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A REVIEW OF SEVEN DOAE PAPERS ON MATHEMATICAL MODELS OF ATTRITI--ETC(U)
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UNCLASSIFIED P-1249 IDA/HQ-76-18957 NL

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A REVIEW OF SEVEN DOAE PAPERS ON MATHEMATICAL MODELS OF ATTRITION

Alan F. Karr

November 1977

INSTITUTE FOR DEFENSE ANALYSES
PROGRAM ANALYSIS DIVISION

IDA Log No. HQ 76-18957

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Nov[redacted]77

54p.

Final rept.,

IDA/HQ, SBIE

76-18957

AD-E500 004

IDA
INSTITUTE FOR DEFENSE ANALYSES
PROGRAM ANALYSIS DIVISION
400 Army-Navy Drive, Arlington, Virginia 22202

IDA Independent Research Program

403 219

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE — READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: P-1249
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): A Review of Seven DOAE Papers on Mathematical Models of Attrition
5. TYPE OF REPORT & PERIOD COVERED: Final
6. PERFORMING ORG. REPORT NUMBER: IDA P-1249
7. AUTHOR(s): Alan F. Karr
8. CONTRACT OR GRANT NUMBER(s): IDA Independent Research
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Institute for Defense Analyses, Program Analysis Division, 400 Army-Navy Drive, Arlington, VA 22202
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS:
11. CONTROLLING OFFICE NAME AND ADDRESS:
12. REPORT DATE: November 1977
13. NUMBER OF PAGES: 50
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS (of this report): Unclassified
15a. DECLASSIFICATION DOWNGRADING SCHEDULE: NA
16. DISTRIBUTION STATEMENT (of this Report): This document is unclassified and suitable for public release.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number): Stochastic attrition process, attrition equation, combat simulation, Lanchester attrition process, battle termination, battle duration, Markov attrition process

20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

This paper is a descriptive and critical review of seven papers on the mathematics of combat attrition prepared by staff members of the Defence Operational Analysis Establishment of the U.K. The papers reviewed are *A Bivariate Probability Distribution; The Moments of the Distribution of Battle States; Homogeneous Battles with General Attrition* → next page

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Functions; and *The Distribution of the Duration of Battle;* ~~all by T.G. Weale~~; *Approximate Moments of the Distribution of States of a Simple Heterogeneous Battle*; and *Stochastic 'Linear Law' Battles* ~~by N. Jennings~~; and *Moments of the Distribution of States for a Battle with General Attrition Functions*, ~~by T.G. Weale and E. Peryer~~. Emphasis of the reviews is on mathematical aspects of the stochastic attrition process treated in the papers.

PREFACE

This paper is one of a series of IDA papers on combat modeling, sponsored by the IDA Independent Research Program. Mathematical attrition processes are fundamental to the combat simulation models used in many studies of defense problems. The DOAE research on attrition processes is a significant body of work, and the comments of this paper are intended to present and evaluate the DOAE results in the context of other mathematical, computational and applied work in combat modeling.

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1. INTRODUCTION

This paper is a review and summary of the following memoranda prepared by the Defence Operational Analysis Establishment (DOAE) of the UK:

(1) The Mathematics of Battle I: A Bivariate Probability Distribution, by T.G. Weale;

(2) The Mathematics of Battle II: The Moments of the Distribution of Battle States, by T.G. Weale;

(3) The Mathematics of Battle III: Approximate Moments of the Distribution of States of a Simple Heterogeneous Battle, by N. Jennings;

(4) The Mathematics of Battle IV: Stochastic "Linear Law" Battles, by N. Jennings;

(5) The Mathematics of Battle V: Homogeneous Battles with General Attrition Functions, by T.G. Weale;

(6) The Mathematics of Battle VI: The Distribution of the Duration of Battle, by T.G. Weale;

(7) The Mathematics of Battle VII: Moments of the Distribution of States for a Battle with General Attrition Functions, by T.G. Weale and E. Peryer.

These papers are hereafter cited as references [13], [14], [5], [6], [15], [16], and [17], respectively.

Common to all seven papers is an attempt to deal on a computational basis with certain stochastic attrition processes of the sort discussed in [10]. Not all processes discussed in the DOAE memoranda are mentioned explicitly in [10]; indeed some of the work in [15, 16, 17], wherein essentially arbitrary attrition functions are permitted, is not physically justified. There is no doubt that Markov processes exist with the generators indicated (cf. Section 6 for details); whether such processes correspond to physically plausible (or even physically definable) sets of assumptions is uncertain. Further remarks concerning this point, which is of considerable importance, may be found in Sections 6 and 9; cf. also [10].

The emphasis in this review is on assumptions underlying the attrition processes discussed, on mathematical computations and approximations presented in the papers, and on probabilistic derivations and interpretations of certain results that have been derived in the DOAE papers using methods from the field of different equations. In particular, we have not reviewed carefully the computer programs included in the papers, nor have we analyzed either sample outputs included in the papers or (except in preparation of the related papers [8] and [9]) a large number of additional outputs generously provided to us by DOAE. The few analyses performed have shown DOAE results to be consistent with those the present author reported in [8] and [9].

We are aware that the main purpose of the DOAE work was to produce computer programs with which numerical conclusions could be obtained, and that much of our criticism is directed at other aspects of the work. Even so, it is useful to have available a careful discussion of the foundations of the processes under analysis, which is what we have attempted to provide.

We wish, at the beginning of this review, to commend the authors of all seven DOAE papers for uniformly high technical and expository qualities of their work. The papers were a pleasure to read.

Throughout this paper our notation and terminology concerning Markov attrition processes are those of [10].

2. REVIEW OF "A BIVARIATE PROBABILITY DISTRIBUTION"

This paper reports efforts to compute and approximate the transition function of a stochastic attrition process that is analogous to the homogeneous Lanchester square differential equations of combat. That is, the author deals with Homogeneous Process 1 of [10, p. 17], which has independent engagement initiation and single kills. Referring to [10], we note that this Markov attrition process, which is denoted by $((B_t,R_t))_{t\geq 0}$, has infinitesimal generator A given by

$$(1)\qquad \begin{aligned} A((i,j),(i,j-1)) &= ic_B \\ A((i,j),(i,j)) &= -(ic_B+jc_R) \\ A((i,j),(i-1,j)) &= jc_R , \end{aligned}$$

where c_B, c_R are positive constants, jump function λ given by

$$(2)\qquad \lambda(i,j) = ic_B + jc_R ,$$

and transition matrix Q of the embedded Markov chain given by

$$(3)\qquad \begin{aligned} Q((i,j),(i,j-1)) &= \frac{ic_B}{ic_B + jc_R} \\ Q((i,j),(i-1,j)) &= \frac{jc_R}{ic_B + jc_R} . \end{aligned}$$

Rather than allow the combat to continue until one side is annihilated, the author imposes termination levels m_B for Blue and m_R for Red. That is, all states of the forms (i,m_R) and (m_B,j) are absorbing. For such states α, expressions (1), (2), and (3) are not valid and one has, instead, that for each state β

$$A(\alpha,\beta) = 0$$

$$\lambda(\alpha) = 0$$

$$Q(\alpha,\beta) = I(\alpha,\beta),$$

where I is the identity matrix. Observe that the state (m_B, m_R) is almost surely never entered.

Let us denote by (P_t) the transition function of this attrition process; cf. [10, p.10]. In [13], Weale is concerned mainly with computation and approximation of the transition function (P_t); that is, computation and approximation, for each fixed t, of the Markov matrix P_t. Another interest is computation of the limit $P_\infty = \lim_{t\to\infty} P_t$, which obviously exists and is the distribution of the terminal state of the process.

Weale begins by deriving, in a heuristic manner based on a physical interpretation of the attrition coefficients c_B and c_R, the forward equation for the process, namely

$$
\begin{aligned}
(4)\qquad P_t'((i,j),(k,\ell)) &= c_R \ell P_t((i,j),(k+1,\ell)) \\
&\quad + c_B k P_t((i,j),(k,\ell+1)) \\
&\quad - (c_B k + c_R \ell) P_t((i,j),(k,\ell)) \\
&= P_t A((i,j),(k,\ell)) \; .
\end{aligned}
$$

The expression (4) is valid if (k,ℓ) is not an absorbing state; for absorbing states (k,m_R) we have

$$(5a)\qquad P_t'((i,j),(k,m_R)) = c_B k P_t((i,j),(k,m_R+1)) \; ,$$

while for states of the form (m_B,ℓ)

$$(5b)\qquad P_t'((i,j),(m_B,\ell)) = c_R \ell P_t((i,j),(m_B+1,\ell)) \; .$$

That these equations are valid is an immediate and rigorous consequence of the probabilistic derivation appearing in [10]; cf. also the Appendix to [7], where a detailed proof is given.

