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FOREIGN TECHNOLOGY DIVISION

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PRECISE RESEARCH OF ASYMPTOTIC RELATIONS BETWEEN
THE AVERAGE EXPONENTIAL AND CHEBYSHEV APPROXIMATIONS
(REPORT II)

by

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PRECISE RESEARCH OF ASYMPTOTIC RELATIONS BETWEEN THE AVERAGE
EXPONENTIAL AND CHEBYSHEV APPROXIMATIONS (REPORT II *).

Ye. Ya. Remez.

FOOTNOTE *. The first part of the work is printed in Ukrainian
language in the previous issue of this collection. See also three
author's compressed report/communications on these questions in the
PAS of the USSR (1947), of Vol. LVIII.

The being encountered in this article literary references,
marked by numerals in brackets, are related to the list of the cited
literature of the first part of the work. ENDFOOTNOTE.

§13. Subsequently we are intended to place in parallel with the
results, obtained in §§3-11 the previous report/communication, the
analogous facts, which concern the processes interpolational,
connected with construction for the assigned on segment [a, b], set

of functions $\{v_i(x)\}$ the solutions of the problem of the average exponential approach/approximation on the alternating/variable finite set N of the points of segment [a, b], when a number of points consecutively and unlimitedly grows.

Let us begin from the establishment of some basic common/general/total facts.

In the same way as in §3 and the following after it, let us be hearth v_0 (x), v_1 (x). ..., $v_n(x)$ to understand certain assigned and fixed/recorded system real, continucus cn the segment [a, b] (b — a = l) and of linearly independent on it functions, and frcm the common/general/total set of polynomials $\Omega(x) = \sum_{i=0}^{n} c_i v_i(x)$ let us isolate as the permissible polynomials those, for which $c_0 = 1$, i.e.,

$$\Phi(x) = v_0(x) + \sum_{i=1}^{n} c_i v_i(x).$$

Let us place in the correspondence to each positive integer value of number N ⩾ n + 1 certain specific totally disconnected set $E_N \subset [a, b]$ consisting of N of points [33]:

$$x_{N1}, x_{N2},..., x_{NN} \subset [a, b]. \qquad\qquad (134)$$

FOOTNOTE [33]. Essence of further examinations did not change if we gave N not all positive integer values ⩾n + 1, but certain

infinite-increasing particular sequence of the same. ENDFOOTNOTE.

Analogous with problems (9), (10) they are formulated for each of the final totally disconnected sets E_N in question the problem:

$$\delta_{m,N}[\Phi] = \delta_{m,N}(c_1,\ldots,c_n) = \left[\frac{1}{N}\sum_{i=1}^{N}|\Phi(x_{Ni})|^m\right]^{\frac{1}{m}} = \min \quad (m>1) \quad (135)$$

and

$$\delta_{0N}[\Phi] = \delta_{0N}(c_1,\ldots,c_n) = \max_{x \in E_N}|\Phi(x)| = \min, \quad (136)$$

solutions of which (for the second task - not necessarily the only ones) let us designate respectively by $\Phi_{m,N}(x)$ and $\Phi_{0N}(x)$.

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We will assume still:

$$\delta_{0N}[\Phi_{0N}] = \varrho_N, \quad \delta_{0N}[\Phi_{m,N}] = \varrho_N(1+2\alpha_{m,N}) \quad (137)$$

and, on the other hand, preserving designations (10), (13),

$$\delta_0[\Phi_{m,N}] = \varrho + \Delta_{m,N} = \varrho(1+2\beta_{m,N}). \quad (138)$$

The establishment of the corresponding estimate/evaluations for value $\beta_{m,N}$ at the large values of m and N will compose the final goal of further examinations (§15).

In order to ensure the uniqueness of the solution of problem (135) for each [34] value of N, it is sufficient (comp. §1) to agree, during the composition of totally disconnected sets $E_N (N=n+1, \ n+2,...)$, each time to attend to to include/connect in its composition any and the points of segment [a, b], on which the functions $v_1(x),...,v_n(x)$ are linearly independent, which is surely possible [35] in view of the assumed linear independence of functions $v_i(x)$ from segment [a, b].

