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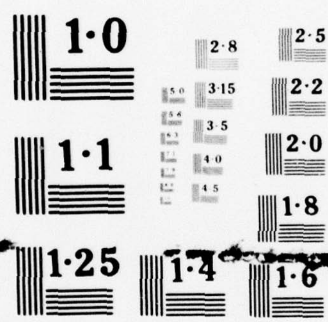

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ON A GENERALIZATION OF FREUND'S BIVARIATE FAILURE MODEL

BY

10 THOMAS J. TOSCH*
PAUL T. HOLMES

9 Technical rept.

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Abstract

A bivariate failure model is proposed in which the residual lifetime of one component is dependent on the working status of the other. General properties of the model are discussed, and the maximum likelihood estimates of the parameters are found in a bivariate exponential-like special case.

Keywords: Bivariate failure model, bivariate exponential distribution, maximum likelihood estimation.

1. INTRODUCTION

A new bivariate failure model is proposed in which the residual lifetime of one component is dependent on the working status of the other component. This is applicable when the failure of one component puts more (possibly less) strain on the remaining components, for example, the kidneys. Section two derives properties of the lifetimes, including their joint Laplace-Stieltjes transform.

In the third section a bivariate exponential-like special case is considered. In this example maximum likelihood estimates of the parameters are obtained and their asymptotic distribution studied. This model is compared with the bivariate exponential models of Freund [3] and Marshall and Olkin [4].

2. MODEL DEFINITION AND GENERAL PROPERTIES

Label the two components of the system A and B with lifetimes S and T respectively. The lifetimes of the two components are dependent, in that the failure of one component affects the residual lifetime of the other. To describe S and T, let X, Y, U, V be non-negative mutually independent random variables with X and Y absolutely continuous. Then we write

$$\begin{aligned} S &= \min(X,Y) + U \cdot I_{\{X > Y\}} \\ T &= \min(X,Y) + V \cdot I_{\{X \leq Y\}} \end{aligned} \qquad (2.1)$$

We will obtain an expression for the joint survival distribution

$\bar{F}(s,t) = \Pr\{S > s,\ T > t\}$. Assuming $s < t$ and conditioning on the values of X and Y, we obtain

$$\bar{F}(s,t) = \int_0^\infty \int_0^\infty \Pr\{S > s,\ T > t \mid X=x,\ Y=y\} dF_X(x) dF_Y(y).$$

Now partition the region $[0,\infty) \times [0,\infty)$ into 12 subregions based on the relative sizes of x,y,s,t. The only non-zero contributions are $\Pr\{V > t-x\}$ on the region $[s,t] \times [x,\infty)$ and 1 on the region $[t,\infty) \times [t,\infty)$. Thus the following can be established.

<u>Theorem 1</u>:

$$\bar{F}(s,t) = \begin{cases} \bar{F}_X(t)\bar{F}_Y(t) + \int_s^t \bar{F}_V(t-x)\bar{F}_Y(x) dF_X(x) & \text{if } s < t \\ \bar{F}_X(s)\bar{F}_Y(s) & \text{if } s = t \\ \bar{F}_X(s)\bar{F}_Y(s) + \int_t^s \bar{F}_U(s-y)\bar{F}_X(y) dF_Y(y) & \text{if } s > t. \end{cases}$$

The joint Laplace-Stieltjes transform of (S,T) is defined to be $f^*(a,b) = \int_0^\infty \int_0^\infty e^{-as-bt} F(ds,dt)$, where $F(ds,dt)$ is the measure determined by the survival function $\bar{F}(s,t)$. This measure can be represented by

$$F(ds,dt) = \begin{cases} \bar{F}_Y(s) dF_V(t-s) dF_X(s) & \text{if } s < t \\ \bar{F}_Y(s) p_V dF_X(s) + F_X(s) p_U dF_Y(s) & \text{if } s = t \\ \bar{F}_X(t) dF_U(s-t) dF_Y(t) & \text{if } s > t, \end{cases} \qquad (2.2)$$

where $p_V = \Pr\{V=0\}$ and $p_U = \Pr\{U=0\}$. This expression is used to evaluate $f^*(a,b)$ and we get

