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INFORMATION PERISHING AND REPLENISHMENT

STANFORD UNIVERSITY

Allen Frederick Grum

ADVANCED DECISION TECHNOLOGY PROGRAM

CYBERNETICS TECHNOLOGY OFFICE
DEFENSE ADVANCED RESEARCH PROJECTS AGENCY
Office of Naval Research • Engineering Psychology Programs

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INFORMATION PERISHING AND REPLENISHMENT.

by

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SUMMARY

The dissertation treats the dynamics of a decision maker's value of information. There are two main parts, a section on the depreciation (perishing) of information and a section on the appreciation (replenishment) of information.

A notion, widely held by decision analysts but tenuously defined, is that the value of any specific information diminishes over time. This concept, termed information perishing, is rigorously defined and illustrated by the use of a Markov model in the first section of the study.

The main assertions of the section are:

1. Information perishing is inevitable (not only for the Markov model of information but for any state of information described by a probability distribution).
2. For the Markov model the absolute value of the largest transient eigenvalue is an upper bound for the rate of information perishing;
3. The rate of perishing is a decreasing function of time.

A short transition section alters the basic decision model to allow an element of uncertainty for the exact timing of the decision. Basically the new model of the decision process recognizes that many decisions in real life are "triggered" by events which may be described by some stochastic process. Without this uncertainty the decision maker could simply discount the value of information because of perishing and would reduce his problem to a static case. However, the uncertainty in timing forces consideration of optimal policies of information replenishment, the second main area of the thesis.

The major results of this section are:

1. Rules of optimality are developed for singly and multiple occurring decisions.
2. The optimality of periodic replenishment (under certain limiting conditions), is established.
3. The suggestion that some of the research results of reliability and maintainability theory may be applied to information replenishment strategy.

The thesis closes with the customary delineation of areas of further application and research.

CONTENTS

FIGURES

Figure		page

TABLES

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Finally, I must single out two people whose forebearance and support have been central to my life in the last three years. My wife has been a pillar of strength and encouragement during many hectic months of thesis preparation. She is the epitome of Euripides' thought that "man's best possession is a sympathetic wife". My advisor, Professor Ronald A. Howard, deserves and receives the customary credit for interesting me in decision analysis, stimulating my curiosity, and encouraging my academic endeavors. However, far beyond these advisor accolades, I thank him for being a friend when a friend was badly needed.

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CHAPTER 1

INTRODUCTION

Man has a propensity to acquire and store items he will need to satisfy future needs. History depicts prehistoric man carefully collecting and hoarding food, stone tools, and animal skins to carry him through an arduous winter. Modern man has perpetuated this characteristic. However, in an age when physical wants are more easily satisfied the emphasis has shifted from the acquisition of material objects. Instead, on an increasing scale, people, organizations and nations are collecting information as a hedge against tomorrow's demands. As Shubik [21]* notes

> There is an old saying in bridge that a peek is worth two finesses. In many instances the major weapon of competition may be special knowledge or information.

McDonough [3] highlights the trend by reporting that over 14% of the total U.S. Labor force is engaged in clerical activities; over 10,000,000 people are directly concerned with the production and processing of information; and at least 50% of the cost of running the economy is information costs.

The very emphasis on information has led to inevitable problems --"... in every .. sphere of modern life, the chronic condition is a surfeit of information, useless, poorly integrated, or lost somewhere in the system" [7]. Wilensky continues with a desiderata for information: clear, timely, reliable, valid, adequate, and wide-ranging--the obvious connotation that these are more noticeable by their absence than by their presence.

These problems arise in part because organizations have not adopted means to rationalize the information process. Decision analysis, among the many quantitative models of decision making, most explicitly treats the value of information and provides a consistent basis for consideration

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*Numbers in square brackets refer to the Bibliography found in rear of the thesis.

of the acquisition and use of information. Expository works by Howard [14,15], North [18], and Raiffa [4], as well as a recent dissertation by Miller [17], are significant buttresses for a methodology of information resource allocation.

However, even these valuable contributions are silent on the dynamics of information. Implicit in many of the qualitative analysis of information acquisition (Wohlstetter [8], Wilensky [7]) and explicit in criticism of national intelligence activities (e.g., post hoc analysis of the Berlin Wall, Tet, and the Yom Kippur war) is a recognition of an information value-time relationship. However, most quantitative analysis of information treats the value of information as static, invariant over time.* This dissertation, building on the seminal foundation of the previous cited works, analyzes the dynamics of information.

Chapter 2 is the framework for the entire thesis. We perhaps all share an intuitive feeling that the value of information decreases with the passing of time. However, exactly what do we mean by information "perishing"? Is this an inevitable phenomenon? How do we measure the rate at which perishing occurs? Is the rate invariant? What is the effect, if any, of risk aversion on this "perishing"?

Chapter 2 treats the depreciation of the value of information over time. The phenomenon is indeed inevitable, and for states of information that can be modeled by a Markov process we have a handy benchmark for the rate of perishing. This yardstick, for the two-state case, is related to the "shrinkage" as defined by Howard [2]. An important result is that the value of information "perishes" at a rate equal to or less than the absolute value of the largest "transient" eigenvalue of the underlying Markov process.

The results of Chapter 2 have merit in their own right. However, an astute analyst, if he knew, for example, the exact timing of a decision could allow the necessary time for information collection, calculate the depreciation of the value, and reduce the problem to essentially a static situation. This, of course, assumes he knows the exact timing

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*Ransom [5] reports that strategic intelligence in wartime depreciates at the rate of 10% per month. This is, at best, an empirical observation which lacks a rigorous definition and quantification of value.

of the implementation of the decision. As illustrated in Chapter 3 many decisions of importance and interest are implemented at an uncertain time in the future. We slightly alter Howard's decision model [15] to introduce an element of uncertainty in the time of occurrence of the decision. Incorporation of this probability into the basic decision model leads to fruitful study.

In particular, Chapter 4 reconsiders the rate of information perishing in light of this uncertainty. We also treat intermediate information acquisition and discounting of rewards as extensions of the basic results of Chapter 2.

In a sense Chapters 3 and 4 serve as a transition from Chapter 2 to Chapter 5, a consideration of the appreciation or replenishment of information. We illustrate the meaning of an optimal policy of information acquisition and determine rules of optimality for single and multiple occurring decisions. In particular a decision occurrence described by a geometric probability distribution serves as a metric for other distributions.

Chapter 6 builds on the results of Chapter 5 and extends the techniques of information appreciation by utilizing results from the established theory of maintainability and reliability. Several of these well-established results lead to extensions of the original conclusions of Chapter 5.

The final chapter summarizes the study and suggests areas for further development and research.

As noted previously this thesis fill a niche in a growing body of work on information value theory. The intelligence agencies of this country as well as analysts of many business firms are faced with a formidable resource allocation problem. There usually exists a multiple array of collection devices, each with its own probability of acquiring various pieces of data. These data in turn result in different updates of prior information that influence one or more of a compendium of decisions. These decisions, likewise, have different associated costs and benefits as well as probabilities of occurrence.

One would be both naive and foolhardy to claim at this stage of development a complete theory of information resource allocation that would aid these decision makers. However, the results of this thesis are a solid groundwork for the much needed follow-on research. The definition and concept of information perishing and the revision of the decision model lead to results that were previously tenuously shared and accepted by many decision analysts but never precisely defined. The theory of appreciation and the optimal policies of information acquisition are new to information value theory and presage even fuller exploitation of reliability theory. While much of the reliability work has to do with statistical inference and parameter estimation there is also a large body of conclusions concerning maintainability and optimal replacement policies. These results have yet to be fully mined for their application to information perishing and replenishment.

The ultimate goal, of course, is a set of allocation rules for the intelligence or information decision maker. This thesis forms a secure stepping stone for reaching that goal.

CHAPTER 2
INFORMATION DYNAMICS

2.1 Purpose

This chapter examines the time variation of the value of information. In particular, we define two key concepts, information perishing and the rate of information perishing. We then proceed to develop several properties of these two essential parameters.

2.2 Introduction

Most of the expository discussions of decision analysis treat the value of information as a static quantity [14,15,19]. Howard's well-known bid problem [14], as an example, computes the expected increased profit to the bidder, given clairvoyance or perfect information about his own cost, to be 1/96 units. However, one may consider two extremes. If the clairvoyant delivers the perfect information too late for the bidder to incorporate the data into his bid, then the expected increase in profit is surely not 1/96. Conversely, one may also argue that if the bidder receives the information much earlier than the date of the bid, he may feel that changing environmental factors would affect the validity of the information. Therefore, the expected increased profit of 1/96 is in a sense a conditional value--a value that is correct if the information is "timely" and "fresh."

We may illustrate the dynamics of the value of the state of information with an example.

2.3 Example 1: A Two-State Markovian Case

We choose the simplest of examples where the decision maker can choose either state "1" or state "2." When the true state of nature is subsequently revealed, he receives a greater reward if he has correctly chosen the state and a lesser reward (perhaps a cost) for an incorrect choice. His state of information is described by a Markov process.

Although not critical to the discussion we could suggest that the situation represents such real-life decisions as stockage of item 1 or item 2 where financial or storage constraints limit the seller's choice to one or the other item; defense of Area 1 or Area 2 against repetitive enemy attacks where the small size of the defending force or a lack of transportation precludes defense of both areas; or even the "pea in a shell" game at the local carnival.

We precisely define the situation as:

1. The decision maker can choose state 1 or state 2 but not both.
2. A Markovian model, Fig. 2-1, represents the model of his information on state occupancy.
3. The decision maker can change his decision prior to each transition. However, he does not observe the process at any time. In other words, he makes a series of decisions, e.g., 1, 1, 2, 1, ..., 2, etc., and at the end of the game is given some reward contingent on the number of correct decisions.
4. The decision-outcome matrix is shown in Table 2-1.

TABLE 2-1

Decision-Outcome Results, Example 1

	Decision	Choose State 1	Choose State 2
True State	State 1	+100	-100
	State 2	-100	+100

2.4 The Optimal Decision with Only Prior Knowledge

Let

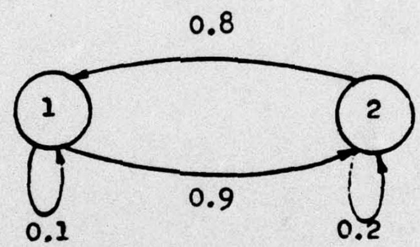
$$\delta(m) = \delta^{(1)}$$

be the decision to choose state 1 at transition m, and

$$\delta(m) = \delta^{(2)}$$

be the decision to choose state 2 at transition m. Assume the game or decision process lasts for M transitions. The decision maker must a priori make a series of M decisions $\{\delta(m)\} = \{\delta(0),\delta(1),\ldots,\delta(M)\}$ such as

$$\{\delta(0) = \delta^{(1)},\delta(1) = \delta^{(2)},\ldots,\delta(M) = \delta^{(1)}\}$$



$$\mathcal{P}\{\text{ state 1 at n} \mid \text{state 1 at n-1, } \epsilon\} = 0.1$$

$$\mathcal{P}\{\text{ state 2 at n} \mid \text{state 1 at n-1, } \epsilon\} = 0.9$$

$$\mathcal{P}\{\text{ state 2 at n} \mid \text{state 2 at n-1, } \epsilon\} = 0.2$$

$$\mathcal{P}\{\text{ state 1 at n} \mid \text{state 2 at n-1, } \epsilon\} = 0.8$$

Figure 2-1 Markovian information model

His prior knowledge is contained solely in the Markov model of Fig. 2-1. Therefore, he would rationally calculate

$$\mathcal{P}r\{s(0)=1|\varepsilon\} = \pi_1 = 8/17$$

and

$$\mathcal{P}r\{s(0)=2|\varepsilon\} = \pi_2 = 9/17$$

From Table 2-1 we may calculate the expected reward at transition $m = 0$, conditioned on the choice of $\delta^{(1)}$, as an example, as

$$\langle v(0)|\delta(0)=\delta^{(1)},\varepsilon\rangle = \underline{\pi_1(0) \quad \pi_2(0)} \begin{bmatrix} +100 \\ -100 \end{bmatrix}$$

$$= -5.88$$

In general, the expected reward at any transition is

$$\langle v(m)|\delta(m)=\delta^{(k)},\varepsilon\rangle = \sum_i \pi_i(m)\, r_i^{(k)} , \quad k = 1,2 \qquad (2.1)$$

However, in the example, with only prior knowledge

$$\pi_i(m) = \pi_i(\infty) = \pi_i \qquad (2.2)$$

and

$$\langle v(m)|\delta(m) = \delta^{(k)},\varepsilon\rangle = \sum_i \pi_i r_i^{(k)} , \quad k = 1,2 \qquad (2.3)$$

The optimal decision is defined by

$$\delta^*(m) = \max_k{}^{-1} \langle v(m)|\delta(m)=\delta^{(k)},\varepsilon\rangle$$

$$= \max_k{}^{-1} \underline{\pi_1 \quad \pi_2} \begin{bmatrix} \delta^{(1)} & \delta^{(2)} \\ +100 & -100 \\ -100 & +100 \end{bmatrix}, \quad k = 1,2 \qquad (2.4)$$

or

$$\delta^*(m) = \delta^{(2)} , \quad m = 0,1,2,\ldots,M$$

The optimal decision, in effect, is no more than the optimal choice of a column from the reward matrix of Table 2-1. Corresponding to this optimal decision is a reward

$$\langle v(m)\,|\,\delta^*(m),\varepsilon\rangle = \langle v(m)\,|\,\delta(m)=\delta^{(2)},\varepsilon\rangle = 5.88$$

Some compactness in notation is achieved by defining

$$v^{(k)}(m) = \langle v(m)\,|\,\delta(m)=\delta^{(k)},\varepsilon\rangle \qquad (2.5)$$

and

$$v^*(m) = \langle v(m)\,|\,\delta(m)=\delta^*,\varepsilon\rangle \qquad (2.6)$$

The decision maker's expected future reward is also of interest. We will use "n" to index periods remaining and define the expected future rewards with n time periods remaining as

$$\langle v(n)\,|\,\delta^*(n),\varepsilon\rangle \qquad (2.7)$$

where $\delta^*(n)$ implies

$$\{\delta^*(n) = \delta^*, \delta^*(n-1) = \delta^*, \ldots, \delta^*(1) = \delta^*\} \qquad (2.8)$$

In the example the optimal decision, as noted, is $\delta^{(2)}$ for every transition. Therefore, the expected future reward has a particularly simple form

$$\langle v(n)\,|\,\delta^*(n),\varepsilon\rangle = \langle v(n)\,|\,\delta^*(n)=\delta^{(2)},\varepsilon\rangle = n(5.88)\ ,$$

$$n = M, M-1, M-2, \ldots, 2, 1, 0 \qquad (2.9)$$

This "ramp" is plotted in Fig. 2-3a for $M = 10$.

Again compactness is realized by defining

$$v^*(n) = \langle v(n)\,|\,\delta^*(n),\varepsilon\rangle \qquad (2.10)$$

for the expected future reward conditioned on the decision maker electing the optimal decision at each transition.

(The indices "n" for periods to go and "m" for periods past imply that $n+m = M$ for a process with horizon M . See Fig. 2-2.)

We may summarize the example. Based on a prior state of knowledge contained in the Markov model of Fig. 2-1 the decision maker should choose state 2 for the entire sequence. His expected reward per transition is +5.88.

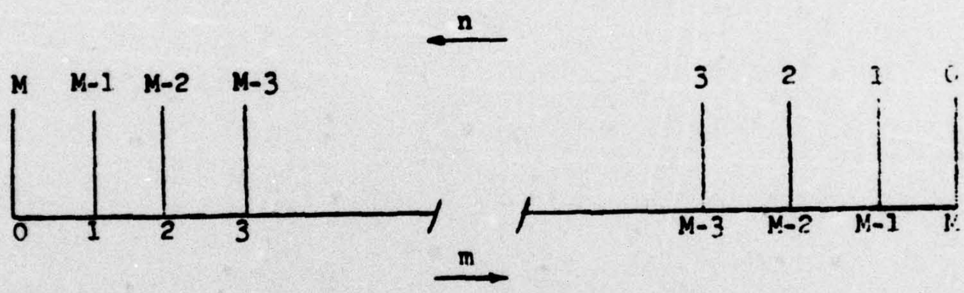


Figure 2-2 Indexing schematic

2.5 Optimal Strategy with Perfect Information

What changes would the decision maker effect if he were to receive perfect information on the initial state (all other assumptions of the example remaining the same)?