The author proceeds to note that for fixed initial conditions (i,j) there apparently exists no simple closed-form solution to the forward equations (4)-(5). To the reviewer's knowledge this is still so, although we note below some steps in this direction hinted at, but not fully identified, by Weale. That an explicit solution to (4)-(5) is hard to obtain is not, of course, a new piece of knowledge. R.N. Snow, who originally proposed this particular model [12], was aware of the difficulty.

With (i,j) fixed, certain of the forward equations can be solved in closed form. For example, it is immediate that

$$P_t((i,j),(i,j)) = e^{-(c_B i + c_R j)t} \quad ;$$

this follows from (2) and properties of continuous time Markov processes with finite state space; cf. [1] and [10, p.11]. The author gives a nonprobabilistic derivation from the forward equation, using differential equation methods.

Also, the author obtains the following explicit solutions by nonprobabilistic reasoning. For $m_R < \ell \le j$,

$$\text{(6a)} \quad P_t((i,j),(i,\ell)) = \frac{1}{(j-\ell)!}\left(\frac{c_B i}{c_R}\right)^{j-\ell} e^{-(c_B i + c_R \ell)t}\left(1-e^{-c_R t}\right)^{j-\ell} ,$$

while for $m_B < k \le i$,

$$\text{(6b)} \quad P_t((i,j),(k,j)) = \frac{1}{(i-k)!}\left(\frac{c_R j}{c_B}\right)^{i-k} e^{-(c_B k + c_R j)t}\left(1-e^{-c_B t}\right)^{i-k} .$$

These equations, which correspond to one side's having suffered no casualties, do not seem to have appeared elsewhere. Weale's derivation, as noted above, is based on properties of the forward equations as differential equations, so it seems worthwhile to sketch a probabilistic derivation. For fixed initial conditions (i,j) and a fixed time t

$$\text{(7)}\quad P_t((i,j),(i,\ell)) = P^{(i,j)}\{\text{first } j-\ell \text{ transitions are kills of Reds, } (j-\ell)^{th} \text{ transition occurs before } t,\ (j-\ell+1)^{st} \text{ transition occurs after } t\}$$

$$= P^{(i,j)}\{\text{first } j-\ell \text{ transitions are kills of Red}\}$$

$$\times\ P^{(i,j)}\ \{\text{time for } (j-\ell) \text{ transitions is } \le t, \text{ time for } (j-\ell+1) \text{ transitions is } > t \mid \text{first } (j-\ell) \text{ transitions are kills of Reds}\}$$

$$= \prod_{p=\ell+1}^{j} \frac{c_B i}{c_B i + c_R p}$$

$$\times \int_0^t (c_B i + c_R j) e^{-(c_B i + c_R j)t_1}\, dt_1$$

$$\int_0^{t-t_1} (c_B i + c_R (j-1)) e^{-(c_B i + c_R (j-1))t_2}\, dt_2$$

$$\int_0^{t-u_2} (c_B i + c_R (j-2)) e^{-(c_B i + c_R (j-2))t_3}\, dt_3$$

$$\vdots$$

$$\int_0^{t-u_{j-\ell+1}} \left\{ (c_B i + c_R(\ell+1)) e^{-(c_B i + c_R(\ell+1))t_{j-\ell}} \times e^{-(c_B i + c_R \ell)(t-u_{j-\ell})} \right\} dt_{j-\ell}\ ,$$

where $P^{(i,j)}$ is the probability law of the attrition process subject to $P^{(i,j)}\{(B_0,R_0)=(i,j)\} = 1$ and $u_p = (t_1 + \ldots + t_p)$.

In arriving at this expression the first terms are obtained from the transition matrix Q of the embedded Markov chain and the integral term arises from the jump function λ. One must invoke the characterization given in Corollary (8.3.11) of [1]. Tedious but straightforward calculations transform (7) to (6).

The author takes note of the further facts that for (k,ℓ) not an absorbing state there exists the representation

$$(8) \quad P_t((i,j),(k,\ell)) = \sum_{\rho=k}^{i} \sum_{\sigma=\ell}^{j} c((i,j),(k,\ell),(\rho,\sigma))e^{-(c_B\rho+c_R\sigma)t} ,$$

where the $c((i,j),(k,\ell),(\rho,\sigma))$ are constants, and that, for $m_B < k \le i$,

$$(9) \quad P_t((i,j),(k,m_R)) = P_\infty((i,j),(k,m_R)) + \sum_{\rho=k}^{i} \sum_{\sigma=m_R+1}^{j} c((i,j),(k,m_R),(\rho,\sigma))e^{-(c_B\rho+c_R\sigma)t}$$

with an analogous expression holding for states of the form (m_B,ℓ). These expressions are derived from general considerations about solutions of systems of differential equations. If the constants appearing in these expressions were known in closed form, computional problems would be solved. Weale asserts [13, p.11] that the constants can be calculated recursively, but does not elaborate; we sketch below a possible way of performing this computation. For ease of exposition fix (i,j) and consider $(k,\ell) = (i,j-1)$. We need to calculate $c((i,j),(i,j-1),(i,j))$ and $c((i,j),(i,j-1),(i,j-1))$, assuming that $(i,j-1)$ is not an absorbing state. Differentiating (8) and setting $t = 0$ yields, on the basis of the fundamental relation $A = P_0'$, the equation

$$(10a) \quad \begin{aligned} A((i,j),(i,j-1)) = &-(c_Bi+c_Rj)c((i,j),(i,j-1),(i,j)) \\ &-(c_Bi+c_R(j-1))c((i,j),(i,j-1),(i,j-1)) , \end{aligned}$$

while setting t = 0 in (8) gives

$$(10b) \quad c((i,j),(i,j-1),(i,j)) + c((i,j),(i,j-1),(i,j-1)) = P_0((i,j),(i,j-1)) = 0 \; .$$

One may then solve (10) for $c((i,j),(i,j-1),(i,j))$ and $c((i,j),(i,j-1),(i,j-1))$. More equations are necessary when (i,j) and (k,ℓ) are not adjacent states; exactly how these are to be obtained is not clear and is a problem worthy of further research.

Concerning computation of the limit (and terminal) distribution P_∞, the author obtains the recursion relations

$$(11a) \quad P_\infty((i,j);(k,m_R)) = \frac{c_B k}{c_B k + c_R(m_R+1)} P^*_\infty((i,j);(k,m_R+1))$$

$$(11b) \quad P_\infty((i,j);(m_B,\ell)) = \frac{c_R \ell}{c_B(m_B+1) + c_R \ell} P^*_\infty((i,j);(m_B+1,\ell)) \; ,$$

which are valid for $m_B < k \leq$ and $m_R < \ell \leq j$, where the asterisk in the right-hand side of (11) implies that the probabilities in question are computed with respect to termination levels (m_B,m_R+1) in (11a) and (m_B+1,m_R) in (11b), respectively. The result in (11) is not a closed-form solution but is certainly feasible for numerical computations even though to compute the termination distribution for one set of termination levels, the termination distribution must first be computed for every set of *higher* termination levels.

Two alternatives are available for performing numerical computations. First, one can introduce transforms in the following manner: define the *resolvent* $(R_\lambda)_{\lambda>0}$ of the transition function (P_t) by

$$R_\lambda = \int_0^\infty e^{-\lambda t} P_t \, dt \; ,$$

where matrix integration is performed componentwise. The matrix R_λ is called the λ-potential matrix of the transition function (P_t). Then either directly from (8)-(9) or by virtue of a standard Tauberian theorem [4, p. 421] it follows that

(12) $$P_\infty = \lim_{\lambda \downarrow 0} \lambda R_\lambda .$$

It is known [1,2] that for each $\lambda > 0$ the λ-potential matrix R_λ is given explicitly by

(13) $$R_\lambda = (\lambda I - A)^{-1}$$

and the infinitesimal generator A is known in closed form, so by (12) for small λ, λR_λ is a good approximation to P_∞. Second, since all recurrent states are absorbing,

(14) $$P_\infty = \lim_n Q^n ,$$

where Q is the transition matrix of the embedded Markov chain. Hence large powers of Q are good approximations to P_∞ and, indeed, the convergence in (14) takes place geometrically fast. One could thus compute, for example, $Q^2, Q^4 = (Q^2 \times Q^2), Q^8, \ldots$ to quickly obtain an approximation to $Q^\infty = P_\infty$. Here, moreover, the computational work is less than for the method based on (13), for the latter requires a time-consuming matrix inversion, while the method based on (14) requires relatively few matrix multiplications. Neither of the above methods is, however, numerically exact, whereas the method using (11) is.