FOOTNOTES [34]. For sufficiently large values of N the linear independence of functions $v_i(x)$ for E_N [including even $v_0(x)$] and, therefore, the uniqueness of the solution of problem (135) in actuality turn out to be automatically provided for under condition (139) already on the strength of the assumed linear independence of continuous functions $v_i(x)$ on segment [a, b] as of this it is possible to be convinced by the reasoning, analogous to given below in the proof of theorem (comp. also observation to theorem).

[35]. Comp. chapter VIII my monograph [10]. ENDFOOTNOTES.

Very significant for future reference it turns out to be,

furthermore, the requirement in order that during growth in number N
of the point of set E_N they would be distributed over segment [a, b]
at least approximate-evenly. This requirement we can understand here
in the following sufficiently soft sense:

for any fixed/recorded particular segment $[\alpha, \beta] \subset [a, b]$ must exist
this positive number x, independent of N, but which is powerful of
depending on the segment itself $[\alpha, \beta]$ in order that designate
through $\mathcal{R}_{\alpha, \beta, N}$ a number of points of set E_N, which fall on this
particular segment $[\alpha, \beta]$ would satisfy the inequality:

$$\frac{\mathcal{R}_{\alpha, \beta, N}}{N} > x \frac{\beta - \alpha}{b - a} \quad \text{for sufficiently large N.} \qquad (139)$$

By assuming these conditions made, we now let
establish/install the following, that has fundamental value for
further conclusion/derivations the theorem, analogous by that, that
was establish/installed in §1 for the in principle simpler case of
the fixed/recorded multitude E.

Theorem. The coefficients $c_{m, N, i}$ the polynomial

$$\Phi_{m, N}(x) = v_0(x) + \sum_{i=1}^{\cdot} c_{m, N, i} v_i(x). \qquad (140)$$

employee by the solution of problem (135), are uniformly bounded in
their set for all values of $N \geqslant n + 1$ and $m > 1$ in question.

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Proof. For any that which was fix/recorded E_N and, therefore, generally, for limited $N \leqslant \bar{N}$ the limitedness of coefficients $c_{m,N,i}$ escape/ensues from theorem of §1. Therefore, proving this theorem of the contrary, it suffices to assume existence of the sequence of the permissible polynomials - the solutions of the corresponding problem of average exponential approach/approximation (135):

$$\Phi_I = \Phi_{m_v, N_i}(x), \quad \Phi_{II} = \Phi_{m_v, N_i}(x), \dots, \quad \Phi_{(v)} = \Phi_{m_v, N_v}(x), \dots, \qquad (141)$$

for which simultaneously they would take place of the relationship:

$$N_v \to \infty, \quad L_v = L[\Phi_{(v)}] \to \infty \quad \text{with} \quad v \to \infty, \qquad (142)$$

where L it designates value (3).

Let us consider (generally, not belonging already to the class "permissible") the "given polynomials":

$$\Omega_v(x) = \Omega_v = \frac{\Phi_{(v)}}{L_v} \quad (v = 1, 2, 3, \dots), \qquad (143)$$

satisfying, obviously, the condition

$$L[\Omega_v] = 1 \quad (v = 1, 2, 3 \dots). \qquad (144)$$

Let it will be

$$\delta_{m_v, N_v}[\Phi_{(v)}] = \left[\frac{1}{N_v} \sum_{j=1}^{N_v} |\Phi_{v(}(x_{N_v})|^{m_v} \right]^{\frac{1}{m_v}} = M_v \quad (v = 1, 2, 3, \dots). \qquad (145)$$