We may define the expected reward at transition m given the starting state i as

$$\langle v(m) \mid \delta(m)=\delta^*, s(0)=i, \varepsilon\rangle , \quad i = 1,2 \tag{2.11}$$

or compactly as

$$v^*_{i(0)}(m) = \langle v(m) \mid \delta(m)=\delta^*, s(0)=i, \varepsilon\rangle \tag{2.12}$$

The equivalent relationships for expected future rewards are

$$\langle v(n) \mid \underbrace{\delta(n)}=\delta^*, s(0)=i, \varepsilon\rangle , \quad i = 1,2 \tag{2.13}$$

and

$$v^*_{i(0)}(n) = \langle v(n) \mid \underbrace{\delta(n)}=\delta^*, s(0)=i, \varepsilon\rangle \tag{2.14}$$

Finally, for perfect information at time $m = 0$, [PI(0)] , the expected reward at any transition is

$$\langle v(m) \mid \delta(m)=\delta^*, PI(0), \varepsilon\rangle$$

$$= \pi_1\langle v(m) \mid \delta(m)=\delta^*, s(0)=1, \varepsilon\rangle + \pi_2\langle v(m) \mid \delta(m)=\delta^*, s(0)=2, \varepsilon\rangle \tag{2.15}$$

We economize further on notation by writing

$$v^*(m) = v^*_{PI(0)}(m) = \sum_i \pi_i v^*_{i(0)}(m) , \quad i = 1,2 \tag{2.16}$$

Analogously we have for expected future rewards

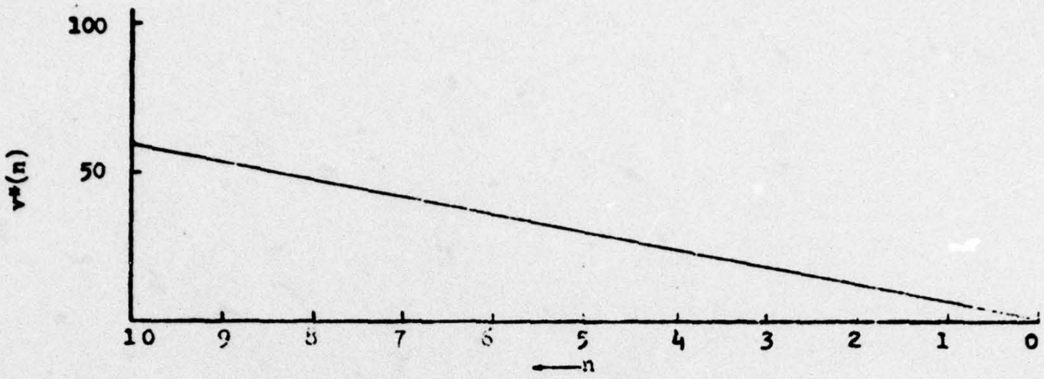
$$v^*(n) = \sum_i \pi_i v^*_{i(0)}(n) \tag{2.17}$$

We may use these results and the usual Markov matrix mechanics (Howard [2]) to calculate the value of perfect information as shown in Table 2-2. The values in columns (5) and (9) are plotted as Fig. 2-3b.

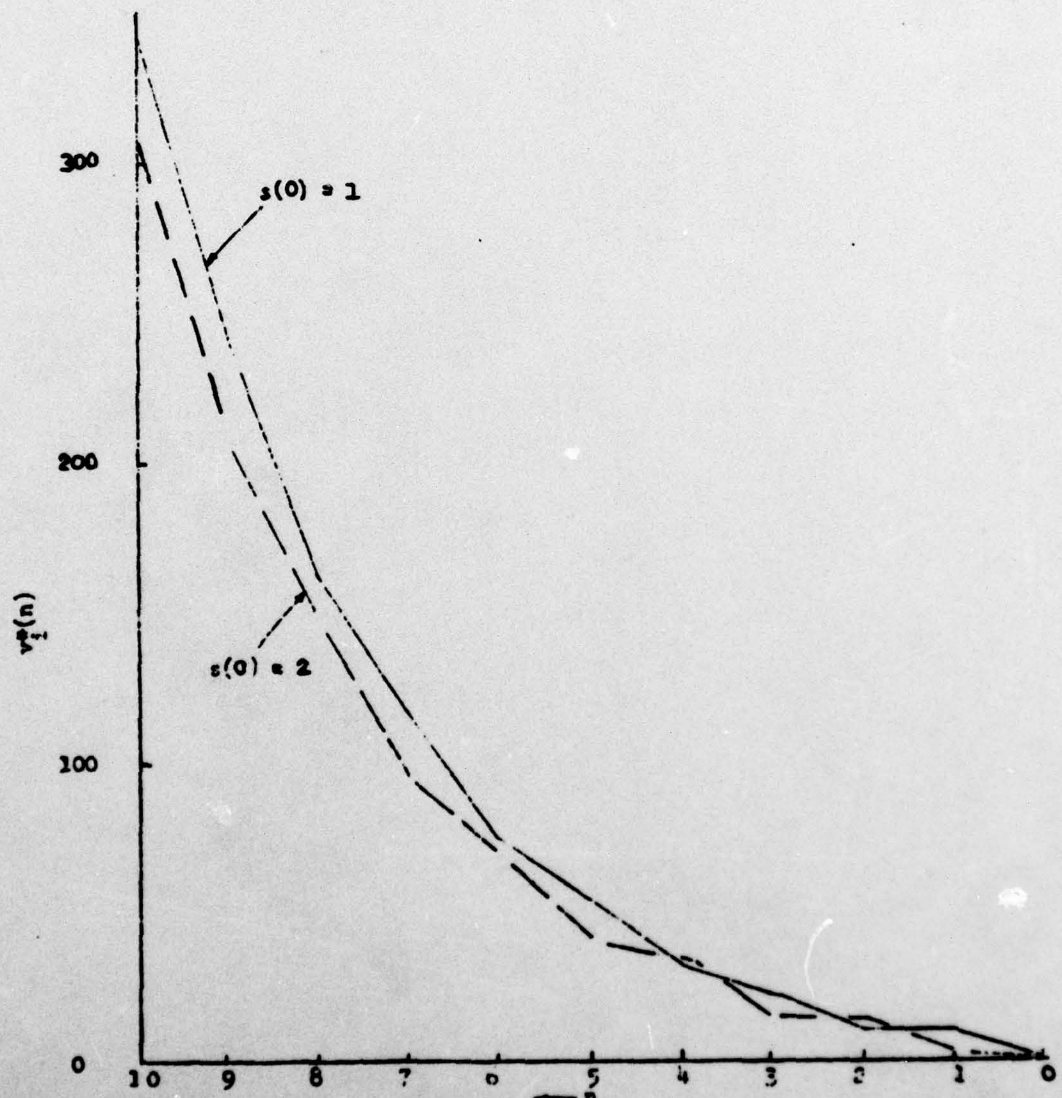
TABLE 2-2

Rewards and Probabilities, Example 1

(1) m	(2) $\varphi_{11}(m)$	(3) $\delta^*(m)$	(4) $v^*_{1(0)}(m)$	(5) $v^*_{1(0)}(n)$	(6) $\varphi_{21}(m)$	(7) $\delta^*(m)$	(8) $v^*_{2(0)}(m)$	(9) $v^*_{2(0)}(n)$	(10) $v^*_{PI(0)}(n)$	(11) n
1	1.000	1	100.0	342.8	0.000	2	100.0	309.0	324.9	10
2	0.100	2	80.0	242.8	0.800	1	60.0	209.0	224.9	9
3	0.730	1	46.0	162.8	0.240	2	52.0	149.0	155.5	8
4	0.289	2	42.2	116.8	0.632	1	26.4	97.0	106.3	7
5	0.598	1	19.6	74.6	0.358	2	28.4	70.6	72.5	6
6	0.382	2	23.6	55.0	0.550	1	10.0	42.2	48.2	5
7	0.533	1	6.6	31.4	0.415	2	17.0	32.2	31.8	4
8	0.427	2	14.6	24.8	0.509	1	1.8	15.2	19.7	3
9	0.501	1	0.0	10.2	0.443	2	11.4	13.4	11.9	2
10	0.449	2	10.2	10.2	0.490	2	2.0	2.0	5.9	1



(a) v*(n) , no information case



(b) $v^*_{i(0)}(n)$, known starting state case

Figure 2-3 Rewards for Example 1

Figure 2-4a shows the expected future reward conditioned on receipt of perfect information at $m = 0$ and the expected future rewards conditioned on only the prior state of knowledge. Figure 2-4b indicates the difference between these two quantities

$$\Delta(n) = v^*(n) - v^*(n) \tag{2.18}$$

Examination of Fig. 2-4b reveals that although the perfect information acquired at $m = 0$ initially places the decision maker in a relatively favorable position this advantage diminishes over time and by the eighth transition the advantage has disappeared. This decrement, which we shall shortly define as information perishing, has a natural interpretation in terms of response time. If the decision maker requires one period to adjust his strategy to the receipt of perfect information at transition zero, the value of this clairvoyance is 171.98; if he requires over eight periods to react, then the information has no value.

The rate of decline of this relative advantage is also of interest. We define

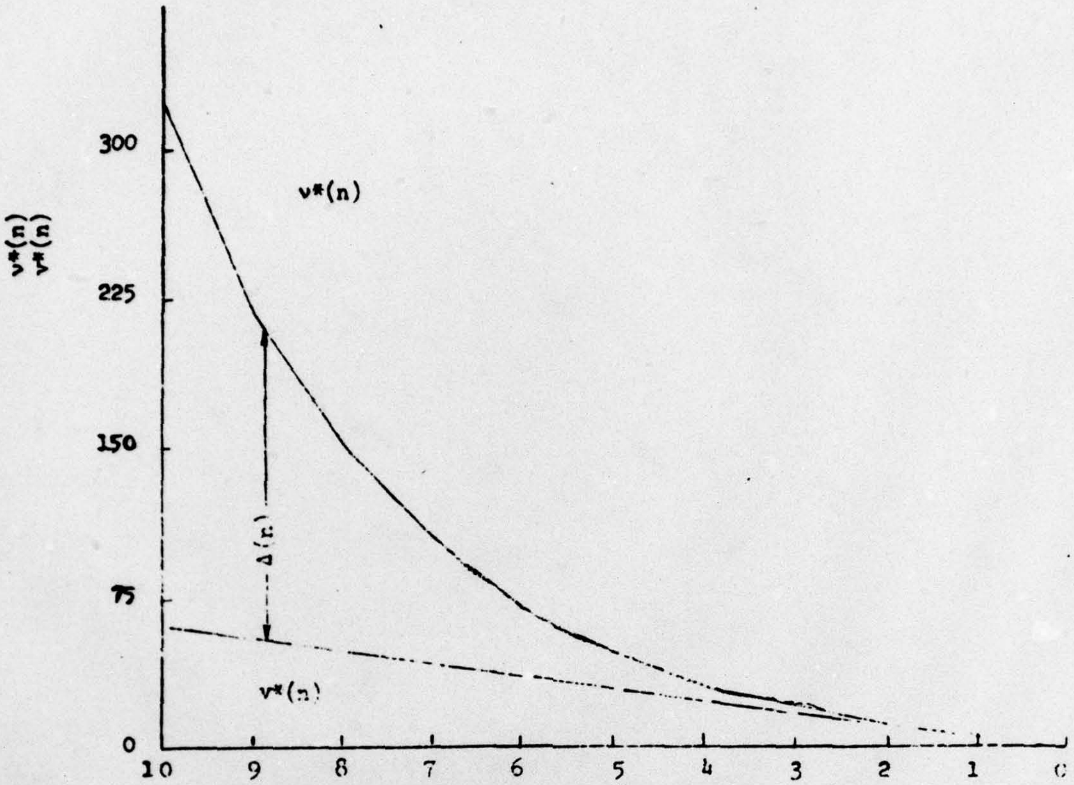
$$\begin{aligned} \rho(n) &= \frac{\Delta(n-1)}{\Delta(n)} \quad \text{for} \quad \Delta(n) \neq 0 \\ &= 0 \quad \text{for} \quad \Delta(n) = 0 \end{aligned} \tag{2.19}$$

This quantity is plotted in Fig. 2-5.

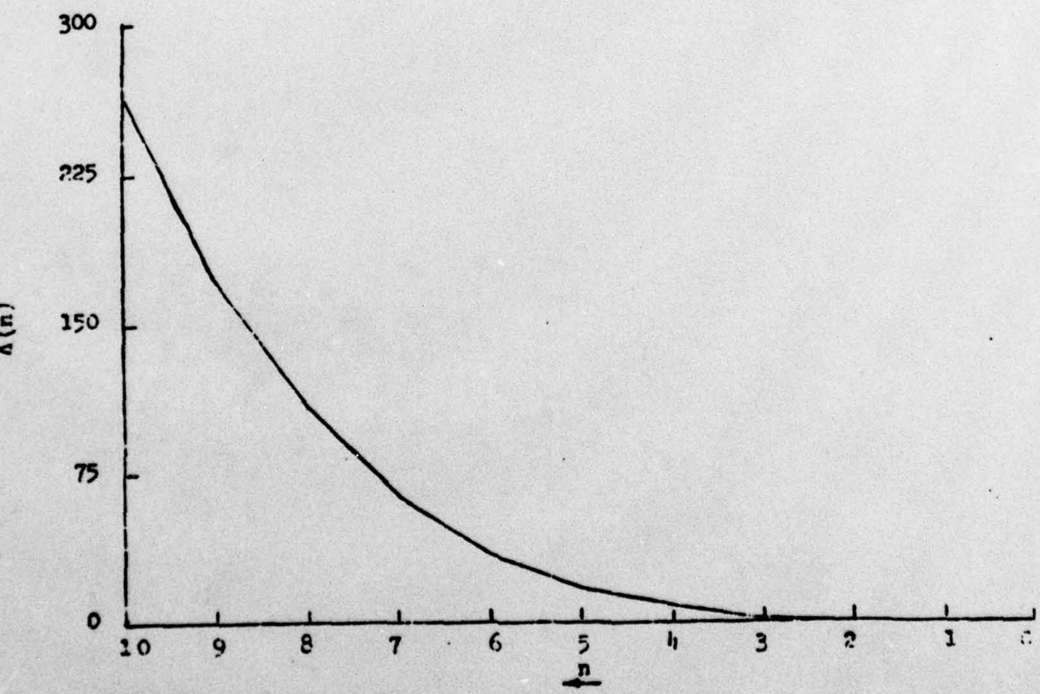
2.6 Basic Definitions

The phenomenon of the degradation of the value of information over time, while apparently a characteristic of many real-life decision problems, is not extensively treated in the literature. North [18], Smallwood [22], and Howard [13] discuss aspects of inference in a dynamic situation while Robinson [20] reports on the practical difficulties of estimating time varying probabilities. However, these articles are limited to problems of inference without consideration of the value of the information. The concept of information "perishing" appears more general and powerful than implied by this literature.

We as a first step must agree on a definition of information "perishing." Information may, of course, evolve over time without



(a) ν*(n) and ν*(n) as a function of n



(b) Δ(n) as a function of n

Figure 2-4 ν*(n) , ν*(n) , and Δ(n) as a function of n

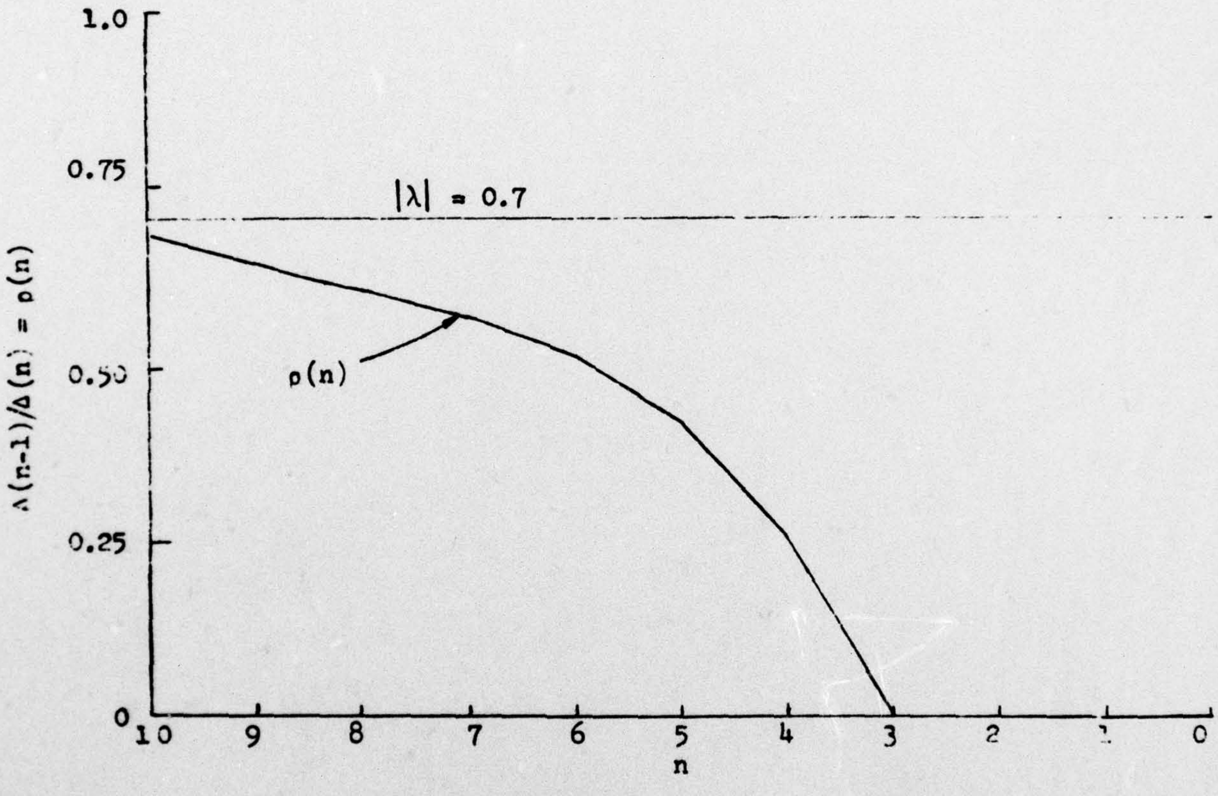


Figure 2-5 ρ(n) as a function of n

affecting the choice of decisions. As an example, suppose there exists some vector valued state of information $\underline{s}$ which is a function of time. Let $\underline{s}(t)$ represent this functional dependence and assume that $\underline{s}(t) \subset S$, a set of possible states of information. Then if $\delta^* = \delta^{(0)}$ for all $\underline{s}(t) \subset S$, that is, the optimal decision is the same for all states of information, would one characterize information acquired at $m = 0$ as perishing? or is this instant perishability?