The paper also deals with numerical solution of the forward equations (4)-(5); this is a further useful contribution and we believe that the associated computer program has received less attention than it deserves. However, the usual difficulties of extrapolating numerical results and of extracting theoretical insight from empirical data remain. Nonetheless these programs seem to represent a significant contribution to qualitative understanding of this stochastic attrition process.

3. REVIEW OF "THE MOMENTS OF THE DISTRIBUTION OF BATTLE STATES"

The principal objective of this paper [14] by T.G. Weale is to study time-dependent behavior of moments of the stochastic attrition process $((B_t,R_t))$ with generator A given by

$$A((i,j),(i,j-1)) = c_B i$$

$$(15) \qquad A((i,j),(i,j)) = -(c_B i + c_R j)$$

$$A((i,j),(i-1,j)) = c_R j \,;$$

i.e., the same homogeneous process discussed in [13], which was reviewed in the preceding section.

For initial states (i,j), points $(x,y) \in R^2$, nonnegative integers r,s, and times $t \geq 0$, let

$$(16) \quad M_{i,j}(r,s(x,y);t) = E^{(i,j)}[(B_t-x)^r(R_t-y)^s]$$

$$= \sum_{k=m_B}^{i} \sum_{\ell=m_R}^{j} P_t((i,j),(k,\ell))(k-x)^r(\ell-y)^s \,,$$

where (P_t) is the transition function of the attrition process and m_B, m_R are the termination thresholds for Blue and Red, respectively. If the transition function (P_t) were known explicitly or a good numerical approximation were available, then (16) could be used directly to compute all moments desired. In the absence of a closed-form expression for the transition function, it is useful--from both theoretical and practical points of view--to approximate moments, to derive relations among moments, and to seek qualitative information about moments by studying (16) directly. Such is the author's main goal and contribution; we now proceed to describe his work in more detail.

Assuming that (x,y) is not a function of t, one may differentiate (16) to obtain

$$(17)\quad M'_{i,j}(r,s,(x,y);t) = \sum_k \sum_\ell P'_t((i,j),(k,\ell))(k-x)^r(\ell-y)^s$$

$$= \sum_k \sum_\ell P_t A((i,j),(k,\ell))(k-x)^r(\ell-y)^s$$

$$= \sum_k \sum_\ell AP_t((i,j),(k,\ell))(k-x)^r(\ell-y)^s ,$$

where A is given by (15) and where the second and third equalities in (17) hold by virtue of the forward equation (4) and the related backward equation (cf.[1]), respectively. Direct substitution for A in either of the latter two equalities in (17) does not lead to a tractable system of equations. Indeed, use of AP_t produces a system that is not closed with respect to the initial conditions (i,j); i.e., solutions for (i,j+1) and (i+1,j) are needed in order to obtain the solution for (i,j). Use of P_tA leads to a closed system that, however, appears very difficult to solve; in particular one does not obtain a differential equation for $M_{i,j}(r,s,(x,y);\cdot)$. The author circumvents these obstacles by introducing the truncated moments

$$(18)\quad \tilde{M}_{i,j}(r,s,(x,y);t) = E^{(i,j)}[(B_t-x)^r(R_t-y)^s;\{T>t\}]$$

where T is the random time of termination of the battle, at which either $B_T = m_B$ and $R_T > m_R$ or $B_T > m_B$ and $R_T = m_R$. The author obtains the following equation (equation (12) of [14]), which is his main theoretical result:

(19) $M'_{i,j}(r,s,(x,y);t)$

$$= - c_B \sum_{\sigma=1}^{s} (-1)^{\sigma-1}\binom{s}{\sigma}[\tilde{M}_{i,j}(r+1,s-\sigma,(x,y);t) + x\tilde{M}_{i,j}(r,s-\sigma,(x,y);t)]$$

$$- c_R \sum_{\rho=1}^{r} (-1)^{\rho-1}\binom{r}{\rho}[\tilde{M}_{i,j}(r-\rho,s+1,(x,y);t) + y\tilde{M}_{i,j}(r-\rho,s,(x,y);t)].$$

If $r = 0$ or $s = 0$ the corresponding summation (taken over an empty index set) is zero. Except as a means of deriving relations among moments and truncated moments, (19) seems to have little application. From the standpoint of numerical calculations, truncated moments are rather difficult to obtain and even then (19) involves truncated moments of all lower orders and produces only the derivative M'. It might be better--if one's objective were numerical results--first to produce numerical approximations to the transition function (P_t) and then to employ (16) directly.

This leaves the possibility that special cases of (19) may lead to relations among the moments M and truncated moments $\tilde{M}$ that extend one's understanding of the attrition process--at least in a qualitative sense. The author, therefore, proceeds to consider several special cases.

Taking $r = 1$, $s = 0$, $x = 0$, $y = 0$ leads to the equation

$$(20) \qquad \frac{d}{dt} E^{(i,j)}[B_t] = - c_R E^{(i,j)}[R_t;\{T>t\}] .$$

This famous equation, derived originally by R.N. Snow [12] states that for small values of t the expectations of the stochastic attrition process approximately satisfy Lanchester's square-law differential model of combat, since an analogous expression is valid for $E^{(i,j)}[R_t]$.

To our knowledge, a completely probabilistic derivation of (20) has not been given before. The Theorem below is of interest, therefore, not only for its novelty, but also because it both simplifies and extends the results obtained by Weale in [14].

(21) **THEOREM.** Let f be a function on the state space E of the attrition process. Then for each t

$$(22)\qquad \frac{d}{dt}E^{(i,j)}[f(B_t,R_t)] = E^{(i,j)}[Af(B_t,R_t);\ \{T>t\}]\ ,$$

where A is the infinitesimal generator given by (15).

PROOF. By virtue of the forward equation (4)

$$(23)\qquad \frac{d}{dt}\,E^{(i,j)}[f(B_t,R_t)] = \frac{d}{dt}\sum_{\alpha} P_t((i,j),\alpha)f(\alpha)$$

$$= \sum_{\alpha}\left(\frac{d}{dt}P_t((i,j),\alpha)\right)f(\alpha)$$

$$= \sum_{\alpha} P_t A((i,j);\alpha)f(\alpha)$$

$$= \sum_{\alpha}\sum_{\beta} P_t((i,j);\beta)A(\beta,\alpha)f(\alpha)$$

$$= \sum_{\beta} P_t((i,j);\beta)\sum_{\alpha} A(\beta,\alpha)f(\alpha)$$

$$= \sum_{\beta} P_t((i,j);\beta)Af(\beta)$$

$$= E^{(i,j)}[Af(B_t,R_t)]\ .$$

But since $Af(\beta) = 0$ when β is an absorbing state,

$$(24)\qquad E^{(i,j)}[Af(B_t,R_t)] = E^{(i,j)}[Af(B_t,R_t);\{T>t\}]\ .$$

The proof follows by combination of (23) and (24). ∎

The reason for using the right-hand side of (24) in (22), rather than the left-hand side, is that in particular cases the function Af will have a closed-form expression for nonabsorbing states which is invalid for absorbing states. Use of the expression in the right-hand side of (24) is permissible, but one cannot do this on the left-hand side of (24). The examples below illustrate.

EXAMPLE. For $f(i,j) = i$ one has for nonabsorbing (i,j),

$$Af(i,j) = -jc_R$$

and (22) becomes (20). Similarly, for $f(i,j) = j$, one has

$$Af(i,j) = -ic_B$$

and obtains the corresponding equation for $\frac{d}{dt} E^{(i,j)}[R_t]$.

