v_n}^\ell\|$  must under these conditions tend to  $+\infty$  as  $\ell \rightarrow +\infty$ . Next, define the sequence of rays  $\Gamma^\ell := \{\lambda \cdot x^\ell \mid \lambda > 0\}$ . Since  $(x_{v_1}^\ell, x_{v_2}^\ell, \dots, x_{v_k}^\ell)$  is bounded,  $\lim_{\ell \rightarrow +\infty} \Gamma^\ell \in D(v_1, v_2, \dots, v_k)$ , contradicting the condition  $K \cap D(v_1, v_2, \dots, v_k)$  empty. Hence,  $S^0$  is a compact subset of  $\mathbb{R}_+^n$  and since the bound on  $(x_{v_1}^\ell, x_{v_2}^\ell, \dots, x_{v_k}^\ell)$  was arbitrarily chosen the lemma is proved. Q.E.D.

To prove Proposition 5, assume that the factor combination  $\{v_{k+1}, v_{k+2}, \dots, v_n\}$ ,  $1 \leq k < n$ , is null joint with  $\{v_1, v_2, \dots, v_k\}$  then  $\overline{C(1)} \cap D(v_1, v_2, \dots, v_k)$  is empty (see Proposition 1 in [1]). From the second part of property  $\phi.1$  of the production function, there is  $z^0 \in \overline{C(1)}$  such that  $\phi(z^0) > 0$ . Now,  $\overline{C(1)}$  is a closed cone with  $0 \in \overline{C(1)}$  and hence the conditions of the lemma are satisfied. Consequently there is an input vector  $x^0 := z^0 + y$  for  $y \in D(v_1, v_2, \dots, v_k)$  such that  $\phi(x) = 0$  for all  $x \in \left\{ \{x \mid \|x\| > \|x^0\|\} \cap \{x \mid x \geq 0\} \right\}$ ,  $(x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) = (x_{v_1}^o, x_{v_2}^o, \dots, x_{v_k}^o)$ , proving the first part of Proposition 5.

To prove the converse, assume there is an input vector  $x^0 \in A(OP)$  such that  $\phi(z^0) > 0$  for some  $z^0 := x^0 - y$ ,  $y \in D(v_1, v_2, \dots, v_k)$  and  $\left\{ \{x \mid \|x\| > \|x^0\|\} \cap \{x \mid x \geq 0\} \right\} = (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \subset A(OP)$ . From Proposition 3 and property  $\phi.3$  of the

production function,  $\phi(x) = 0$  for all  $x \in \left\{ \{x \mid \|x\| > \|x^0\|\} \cap \left\{ x \mid x \geq 0, (x_{v_1}, x_{v_2}, \dots, x_{v_k}) \leq (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \right\} \right\}$ . Clearly, since  $\overline{C(1)} = \overline{C(\phi(z^0))}$  is a closed convex cone,  $\overline{C(1)} \cap D(v_1, v_2, \dots, v_k)$  is empty and hence by Proposition 1 in [1], the factor combination  $\{v_{k+1}, v_{k+2}, \dots, v_n\}$  is null joint with  $\{v_1, v_2, \dots, v_k\}$ , proving the proposition.

Note that quasi-concavity is not used in the first part of the proof of Proposition 5, but as the following example (found in [1]) shows it is essential for the second.

$$(6) \quad \phi(x_1, x_2) := \min \left\{ x_1, \max \left\{ 0, x_2 - x_1^{\frac{1}{2}} \right\} \right\}.$$

This production function is clearly not quasi-concave but it satisfies the conditions for the second part of the proof, i.e.,  $\phi(1, 2) > 0$ ,  $\phi(1 + y_1, 2) = 0$  for all  $y_1 \geq 4$ . However as shown in [1],  $x_1$  is not null joint with  $x_2$ .

Also note that the condition  $\phi(z^0) > 0$  for some  $z^0 := x^0 - y$ ,  $y \in D(v_1, v_2, \dots, v_k)$  is essential for the second part of the proof. Example (3) above shows this.

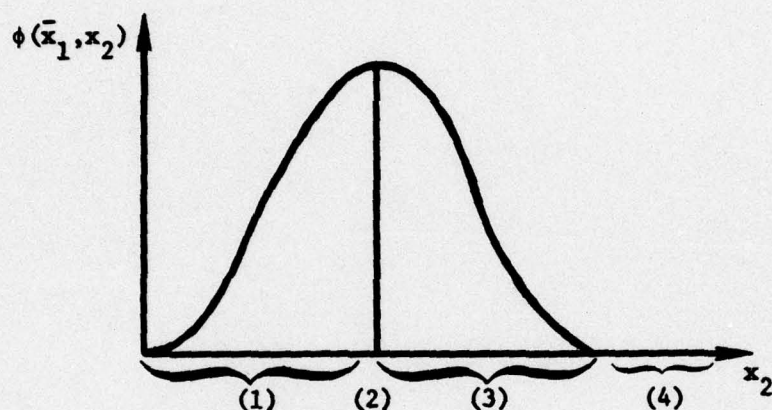
Finally, as the next example shows, the condition,  $\left\{ \{x \mid \|x\| > \|x^0\|\} \cap \left\{ x \mid x \geq 0, (x_{v_1}, x_{v_2}, \dots, x_{v_k}) \leq (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \right\} \right\} \subset A(OP)$  is also essential for the second part of the proof.