From the requirement

$$M_\nu < \delta_{m_\nu, N_\nu}[\Phi_\nu] < \rho \qquad (146)$$

it follows immediately:

$$\left[\frac{1}{N_\nu} \sum_{j=1}^{N_\nu} |\Omega_\nu(x_{j\nu})|^{m_\nu}\right]^{\frac{1}{m_\nu}} = \frac{M_\nu}{L_\nu} < \frac{\rho}{L_\nu} \to 0 \quad \text{with} \quad \nu \to \infty. \qquad (147)$$

On the other hand, as a result of the limitedness of the coefficients of the given polynomials Ω_ν, expressed by condition (144), there is a particular sequence of indices ν_s (s = 1, 2, 3, ...), for which all n + 1 coefficients of polynomial Ω_ν simultaneously approach the limiting values, which determine certain polynomial $\Omega_0(x)$.   The latter satisfies, obviously, the same condition (144) (has a coefficient with $\nu_0(x)$ the knowingly equal to zero).

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Restoring, for larger simplicity, a designation ν, instead of ν_s, for the index of this particular sequence, we have, therefore,

$$\lim_{\nu \to \infty} \Omega_\nu(x) = \Omega_o(x) \text{ evenly for } a < x < b, \tag{148}$$

$$\mathcal{U}[\Omega_o] = 1. \tag{149}$$

It is obvious with the fact that relationship (147) remains valid and for this particular sequence. Here, however, immediately is reveal/detected contradiction.

Actually, from (149), taking into account the linear independence of functions $v_i(x)$ on [a, b], we include that the polynomial $\Omega_o(x)$ is different of identical zero on [a, b], and its module/modulus, being continuous function of x, it has certain positive minimum 2μ of any particular segment $[\alpha, \beta] \subset [a, b]$ length $\beta - \alpha = \lambda$. On the same segment $[\alpha, \beta]$ the given polynomials Ω_ν at sufficiently large values $\nu > \bar{\nu}$ satisfy the condition

$$\min_{\alpha < x < \beta} |\Omega_\nu(x)| > \mu \text{ for } \nu > \bar{\nu}. \tag{150}$$

But in that case, taking into account condition (139), which will be made along with (150) for sufficiently large $\nu > \bar{\bar{\nu}} > \bar{\nu}$, we for all such values ν must have:

$$\left[\frac{1}{N_\nu} \sum_{i=1}^{N_\nu} |\Omega_\nu(x_{N_\nu i})|^{m_\nu} \right]^{\frac{1}{m_\nu}} > \mu \left(N \frac{\lambda}{l} \right)^{\frac{1}{m_\nu}} > \mu N \frac{\lambda}{l}, \tag{151}$$

that, on comparison with (147), and it completes proof by contradiction.

Observation. By using the very light/lung modification of the given here reasonings, it is possible to be convinced analogously of the uniform limitedness of the coefricients of the polynomial $\Phi_{0N}(x)$, and also of absolute limitedness frcm below value ϱ_N at

$$N \to \infty: \varliminf_{\overline{N \to \infty}} \varrho_N > 0.$$

§14. We can now nearer approach the resolution of the question concerning the establishment of estimate/evaluations for values $\Delta_{m,N}, \beta_{m,N}$ [(138)].

Examining preliminarily problems (135) and (136) on any fixed/recorded multitude E_N [(134)] and completely without taking into account the behavior of functions $v(x)$ and cf polynomial $\Phi(x)$ on the remaining part of the segment [a, b], we we can consider value $\eta_{m,N}$ [(137)], on the basis of the relationship

$$(\varrho_N)^m (1 + 2\alpha_{m,N})^m < M(\varrho_N)^m, \qquad (152)$$

escape/ensuing from the requirement

$$\delta_{m,N}[\Phi_{m,N}] < \delta_{m,N}[\Phi_{0N}].$$

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Logarithmizing (152) (after decrease) and introducing, analogous
to (18) and (20), the value:

$$\frac{\log(1+2a_{m,N})}{2a_{m,N}} = \sigma_{m,N}, \quad \inf_m \sigma_{m,N} = \underline{\sigma}_N, \qquad (153)$$

we will obtain:

$$a_{m,N} < \frac{1}{2\sigma_{m,N}} \frac{\log N}{m}, \qquad (154)$$

and, which means, themes it is more

$$a_{m,N} < \frac{1}{2\underline{\sigma}_N} \frac{\log N}{m}. \qquad (155)$$

FOOTNOTE ^{36}. In the case $a_{m,N}=0,$ or even $\varrho_N=0,$ which also is
not excluded by the general conditions, accepted in previous
paragraph (at least, for too large values of N), we will set/assume
$\sigma_{m,N}=1,$ and (154) surely it will remain valid, if we with $\varrho_N=0$
count always $a_{m,N}=0.$ ENDFOOTNOTE.

A number $\underline{\sigma}_N$ here surely is positive in view of limitedness $a_{m,N},$
which with fixed E_N and alternating/variable $m > 1$ it is the
corollary of the uniform (relative to m) limitedness of the
coefficients of the polynomial $\Phi_{m,N}(x),$ resulting from the
general theorems of §1. In view of the clearly adjustable by

relationship (155) infinite smallness of $a_{m,N}$ when $m \to \infty$ (which,
however, and priori it had to in this case to occur on the strength
of common/general/total theorem of §2) we come further for the large
values of m to the obvious precise conclusions:

$$\lim_{m \to \infty} \sigma_{m,N} = 1, \; \overline{\lim_{m \to \infty}} (m a_{m,N}) < \frac{1}{2} \log N. \qquad , \qquad (156)$$

If we begin now to vary in (152) not only m, but also N, then we
logically will arrive at the need for introducing to examination
value

$$\inf_{m,N} \sigma_{m,N} = \underline{\sigma}. \qquad (157)$$

The having most essential value for the justifiability of
further conclusion/derivations fact of the positivity of this value
is establish/installed analogously to fact $\underline{\sigma}_N > 0$,   but no longer on
the basis of theorem of §1, but on the basis of the theorem,
demonstrated in the previous paragraph, and observation to it. On the
strength of (154) we have directly

$$a_{m,N} < \frac{1}{2\underline{\sigma}} \frac{\log N}{m}. \qquad (158)$$

In this relationship of parameters m and N (m > 1, N $\geqslant$ n + 1)
they can be varied completely independently of each other.

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Let us allow however, that, increasing m, we simultaneously change and N as certain function of m [37], thus, that with m → ∞ we have log N/m → 0.

FOOTNOTE [37]. We make this assumption for simplicity. It would be possible to visualize in more common/general/total form that m and N are changed simultaneously depending on any third value, which plays the role of their common/general/total argument. ENDFOOTNOTE.

Then value $\alpha_{m,N}$ turns out to be on the strength of (158) infinitesimal, and this it imply, according to (153), (154), the relationship:

$$\lim_{\frac{\log N}{m} \to 0} \sigma_{m,N} = 1, \quad \overline{\lim_{\frac{\log N}{m} \to 0}} \left(\alpha_{m,N} : \frac{\log N}{m} \right) < \frac{1}{2}. \qquad (159)$$

Finally, before going further, we will refine somewhat introduced by us under §13 condition (139) about approximate- the even distribution of the points of set E_N on segment [a, b] (b - a = 1) during the large values of N. Specifically,, considering numbers (134) arrange/located in ascending order and designating

$$\lambda_N = \max (x_m - a, x_m - x_m, x_m - x_m, \ldots, b - x_{NN}), \qquad (160)$$

we let us assume made the following condition of the extremity of two values s, s': [38]

$$s = \varlimsup_{N \to \infty} \left(\lambda_N : \frac{l}{N} \right) < \sup_{N \geq n+1} \left(\lambda_N : \frac{l}{N} \right) = s' < \infty. \tag{161}$$

FOOTNOTE [38]. It is not difficult to comprehend that always $s \geqslant 1$. ENDFOOTNOTE.

It is easy to understand that this last/latter condition is certain intensification of condition (139): under condition (161) surely turns out to be that which was made condition (139), besides with the constant value of parameter x (independent of the selection of the particular segment $[\alpha, \beta]$ on $[a, b]$), for such it will be possible to take any fixed/recorded positive number less than 1/s.