EXAMPLE. For $f(i,j) = ij$ we have, provided (i,j) is not absorbing,

$$Af(i,j) = -c_B i^2 - c_R j^2$$

and (22) becomes

$$\frac{d}{dt} E^{(i,j)}[B_t R_t] = -c_B E^{(i,j)}[B_t^2; \{T>t\}]$$
$$-c_R E^{(i,j)}[R_t^2; \{T>t\}] .$$

EXAMPLE. For $f(i,j) = i^2$ and (i,j) nonabsorbing,

$$Af(i,j) = -2c_R ij + c_R j$$

and one obtains the equation

$$\frac{d}{dt} E^{(i,j)}[B_t^2] = -2c_R E^{(i,j)}[B_t R_t; \{T>t\}]$$
$$+ c_R E^{(i,j)}[R_t; \{T>t\}] .$$

One may then derive expressions for derivatives of variances, and so on.

We remark--and this observation is of some importance--that (22) is valid if T is replaced by any stopping time S such that $(B_t, R_t) = (B_S, R_S)$ on $\{S \leq t\}$. S need not be defined in terms of absolute thresholds, for example.

One can also use Theorem (21) to derive the general equation (19) as well as expressions for derivatives of functions not expressible in the form $f(i,j) = (i-x)^r (j-y)^s$. We plan to report more fully on consequences of our Theorem in a forthcoming paper [11].

Most of the remainder of the memorandum [14] is occupied with computations similar to (19). Many of the author's results are worked out in painstaking detail and in this respect he has made a meaningful contribution. We know of no other reference for many of these calculations.

In the very last part of the paper, Weale performs a similar analysis of an analogous heterogeneous attrition process. The process considered is similar but not identical to Heterogeneous Process 1 of [10], with independent engagement initiation and single kills. If there are M weapon types on the Blue side and N weapon types on the Red side and if the states of the process are represented as vectors of the form

$$(x,y) = (x_1, \ldots, x_M; y_1, \ldots, y_N) ,$$

then the infinitesimal generator A of the associated (vector-valued) attrition process $((B_t, R_t))$ is given by

$$A((x,y),(x;y_1,\ldots,y_j - 1,\ldots,y_N)) = \sum_{i=1}^{M} c_B(i,j) x_i$$

$$\text{(25)} \quad A((x,y),(x,y)) = - \sum_{i=1}^{M} \sum_{j=1}^{N} (c_B(i,j) x_i + c_R(j,i) y_j)$$

$$A((x,y),(x_1,\ldots,x_i - 1,\ldots,x_M;y)) = \sum_{j=1}^{N} c_R(j,i) y_j .$$

The reader is referred to [7] and [10] for details concerning this process. Termination states are, not entirely reasonably, taken to be states (x,y) for which either

$$\sum_{i=1}^{M} \alpha_i x_i \leq m_B \tag{26a}$$

or

$$\sum_{j=1}^{N} \beta_j y_j \leq m_R \,, \tag{26b}$$

where the α_i, the β_j, m_B and m_R are prescribed in advance. In such states boundary conditions of the form (5) are valid, rather than (25). Certain modifications are necessary because some of the x_i or y_j may be zero without either (26a) or (26b) being satisfied. The author then notes the forward equation

$$P_t' = P_t A$$

for the transition function (P_t) of this attrition process, states (seemingly correctly) that it cannot easily be solved in closed form, and remarks about its general form; cf. equations (8) and (9) and associated comments in Section 2.

Finally, he derives equations analogous to (19) for various moments. Appropriately extended, Theorem (21) may more easily yield the same results. Nonetheless, the equations presented stand as a contribution because of their explicitness and the care used in their derivation.

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4. REVIEW OF "APPROXIMATE MOMENTS OF THE DISTRIBUTION OF STATES OF A SIMPLE HETEROGENEOUS BATTLE"

This paper [5] by N. Jennings is not of the same interest or importance as the papers reviewed in the two preceding sections; much of it, indeed, concerns calculation of error bounds whose theoretical and practical value is uncertain. In addition, the author seems not to distinguish relatively important ideas from essentially uninteresting (albeit involved and complicated) computations.

Consider the heterogeneous stochastic attrition process with the generator A given by expression (25) in Section 3; this process is discussed in more detail there and in [7,10]. Instead of the unrealistic termination rule embodied in (26), the author chooses the following rule: for each i = 1, ..., M there is a threshold $m_B(i)$ for Blue weapons of type i and for each j = 1, ..., N there is a threshold $m_R(j)$ for Red weapons of type j. The battle terminates if $B_t(i) \leq m_B(i)$ for any i or $R_t(j) \leq m_R(j)$ for any j. That is, the termination time T is given by

(27) $T = \inf\{t: B_t(i) = m_B(i) \text{ for some } i \text{ or } R_t(j) = m_R(j) \text{ for some } j\}$.

This termination rule also is subject to criticism, but seems clearly more reasonable than that of (26).

The author begins with a heuristic derivation of the forward equation

$$P'_t = P_t A$$

for this process; for a rigorous derivation from a set of carefully stated hypotheses the reader is referred to the Appendix of [7]. Jennings proceeds to consideration of moments of first and second orders, namely the expectations and covariances of numbers of survivors given by

(28a) $M_B(\alpha;\ i,t) = E^{\alpha}[B_t(i)]$

(28b) $M_R(\alpha;\ j,t) = E^{\alpha}[R_t(j)]$

(28c) $C_B(\alpha;i,i',t) = E^{\alpha}[(B_t(i)-E^{\alpha}[B_t(i)])(B_t(i')-E^{\alpha}[B_t(i')])]$

(28d) $C_R(\alpha;j,j',t) = E^{\alpha}[(R_t(j)-E^{\alpha}[(R_t(j)])(R_t(j')-E^{\alpha}[R_t(j')])]$

and

(28e) $C_{BR}(\alpha;i,j,t) = E^{\alpha}[(B_t(i)-E^{\alpha}[B_t(i)])(R_t(j)-E^{\alpha}[R_t(j)])]$,

where i,i' = 1, ..., M and j,j' = 1, ..., N. In particular,

$$C_B(\alpha;i,i,t) = Var^{\alpha}(B_t(i))$$

and

$$C_R(\alpha;j,j,t) = Var^{\alpha}(R_t(j))$$

for each i and j.

Jennings obtains differential equations involving these moments that resemble the Lanchester system of equations to which the stochastic process is analogous. "Resemble" in this case means that the equations derived in [5] are qualitatively of the same form as the Lanchester equations, up to an error term which in some sense is small if the time t is small.

For example, it follows from Theorem (21) that for the function f given by $f(\alpha) = x_i$,

$$(29)\quad \frac{d}{dt} M_B(\alpha;i,t) = \frac{d}{dt} E^{\alpha}[f(B_t,R_t)]$$

$$= E^{\alpha}[Af(B_t,R_t);\{T>t\}]$$

$$= E^{\alpha}\left[-\sum_{j=1}^{N} c_R(j,i)R_t(j);\{T>t\}\right]$$

$$= -E^{\alpha}\left[\sum_{j=1}^{N} c_R(j,i)R_t(j)\right]$$

$$+ E^{\alpha}\left[\sum_{j=1}^{N} c_R(j,i)R_t(j);\{T\le t\}\right]$$

$$= -\sum_{j=1}^{N} c_R(j,i)M_R(\alpha;j,t)$$

$$+ E^{\alpha}\left[\sum_{j=1}^{N} c_R(j,i)R_t(j);\{T\le t\}\right].$$

Similar, but more complicated, equations are valid for variances and covariances; together these constitute expression (10) of Jennings' paper. He gives no derivation, nor is there explicit calculation of error terms, such as appears in (29); we will treat probabilistic derivation of such equations--based on Theorem (21)--and probabilistic error estimation, in a forthcoming paper [11].

The author of [5] seems aware that the error

$$\Delta = \frac{d}{dt} M_B(\alpha,i;t) - \left[-\sum_{j=1}^{N} c_R(j,i)M_R(\alpha;j,t)\right]$$

is related to the termination probability $P^{\alpha}\{T\le t\}$, a relation first observed in the homogeneous case by Snow [12], but seems not to know the precise nature of the relation, which is given in (29). He proceeds, nonetheless, to attempt to compute $P^\alpha\{T \leq t\}$ for each t or, actually, to approximate this probability. From a computational standpoint, approximation of these probabilities seems to us to be of limited value. Such approximations can, at best, warn one about values of t for which that approximation

$$(30) \qquad E^\alpha[B_0(i)] - E^\alpha[B_t(i)] \simeq \sum_{j=1}^{n} c_R(j,i) \int_0^t E^\alpha[R_u(j)]du$$

is grossly invalid. While this constitutes useful information for very detailed combat models in which the time increment is small, it is precisely in such instances that still more careful error estimation is necessary. At the other extreme, for highly aggregated models of large scale combat, it is likely that t will be sufficiently large that one incurs substantial errors by use of (30), but no alternatives seem to exist. In both cases, more accurate error estimates are required.