$$(7) \quad \phi(x_1, x_2, x_3) := \begin{cases} \min \{x_1, x_2\} & \text{for } x \in D(3) \\ 0 & \text{otherwise.} \end{cases}$$

This production function is quasi-concave. However,  $\phi(1,1,0) = 1$  and  $\phi(1,1,1) = 0$ , i.e., it does not meet the above requirement. Consequently the factor combination  $\{x_1, x_2\}$  is not null joint with  $x_3$ , as would have been the case if in addition the above condition was met.

### 5. A LAW OF VARIABLE PROPORTION

Consider the following illustration of a two factors production function  $\phi(x_1, x_2)$ , where the first factor is kept constant while the second may vary.



Four phases are distinguished. The first (1) when output is increasing, the second (2) where it reaches its maximum and the third when output decreases with increases of  $x_2$  and finally the fourth (4) when output is null.

Over the four regions the factor proportion  $(x_1/x_2)$  is changing and production variation like this has become known as a law of variable proportion.

In relation to the above forms of congestion and Proposition 5, phases (2), (3) and (4) are of interest. Note that in Proposition 5,  $z^0$  may be chosen so that  $\phi(z^0) = \max \left\{ \phi(x) \mid x \in \{x \mid \|x\| \leq \|x^0\|\} \cap \left\{ x \mid x \geq 0, (x_{v_1}, x_{v_2}, \dots, x_{v_k}) = (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \right\} \right\}$ , since the production function is upper semi-continuous (property  $\phi.5$ ). Also since

$\phi(x) = 0$  for all  $x \in \left\{ \{x \mid \|x\| > \|x^0\|\} \cap \left\{ x \mid x \geq 0, (x_{v_1}, x_{v_2}, \dots, x_{v_k}) = (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \right\} \right\}$ , both phase (2) and (4) are described in Proposition 5.

Consequently, if  $\{x \mid \phi(x) = \phi(z^0)\}$  is contained in  $A(\text{MOL})$  a law of variable proportion is given. Formally:

**Definition 5:**

A law of variable proportion is defined for the factor combination  $\{v_1, v_2, \dots, v_k\}$ ,  $1 \leq k < n$ , if there is a  $x^0 \in A(\text{OP})$  such that  $\phi(z^0) > 0$ , where  $\phi(z^0) := \max \left\{ \phi(x) \mid \|x\| \leq \|x^0\| \cap \left\{ x \mid x \geq 0, (x_{v_1}, x_{v_2}, \dots, x_{v_k}) = (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \right\} \right\}$  and  $\{x \mid \phi(x) = \phi(z^0)\} \subset A(\text{MOL})$  and where  $\left\{ \{x \mid \|x\| > \|x^0\|\} \cap \left\{ x \mid x \geq 0, (x_{v_1}, x_{v_2}, \dots, x_{v_k}) = (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \right\} \right\} \subset A(\text{OP})$ .

From Proposition 5 the following corollary is obvious:

**Corollary 1:**

Let the production function  $\phi$  be quasi-concave and  $A(\text{OL}) = A(\text{MOL})$ , then there is a law of variable proportion for the factor combination  $\{v_1, v_2, \dots, v_k\}$ ,  $1 \leq k < n$ , if and only if the factor combination  $\{v_{k+1}, v_{k+2}, \dots, v_n\}$  is null joint with  $\{v_1, v_2, \dots, v_k\}$ .

As an immediate consequence of this corollary and Proposition 1 one has:

**Corollary 2:**

Let the production function  $\phi$  be quasi-concave and  $D(v_1, v_2, \dots, v_k)$  be one dimensional, then there is a law of variable proportion for the factor combination  $\{v_1, v_2, \dots, v_k\}$ ,  $1 \leq k < n$ , if and only if the factor combination  $\{v_{k+1}, v_{k+2}, \dots, v_n\}$  is null joint with  $\{v_1, v_2, \dots, v_k\}$ .

## REFERENCES

- [1] Färe, R. and L. Jansson, "Joint Inputs and the Law of Diminishing Returns," Zeitschrift für Nationalökonomie, Vol. 36, (1976).
- [2] Shephard, R. W., "Proof of the Law of Diminishing Returns," Zeitschrift für Nationalökonomie, Vol. 30, (1970).
- [3] Shephard, R. W., "Semi-Homogeneous Production Functions," in PRODUCTION THEORY, Lecture Notes in Economics and Mathematical Systems, Vol. 99, Berlin, (1974).