Theorem 2:

$$f^{*}(a,b) = f_{V}^{*}(b)\int_{0}^{\infty} e^{-(a+b)s}\overline{F}_{Y}(s)\,dF_{X}(s) + f_{U}^{*}(a)\int_{0}^{\infty} e^{-(a+b)s}\overline{F}_{X}(s)\,dF_{Y}(s).$$

These integrals cannot be evaluated in general, but can be for certain important special cases. In particular we have

Corollary 3: If Y has an exponential distribution with parameter β, then

$$f^{*}(a,b) = f_{V}^{*}(b) f_{X}^{*}(a+b+\beta) + (f_{U}^{*}(a)\beta/a+b+\beta)[1-f_{X}^{*}(a+b+\beta)].$$

Moments can be calculated by differention using this expression.

3. ESTIMATION IN A SPECIAL CASE

As a special case suppose that X and Y are exponentially distributed with parameters α and β respectively, $\Pr\{U > t\} = qe^{-\alpha' t}$, $\Pr\{V > t\} = qe^{-\beta' t}$, where $\alpha, \beta, \alpha', \beta' > 0$, $0 \leq q \leq 1$, $t \geq 0$. Freund's [3] model corresponds to the case $q = 1$. The parameter q allows for simultaneous failure of the components, since $\Pr\{S=T\} = 1-q = p$. Using results from section 2, properties (including moments) can be derived. In particular,

$$\overline{F}(s,t) = \begin{cases} e^{-(\alpha+\beta)t} + \dfrac{q\alpha e^{-\beta' t}}{\alpha+\beta-\beta'}\left[e^{-(\alpha+\beta-\beta')s} - e^{-(\alpha+\beta-\beta')t}\right] & \text{if } s < t \\ e^{-(\alpha+\beta)s} & \text{if } s = t \\ e^{-(\alpha+\beta)s} + \dfrac{q\beta e^{-\alpha' s}}{\alpha+\beta-\alpha'}\left[e^{-(\alpha+\beta-\alpha')t} - e^{-(\alpha+\beta-\alpha')s}\right] & \text{if } s > t. \end{cases} \tag{3.1}$$

Marshall and Olkin [4] defined a bivariate lack of memory property by $Pr\{S > s+\Delta, T > t+\Delta\} = Pr\{S > s, T > s\} \cdot Pr\{S > \Delta, T > \Delta\}$ for all s, t, $\Delta \geq 0$. The survival distribution (3.1) possesses this property and has mixtures of exponential distributions as marginals.

The measure determined by (3.1) however is not absolutely continuous. If we let μ_i represent Lebesgue measure on R_i, then the measure defined by $\mu(A) = \mu_2(A) + \mu_1\{x:(x,x) \in A\}$ is a suitable dominating measure for maximum likelihood estimation. This is the same dominating measure used by Bhattacharyya and Johnson [1].

Consider a sample of N independent observations on (S,T), $\{(S_1,t_1),\ldots,(S_N,t_N)\}$. The following notation simplifies the likelihood function. Let $A_1 = \{(S_i,t_i) | s_i < t_i\}$, $A_2 = \{(s_i,t_i) | s_i > t_i\}$ and $A_3 = \{(s_i,t_i) | s_i = t_i\}$. Further let $N_1 = \#A_1$, $N_2 = \#A_2$, $N_3 = \#A_3$, $S_1 = \sum_{A_1} s_i$, $S_2 = \sum_{A_2} s_i$, $R = \sum_{A_3} s_i$, $T_1 = \sum_{A_1} t_i$, $T_2 = \sum_{A_2} t_i$. Here #A is the number of items in A. Then L, the likelihood function, can be expressed as

$$L = (\alpha\beta')^{N_1}(1-p)^{N_1+N_2}p^{N_3}(\alpha+\beta)^{N_3}(\beta\alpha')^{N_2}\exp[-(\alpha+\beta)(S_1+R+T_2)- \alpha'(S_2-T_2)-\beta'(T_1-S_1)]. \tag{3.2}$$