However, estimates of $P^\alpha\{T \leq t\}$ are better than no error estimates at all, so the author deserves credit at least for having performed some preliminary work. There appears to be a slight circularity to his method, which consists in the following steps:

(1) Approximate $M_B(\alpha;i,t)$ as in (30), and perform similar approximations of other moments defined in (28);

(2) Assume that with respect to P^α, the random vector (B_t, R_t) is normally distributed with mean vector given by (28a,b) and covariance matrix given by (28c-e);

(3) Compute the mass of the joint normal distribution specified in (2) that lies outside the set of absorbing states

(i.e., in the set of transient states in which the battle continues) and take this as an approximation to $P^{\alpha}\{T>t\}$.

In truth, the author's approach is slightly different: he fixes in advance an upper threshold value of $P^{\alpha}\{T\leq t\}$ and seeks the minimal t at which the threshold is exceeded. He effects this computation by calculating the mass of a normal distribution lying within an ellipsoid on which the joint normal density is constant, finds the time-dependent constant density ellipsoid corresponding to the threshold probability, and then calculates the minimal time at which the ellipsoid intersects the termination set.

Virtually all these steps involve further approximations, some of which appear unavoidable from a computational point of view, at least given the author's objective. The circularity alluded to above is that the assumption of a joint normal distribution for (B_t, R_t) is valid only if $P^{\alpha}\{T\leq t\}$ is already very small, and so may not produce an accurate estimate of the termination probability. Nearly all of the paper is concerned with computations arising in this approximation scheme; the reader is referred there [5] for further details.

5. REVIEW OF "STOCHASTIC 'LINEAR LAW' BATTLES

This paper [6], also by N. Jennings, provides a treatment similar to that accorded stochastic "square law" battles in the paper reviewed in Section 2. The particular stochastic attrition process analyzed is Homogeneous Process 2 of [10], with proportional engagement initiation and single kills. The infinitesimal generator A is given by

$$A((i,j),\ (i,j-1)) = ijc_B$$

(31)
$$A((i,j),\ (i,j)) = -\ ij(c_B+c_R)$$

$$A((i,j),\ (i-1,j)) = ijc_R\ ,$$

where c_B, c_R again denote positive constants, but not with the same dimensions as the constants appearing in Sections 2-4. The jump function λ is given by

$$\lambda(i,j) = ij(c_B+c_R)$$

and the transition matrix Q of the embedded Markov chain is given by

$$Q((i,j),\ (i,j-1)) = \frac{c_B}{c_B + c_R}$$

(32)
$$Q((i,j),\ (i-1,j)) = \frac{c_R}{c_B + c_R}\ .$$

In particular, the embedded Markov chain is a spatially homogeneous random walk, which is computationally tractable.

Boundary conditions in the form of thresholds m_B for Blue and m_R for Red are imposed.

The author first presents the forward equation for the transition function (P_t) of this attrition process. He then derives differential equations for moments of the stochastic attrition process $((B_t,R_t))$. For example, he presents the equation

$$
\begin{aligned}
(33) \qquad \frac{d}{dt} E^{(i,j)}[B_t] &= - c_R E^{(i,j)}[B_t R_t] \\
&\quad + m_R c_R E^{(i,j)}[B_t ; \{R_t = m_R\}] \\
&\quad + m_B c_R E^{(i,j)}[R_t ; \{B_t = m_B\}] \\
&= - c_R E^{(i,j)}[B_t R_t] \\
&\quad + c_R E^{(i,j)}[B_t R_t ; \{T \le t\}] \\
&= - c_R E^{(i,j)}[B_t R_t ; \{T > t\}] ,
\end{aligned}
$$

where T is the termination time of the engagement. The author provides only the first equality in (33); we have provided the other two in order to show that this equation and similar equations for second moments can be obtained from Theorem (21), since if (i,j) is not an absorbing state and $f(k,\ell) = k$, then

$$Af(i,j) = - c_R ij .$$

The author's equations appear correct (except for typographical errors) and represent a useful set of facts. We refer the reader to [6] for Jennings' actual results.

The author demonstrates that

$$(34) \qquad E^{(i,j)}[c_B B_t - c_R R_t] = c_B i - c_R j$$

for all t, a relation which indicates a plausible equilibrium property of this stochastic attrition process. No analogous property holds, however, for the independent engagement initiation process discussed in Sections 2 and 3; cf. [11].

Brief mention is made of a computer program developed to solve the forward equation $P_t' = P_t A$ numerically; when such a solution is obtained, moments can be calculated at once. Of course, this approach does not lead to a general understanding of the process in the way that theoretical approaches may. Nonetheless, existence of this program is a useful contribution, and, indeed, may stimulate and support further theoretical developments.

Finally, the author considers the distribution of the termination state (B_T, R_T). Let

$$q((i,j),(k,\ell)) = P^{(i,j)}\{(B_T, R_T) = (k,\ell)\} ,$$

dependence of q on the termination thresholds m_B, m_R exists but is suppressed from the notation. Clearly $q(\cdot, (k,\ell)) = 0$ unless $k = m_B$ or $\ell = m_R$, but not both. At this point the author's non-probabilistic approach leads him to an unnecessarily complicated derivation of this distribution. Let

$$p = \frac{c_R}{c_B + c_R}$$

be the probability that each given casualty represents a Blue loss. By (32) it is immediate that for fixed initial conditions (i,j), $(B_T, R_T) = (k, m_R)$ if and only if the first $(i-k+(j-m_R)-1)$ casualties represent i-k losses to Blue and $(j-m_R)-1$ losses to Red and the next ($((i-k)+(j-m_R))^{st}$) casualty is a loss to Red. Since different casualties are Blue or Red with probabilities p and (1-p), respectively, and are mutually independent by virtue of the form of the embedded Markov chain, we see at once that

$$(35a)\quad q((i,j),(k,m_R)) = \binom{(i-k)+(j-m_R)-1}{i-k} p^{i-k}(1-p)^{j-m_R-1}(1-p)$$

$$= \binom{(i-k)+(j-m_R)-1}{i-k} p^{i-k}(1-p)^{j-m_R},$$

and analogously, that

$$(35b)\quad q((i,j),(m_B,\ell)) = \binom{(i-m_B)+(j-\ell)-1}{j-\ell} p^{i-m_B}(1-p)^{j-\ell}.$$

This agrees with the result obtained by the author--his equation (42)--but provides more understanding. A probabilistic derivation is illuminating not only in the result obtained but also in each step and in the overall pattern of reasoning, in a manner that an analytical derivation is not.

Let us return now to consideration of the terminal distribution. The distributions appearing in (35) are truncated negative binomial distributions, further properties of which are discussed, e.g., in [3]. Having derived the form of the terminal distribution, the author calculates the most probable numbers of survivors on each side, given that the other side has been forced to its threshold and then considers a normal approximation to the terminal distribution, which appears to be of limited value. In any case, cf. [3], a Poisson approximation may be more appropriate. We refer the reader to Appendix A of [6] for details of the results Jennings obtains. Although these results are not discussed here, we do not intend to imply that they are not useful in the practical sense.

6. REVIEW OF "HOMOGENEOUS BATTLES WITH GENERAL ATTRITION FUNCTIONS"

This paper [15] by T.G. Weale continues the line of development represented by the four papers reviewed in Sections 2-5 by extending some results contained in the papers reviewed in those sections to the case of "general attrition functions," a particular development he pursues in [16,17]. In our opinion, this particular form of generalization has pitfalls and, indeed, can be interpreted as an attitude which we have long criticized. Except in particular cases such as those considered in [10] that lead, for example, to the processes discussed in Sections 2-5, there need not exist a set of physical assumptions leading to a given pair of attrition functions. Mathematical treatment of processes not verifiably arising from physical assumptions about individual weapons systems and their interactions may create the (possibly) false impression that the resultant attrition process is of value as a model of combat.

The author treats a homogeneous stochastic attrition process $((B_t,R_t))_{t\geq 0}$ with infinitesimal generator A given by

$$
\begin{aligned}
A((i,j),(i,j-1)) &= \varphi_B(i,j) \\
A((i,j),(i,j)) &= -[\varphi_B(i,j)+\varphi_R(i,j)] \qquad (36) \\
A((i,j),(i-1,j)) &= \varphi_R(i,j) ,
\end{aligned}
$$

where φ_B and φ_R are arbitrary nonnegative functions on the state space of the attrition process. In addition, termination thresholds m_B for Blue and m_R for Red are prescribed in the manner of Sections 2 and 5.