As a second example we consider the case where the decision maker receives clairvoyance at $m = 0$ and also at $m = m_1$, $m_1 > 0$. Although we shall analyze this situation in some detail in Chapter 4 it is perhaps intuitively obvious that the second acquisition of clairvoyance "wipes out" the value of the first disclosure of perfect information. Is this information perishing?

We precisely define information perishing. Let $\nu^*(n)$ be the expected future rewards with n periods to go conditioned on acquisition of information (perfect or imperfect) at $m = 0$. Let $v^*(n)$ represent the expected future reward based solely on prior knowledge at $m = 0$. Let $\Delta(n) = \nu^*(n) - v^*(n)$. If $\Delta(n)$ is a non-increasing function of n without benefit of test, observation, experiment, or other information acquisition, then the information acquired at $m = 0$ is perishing.

If $\Delta(n) = 0$, the information has perished. The rate of perishing, $\rho(n)$, is defined by (2.19), i.e.,

$$\rho(n) = \frac{\Delta(n-1)}{\Delta(n)}$$

2.7 Generalizations

We now rigorously prove several properties of information perishing and the rate of perishing.

2.7.1 The Reward Structure

We have previously defined the expected reward at any transition m as

$$\langle v(m) \mid \delta(m)=\delta^{(k)}, \varepsilon \rangle = \sum_i \pi_i(m)\; r_i^{(k)} \qquad (2.20)$$

There are two other forms of this expression that will be useful. The first is

$$\langle v(m)\,|\,\delta(m)=\delta^{(k)},\epsilon\rangle = \underbrace{\pi(0)}\;[P]^m\;\underbrace{r_i^{(k)}}^{T} \qquad (2.21)$$

The second results from the expansion of $[P]^m$ into a series of N differential matrices as

$$[P]^m = [Q_0] + \lambda_1^m[Q_1] + \lambda_2^m[Q_2] + \ldots + \lambda_{N-1}^m[Q_{N-1}] \qquad (2.22)$$

Substituting (2.22) into (2.21) yields

$$\langle v(m)\,|\,\delta(m)=\delta^{(k)},\epsilon\rangle = C_0^{(k)} + C_1^{(k)}\,\lambda_1^m + \ldots + C_{N-1}^{(k)}\,\lambda_{N-1}^m \qquad (2.23)$$

Figure 2-6a plots the reward structure of the two-state example we have been considering, while Fig. 2-6b represents a general two-state $\mathcal{N}$ decision model. The extension to $\mathcal{N}$ state is obvious but not representable.

2.7.2 The Inevitability of Information Perishing

We have seen in the simple example that information perishes. However, we can establish this result for a far more general case.

Theorem 2.1

For any N-state Markov decision process where the decision maker may choose both the transition matrix $[P^{(l)}] = \{p_{ij}\}^{(l)}$ and the reward from some constant reward matrix $[R] = \{r_i\}^{(k)}$ (k and l contained in index sets, K and L of decisions and transition matrices, respectively), $\Delta(n)$ is a non-increasing function of n .

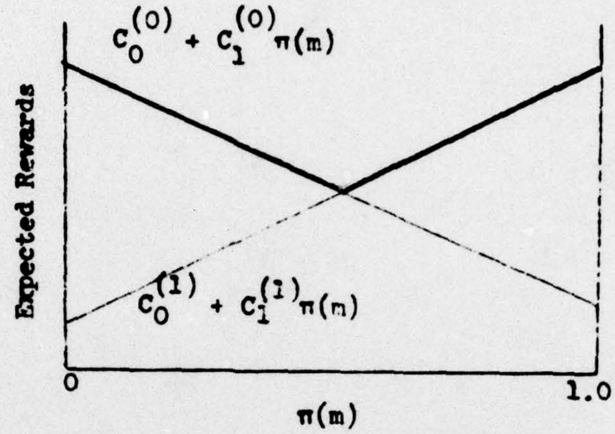
Proof

Let $\delta^{(1,1)}$, as an example, represent the decision to choose the first transition matrix and the first column of the reward matrix. There are three cases to be considered:

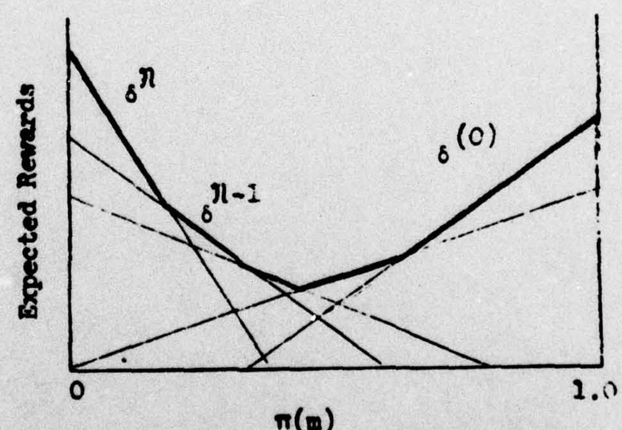
Case 1. We may consider first the trivial case of some decision, say $\delta^{(0,0)}$, being completely dominant, i.e., both $v^*(m)$ and $v^*(m)$ imply $\delta^* = \delta^{(0,0)}$ for all m . Therefore, $\Delta(n) = v^*(n) - v^*(n) = 0$, and $\Delta(n)$ is obviously non-increasing.

Case 2. Partial dominance may exist in the sense that $v^*(m)$ and $v^*(m)$ both imply $\delta^* = \delta^{(0,0)}$ for some $m \geq m_{cr}$. If this be true, and if $n \leq M - m_{cr}$, then $\Delta(n) = 0$ again, and the theorem is true.

Case 3. The interesting case is the case of no dominance. We proceed by induction. With one time period remaining, to show that



Two State, Two Decision Case



Two State, η Decision Case

Figure 2-6 Reward structures

$\Delta(n)$ is a non-increasing function of n is equivalent to showing

$$\Delta(1) - \Delta(0) \geq 0 \tag{2.24}$$

However,

$$\begin{aligned}\Delta(1) - \Delta(0) &= [\nu^*(1) - v^*(1)] - [\nu^*(0) - v^*(0)] \\ &= [\nu^*(1) - \nu^*(0)] - [v^*(1) - v^*(0)]\end{aligned}$$

Counting <u>forward</u> we may write (see Fig. 2-2)

$$v^*(1) = \nu^*(M-1) + \nu^*(M) \tag{2.25}$$

and

$$\nu^*(0) = \nu^*(M) \tag{2.26}$$

Similarly we may express the other two terms as

$$v^*(1) = v^*(M-1) + v^*(M) \tag{2.27}$$

and

$$v^*(0) = v^*(M) \tag{2.28}$$

Performing the obvious subtraction we can express (2.24) as

$$\Delta(1) - \Delta(0) = \nu^*(M-1) - v^*(M-1) \tag{2.29}$$

which is obviously greater or equal to zero. Assume the induction hypothesis holds for $n-1$ time periods. It remains to show that the theorem holds for n time periods to go, or that $\Delta(n) - \Delta(n-1) \geq 0$.

$$\Delta(n) = [\nu^*(n-1) + \nu^*(\eta)] - [v^*(n-1) + v^*(\eta)] \tag{2.30}$$

where "η", counting <u>forward</u>, is the transition at which there are n transitions <u>to go</u>. Therefore,

$$\Delta(n) = \Delta(n-1) + \nu^*(\eta) - v^*(\eta)$$

or

$$\Delta(n) - \Delta(n-1) = \nu^*(\eta) - v^*(\eta) \geq 0$$

which completes the proof.

We may extend these results by consideration of a continuous time process. We shall use "τ" to indicate time starting from time "zero" and "t" to indicate time to go. We shall assume a constant but completely general generation of rewards as shown in Fig. 2-7.

To parallel (2.18) we define

$$\Delta(t) = \int_t^T \nu^*(\tau)\, d\tau - \int_t^T v^*(\tau)\, d\tau \qquad (2.31)$$

$$= \int_t^T [\nu^*(\tau) - v^*(\tau)]\, d\tau$$

To show that the information is perishing we show $\partial\Delta(t)/\partial(t) \le 0$, or

$$\frac{\partial\Delta(t)}{\partial t} = \frac{\partial}{\partial t}\int_t^T [\nu^*(\tau) - v^*(\tau)]\, d\tau \le 0 \qquad (2.32)$$

$$= -[\nu^*(t) - v^*(t)] \qquad \le 0 \quad \text{for all} \quad t \le T$$

Thus, we see that in a decision process that continues over some period of time that any information is perishing. We emphasize this result by stating Theorem 2.2.

Theorem 2.2

All information is perishing (assuming the reward structure is constant over time).

2.7.3 The Rate of Information Perishing

a. Introduction. We have noted in Fig. 2-5 that the rate of information perishing as defined by (2.19) was always less than 0.7, the absolute value of the transient* eigenvalue. Is this result always true?

b. Initial Result. We may show that this result holds not only in the example but in a far more general case.

The case we shall consider is this:

(1) N-state process

(2) k decisions possible with reward matrix

*Transient connotes eigenvalues not equal to one.

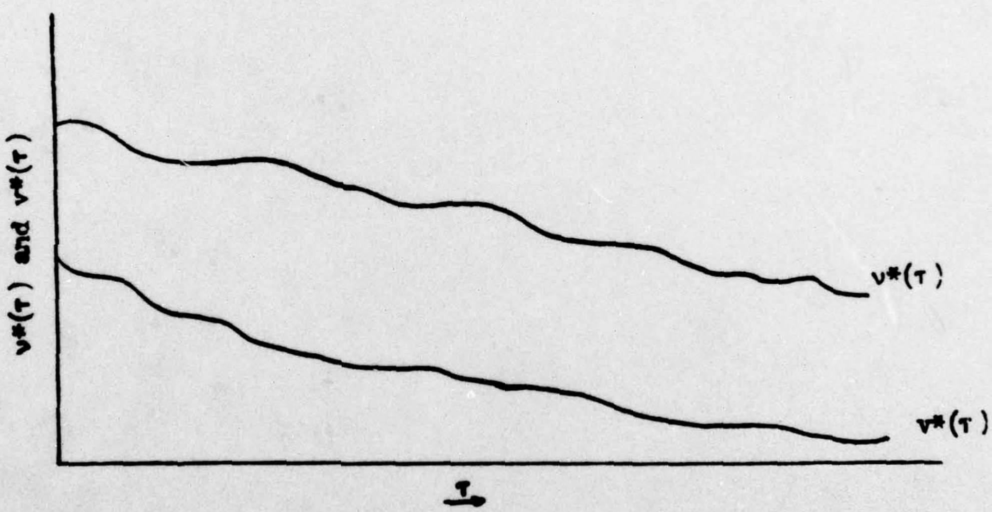


Figure 2-7 General reward structure

$$[R] = \begin{bmatrix} r_1^{(0)} & r_1^{(1)} & \cdots\cdots\cdots & r_1^{(k-1)} \\ r_2^{(0)} & r_2^{(1)} & \cdots\cdots\cdots & r_2^{(k-1)} \\ \cdots & \cdots & \cdots & \cdots \\ r_n^{(0)} & r_n^{(1)} & \cdots\cdots\cdots & r_n^{(k-1)} \end{bmatrix} \tag{2.33}$$

Let $\{r_\ell\}^{(k)}$ be a general element of [R] .

(3) The transition matrix [P] is not part of the decision. In other words, [P] is invariant.

(4) The decision maker receives perfect information at transition zero, but there is no observation or information after this.

(5) The decision maker has no risk aversion.

(6) A decision is possible at each transition.

We shall prove that $\rho(n) \leq |\lambda_1|$ where λ_1 is the maximum in absolute value of transient eigenvalues associated with the transition matrix [P] .

<u>Theorem 2.3</u>

For the n-state Markov process with k reward decisions and an invariant transition matrix, $\rho(n) \leq |\lambda_1|$, the absolute value of the largest transient eigenvalue.

The proof of this theorem is of such length that it is reserved to Appendix A.

c. <u>An Extension</u>. We had limited the previous proof to decision situations where the decision was limited to a choice of state, and the transition matrix was invariant. However, we may also extend the result to the situation where the decision maker may elect not only the reward structure but also the transition matrix.

<u>Theorem 2.4</u>

For a n-state Markov process let $k \subset K$ represent an index set of reward decisions and $l \subset L$ represent an index set of transition matrix decisions. Then

$$\rho(n) \le |\lambda_1|'$$

where $|\lambda_1|' = \max \{\lambda_1^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_1^{(L)}\}$, $\lambda_1^{(1)}$ being the greatest absolute value of the transient eigenvalues of $[P^{(1)}]$.

The proof of this theorem will also be found in Appendix A.

d. <u>The Acceleration of Information Perishing</u>. Figure 2-5 shows that $\rho(n)$ is a decreasing function of n or that information perishes more rapidly with the passage of time. We may show that this is a general result for those decision situations where the transition matrix is invariant. We first need to prove a lemma concerning the reward structure.

<u>Lemma 2.1</u>

For a N-state Markov process where the decision maker's alternatives are limited to choice of columns from the reward matrix there exists for some starting state, say $s(0) = i$, at most three optimal policies. Further, if all the eigenvalues are positive, there exists at most two optimal policies.

<u>Proof</u>

We use (2.22) to write

$$\langle v(M) | s(0)=i \rangle = \max_k \sum_{j=0}^{N-1} \lambda_j^M \sum_{\ell=1}^{N} {}_j q_{i\ell} \, r_\ell^{(k)} \qquad (2.34)$$

where $\lambda_0 = 1$. Let $M \to \infty$ so that $\lambda_j^M \to 0$, $j \neq 0$. Obviously,

$$\langle v(M) | s(0)=i \rangle \to \max_k \sum_{\ell=1}^{N} {}_0 q_{i\ell} \, r_\ell^{(k)} \qquad (2.35)$$

the "stationary" policy noted in Howard [2]. This is the first of the two or three policies. Now assume $\lambda_j > 0$, all j . We represent the scalar product of (2.35) by ${}_j C^{(k)}$ so that

$$\langle v(M) | s(0)=i \rangle = \max_k \sum_{j=0}^{N-1} \lambda_j^M \, {}_j C^{(k)} \qquad (2.36)$$

Assume that $k = 1$ for the M^{th} transition and that for the $M + \alpha$ transition, $-M \le \alpha < \infty$, $k = 2$. For the two decisions ${}_j C$ must differ in at least one term. We will let $j = 1$ be that term. By the assumed optimality

$$\lambda_1^{M+\alpha}\ {}_1C^{(1)} + \sum_{\substack{j=0 \\ j\neq 1}}^{N-1} \lambda_j^{M+\alpha}\ {}_jC^{(1)} \le \lambda_1^{M+\alpha}\ {}_1C^{(2)} + \sum_{\substack{j=0 \\ j\neq 1}}^{N-1} \lambda_j^{M}\ {}_jC^{(2)} \quad (2.37)$$

$$\le \lambda_1^{M+\alpha}\ {}_1C^{(2)} + \sum_{\substack{j=0 \\ j\neq 1}}^{N-1} \lambda_j^{M}\ {}_jC^{(1)} \quad (2.38)$$

or

$$\lambda_1^{M+\alpha}\ {}_1C^{(1)} \le \lambda_1^{M+\alpha}\ {}_1C^{(2)} \quad (2.39)$$

and

$${}_1C^{(1)} \le {}_1C^{(2)} \quad (2.40)$$

However, if ${}_1C^{(2)} \ge {}_1C^{(1)}$, then decision 1 would be improved by switching to decision 2 as all the other C's are the same, and all the eigenvalues are positive. Therefore, there cannot be two decisions that are optimal for the different transitions. Similar reasoning prevails if $\lambda_j \le 0$ for some j except now the optimal decisions may switch from odd to even transitions.