§15. Now we can be converted already directly to values (138), for the purpose of evaluating the uniform on all segment $[a, b]$ approach/approximation, reached by polynomial $\Phi_{n, N}(x)$ - by the solution of the problem of the average exponential

approach/approximation on the final point set E_N (134) the same

segment, comparing this polynomial with polynomial $\varphi_0(x)$ [(12)],

that represent the best uniform approximation (smallest uniform

deviation from zero) on segment [a, b].

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With designations (10) - (13) and (135) - (138) we have first of

all:

$$\Delta_{m,N} = \delta_0[\Phi_{m,N}] - \varrho \leqslant \{\delta_0[\Phi_{m,N}] - \delta_{0N}[\Phi_{m,N}]\} + \{\delta_{0N}[\Phi_{m,N}] - \varrho_N\}. \quad (162)$$

Assuming functions $b_i(x)$ (i = 0, 1, ..., n) belonging to

certain class, characterized by conditions of the type (61) with

specific majorant function $\bar{\omega}(\delta)$,   polynomials $\Phi_{m,N}(x)$ on the

strength of the demonstrated uniform limitedness of their

coefficients will everything satisfy the same condition (61) with

certain constant value of coefficient of K, depending neither on m

nor of N, since we thus far leave aside the question concerning the

refinement of the value of this coefficient. In that case we from

(162) is easily concluded:

$$\Delta_{m,N} < K\bar{\omega}(\lambda_N) + 2a_{m,N}\varrho_N < K\bar{\omega}\left(s_N \frac{l}{N}\right) + \frac{\varrho}{2}\frac{\log N}{m}, \quad (163)$$

where value $s_N = \lambda_N : \dfrac{l}{N}$ it is bounded from two sides:

$$\frac{2}{3} < \frac{N}{N+1} < s_N = \lambda_N : \frac{l}{N} < \delta'. \quad (164)$$

Let us introduce into more specific type examination the interpolational process, in which the selection of the final totally disconnected set E_N (134), with increasing m, is subordinated, besides condition (161), to the also following condition, which establish/installs the general nature of dependence between parameters m and N:

$$\lambda_N \equiv s_N \frac{l}{N} = \chi[\varphi(qm)]. \tag{165}$$

Here, with the preservation/retention/maintaining of designations of §§9 and 10, $q = q_m$ it designates certain remaining at our disposal (for providing a larger flexibility of algorithm) positive parameter, which we at first, for a definition, also will assume bilaterally limited [with subsequent replacement of this requirement by more general conditions (172)]:

$$0 < c_1 < q < c_2. \tag{166}$$

From condition (165) we obtain, taking into account (93) and (95):

$$\log \frac{N}{s_N l} = \log \frac{1}{\chi[\varphi(qm)]} = qm\varphi(qm), \tag{167}$$

that directly it gives to us, by the way, convenient for many cases explicit expression for N in the form

$$N = s_N l e^{qm\varphi(qm)}, \tag{168}$$

showing that with m $\rightarrow$ ∞ number N will grow always slower, in any case, than e^m.

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On the other hand, then (167), taking into account (166), we have, obviously,

$$\log N = (1+\eta) \log \frac{1}{\chi[\varphi(qm)]} = (1+\eta)\, qm\varphi\,(qm) \qquad (167')$$

$$(\eta \rightarrow 0 \text{ with } m \rightarrow \infty).$$

On the basis (165) and (167') relationship (163) assumes the form:

$$\Delta_{m,N} < K\varphi\,(qm) + \frac{e(1+\eta)}{e}\, q\varphi\,(qm). \qquad (169)$$