That there exists a Markov process whose infinitesimal generator is given by (36) is, of course, true. What *is* uncertain is existence of a plausible or even definable set of assumptions concerning physical behavior of combatants--both individually and interactively--that leads to an attrition process with the generator A given by (36). For certain cases, the author of this review has shown the existence of such sets of assumptions; cf. [7,10]. These assumptions concern the qualitative and quantitative probabilistic nature of engagement initiation by combatants and are stated in a form in which a potential applier of stochastic attrition processes (to computerized combat simulations, for example) can readily verify their plausibility, or at least choose one process among several alternatives.

Even from a mathematical standpoint, the arbitrary generator approach is subject to criticism. Except for the restriction that sample paths be componentwise nonincreasing and decrease only by jumps of size one in one component, the infinitesimal generator A given by (36) is perfectly general. Specific computations are, therefore, likely to be impossible to perform, as the main body of [15] confirms. Theoretical results of sufficient specificity to be of interest are likewise difficult to obtain. The role of (restrictive) assumptions in mathematics is to sufficiently limit the class of objects under study that nontrivial statements become possible. Furthermore, consideration of problems that are too general denies one use of both intuition and methods of analysis that exploit the special structure of specific problems.

Finally, one does not deal in practice with an "arbitrary" generator A but chooses some specific form. A set of physical assumptions leading to a given form of generator contains within itself the appropriate suggestions for generalization. If a certain assumption is believed to be implausible, one can modify it to be acceptable in physical terms and then derive--possibly not without difficulty--the generator resulting from the new set of assumptions. On the other hand, a methodology based on direct and arbitrary choice of the generator is inherently self-limiting in that it admits no such potential for generalization.

For all these reasons, we believe that the research reported by Weale in this paper has, in its present form, little significant practical or mathematical implication. Therefore, the following description of the contents of the paper is quite brief.

Weale takes note of the forward equation

$$P_t' = P_t A \tag{37}$$

with suitable boundary conditions, for the transition function (P_t) of the attrition process. No derivation is required, for Markov process theory ensures the validity of the forward equation. For reasons that are obscure to us, the author of [15] thereafter considers higher derivatives of the transition function and obtains various relations involving them. It is evident from (37) that

$$P_t'' = P_t' A = P_t A^2$$

and that, more generally, for each n

$$P_t^{(n)} = P_t^{(k)} A^{n-k} \tag{38}$$

for all t and k = 0, ..., n, where $P^{(k)}$ is the k^{th} derivative of the transition function and, by convention $P_t^{(0)} = P_t$. The utility of (38) in the context of stochastic attrition processes is not apparent to us, as (37) is quite sufficient to specify (P_t) uniquely. For general attrition functions further information given by (38) is not sufficiently specific to be of value. In special cases, of course, a useful Taylor expansion of P_t might be obtainable, but the expansion follows more easily from the well-known property that $P_t = \exp(tA)$.

The author includes some remarks concerning the qualitative form of the attrition functions φ_B, φ_R. For example, both should be increasing in each variable separately and (possibly) strictly increasing in the variable representing the opposition. For example $\varphi_R(i,j)$, which corresponds to Red kills of Blue weapons should be nondecreasing in i (which represents the Blue side) and strictly increasing in j. The author proposes that $j \to \varphi_R(i,j)$ be a function with an S-shaped graph, a property not possessed by any of the attrition functions so far derived from physical assumptions (which are linear in j), but certainly plausible.

A discussion is given of the general form of the solution to (37); cf. Section 2 for a similar treatment of a specific case. Even in that specific case no concrete results are obtained; in this general case nothing of interest is presented.

Weale mentions in the main text, and presents in Appendices, computer programs for numerical integration of the forward equation (37) together with some results obtained therefrom. The attrition functions used in the sample program are given by

$$\varphi_B(i,j) = i(c_B + c'_B j)$$

and

$$\varphi_R(i,j) = j(c_R + c'_R i)$$

where c_B, c'_B, c_R, c'_R are constants. We emphasize that to our knowledge no set of physical assumptions leading to these attrition functions exists. That such assumptions do exist is, indeed, quite possible; verification of such matters is the problem of interest and importance.

Finally, the author treats the terminal distribution by the same Tauberian methodology used in the paper reviewed in Section 2. No specific analytical results are obtained, although many numerical explorations are possible.

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7. REVIEW OF "THE DISTRIBUTION OF THE DURATION OF BATTLE"

In this paper [16] T.G. Weale continues his analysis of the homogeneous attrition process $((B_t, R_t))_{t \geq 0}$ introduced in [15] (cf. Section 6 for a review thereof): namely, the Markov attrition process with infinitesimal generator A given by

$$
\begin{aligned}
A((i,j),(i,j-1)) &= \varphi_B(i,j) \\
A((i,j),(i,j)) &= -[\varphi_B(i,j) + \varphi_R(i,j)] \qquad (39) \\
A((i,j),(i-1,j)) &= \varphi_R(i,j) ,
\end{aligned}
$$

where φ_B, φ_R are arbitrary, but fixed, nonnegative functions. As we have observed in Section 6, this general approach has difficulties: for most functions φ_B, φ_R there is no known set of underlying physical assumptions that leads in the manner of [7,10] to an attrition process with the generator A given by (39). Moreover the general approach represented by [15,16,17] appears unlikely to yield results of sufficient specificity to be really useful.

Nonetheless the particular problem studied in [16] is of some interest--especially in physically justifiable special cases such as the processes discussed in the DOAE papers [13], [14], and [6] that are reviewed in Sections 2, 3, 5, respectively. That problem is the following: let m_B and m_R be termination levels (possibly but not necessarily zero) for the Blue and Red sides, respectively, so that the combat ceases when $B_t = m_B$ or $R_t = m_R$. Let T be the duration of the battle, given by

$$T = \inf\{t: B_t = m_B \text{ or } R_t = m_R\} .$$

One then wishes to compute, characterize, or approximate, for each initial state (i,j), the distribution of T under the probability measure $P^{(i,j)}$, i.e., to study the function

$$(40) \qquad F(i,j;t) = P^{(i,j)}\{T \le t\} .$$

As observed in Section 4 (cf. page 21) such probabilities are of interest in the context of the moment equations derived in Theorem (21) in Section 3 and are also, of course, of intrinsic interest as properties of the combat.

Weale's approach to the problem is the following: if B_T denotes the state of the process at the time of termination and if $i > m_B$, $j > m_R$, then

$$P^{(i,j)}\{B_T \in D\} = 1 ,$$

where

$$D = \{(k,m_R): m_B < k\} \cup \{(m_B,\ell): m_R < \ell\} .$$

Moreover, since each state in D is by definition an absorbing state, it follows that

$$(41) \quad F(i,j;t) = P^{(i,j)}\{B_t \in D\}$$

$$= \sum_{k=m_B+1}^{i} P_t((i,j),(k,m_R)) + \sum_{\ell=m_R+1}^{j} P_t((i,j),(m_B,\ell)) .$$

The expression given in (41) is valid in general, whether the generator A arises from explicit physical assumptions or not. Therefore, if the transition function (P_t) of the process were known, then the functions $F(i,j;\cdot)$ would be completely determined. Unfortunately, however, this is not so even for the special cases discussed in previous sections.

Weale partitions the termination set D into three subsets consisting of

$$D_1 = \{(k,m_R): k \geq \tilde{m}_B\} ,$$

$$D_2 = \{(m_B,\ell): \ell \geq \tilde{m}_R\}$$

and

$$D_3 = D - (D_1 \cup D_2) ,$$

where $\tilde{m}_B \geq m_B$ and $\tilde{m}_R \geq m_R$ are also prescribed in advance. States in D_1 are in Weale's terminology "Blue victory states," those in D_2 are "Red victory states," and those in D_3 are "draw states." The intention and interpretations are apparent. Clearly

$$P^{(i,j)}\{B_T \in D_1\} = \sum_{k=\tilde{m}_B}^{i} P_\infty((i,j),(k,m_R)) ,$$

(42)
$$P^{(i,j)}\{B_T \in D_2\} = \sum_{\ell=\tilde{m}_R}^{j} P_\infty((i,j),(m_B,\ell)) ,$$

$$P^{(i,j)}\{B_T \in D_3\} = \sum_{k=m_B+1}^{\tilde{m}_B-1} P_\infty((i,j),(k,m_R)) + \sum_{\ell=m_R+1}^{\tilde{m}_R-1} P_\infty((i,j),(m_B,\ell))$$

$$= 1 - P^{(i,j)}\{B_T \in D_1 \cup D_2\} ,$$

where

$$P_\infty = \lim_{t\to\infty} P_t$$

and evidently exists provided that $\varphi_B(i,j) + \varphi_R(i,j) > 0$ whenever $i \neq m_B$ and $j \neq m_R$ (cf. pp. 8-9 above for some remarks relevant to numerical computation of P_∞).