This completes the proof of the lemma and allows us to state the following corollary.

Corollary 2.1

For the N-state Markov process with an invariant transition matrix

$$\frac{\Delta(n-1)}{\Delta(n)} \ge \frac{\Delta(n-2)}{\Delta(n-1)}\ , \quad n \ge 2 \quad (2.41)$$

Proof

The corollary requires that

$$\Delta^2(n-1) \ge \Delta(n)\ \Delta(n-2) \quad (2.42)$$

The proof follows by induction on n using the expressions for $\Delta(n)$, $\Delta(n-1)$, and $\Delta(n-2)$ developed in (A.6), (A.7), and (A.26).

Then Lemma 2.1 makes possible term by term comparisons. The details are an exercise in tedium rather than enlightment and are omitted.

2.7.5 The Effect of Risk Aversion

a. Definitions. To this point we have tacitly assumed that the decision maker based his decision on expected values. A logical next step is consideration of the effects of risk aversion. We shall limit the discussion to exponential utility functions.

A natural extension of the defining equation for $\Delta(n)$ [(2.18)] is

$$\tilde{\Delta}(n) = \tilde{v}^*(n) - \tilde{v}^*(n) \tag{2.43}$$

where $\tilde{v}^*(n)$ represents the certain equivalent with n periods to go conditioned on receipt of perfect information at $m = 0$, and $\tilde{v}^*(n)$ represents the certain equivalent with n periods to go based solely on prior information. Howard and Matheson [16] have shown that the "delta property" of the exponential utility function allows summation of the certain equivalents.

Analogous to Eq. (2.19) is

$$\tilde{\rho}(n) = \frac{\tilde{\Delta}(n-1)}{\tilde{\Delta}(n)} \;, \quad \tilde{\Delta}(n) \neq 0$$
$$= 0 \qquad\quad , \quad \tilde{\Delta}(n) = 0 \tag{2.44}$$

b. An Example. Assume that the decision maker in the basic example has a risk aversion coefficient, $\gamma = 0.001$. We may calculate $\tilde{\Delta}(n)$ and $\tilde{\rho}(n)$ which are plotted in Figs. 2-8 and 2-9 (along with the comparable values of $\Delta(n)$ and $\rho(n)$).

c. Generalizations. Comparisons of $\tilde{\Delta}(n)$ and $\Delta(n)$ and $\tilde{\rho}(n)$ and $\rho(n)$ for the general Markov case are made possible by use of the approximation

$$\tilde{v}(m) \approx \bar{v}(m) - \frac{1}{2} \gamma \check{v}(m) \tag{2.45}$$

where $\tilde{v}(m)$, $\bar{v}(m)$, and $\check{v}(m)$ represent the certain equivalent, mean, and variance, respectively, of the profit lottery on the m^{th} transition. (The approximation results in an error of less than 0.2% in the example.)

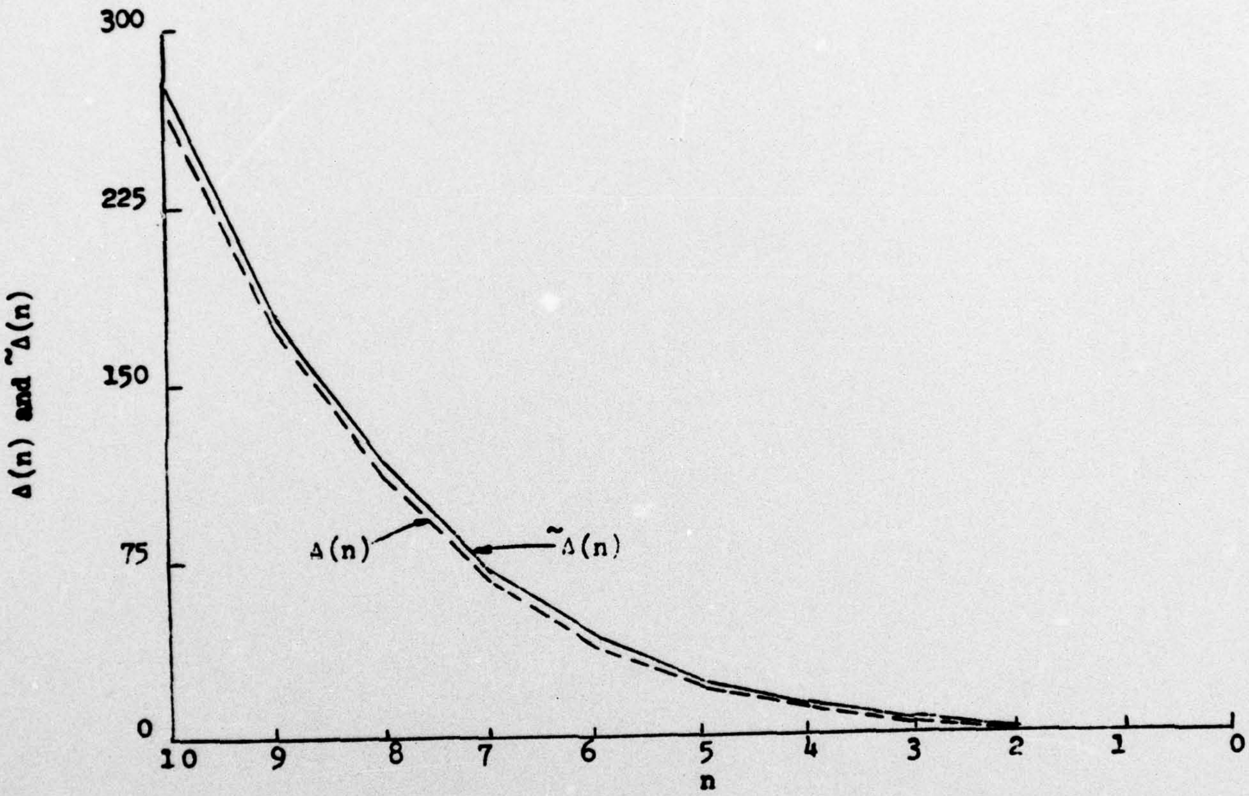


Figure 2-8 Comparison of Δ(n) and ~Δ(n)

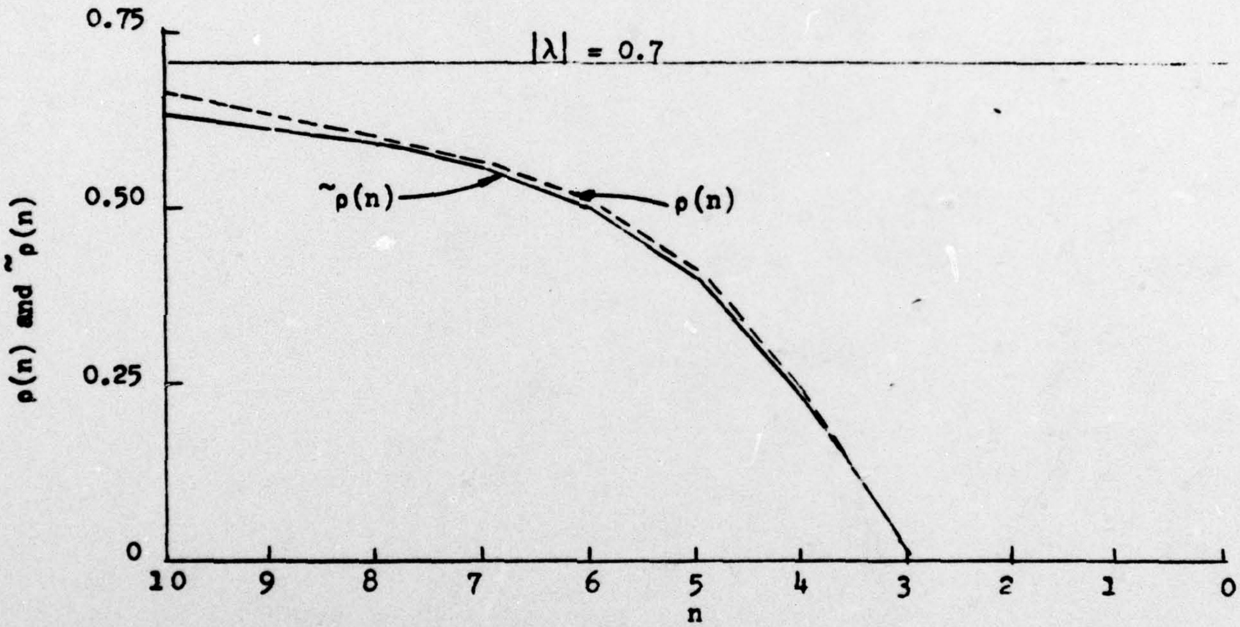


Figure 2-9 Comparison of ρ(n) and ~ρ(n)

Approximation (2.45) allows (2.43) to be rewritten as

$$\tilde{\Delta}(n) = \sum_{\ell=M-n}^{M} \left\{ \tilde{v}^*(\ell) - \frac{1}{2}\gamma\left[\tilde{\tilde{v}}^*(\ell)\right] - v^*(\ell) + \frac{1}{2}\gamma\left[\check{v}^*(\ell)\right]\right\} \quad (2.46)$$

$$= \sum_{\ell=M-n}^{M} \left\{ \tilde{v}^*(\ell) - v^*(\ell) + \frac{1}{2}\gamma\left[\check{v}^*(\ell) - \check{\tilde{v}}^*(\ell)\right]\right\} \quad (2.47)$$

$$= \Delta(n) + \sum_{\ell=M-n}^{M} \frac{1}{2}\gamma\left[\check{v}^*(\ell) - \check{\tilde{v}}^*(\ell)\right] \quad (2.48)$$

For a "symmetric" reward matrix of the form

$$[R] = \begin{bmatrix} +r & -r & \cdots & -r \\ -r & +r & \cdots & -r \\ \cdots & \cdots & \cdots & \cdots \\ -r & -r & & +r \end{bmatrix} \quad (2.49)$$

the variance with information is less than the variance without information and we conclude the following theorem.

<u>Theorem 2.5</u>

For a symmetric reward matrix $\tilde{\Delta}(n) \geq \Delta(n)$. (For a general reward matrix one may construct counter-examples to Theorem 2.5.)

We may also show

<u>Theorem 2.6</u>

For the symmetric reward matrix $\tilde{\rho}(n) \leq \rho(n)$. By the use of (2.48) we may write

$$\tilde{\rho}(n) = \frac{\Delta(n-1) + \sum_{\ell=M-(n-1)}^{M} \frac{1}{2}\gamma\left[\check{v}^*(\ell) - \check{\tilde{v}}^*(\ell)\right]}{\Delta(n) + \sum_{\ell=M-n}^{M} \frac{1}{2}\gamma\left[\check{v}^*(\ell) - \check{\tilde{v}}^*(\ell)\right]} \leq \frac{\Delta(n-1)}{\Delta(n)} \quad (2.50)$$

We may simplify this expression considerably by letting

$$\left.\begin{array}{l} \check{v}^*(\ell) = S \text{ (as the variant without information is a constant)} \\ \check{\nu}^*(\ell) = S(\ell) \text{ , and} \\ L = \text{transition with } n \text{ to go, and} \\ V(L) = \nu^*(L) - v^* \end{array}\right\} \quad (2.51)$$

Substituting these into (2.50), cross-multiplying, transposing and simplifying yields

$$\Delta(n) \sum_{\ell=M-(n-1)}^{M} [S-S(\ell)] \leq \Delta(n-1) \sum_{\ell=M-n}^{M} [S-S(\ell)] \quad (2.52)$$

$$[V(L) + \Delta(n-1)] \sum_{\ell=M-(n-1)}^{M} [S-S(\ell)]$$

$$\leq \Delta(n-1)\left\{S-S(L) + \sum_{\ell=M-(n-1)}^{M} [S-S(\ell)]\right\} \quad (2.53)$$

or

$$V(L)(n-1)S + \Delta(n-1)S(L) \leq \Delta(n-1)S + V(L) \sum_{\ell=M-(n-1)}^{M} S(\ell) \quad (2.54)$$

Dividing by $V(L)\ S(L)$ results in

$$\frac{(n-1)S}{S(L)} + \frac{\Delta(n-1)}{V(L)} \leq \frac{\Delta(n-1)S}{V(L)S(L)} + \frac{\sum_{\ell=M-(n-1)}^{M} S(\ell)}{S(L)} \quad (2.55)$$

or

$$\frac{\Delta(n-1)}{V(L)}\left[1 - \frac{S}{S(L)}\right] \leq \frac{\left[\sum_{\ell=M-(n-1)}^{M} S(\ell)\right] - (n-1)S}{S(L)} \quad (2.56)$$

As

$$\frac{\Delta(n-1)}{V(L)} \leq \frac{(n-1)\ V(L)}{V(L)} = n-1 \quad (2.57)$$

we may prove the inequality by proving

$$(n-1)\left[1 - \frac{S}{S(L)}\right] \le \frac{\left[\sum_{\ell=M-(n-1)}^{M} S(\ell)\right] - (n-1)S}{S(L)} \tag{2.58}$$

or

$$1 - \frac{S}{S(L)} \le \frac{\left[\sum_{\ell=M-(n-1)}^{M} \frac{S(\ell)}{n-1}\right] - S}{S(L)} \tag{2.59}$$

$$S(L) - S \le \sum_{\ell=M-(n-1)}^{M} \left[\frac{S(\ell)}{n-1}\right] - S \tag{2.60}$$

and

$$S(L) \le \sum_{\ell=M-(n-1)}^{M} \frac{S(\ell)}{n-1} \tag{2.61}$$

But S(L) is the minimum variance so that

$$S(L) \le \frac{n-1}{n-1} S(L) \le \sum_{\ell=M-(n-1)}^{M} \frac{S(\ell)}{n-1} \tag{2.62}$$

This completes the proof.

2.7.6 Transient Processes

The previous examples involved only Markov chains with recurrent states. We briefly digress to consider the transient chain shown in Fig. 2-10. We shall assume a reward matrix

$$[R] = \begin{bmatrix} +100 & -100 & -100 \\ -100 & +100 & -100 \\ -100 & -100 & +100 \end{bmatrix}$$

If the process had run for some length of time, then

$$\pi_1 = 0\ , \quad \pi_2 = 0\ , \quad \text{and} \quad \pi_3 = 1.0$$

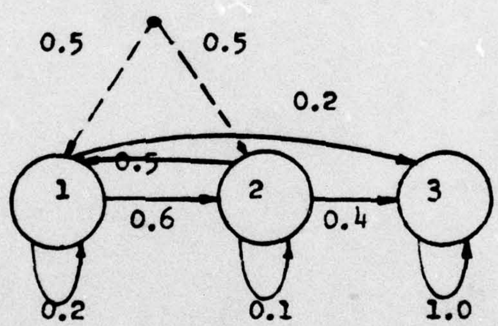


Figure 2-10 A transient state example

In this case $\delta^*(m) = \delta^{(3)}$, and $v^*(m) = 0(-100) + 0(-100) + 1(100) = 100$. $\Delta(n)$ and $\rho(n)$ are both trivially equal to zero.

We may create a more interesting example by assuming that the process has just begun and that some outside probability mechanism such as the flip of a fair coin determines if state 1 or state 2 is the initial state. In other words, as shown in Fig. 2-10,

$$\mathcal{P}\{s(0)=1|\varepsilon\} = \mathcal{P}\{s(0)=2|\varepsilon\} = 0.5$$

$\Delta(n)$ and $\rho(n)$ are plotted in Fig. 2-11. The figure confirms that $\Delta(n)$ is a decreasing function of n and $\rho(n) \leq \lambda_1 = +0.7$.

2.8 <u>Summary</u>

This chapter has developed the fundamental concepts and results necessary for an understanding of the dynamics of the value of information. The most important result was the inevitability of information perishing. Equally significant is the result that the value of information for a Markov process perishes at a rate that exceeds the shrinkage of the underlying process. The chapter also considered the effect of risk aversion where the utility function can be modeled by an exponential expression. The following chapter extends these results by a slight alteration of the basic decision model.