However, we can be freed from denominator e in the second member in the form of the established/installed now infinite smallness of relation $\frac{\log N}{m} = (1+\eta)q\varphi(qm)$; being returned to (163) and using now inequality (154), instead of (158), taking into account (159), we will assume $(1+\eta): e_{m,N} = 1+\varepsilon,$ that it will give to us:

$$\Delta_{m,N} < K\varphi\,(qm) + q\varphi\,(1+\varepsilon)\varphi\,(qm) \qquad (\lim_{m \to \infty} \varepsilon = 0) \qquad (170)$$

or, dividing on 2ϱ and remembering that $\Delta_{m,N} = \varrho \cdot 2\beta_{m,N},$

$$\beta_{m,N} < \frac{K}{2\varrho}\varphi(qm) + \frac{1+\varepsilon}{2}q\varphi(qm) \quad (\varepsilon = \varepsilon_m \to 0 \text{ with } m \to \infty). \quad (171)$$

This means that taking account (96) that

$$\beta_{m,N} < \left(\frac{K}{2\varrho} + \frac{1+\varepsilon}{2}q\right)q^{-\theta}\varphi(m) \quad (0 < \theta = \theta_m < 1). \quad (171')$$

Value $\beta_{m,N}$ **characterizes,** according to (138), the uniform on segment [a, b] approach/approximation, reached by the interpolational polynomial of the average exponential approach/approximation $\Phi_{m,N}(x)$. Equate/comparing obtained for it estimate/evaluation (171) - (171') with estimate/evaluation (104) - (104') - (121) - (123) analogous value α_m for the polynomial of the average exponential approach/approximation on the entire segment $\Phi_m(x)$, taking into account the making more precise observations in §11, we first of all we can state/establish that significant circumstance that the replacement $\Phi_m(x)$ by interpolational polynomial $\Phi_{m,N}(x)$, with the observance of conditions (165) - (166) and (161) not only does not break the very fact of convergence to Chebyshev approach/approximation in the sense of the tendency toward zero of both differences

$$\delta_0[\Phi_m] - \varrho \quad \text{and} \quad \delta_0[\Phi_{m,N}] - \varrho,$$

but also leaves (in the general case) by constant the order of magnitude of the difference in question, which in both problems is

determined by the order of magnitude of function ϕ (m).

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Being converted further to the examination of the coefficients
with ϕ (m) in the compared estimates, we let us note, in the first
place, that when Chebyshev problem (10) has infinite solution set
whose coefficients can in that case be treated as coordinates of
point $(c_1, c_2, ..., c_n)$, that which pass certain limited, closed, and
convex totally disconnected set H_0 in the appropriate N-dimensional
euclidean space [39], about coefficient of K in (171) is represented
possible to only say that to it can be assigned (at the large values
of m) value, how conveniently close to the face side of the same
structural coefficient K in relationship (61) for the mentioned set
(H_0) of Chebyshev polynomials $\{\Phi_N(x)\}$ [40].

FOOTNOTES [39]. Comp. chapter VIII my, cited monograph [10].

[40]. It is not difficult to see that the point $(c_{-N,1}, c_{-N,2}, ..., c_{m,N,1})$
    in the mentioned N-dimensional euclidean space infinite
approaches, in the course of interpolation process in question,
totally disconnected set H_0. On the other hand, it is easy to
comprehend that the order of the smallness of the distance of the

point indicated from set H_0 noncan never to be higher than the order
of the smallness of values $\sigma_{m,N}$ and $\beta_{m,N'}$ consequently in the
general case there cannot be higher than the order of the smallness
of value ϕ (m) at a $\rightarrow$ ∞. ENDFOOTNOTES.

But when solution $\phi_0(x)$ Chebyshev problem (10) turns out to be
only, and specifically, consequently, in all cases when functions
$v_1(x),....v_n(x)$ compose on segment [a, b] the set of functions of
Chebyshev [*] [in other words - when is satisfied the Haar condition
[11], mentioned in footnote 5 §2], it will be possible to count
coefficient of K (171) how convenient to close to the value of the
mentioned structural coefficient of K for this only Chebyshev
polynomial $\phi_0(x)$.

FOOTNOTE [*] S. N. Bernshteyn, [21], Chapter 1. ENDFOOTNOTE.