Weale then proceeds to consider the distribution of T conditioned on the termination state; that is, he considers the conditional distribution functions

$$F_1(i,j;t) = P^{(i,j)}\{T \leq t | B_T \in D_1\}$$

$$F_2(i,j;t) = P^{(i,j)}\{T \leq t | B_T \in D_2\}$$

$$F_3(i,j;t) = P^{(i,j)}\{T \leq t | B_T \in D_3\} .$$

Formally, these functions can be computed directly from (41) and (42) using only elementary probability; for example

$$(43) \qquad F_1(i,j;t) = \frac{P^{(i,j)}\{B_t \in D_1\}}{P^{(i,j)}\{B_T \in D_1\}}$$

$$= \frac{\sum_{k=\tilde{m}_B}^{i} P_t((i,j),(k,m_R))}{\sum_{k=\tilde{m}_B}^{i} P_\infty((i,j),(k,m_R))} .$$

Analogous expressions, which need not be included here, exist for $F_2(i,j;t)$ and $F_3(i,j;t)$. From (43) it is clear that the $F_q(i,j;t)$ can be computed in closed form if the transition function (P_t) is known, but seem inaccessible otherwise.

To continue our review of [16], Weale next discusses the density functions of the distributions $F(i,j;t)$. From (41) and the forward equation Weale derives the relation

$$(44)\quad \frac{d}{dt}F(i,j;t) = \sum_{k=m_B+1}^{i} P_t'((i,j),(k,m_R)) + \sum_{\ell=m_R+1}^{j} P_t'((i,j),(m_B,\ell))$$

$$= \sum_{k=m_B+1}^{i} P_t A((i,j),(k,m_R)) + \sum_{\ell=m_R+1}^{j} P_t A((i,j),(m_B,\ell))$$

$$= \sum_{k=m_B+1}^{i} P_t((i,j),(k,m_R+1))\varphi_B(k,m_R+1)$$

$$+ \sum_{\ell=m_R+1}^{j} P_t((i,j),(m_B+1,\ell))\varphi_R(m_B+1,\ell) \; .$$

Analogous expressions are also given for the densities of the conditional distributions F_1, F_2, F_3. Like other relations derived earlier in the paper these formulas give closed form results if the transition function (P_t) is known in closed form and not much information in other cases.

One exception to the latter assertion is when the forward equation is integrated numerically for given initial conditions and attrition functions; this is a principal objective of the computer programs that comprise the larger part of not only [16] but also several of the papers reviewed in preceding sections of this paper. In this situation the formulas (41), (42), (43), and (44) are all applicable and yield good numerical approximations to the probabilities in question. The work and results concerning numerical computations are, as we discuss further in Section 9, a significant contribution of at least some of these DOAE papers. This method of development for all its usefulness, however, does not yield closed-form analytical expressions by means of which one can fully understand and describe the attrition processes under study. Neither does it produce qualitative insights that can be extrapolated beyond the (necessarily limited) numbers of numerical inputs that are actually treated;

of course many useful qualitative insights result nevertheless.

Finally, Weale deals with expectations, medians, and modes of the termination time distributions. Since, for example, for each (i,j)

$$(45) \qquad E^{(i,j)}[T] = \int_0^\infty (1-F(i,j;t))dt$$

$$\sim \int_0^u (1-F(i,j;t))dt$$

$$= u - \int_0^u F(i,j;t)dt$$

for large values of u, one can use the first equality in (45) to compute $E^{(i,j)}[T]$ if $F(i,j;\cdot)$ is known in closed form or can use the approximation contained in (45) if (as is the case in [16])a numerical approximation to $F(i,j;\cdot)$ is available. Since

$$u - \int_0^u F(i,j;t)dt \leq u,$$

in order that the approximation in (45) be even reasonably accurate one must have

$$u \geq E^{(i,j)}[T].$$

This point is not mentioned in [16]; it is easy, however, to estimate $E^{(i,j)}[T]$. If, as the physics of combat almost certainly require, the attrition functions φ_B, φ_R are both nonincreasing in each argument, then evidently

$$(46) \qquad E^{(i,j)}[T] \leq (i-m_B+j-m_R-1)[\varphi_B(m_B+1,m_R+1)+\varphi_R(m_B+1,m_R+1)]^{-1}.$$

The reasoning underlying (46) is the following: at most $(i-m_B + j-m_R - 1)$ casualties can occur before termination and each interval between casualties is exponentially distributed with expectation not exceeding $[\varphi_B(m_B+1,m_R+1) + \varphi_R(m_B+1,m_R+1)]^{-1}$. One may then use (46) to determine a sufficiently large value of u for use in (45). In the computer program appended to [16] the integration in (45) is performed using Simpson's rule, which seems eminently reasonable.

Computation of the median and mode of the distribution of T is also considered in [16]. Also, the author provides similar treatments of the conditional distributions F_1, F_2, F_3; further details are not necessary here.

As we mentioned before, the distribution of the terminal time T is of some interest, particularly in the context of the differential equations discussed in Section 3; see also [11] where we consider the problem in some detail. One approach that may be superior to that of Weale not only for derivation of analytical results but also for certain computational applications is the following recursive method.

(47) PROPOSITION. For each nonabsorbing state (i,j)

$$(48)\quad F(i,j;t) = \varphi_B(i,j)\int_0^t F(i,j-1;t-u)e^{-[\varphi_B(i,j)+\varphi_R(i,j)]u}\,du$$

$$+ \varphi_R(i,j)\int_0^t F(i-1,j;t-u)e^{-[\varphi_B(i,j)+\varphi_R(i,j)]u}\,du\ .$$

PROOF. Let T_1 be the first change of state of the attrition process $((B_t,R_t))$ and X_1 the state entered at time T_1. By Theorem (8.3.3) of [1],

$$P^{(i,j)}\{T_1\le t, X_1=(i,j-1)\} = \frac{\varphi_B(i,j)}{\varphi_B(i,j)+\varphi_R(i,j)} \int_0^t [\varphi_B(i,j)+\varphi_R(i,j)]e^{-[\varphi_B(i,j)+\varphi_R(i,j)]u}\,du$$

$$= \varphi_B(i,j)\int_0^t e^{-[\varphi_B(i,j)+\varphi_R(i,j)]u}\,du$$

and, in the same way,

$$P^{(i,j)}\{T_1\le t; X_1=(i-1,j)\} = \varphi_R(i,j)\int_0^t e^{-[\varphi_B(i,j)+\varphi_R(i,j)]u}\,du\ .$$