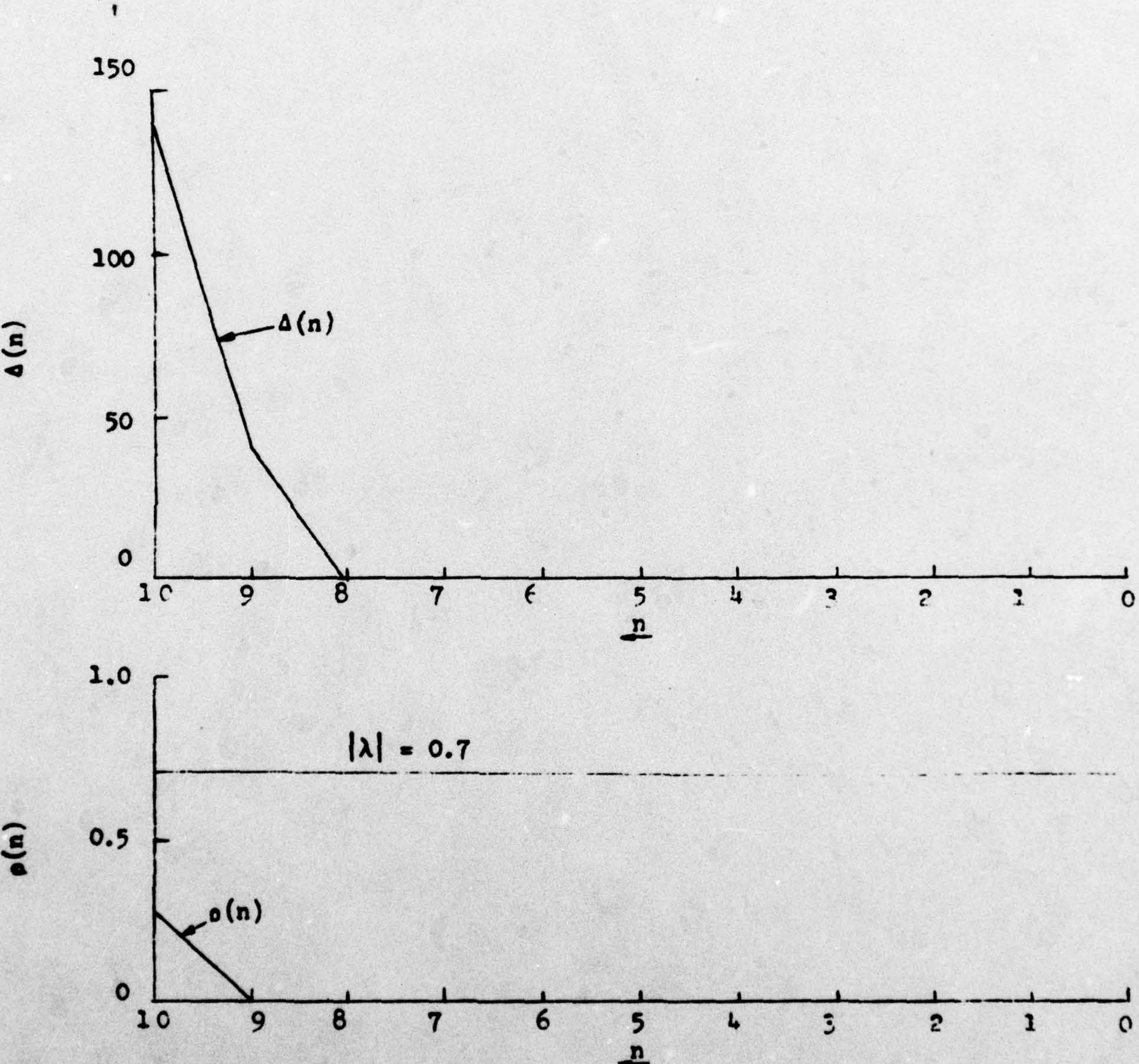


Figure 2-11 Δ(n) and ρ(n) for the transient state example

CHAPTER 3
THE DECISION MODEL

3.1 Purpose

This chapter describes the decision model that will be used throughout the remainder of the thesis.

3.2 An Historical Example

During the 1960s the United States, as a portion of its NATO strategy, pre-stocked the equipment for several U.S. Army divisions in Western Europe. This equipment was matched to designated units based within the United States. The anticipated mode of employment was an airlift of personnel to Western Europe, "marrying up" with the equipment, and subsequent deployment in defense of NATO allies. The motivation for this plan was to cut the reaction time in countering any Russian agression. The concept was tested during the 1960s in a series of exercises dubbed "Reforger."

3.3 Comparison with the Extant Decision Model

A comparison of this strategy with the "usual" decision model reveals some subtle differences.

The existing model [15,19], depicted in Fig. 3-1, implicitly recognizes a random event, "A decision is needed." The entire analysis and interest then follows this random event. There is no subsequent uncertainty concerning the occurrence of the decision.

In the cited historical example there is some probability that the Russians will never attack Western Europe and that a decision, in the sense of tactical deployments, will never be made. The U.S. strategy in Europe is assuredly a complex set of supporting decisions. However, the essence of the approach is shown in Fig. 3-2. The significant difference is the recognition of uncertainty in the occurrence of the ultimate decision (the method of defending Europe).

This leads to these metaphores:

1. Reaction decision making. The decision maker sets his decision vector after the need for a decision is recognized as a certainty or near-certainty.

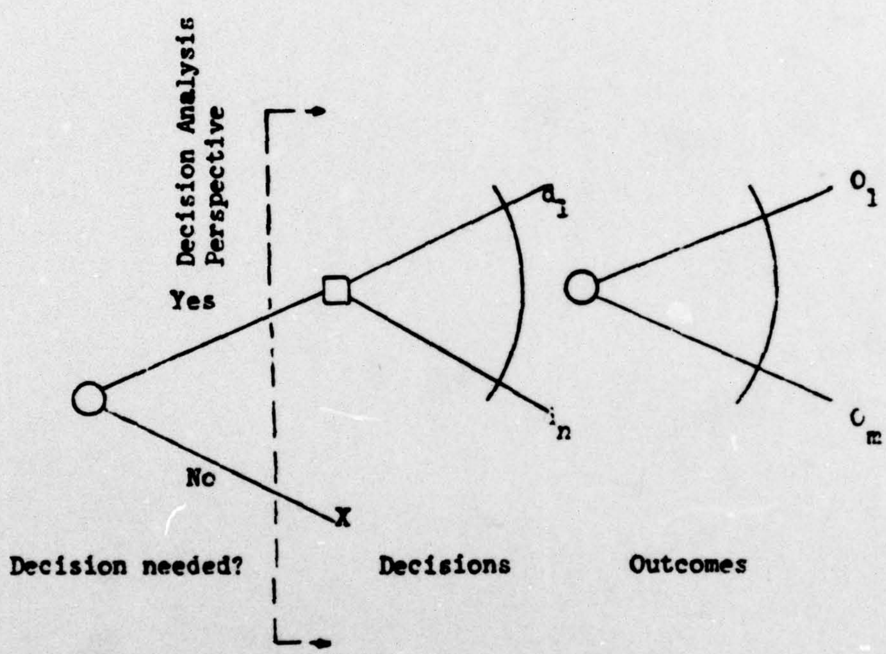


Figure 3-1 A decision-making model

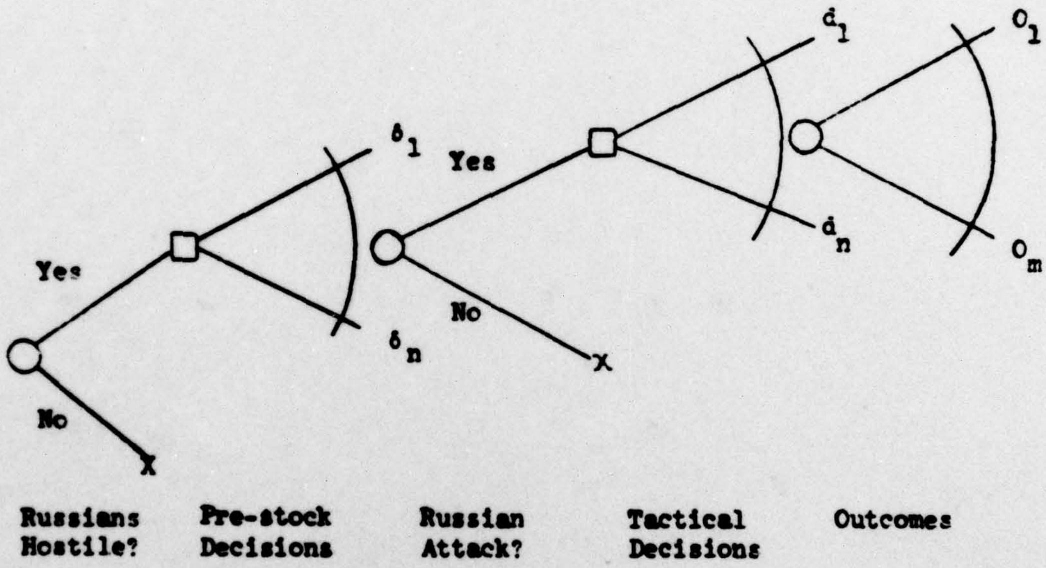


Figure 3-2   U.S. strategy in Europe

2. Contingency decision making. The decision maker partially or completely sets his decision vector before the need for a decision is certain.

Figures 3-3a and 3-3b illustrate the two concepts and suggest a fundamental hypothesis: The set of alternatives available to the contingency decision maker is at least as great if not greater than the set available to the reaction decision maker.

3.4 The Contingency Decision Model

Figure 3-2 does not completely tell the story of the European pre-stock strategy. As we noted in describing the example the United States periodically tested the plan, incurring some costs. In addition, the type and amount of pre-stocked supplies might vary depending on the U.S. state of information, and the final decision is obviously a function of this initial decision. These nuances are depicted in Fig. 3-4.

We will find it helpful in our subsequent analysis to characterize the event "Russian Attack" as a binary "outcome switch." In the "on" position the decision maker completes his decision, if necessary, and receives the reward from his lottery. In the "off" position the decision maker does not receive the outcome of his lottery but recycles to reconsider his pre-set decision.

The setting of the outcome switch may be affected by:

1. Competitive or Gaming Factors.

   Example: The deployment of U.S. troops is contingent on the exact timing of the Russian attack.

2. Environmental Factors.

   Example: The decision maker will buy a new car when his present one requires a new motor.

3. Factors within the Control of the Decision Maker.

   Example: The decision maker will buy a new car in 1976.

4. A Combination of Previous Factors.

   Example: The decision maker will buy a new car in 1976 unless his present one requires a new motor prior to that date.

There are also situations where the decision may be repetitive, and

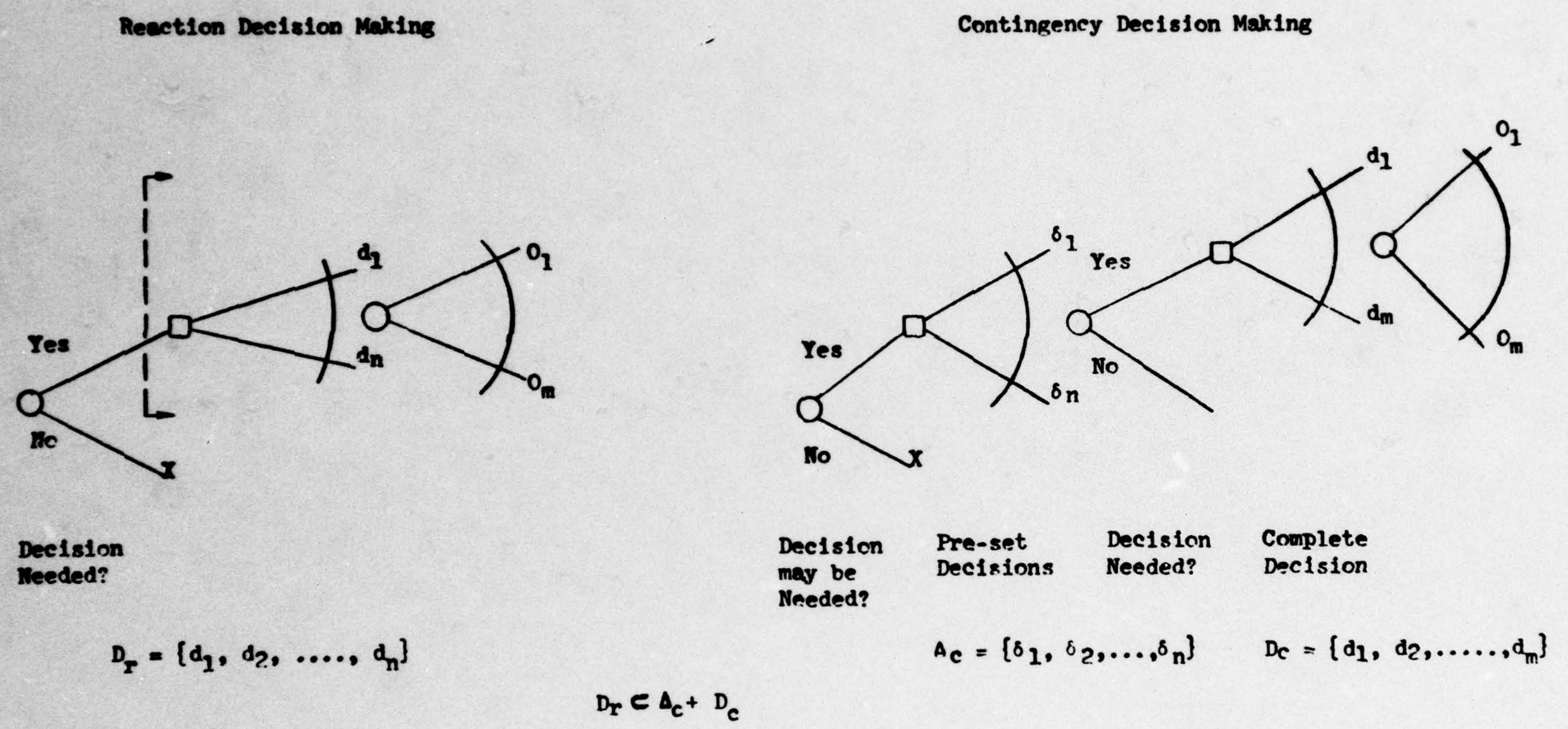


Figure 3-3 Two decision-making models

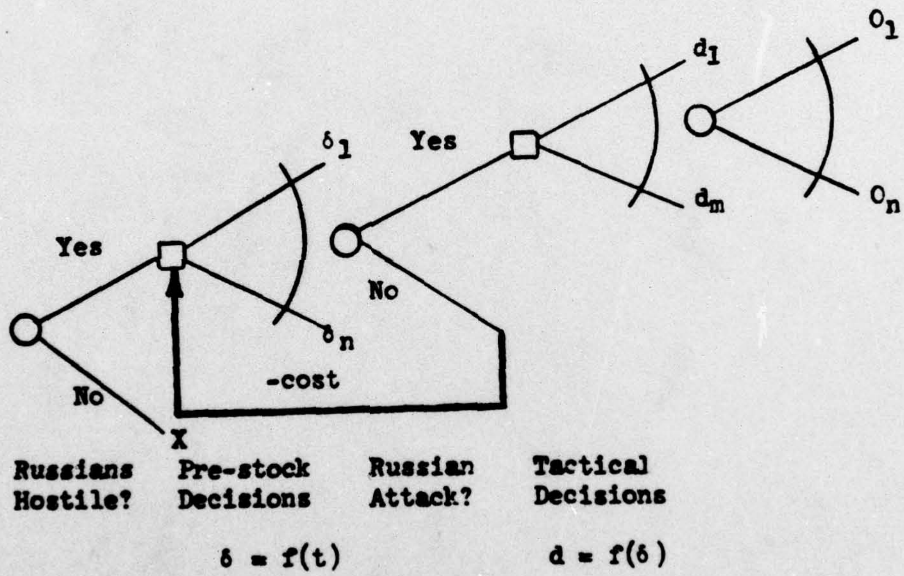


Figure 3-4 Altered decision model

the decision maker would recycle to reconsider his decision after receiving the results of an outcome lottery.

Figure 3-5 reflects these concepts and is the decision model that will be used throughout the remainder of this study.