In these cases we can identify coefficient of K in (171) with
coefficient of k in (104) - (121), also, on this basis/base make a
more precise quantitative comparison of estimate/evaluations (171)
and (104) - (121). Relationship (171) can in actuality serve as
starting point for different posing of the question concerning the

most advantageous possible selection of the value of the parameter
$q=q_m$ in the determined by us interpolational process. We here will
pause, mainly, at one of them, examining question from the standpoint
of the accessible here maximum refinement of interpolational
algorithm and planned more precise comparison of convergence
properties of both processes with $m \rightarrow \infty$.

In the individualized study of problem for the different
concrete/specific/actual classes of functions, we sometimes will have
the capability, always retaining condition (161) and (165),
substantial to widen condition (166), by set/assuming $c_1 = 0$ or by
substituting c_2 on $+\infty$, by requiring only satisfaction of the
conditions

$$\left.\begin{array}{l} \varphi(qm) = O[\varphi(m)] \\ q\varphi(qm) = O[\varphi(m)] \end{array}\right\} \ (q=q_m, \ m \rightarrow \infty). \qquad (172)$$

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With the observance of the first of these conditions (172) we
surely will have $qm \rightarrow \infty$, and this - only, that is required for
provision (167') and all remaining relationships from (167) to (171)
inclusively also, as easily seen, they will remain completely in
force under the conditions (172).

1. Case of functions $v(x)$, satisfying the Lipshitz condition

of order r $(0 < r \leq 1)$.

On the strength of (124) we have here under condition (166) [2]:

$$\varphi(qm) = \frac{1+\epsilon'}{q}\varphi(m); \qquad (173)$$

$$\beta_{m,N} < \frac{K}{2q}\frac{1+\epsilon'}{q}\varphi(m) + \frac{1+\epsilon''}{2}\varphi(m). \qquad (174)$$

FOOTNOTE [2]. By $\epsilon, \epsilon', \epsilon_1, \epsilon_1$ and, etc subsequently, without specifying this, we understand some functions cf m, which approach 0 themselves when $m \to \infty$. ENDFOOTNOTE.

Taking ϵ_1 in (166) by sufficiently large, we we can here do the first member is how many by conveniently small as compared with the second [3].

FOOTNOTE [3]. Upon another posing of the question, without attaining (asymptotic) the minimization of expression (174 in question) [which in this case is reached by the price of an increase in the number of

used abscissas x_N in (135)], but desiring only as far as possible to smooth the values of both members in right side (174), we would be taken $\epsilon \approx \frac{h}{\ell}$.

Let us note still that the ascending order in the number of interpolational abscissas x_N pri, $m \to \infty$ for the taken apart case more convenient anything is obtained not from (168), but directly to (165) in the form:

$$N \sim O\left[\left(\frac{m}{\log m}\right)^{\frac{1}{q}}\right]. \qquad (175)$$

ENDFOOTNOTE.

Consequently, by designating through δ as small as desired, assigned on arbitrariness positive number, let us have, by taking into account again (124):

$$\beta_{m,N} < \frac{1+\delta}{2} \cdot \frac{1}{\tau} \frac{\log m}{m} \quad (m > m_\delta), \qquad (176)$$

and we see directly that the obtained upper boundary for $\beta_{m,N}$ asymptotically differs how conveniently little from the appropriate boundary for α_m that adjustable in this case by most precise relationship (27').

Finally, substituting condition (166) by less constrained

conditions (172) and taking, for example, it is
concrete/specific/actual

$$q = q' \log m \quad (0 < c_1 < q' = q'_m < c_2)^{(**)}. \qquad (177)$$

where, as easily seen, (173) and (174) they preserve their form, we
is already completely reached the asymptotic equality of both
compared boundaries, since arbitrarily the close to unity coefficient

$(1 + \delta)$ in (176) is substituted by coefficient by infinitely close to
unity.