Solving for the maximum likelihood estimates, one obtains

<u>Theorem 4</u>:

(i) If $N_1=N_2=0$, then $\hat{p}=1$ and α, β, α', β' cannot be estimated,

(ii) If $N_1=0$, but $N_2\neq 0$, then $\hat{p}=N_3/N$, $\hat{\alpha}=0$, $\hat{\beta}=N/(S_1+R+T_2)$, $\hat{\alpha}'=N_2/(S_2-T_2)$, and $\hat{\beta}'$ cannot be estimated.

(iii) If $N_1\neq 0$, but $N_2=0$ then $\hat{p}=N_3/N$, $\hat{\beta}=0$, $\hat{\alpha}=N/(S_1+R+T_2)$, $\hat{\beta}'=N_1/(T_1-S_1)$, and α' cannot be estimated.

(iv) If $N_1\neq 0$ and $N\neq 0$ then

$$\hat{\alpha}=(N/S_1+R+T_2)(N_1/N_1+N_2)$$
$$\hat{\beta}=(N/S_1+R+T_2)(N_2/N_1+N_2)$$
$$\hat{\alpha}'=N_2/(S_2-T_2)$$
$$\hat{\beta}'=N_1/(T_1-S_1)$$
$$\hat{p}=N_3/N.$$

It is easy to obtain the biases in these estimators. Using a Lehmann, Scheffé partitioning operation (cf. Zacks [5], p. 50) it can be shown that

<u>Theorem 5</u>: The vector $(N_1, N_2, S_1+R+T_2, S_2-T_2, T_1-S_1)$, (and thus the vector of maximum likelihood estimators) is a minimal sufficient statistic of the sample $\{(s_1,t_1),\ldots,(s_N,t_N)\}$.

To obtain the asymptotic distribution of the maximum likelihood estimates we need to restrict the parameter space as follows. Let $\Omega^* = \{(\alpha,\beta,\alpha',\beta',p)\,|\,0 < t_1 < \alpha < M_1,\ldots,0 < t_5 < p < M_5 < 1\}$. On this space the regularity conditions presented by Chanda [2,p. 56] are satisfied. Thus we have

<u>Theorem 6</u>:

(i) $(\hat{\alpha}, \hat{\beta}, \hat{\alpha}', \hat{\beta}', \hat{p})\rightarrow(\alpha, \beta, \alpha', \beta', p)$ as $N\rightarrow\infty$ with probability one.

(ii) $N^{\frac{1}{2}}(\hat{\alpha}-\alpha,\ \hat{\beta}-\beta,\ \hat{\alpha}'-\alpha',\ \hat{\beta}'-\beta',\ \hat{p}-p)$ is asymptotically distributed as multivariate normal with mean $\underset{\sim}{0}$ and covariance matrix Σ, where

$$\Sigma^{-1} = \begin{bmatrix} \frac{\alpha+\beta-p\beta}{\alpha(\alpha+\beta)^2} & \frac{p}{(\alpha+\beta)^2} & 0 & 0 & 0 \\ \frac{p}{(\alpha+\beta)^2} & \frac{\alpha+\beta-p\alpha}{\beta(\alpha+\beta)^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-p)\beta}{(\alpha')^2(\alpha+\beta)} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-p)\alpha}{(\beta')^2(\alpha+\beta)} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{p(1-p)} \end{bmatrix}.$$

The parameters can also be estimated by maximum likelihood for a censored sample.

We see that this model generalizes Freund's model to include simultaneous failure of both components. It differs from the model of Marshall and Olkin in that the residual lifetime of one component is not independent of the status of the other component. We feel that these features will aid in the application of the model.

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