3.5 Other Examples

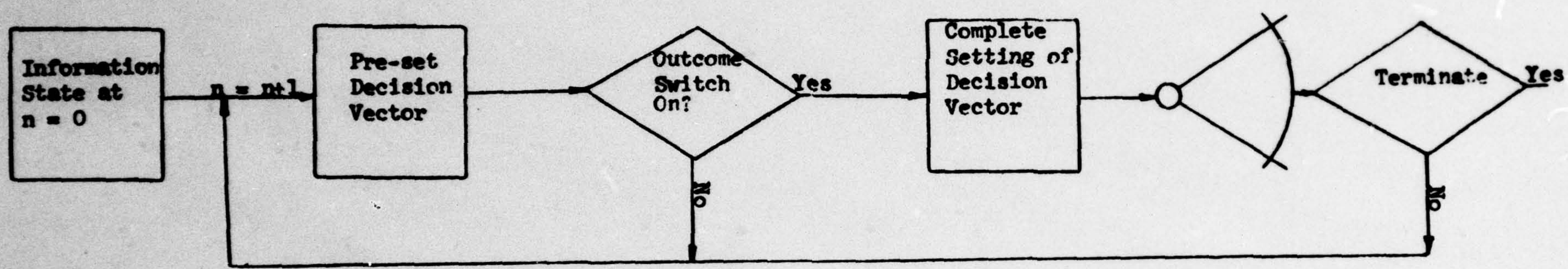
We have concentrated generally on one decision, the U.S. pre-stock of military equipment in Europe. However, other examples of contingency decision making abound. These would include:

1. Military and industrial "intelligence" collection decisions.
2. Put and call market operations.
3. Many R&D decisions.

While any broad generalization is dangerous a common thread is a desire to cut reaction time when a decision is needed. As a consequence, some initial preparation is accomplished before the final decision is taken as a certainty.

3.6 Summary

This chapter has described the decision model that will be used for subsequent analysis.



Examples:

- Simon's Gasoline Rationing Plan
- Pre-stock War Supplies in Europe
- Put and Call Operations
- "Intelligence" Operations

Figure 3-5 Contingency decision-making model

CHAPTER 4

THE CONTINGENCY DECISION MODEL AND INFORMATION DYNAMICS

4.1 Purpose

This chapter extends the results of Chapter 2 by a fuller examination of the contingency decision model.

4.2 Introduction

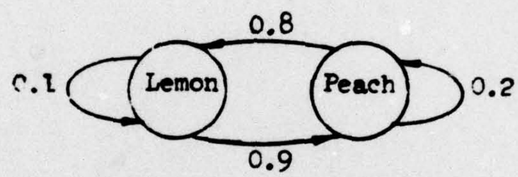
The basic example in Chapter 2 served the purpose of a suitable framework for the development of the basic concepts of information dynamics. However, the scenario--a series of M *a priori* decisions with no opportunity to benefit from the information gained from the intervening outcomes--may be considered a somewhat forced and contrived example of contingency decision making. A second example, more natural than the first, serves to amplify the previous discussions.

4.3 Example Two.

The decision maker is the owner of a model "A" automobile. His mechanic has recently told him that there is a 0.5 probability the car will fail two years hence and have to be replaced. However, if the car does not fail in the second year, then there is a 0.5 probability that it will fail in the fifth year. Peculiar to the model "A" is the fact that if it does not fail by the fifth year it will last forever. Peculiar to the decision maker is his unconcern for events beyond the tenth year.

The decision maker is an advocate of the model "A". Recently he has received the disquieting news that the company has in the past few years developed a pattern of production runs that results in one year's car being a "peach," but in many cases the following year's model is a "lemon." The company engineers feel the Markov model of Fig. 4-1 captures this pattern.

The decision maker also has the option of buying a model "B", a more reliable but more expensive car. After some consideration he has developed a reward structure as shown in Table 4-1.



$\mathcal{P}\{\text{Lemon at year M} \mid \text{Lemon at year M-1}, \epsilon\} = 0.1$
$\mathcal{P}\{\text{Peach at year M} \mid \text{Lemon at year M-1}, \epsilon\} = 0.9$
$\mathcal{P}\{\text{Peach at year M} \mid \text{Peach at year M-1}, \epsilon\} = 0.2$
$\mathcal{P}\{\text{Lemon at year M} \mid \text{Peach at year M-1}, \epsilon\} = 0.8$

Figure 4-1 Model of production runs

TABLE 4-1

Reward Structure, Example 2

	Buy Model "A"	Buy Model "B"
Model "A" a peach	+100	-100
Model "A" a lemon	-100	+100

What now is the value of information concerning the quality of this year's production of Model "A's?" We can recalculate $\nu^*(n)$ and $v^*(n)$ keeping in mind that

$$P\{\text{outcome switch "on" at } m=2 \mid \epsilon\} = 0.5$$

$$P\{\text{outcome switch "on" at } m=5 \mid \text{switch "off" at } m=2, \epsilon\} = 0.5$$

$$P\{\text{outcome switch "on" at } m=5 \mid \epsilon\} = (0.5)(0.5) = 0.25$$

The values of $\nu^*(n)$, $v^*(n)$, $\Delta(n)$, and $\rho(n)$ are plotted in Fig. 4-2.

Figure 3-4 suggests an extension to this example. Assume some cost, say 5 units, is incurred each transition if there is no receipt of the profit lottery. The cost, for instance, might be maintenance and operation of the Model "A". The values of $\nu^*(n)$, $v^*(n)$, $\Delta(n)$, and $\rho(n)$ are plotted in Fig. 4-3. Comparison of the last two figures shows that the alteration of the example changes $\nu^*(n)$ and $v^*(n)$ but not $\Delta(n)$ and $\rho(n)$. We also note in both figures that $\Delta(n)$ is non-increasing over the horizon of interest, but $\rho(n)$ behaves inconsistently with our previous results. What changes are needed to rationalize this behavior?

4.4 The "Outcome Switch"

We first must formalize the probabilistic nature of the occurrence of the decision.

Let

$\Gamma(m) = Y$ be the event that the "outcome switch" is "on" at transition m , i.e., the decision maker receives the reward from his profit lottery at transition m , and

$\Gamma(m) = N$ be the complementary event that the "outcome switch" is "off" at transition m , i.e., the decision maker does not receive his reward.

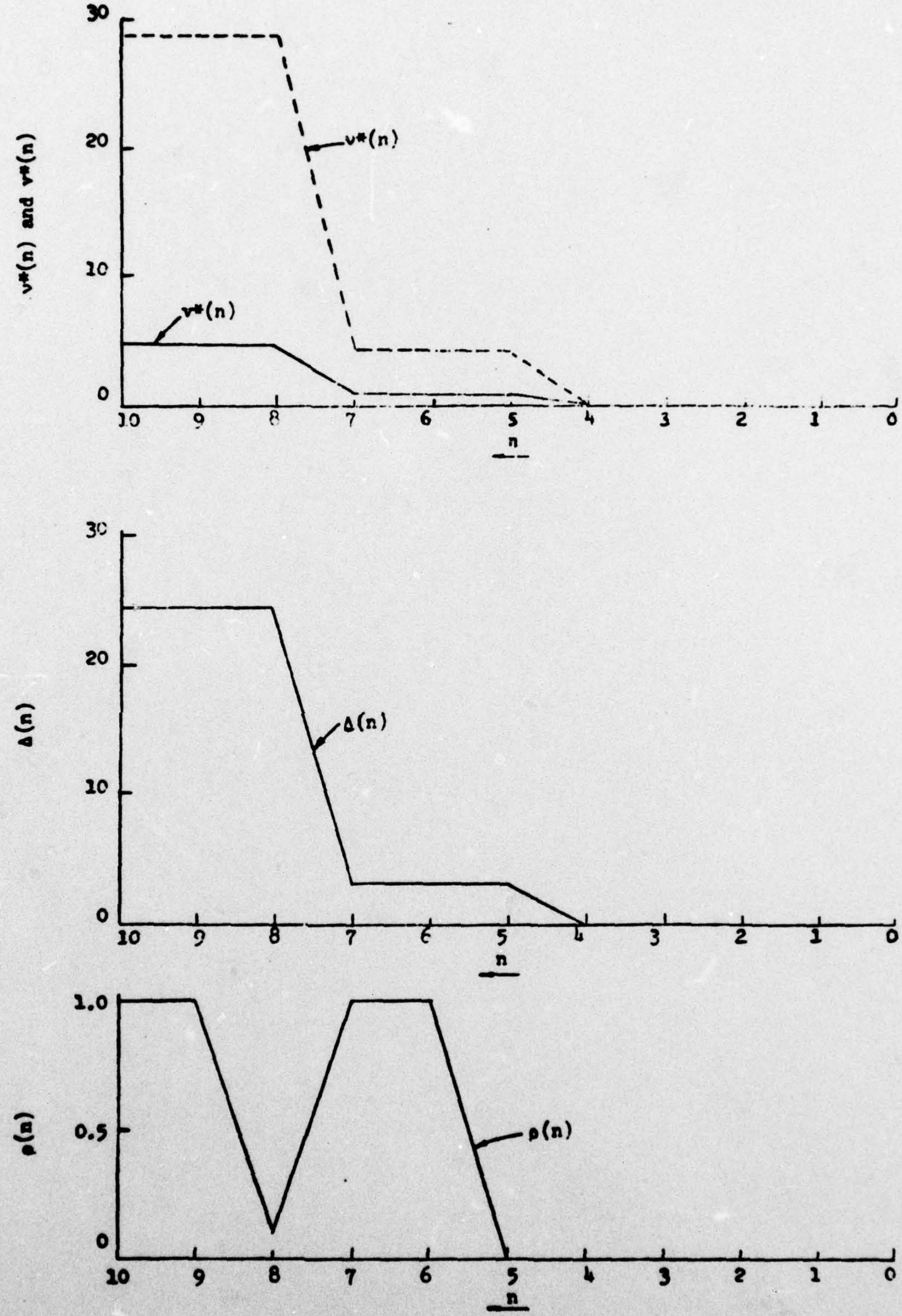


Figure 4-2 Example 2 variables

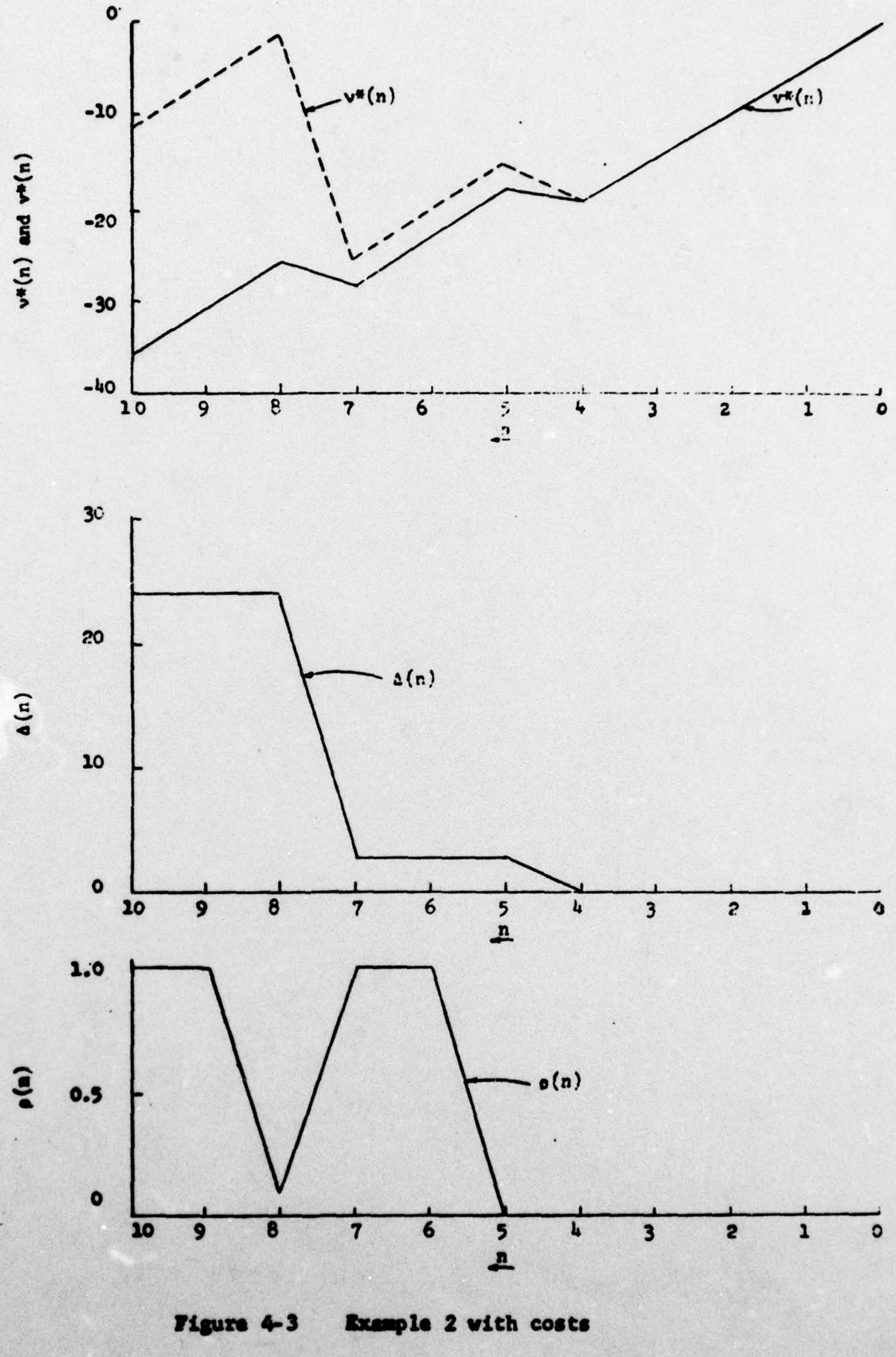


Figure 4-3 Example 2 with costs

Let

$$g(m) = \mathcal{P}\{\Gamma(m)=Y \mid \epsilon\} \tag{4.1}$$

$$1 - g(m) = \mathcal{P}\{\Gamma(m)=N \mid \epsilon\} \tag{4.2}$$

We have discussed in Chapter 3 (see pages 40-41) that the occurrence of the decision may be highly dependent on any number of previous events. Designate such events as $E(1)$, $E(2)$, $E(3)$, ..., $E(m-2)$, $E(m-1)$, $E(m)$. Then

$$g(m) = \mathcal{P}\{\Gamma(m)=Y \mid \epsilon\} = \mathcal{P}\{E(m)=Y \mid E(m-1),\ldots,E(1),\epsilon\}\ \mathcal{P}\{E(m-1) \mid E(m-2),\ldots, E(1),\epsilon\}\ \mathcal{P}\{E(m-2) \mid E(m-3),\ldots,E(1),\epsilon\} \ldots \mathcal{P}\{E(1) \mid \epsilon\} \tag{4.3}$$

We shall assume that the marginal probability, $g(m)$, is always available, either by direct assessment, modelling, computation or by some combination of these techniques.

We can associate a random variable, $X(m)$, with the process such that

$$\begin{aligned} X(m) &= 1 \quad \text{if} \quad \Gamma(m) = Y \\ X(m) &= 0 \quad \text{if} \quad \Gamma(m) = N \end{aligned} \tag{4.4}$$

At any transition m the probability mass function for $X(m)$ is described by Fig. 4-4. In particular, for the example we have just described, the mass functions for $X(m)$ are shown in Fig. 4-5.