Hence,

$$\begin{aligned}
F(i,j;t) &= P^{(i,j)}\{T\le t\} \\
&= E^{(i,j)}[P^{(i,j)}\{T\le t \mid T_1, X_1\}] \\
&= E^{(i,j)}[P^{X_1}\{T\le t-T_1\}] \\
&= E^{(i,j)}[F(X_1;t-T_1)] \\
&= \varphi_B(i,j)\int_0^t F(i,j-1;t-u)e^{-[\varphi_B(i,j)+\varphi_R(i,j)]u}\,du \\
&\quad +\varphi_R(i,j)\int_0^t F(i-1,j;t-u)e^{-[\varphi_B(i,j)+\varphi_R(i,j)]u}\,du\ ,
\end{aligned}$$

where the third equality is by the strong Markov property; cf. [1] or [2]. ∎

To illustrate, we consider the process with

$$\varphi_B(i,j) = c_B i$$

and

$$\varphi_R(i,j) = c_R j\ ,$$

namely the homogeneous square law process discussed in Sections 2 and 3 above. In this and other applications it is more convenient to use (48) in the equivalent form

$$(49)\quad P^{(i,j)}\{T>t\} = \varphi_B(i,j)\int_0^t P^{(i,j-1)}\{T>t-u\}e^{-[\varphi_B(i,j)+\varphi_R(i,j)]u}\,du$$

$$+\varphi_R(i,j)\int_0^t P^{(i-1,j)}\{T>t-u\}e^{-[\varphi_B(i,j)+\varphi_R(i,j)]u}\,du$$

$$+\ e^{-[\varphi_B(i,j)+\varphi_R(i,j)]t}\ .$$

For $i,j \leq 2$ we then have the following exact results:

$$\begin{aligned} P^{(0,0)}\{T>t\} &= P^{(1,0)}\{T>t\} \\ &= P^{(0,1)}\{T>t\} \\ &= P^{(2,0)}\{T>t\} \\ &= P^{(0,2)}\{T>t\} \\ &= 0\ , \end{aligned}$$

while

$$P^{(1,1)}\{T>t\} = e^{-(c_B+c_R)t}\ .$$

Also,

$$P^{(2,1)}\{T>t\} = \frac{c_R}{c_B}e^{-(c_B+c_R)t} + \left(1-\frac{c_R}{c_B}\right)e^{-(2c_B+c_R)t}$$

and by symmetry

$$P^{(1,2)}\{T>t\} = \frac{c_B}{c_R} e^{-(c_B+c_R)t} + (1-\frac{c_B}{c_R})e^{-(c_B+2c_R)t} \quad .$$

Finally,

$$P^{(2,2)}\{T>t\} = 2e^{-(c_B+c_R)t} + 2(\frac{c_R}{c_B} - 1)e^{-(c_B+2c_R)t}$$

$$+ 2(\frac{c_B}{c_R} - 1)e^{-(2c_B+c_R)t}$$

$$+ [3-2(\frac{c_B}{c_R} + \frac{c_R}{c_B})]e^{-(2c_B+2c_R)t} \quad .$$

The remainder of [16] contains descriptions and listings of computer programs that implement various computations described above; these are of significant practical value.

8. REVIEW OF "MOMENTS OF THE DISTRIBUTION OF STATES FOR A BATTLE WITH GENERAL ATTRITION FUNCTIONS"

This paper [17] by T.G. Weale and E. Peryer continues the work of the first author on homogeneous battles with general attrition functions that is reported in [15] and [16], and are reviewed in Sections 6 and 7 above, respectively. The general comments made in those sections (in Section 6 in particular) concerning possible lack of an underlying family of physical assumptions remain relevant. In [17] Weale and Peryer deal with results analogous to those obtained in [5] by Jennings for the heterogeneous square law attrition process; namely, differential equations for expectations of functionals of the attrition process. Many of the comments and analyses presented in Section 4 of this review, in which [5] is reviewed, will also be germane to the discussion of [17].

For the sake of completeness we once again observe that the attrition process treated in [17] is a Markov process $((B_t,R_t))_{t\geq 0}$ with infinitesimal generator A given by

$$A((i,j);(i,j-1)) = \varphi_B(i,j)$$
$$A((i,j);(i,j)) = -[\varphi_B(i,j) + \varphi_R(i,j)]$$
$$A((i,j);(i-1,j)) = \varphi_R(i,j) ,$$

where φ_B and φ_R are nonnegative but otherwise arbitrary functions defined on the state space $E = \underset{\sim}{N} \times \underset{\sim}{N}$ of the attrition process. The authors first take note of the forward equation for the transition function of the process, which in open form is given for nonabsorbing states (k,ℓ) by

$$(50) \quad P_t'((i,j),(k,\ell)) = P_tA((i,j),(k,\ell))$$
$$= P_t((i,j),(k,\ell+1))\varphi_B(k,\ell+1)$$
$$+ P_t((i,j),(k+1,\ell))\varphi_R(k+1,\ell) \ ,$$

for absorbing states (k,m_R), where m_R is the Red termination level, by

$$(51a) \quad P_t'((i,j),(k,m_R)) = P_t((i,j),(k,m_R+1))\varphi_B(k,m_R+1) \ ,$$

and for absorbing states (m_B,ℓ), where m_B is the Blue termination level, by

$$(51b) \quad P_t'((i,j),(m_B,\ell)) = P_t((i,j),(m_B+1,\ell))\varphi_R(m_B+1,\ell).$$

It then follows that if f is a function on the state space E of the attrition process,

$$(52) \quad \frac{d}{dt} E^{(i,j)}[f(B_t,R_t)] = \frac{d}{dt}\left\{\sum_{k=m_B}^{i} \sum_{\ell=m_R}^{j} f(k,\ell)P_t((i,j),(k,\ell))\right\}$$
$$= \sum_{k=m_B}^{i} \sum_{\ell=m_R} f(k,\ell)P_t'((i,j),(k,\ell))$$
$$= \sum_{k=m_B+1}^{i} \sum_{\ell=m_R+1}^{j} f(k,\ell)[\varphi_B(k,\ell+1)P_t((i,j),(k,\ell+1))$$
$$+ \varphi_R(k+1,\ell)P_t((i,j),(k+1,\ell))]$$
$$+ \sum_{k=m_B+1}^{i} f(k,m_R)\varphi_B(k,m_R+1)P_t((i,j),(k,m_R+1))$$
$$+ \sum_{\ell=m_R+1}^{j} f(m_B,\ell)\varphi_R(m_B+1,\ell)P_t((i,j),(m_B+1,\ell)) \ .$$

The expression (52) is essentially identical to equation (8) in [17] which is the main theoretical result therein. Observe that the right-hand side of (52) involves no probabilities of the forms $P_t((i,j),(k,m_R))$ or $P_t((i,j),(m_B,\ell))$. Indeed, elementary calculations verify that (52) is equivalent to the result that Theorem (21)--as suitably extended in [11] to apply to general Markov attrition processes--would yield in this situation.

The authors then proceed to discuss the solution of (52) for functions f of the form

$$f(k,\ell) = (k-i)(k-i+1) \cdots (k-i+r)(\ell-j) \cdots (\ell-j+s) ,$$

in which case the expectations involved are factorial moments. They further assume that φ_B and φ_R are polynomials in their arguments, but even so do not obtain specific results. The difficulty, the reviewer believes, is that the authors' point of view and method of proceeding are analytic rather than physical and probabilistic. From an analytic standpoint there seems to be hope of solving (52) only if the φ_B and φ_R are polynomials, although useful approximations involving polynomials can certainly be made. From the physical, probabilistic standpoint one should attempt to solve (52) only for attrition functions arising from well-defined physical hypotheses, in which case the particular probabilistic structure of the process at hand may aid in obtaining a solution.

Appendices to the paper describe computer programs designed to approximate the solution of (52). There is no doubt that one can solve (52) numerically for essentially any attrition functions that can be programmed into a computer, although for irregular functions the approximation may not be good. For attrition functions verifiably arising from physically definable and plausible assumptions, these computer programs constitute an important analytical and descriptive tool.

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9. CONCLUSIONS

The first four papers presented here represent varying but substantive contributions to the theory of stochastic attrition processes. To the taste of the reviewer these contributions are lessened and obscured by the authors' excessive reliance on analytic approaches since, as we have several places demonstrated, direct probabilistic approaches yield not only more illuminating arguments but also, at least in some cases, more complete or specific results. The computer programs associated with these papers constitute a contribution whose current value, in view of our lack of ability to deal with the processes on a reasonable closed-form basis, is certainly understated by the scant attention devoted to them in this review. In a few places lengthy and unenlightening computations interrupt the development of worthwhile ideas, but in general the papers are of high technical and expository quality, for which their respective authors are to be commended.

On the other hand we believe that the last three papers dealing with battles with general attrition functions make little contribution in the mathematical sense. Since our criticisms are rather strongly worded, let us once more attempt to be specific about our grounds for criticism. The first of these is philosophical but has important practical implications; we strongly believe that one should not deal with attrition processes that cannot be justified in terms of physical assumptions.

But our criticisms are also on mathematical grounds: the great degree of generality involved prohibits the authors' obtaining results that are sufficiently specific to be of mathematical interest even without regard to possible physical interest or applicability.

Of much greater importance and in much greater need, in our opinion, is work aimed at developing physical assumptions that imply forms of attrition functions other than those previously (in [10], e.g.) justified. Better yet, one should strive first to develop plausible sets of physical assumptions and then to derive from these attrition processes that are tractable in terms of applications to combat models.

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