FOOTNOTE **. In this case in relation (175) it is necessary to endow
by divider/denominator log m but with a simultaneous improvement in
the numerical coefficient. ENDFOOTNOTE.

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2. Case of functions $v(x)$,   satisfying the weakened condition
Dinis 1st order (52).

Here we have accurately (comp. §11):

$$\varphi(M) = \frac{1}{\sqrt{M}}, \quad \varphi(qm) = q^{-\frac{1}{2}} \varphi(m). \qquad (178)$$

Thus,

$$\beta_{m,N} < \frac{k}{2q} q^{-\frac{1}{2}} \varphi(m) + \frac{1+\varepsilon}{2} q^{\frac{1}{2}} \varphi(m). \qquad (179)$$

With $q = \dfrac{k}{\varrho}$ we obtain:

$$\beta_{m,N} < \frac{1}{2} \sqrt{\frac{k}{\varrho}} \varphi(m) + \frac{1+\varepsilon}{2} \sqrt{\frac{k}{\varrho}} \varphi(m) =$$

$$= \left(1 + \frac{\varepsilon}{2}\right) \sqrt{\frac{k}{\varrho}} \varphi(m) = \left(1 + \frac{\varepsilon}{2}\right) \sqrt{\frac{k}{\varrho}} \cdot \frac{1}{\sqrt{m}}, \qquad (180)$$

and we again can accurately state/establish asymptotic agreement with upper bound for α_m, adjustable by relationship (59).

3. The case of functions $v_i(x)$, satisfying the weakened condition Dinis (127) arbitrary order $v \geqslant 2$.

Here, as can easily be seen from (128), accepting at first condition (166), we have:

$$\varphi(qm) = (1 + \varepsilon_1) \varphi(m). \qquad (181)$$

Consequently,

$$\beta_{m,N} < \frac{k}{2q} (1 + \varepsilon_1) \varphi(m) + \frac{1+\varepsilon_2}{2} q\varphi(m). \qquad (182)$$

Taking now in (166) c_2 by sufficiently small let us find, it is analogous with that, as this was done in the first case:

$$\beta_{m,N} < \frac{k}{2\varrho}(1+\delta) \cdot \frac{1}{\log^{(r-1)} m} \quad (m > m'_\delta), \qquad (183)$$

understanding by δ as small as desired arbitrarily assigned (fix/recorded) positive number. We have, thus, as in the case 1°, asymptotically how conveniently close agreement of the specific here upper boundary for $\beta_{m,N}$ and with the most precise value of upper bound for α_m according to results of §11.

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But also in this case accessible the full/total/complete asymptotic agreement (asymptotic equality) of both compared boundaries and besides with an advantageous decrease in number N of interpolational abscissas, if we, substituting condition (166) for (172), take concrete/specific/actually at least

$$q = \frac{q''}{\log m} \quad (0 < c_1 < q'' = q'_m < c_s) \qquad (184)$$

with any $r > 2$ [5].

FOOTNOTE [5]. In the relation to an asymptotic decrease in the number of interpolational abscissas here it would be possible to go and

considerably further, reinforcing speed of the decrease of the

parameter $q-q_m$ at $m \rightarrow$ $\bullet$ (especially for $\nu>2$)   - either with

observance (181), or with the replacement of the requirement for

asymptotic equality (181) by certain inequality, let us say,

$$\varlimsup_{m \to \infty} \frac{\varphi(q,m)}{\varphi(m)} < 2, \tag{181'}$$

which, it goes without saying, will make it possible to achieve

further decrease in the ascending order N and, etc. ENDFOOTNOTE.

The aggregate of the results, to which led us in the present

work research presented, has nearest value for the questions of

application/appendix to the Chebyshev problems proposed by me (see

introduction) the method of the approximate construction of the

solutions of the problems of the mean-exponential approximation,

justifying, in particular, the application/use of an interpolational

version of the mentioned method with the elimination of those

difficulties of fundamental nature, into which runs the use in of

this kind the questions of some commonly used in other cases of the

methods of the approximation calculus of integrals.

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