We discern that receipt or non-receipt of the profit lottery in no way affects <u>a priori</u> cerebral consideration of the optimal strategy. Therefore, we redefine (2.5) as

$${}_c v^{(k)}(m) = \langle v(m) \mid \delta(m)=\delta^{(k)}, \Gamma(m)=Y, \epsilon \rangle \tag{4.5}$$

where the subscript "c" emphasizes the contingency decision making. Then the obvious relationships exist

$${}_c v^{(k)}(m) = v^{(k)}(m)\ g(m) \tag{4.6}$$

$${}_c v^{*}(m) = v^{*}(m)\ g(m) \tag{4.7}$$

$${}_c v^{*}(m) = {}_c v^{*}(m)\ g(m) \tag{4.8}$$

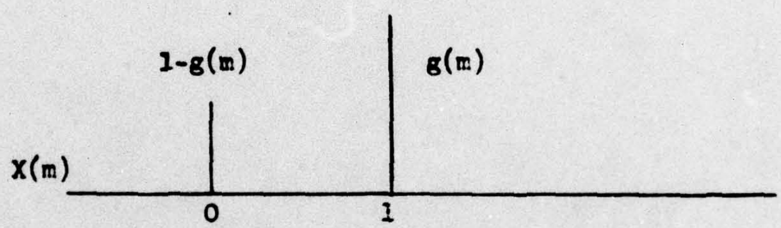


Figure 4-4 Probability mass function for X(m)

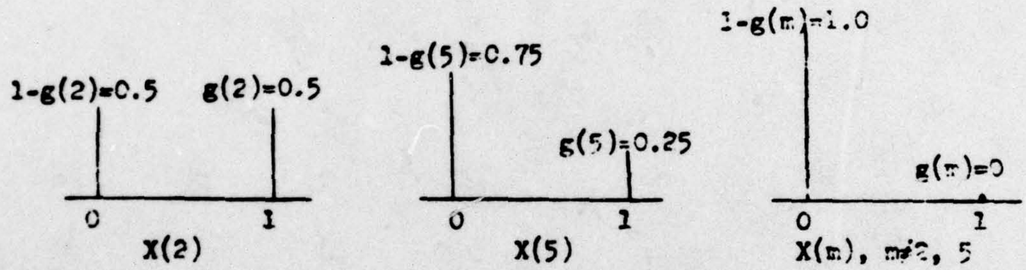


Figure 4-5 Representative probability mass functions, Example 2

4.5 Comparative Results from Contingency Decision Making

4.5.1 The Effect on $\Delta(n)$

The last results of Section 4.4 lead immediately to

Theorem 4.1

$$ {}_c\Delta(n) \le \Delta(n) $$

Proof

$$ \Delta(n) = \sum_{\ell=M-n}^{M} [v^*(\ell) - v^*(\ell)] \tag{4.9} $$

and

$$ {}_c\Delta(n) = \sum_{\ell=M-n}^{M} [v^*(\ell)\, g(\ell) - v^*(\ell)\, g(\ell)] \tag{4.10} $$

Thus,

$$ {}_c\Delta(n) = \sum_{\ell=M-n}^{M} g(\ell)[v^*(\ell) - v^*(\ell)] \le \Delta(n) $$

4.5.2 The Effect on $\rho(n)$

The effect of contingency decision making on $\rho(n)$ is not as obvious as the effect on $\Delta(n)$. In the example,

$$ \rho(n) \text{ is \underline{not} less than } |\lambda_1| , $$

and

$$ \rho(n) \text{ is \underline{not} greater than } \rho(n-1) \text{ for all } n , $$

both in contradiction to previous results.

In Example 1, $g(m) = 1.0$, $0 \le m \le 10$, and $g(m) = 0$, $m > 10$. Moreover, for $m \ge 8$, $v^*(m) - v^*(m) = 0$. In Example 2, $g(m) = 0.5$, for $m = 2$; $g(m) = 0.25$, $m = 5$; and $g(m) = 0$, otherwise. Obviously $g(m)$ is constant in Example 1 (at least for the transitions that cause a contribution to $\Delta(n)$). In Example 2, $g(m)$ is both increasing and decreasing. Consideration of these results leads to a restatement of Theorem 3.3, as

Theorem 4.2

$$ {}_c\rho(n) \le |\lambda_1| \text{ if } g(m) \text{ is non-increasing in } m . $$

Proof

There exists two constants α and β $(1 \geq \alpha\ ,\ \beta \geq 0\ ,\ \alpha > \beta)$, such that $\alpha^m \geq g(m)$ for all m , and $\beta^m \leq g(m)$ for all m . Then one may show that $\Delta(n,\alpha) \geq \Delta(n,\beta) \leq \Delta(n)$. Similar to the proof of Theorem 2.3 we may show that

$$\rho(n,\alpha) = \frac{\Delta(n-1,\alpha)}{\Delta(n,\alpha)} \leq |\lambda_1| \tag{4.11}$$

and

$$\rho(n,\beta) = \frac{\Delta(n-1,\beta)}{\Delta(n,\beta)} \leq |\lambda_1| \tag{4.12}$$

$\Delta(n)$ and $\Delta(n-1)$ are continuous in $g(m)$. Thus, it follows that $\rho(n)$ is also continuous in $g(m)$. From (4.11) and (4.12) we conclude that the theorem holds.

The converse of this statement does not hold in general as can be seen by alteration of Example 1. Suppose $g(0) = 0.5$, $g(1) = 1.0$, and $g(m) = 0$, $m \neq 0,1$. Then $\Delta(10) = 106.35$, and $\Delta(9) = 59.29$, or $\rho(10) = 0.56 \leq |0.7|$. This is true in spite of $g(0) \leq g(1) \geq g(2)$.

4.6 Two Peripheral Issues

4.6.1 Intermediary Information Acquisition

Suppose for Example 3 we alter Example 2 as follows: The decision maker, following any year he purchases an "A" model car, has the probabilities 0.5 and 0.25 of a required successive purchase in the succeeding two and five years. To be specific--if the decision maker buys an "A" model in year 2 then he has a 0.5 chance of requiring a successive purchase in year 4 and a 0.25 chance in year 7. We also assume he receives perfect information if the purchases the car. In other words, he knows at (or immediately after) the time of purchase if the current year's model is a "peach" or a "lemon." He can, of course, use this data in succeeding years. What now is the value of information acquired at $m = 0$, and how do $\Delta(n)$ and $\rho(n)$ change?

Figure 4-6 depicts this behavior. The important consideration is the fact that receipt of a second input of perfect information destroys any residual incremental value from the first acquisition. This will be

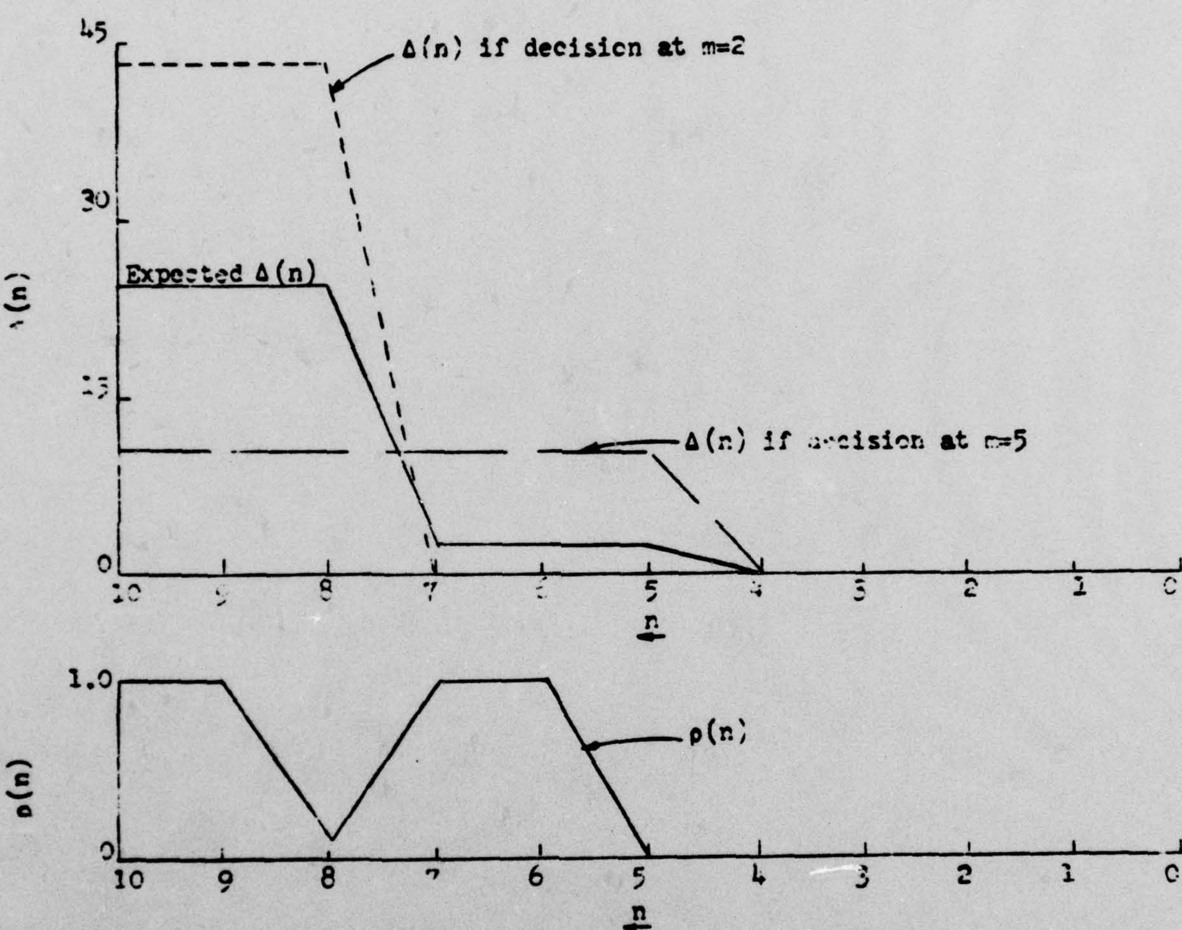


Figure 4-6 Information acquisition

of key concern in the optimal strategy of information acquisition in the following chapter.

4.6.2 Discounted Rewards

What is the effect of discounting on $\Delta(n)$ and $\rho(n)$? Assume in Example 2 that the stream of rewards is discounted by some factor β . In other words, the present value of the expected reward at the m^{th} transition is

$$_{\beta}v^*(m) = \beta^m v^*(m) \ , \quad \beta \le 1.0 \tag{4.13}$$

If, in particular, we assume that for Example 2 $\beta = 0.9$, then the appropriate values for $_{\beta}\Delta(n)$ and $_{\beta}\rho(n)$ are plotted in Figs. 4-7a and 4-7b.

Examination of Fig. 4-7a reveals that $\Delta(n)$ is increasing in violation of Theorem 2.2. However, this illusory appreciation of information is a violation, not of the theorem, but of the assumptions of the theorem. The development in Chapter 2 was premised on a constant reward structure. Once discounting is introduced there is no longer a constant reward over time. Therefore, there is no guarantee that the results of Chapter 2 would remain valid.

4.7 Summary

This chapter has developed the foundations for the contingency decision model and the effects that contingency decision making has on the basic results of Chapter 2. The important result that information perishes within the context of the decision model leads logically to the next chapter, a consideration of the strategy of information replenishment.

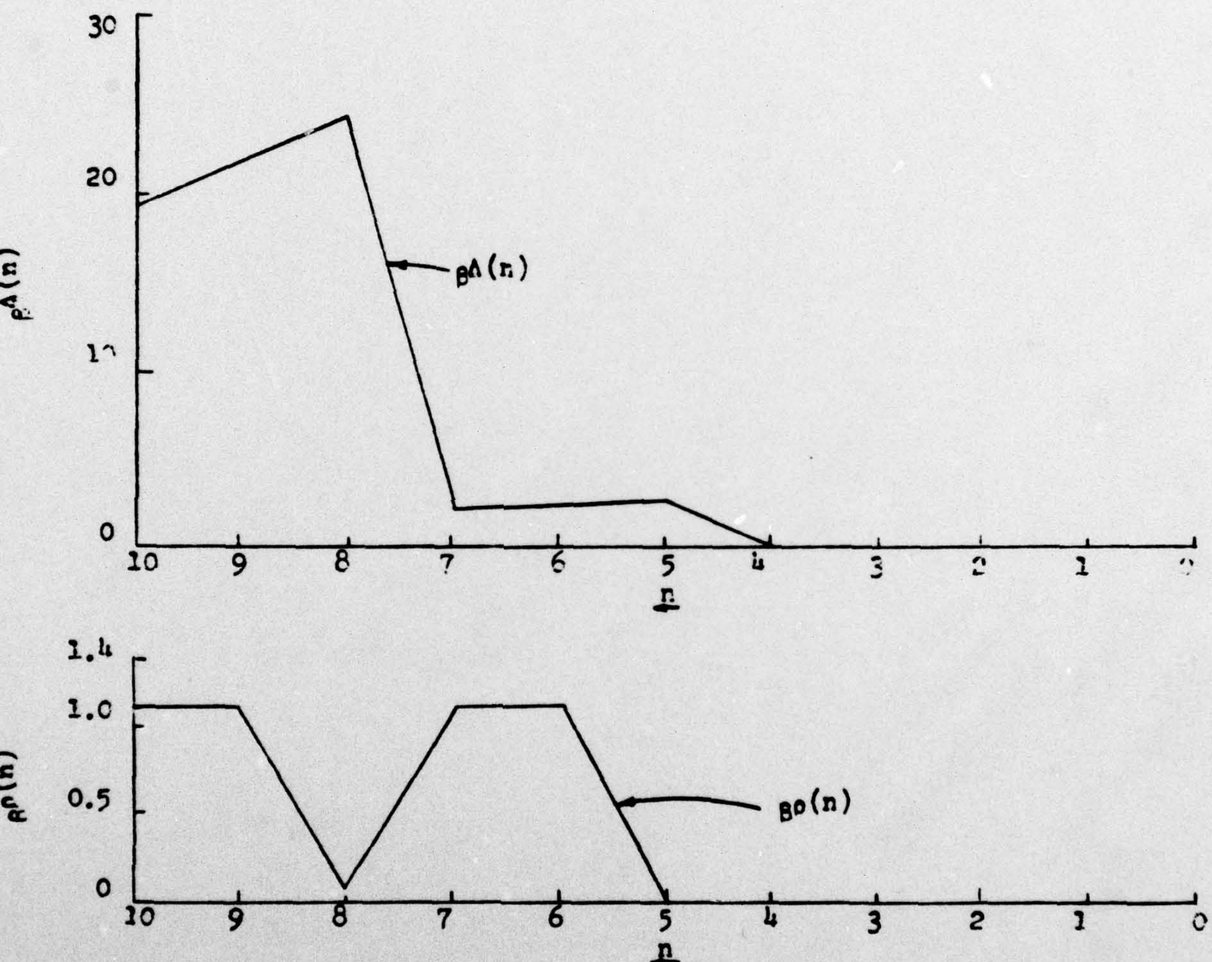


Figure 4-7 Discounted example

CHAPTER 5
INFORMATION REPLENISHMENT

5.1 Introduction

This chapter, in a sense, begins the obverse of the previous results. Information perishes over time. What strategy of information replenishment reverses this perishing?

In the static case the question of information replenishment is straightforward. If the expected value of information is greater than the cost of gaining the information, then the prudent strategy is to acquire the information. However, the issue is far more complex in the contingency decision model. The decision maker faces this paradox. If he acquires information at $m = 0$, the information may have perished by the time the decision actually occurs. Conversely, he may delay his information gathering and be caught short, having to make his decision on a less than complete state of information. We may call this process speculative information acquisition as it has many of the characteristics that futures speculation has in any commodity market.

5.2 Two Examples

5.2.1 A Simple Case--Example 4

We illustrate with a trivial but useful example. Let the state of information be characterized by the two-state Markov model that we have previously used (see Fig. 2-1, page 7).

The alternatives and reward matrix are those used in Example 1 (see pages 5-6). We further assume that the decision occurs with probability 1.0 at $m = 5$ and $m = 8$ and with probability 0 otherwise. Utilizing the notation of Chapter 4 we have

$$\begin{aligned} g(5) &= g(8) = 1.0 \\ g(m) &= 0.0 \ , \quad m \neq 5,8 \end{aligned} \qquad (5.1)$$

The decision maker has a planning horizon of eight transitions.

The decision/information acquisition model is shown in Fig. 5-1. We should particularly note from the model that information acquired at the m^{th} transition is not available for a decision at the m^{th} transition

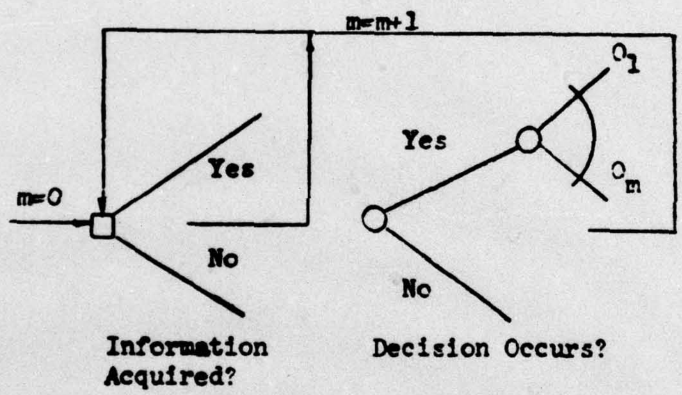


Figure 5-1 Decision/information acquisition model

but is available, at the earliest, at the $m+1^{th}$ transition. At this time it has perished for one unit of time.

To return to the example we may establish a base case that would consist of an expected reward of +5.88 at $m = 5$ (based on prior information only) and an expected reward of 33.83 at $m = 8$ (based on acquisition of perfect information as a result of observing the outcome of the decision at $m = 5$). (These expected rewards were established in Chapter 2.)

It is transparently clear that although the combination and permutations of schemes of data acquisition are almost unlimited that (assuming the cost of information gathering is constant over time) only four merit consideration:

$\delta^{(0)}$--No acquisition;

$\delta^{(1)}$--Acquire perfect information at $m = 4$ only;

$\delta^{(2)}$--Acquire perfect information at $m = 7$ only;

$\delta^{(3)}$--Acquire perfect information at $m = 4$ and $m = 7$.

Table 5-1 summarizes the results for each alternative.

TABLE 5-1

Summary of Expected Rewards

Alternative (1)	Expected Reward at m=5 (2)	Expected Reward at m=8 (3)	Total Expected Reward at m=0 (4)	Incremental Value* at m=0 (5)
$\delta^{(0)}$	5.88	33.84	39.72	Not Appl.
$\delta^{(1)}$	69.41	33.84	103.25	63.53-C
$\delta^{(2)}$	5.88	69.41	75.29	35.57-C
$\delta^{(3)}$	69.41	69.41	138.82	99.1-2C

*C is the cost of one data acquisition.

Figure 5-2 is a plot of Column (5) of Table 5-1.

As is obvious from the figure, $\delta^{(2)}$ is dominated at all costs, C. However, the choice among the other alternatives is a function of the

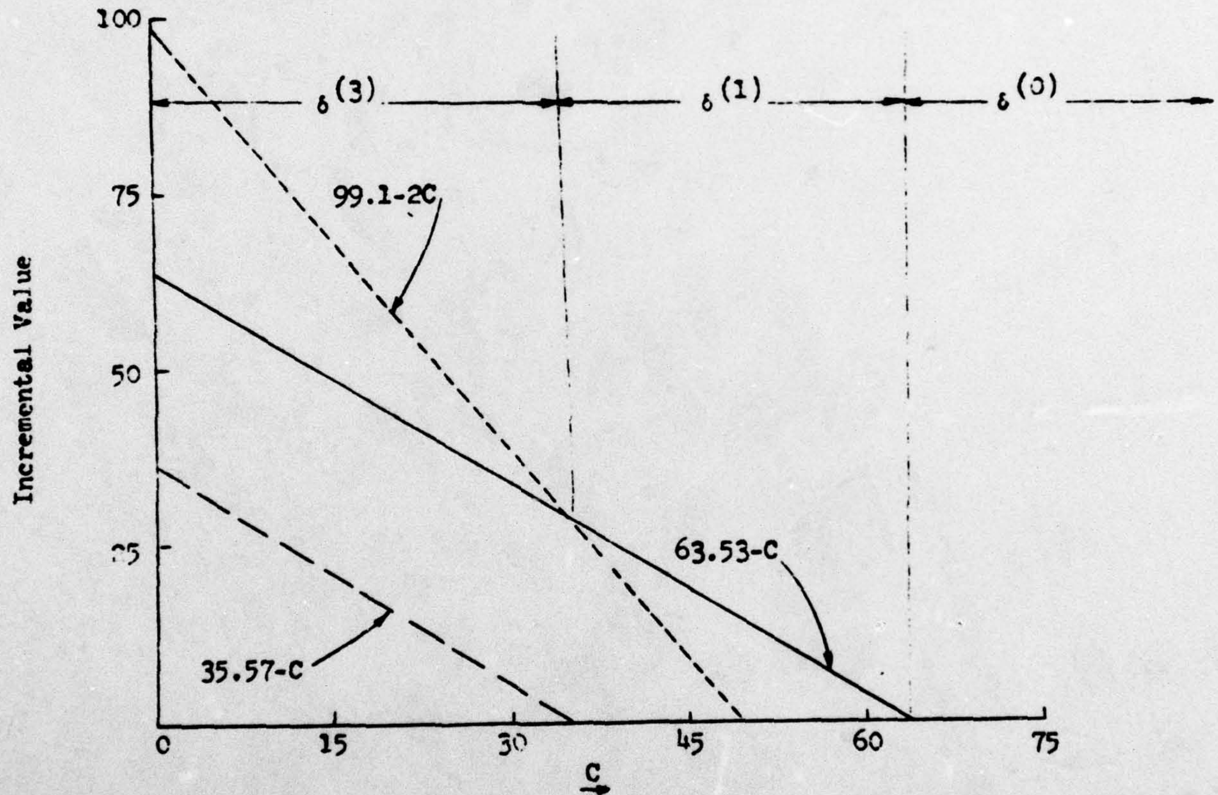


Figure 5-2 Preferred alternatives for Example 4

cost of the information gathering program. At low costs the extensive program is preferred, at intermediate costs the restricted program is dominant, while for costs over 63.53 the best strategy is to forego information acquisition.

In some situations control and planning of such an aperiodic acquisition program would be difficult. A periodic gathering of information, while perhaps less optimal from a strict economic viewpoint, might be far easier to monitor and implement. We shall designate the spacing of the periodic replenishment by the parameter "s". As an example, $s = 8$ signifies acquisition of perfect information at transitions 0, 8, 16, 24, ... We would particularly note that in Example 4 acquisition of perfect information at $m = 5$ and $m = 8$ is redundant and, hence, worthless, as the decision maker will receive perfect information as a result of observing the outcome of his decision at these transitions. Figure 5-3 depicts in a qualitative sense the process of periodic information replenishment.

Table 5-2 summarizes the results of periodic programs applied to Example 4.

TABLE 5-2

Periodic Replenishments, Example 4

S (1)	Total Expected Reward (net) (2)	Number of Acquisitions (3)	Notes (4)
8	0.00	0	Equivalent to Base Case
7	35.57	1	
6	15.34	1	
5	0.00	1	No value for information at 5
4	63.53	1	No acquisition at 8
3	58.64	2	
2	78.87	3	No acquisition at 8
1	99.10	7	

Figure 5-4, the parallel to Fig. 5-2, assists in visualizing the dominant alternatives. Examination of the figure reveals a pattern similar to that of the previous example.

We now turn to a bit more subtle example.

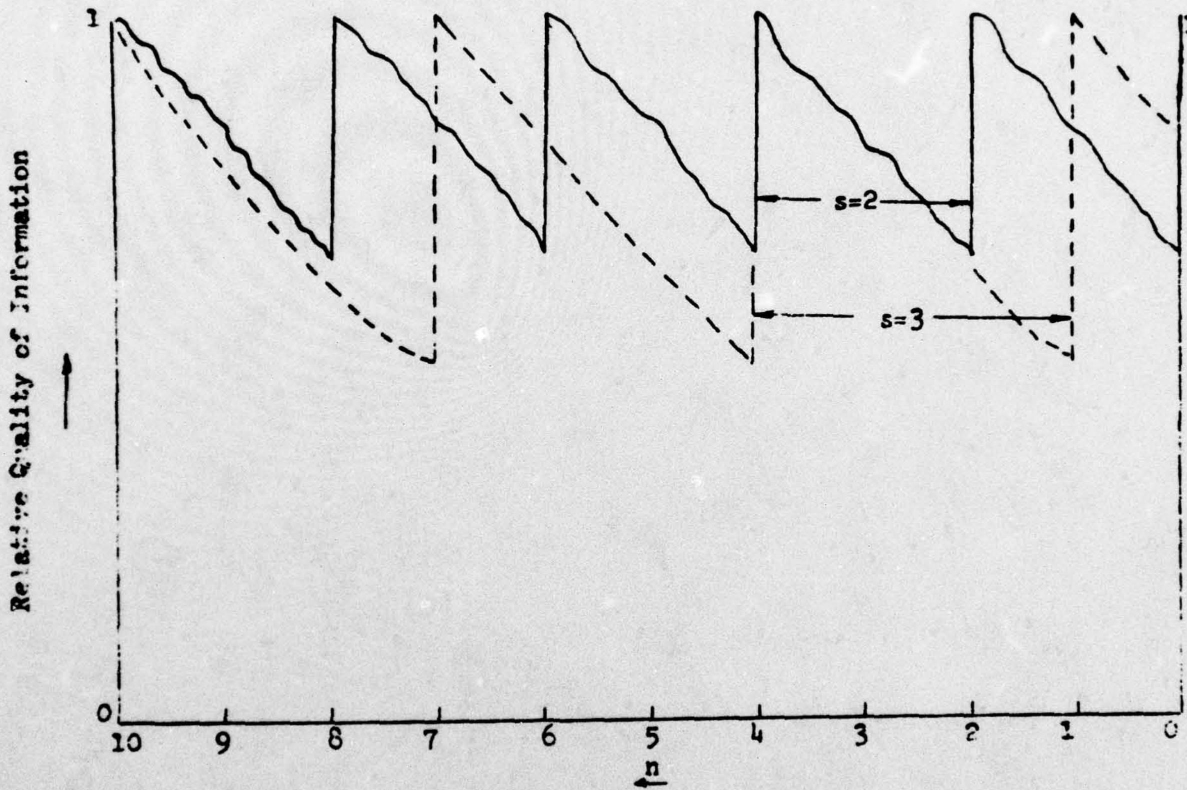


Figure 5-3   Quality of information for two periodic acquisitions

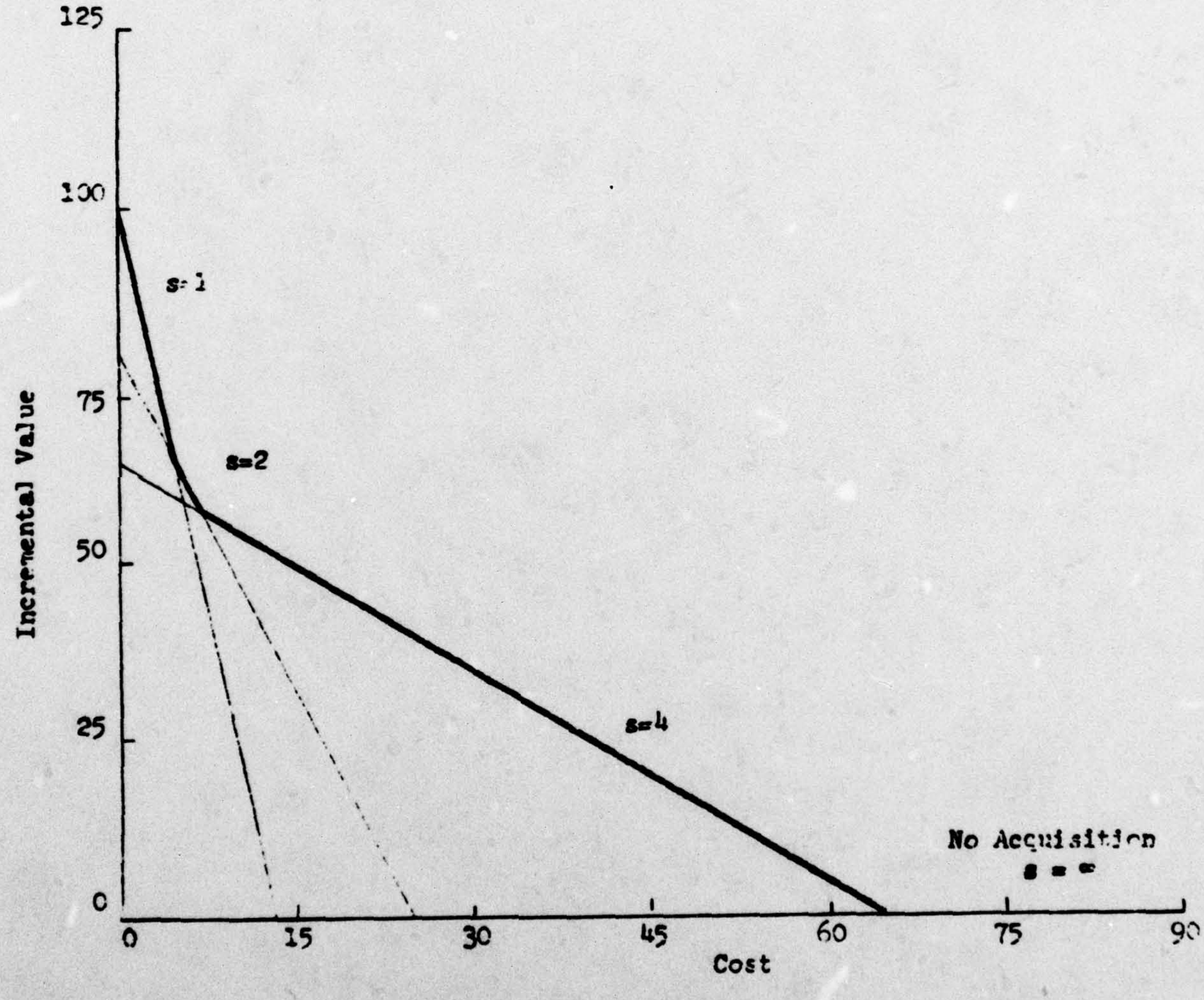


Figure 5-4 Dominant alternatives, periodic acquisition, Example 4

5.2.2 Geometrically Distributed Decision Occurrences

We alter the scenario as follows: Assume the information model, the alternatives, and the rewards remain unchanged. However, $g(m)$ is now a geometric distribution. (The decision maker might elect this model for the decision occurrence because of a feeling that it adequately represents his state of information. In addition, the distribution has some "bench-mark" properties which will be useful in the further development.)

In particular, let

$$\begin{aligned} g(m) &= (0.2)(0.8)^{m-1}, \quad 1 \le m \le 20 \\ &= 0, \quad \text{otherwise} \end{aligned} \tag{5.2}$$

The distribution is represented by Fig. 5-5, and we note that the decision occurs but once.

We limit the replenishment to periodic acquisitions and further assume that the horizon is an integer multiple of the replenishment period. In other words, for the example we are developing acquisition can occur only at periods of 1, 2, 4, 5, and 10 transitions. One can gather information fruitfully only through the nineteenth transition as no decision will occur after $m = 20$.

The situation unfolds as follows: The decision maker may elect some period of information acquisition, say every ℓ^{th} transition. He pays a cost C to acquire information prior to the first transition which enables him to set his decision vector for transitions $1, 2, \ldots, \ell$. He can calculate the expected value of this information. At transition ℓ he again acquires information if the decision has not occurred prior to ℓ. For the geometric distribution in the example the probability is $1 - (0.8)^{\ell}$ that the second acquisition will occur. If the decision maker acquires information the second time, he uses this update to set his decision vector for transitions $\ell+1, \ell+2, \ldots, 2\ell$. The process is repeated through the horizon, M.

We illustrate the numerical approach by creation of Table 5-3. Column (5) of the table is the total expected profit from the information acquired. The net profit, that is the total less the expected cost, we designate by $V_T(n, s=\ell)$ where n is the transitions to go, and $s = \ell$

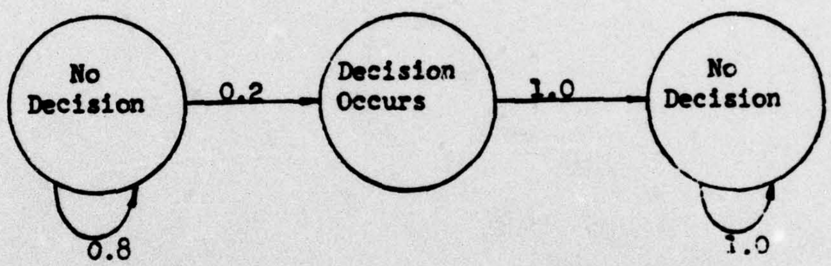


Figure 5-5   Decision occurrence model

TABLE 5-3

Expected Rewards, Geometric Distribution

If Decision Occurs at m = (1)	The Reward is $\nu^*(m)$ (2)	Probability Decision Occurs $g(m)$ (3)	Expected Reward $_c\nu^*(m)$ (4)	Cumulative Reward $\Sigma_c \nu^*(m)$ (5)
1	69.41	0.200	13.88	13.88
2	49.18	0.160	7.87	21.75
3	33.84	0.128	4.33	26.08
4	24.16	0.102	2.46	28.54
5	16.29	0.082	1.33	29.87
6	12.07	0.066	0.79	30.66
7	7.86	0.052	0.41	31.07
8	5.99	0.042	0.25	31.32
9	5.88	0.033	0.20	31.52
10	5.88	0.027	0.16	31.68

implies acquisition of information each ℓ^{th} transition. We may express V_T, as an example, as

$$V_T(20, s=10) = -C + \Sigma[\nu^*(j) - v^*(j)] \Pr\{\text{decision occurs at } j|\epsilon\}$$

$$-(1 - 0.8^{10})C + \Sigma[\nu^*(j) - v^*(j)] \Pr\{\text{decision occurs at } j+10|\epsilon\}$$

$$= -C + \sum_{j=1}^{10} [\nu^*(j) - v^*(j)]g(j)$$

$$+\bar{G}(10)\left\{-C + \sum_{j=1}^{10} [\nu^*(j) - v^*(j)]g(j)\right\} \tag{5.3}$$

We may develop similar expressions for $s = 1, 2, 4,$ and 5. These are plotted in Fig. 5-6.

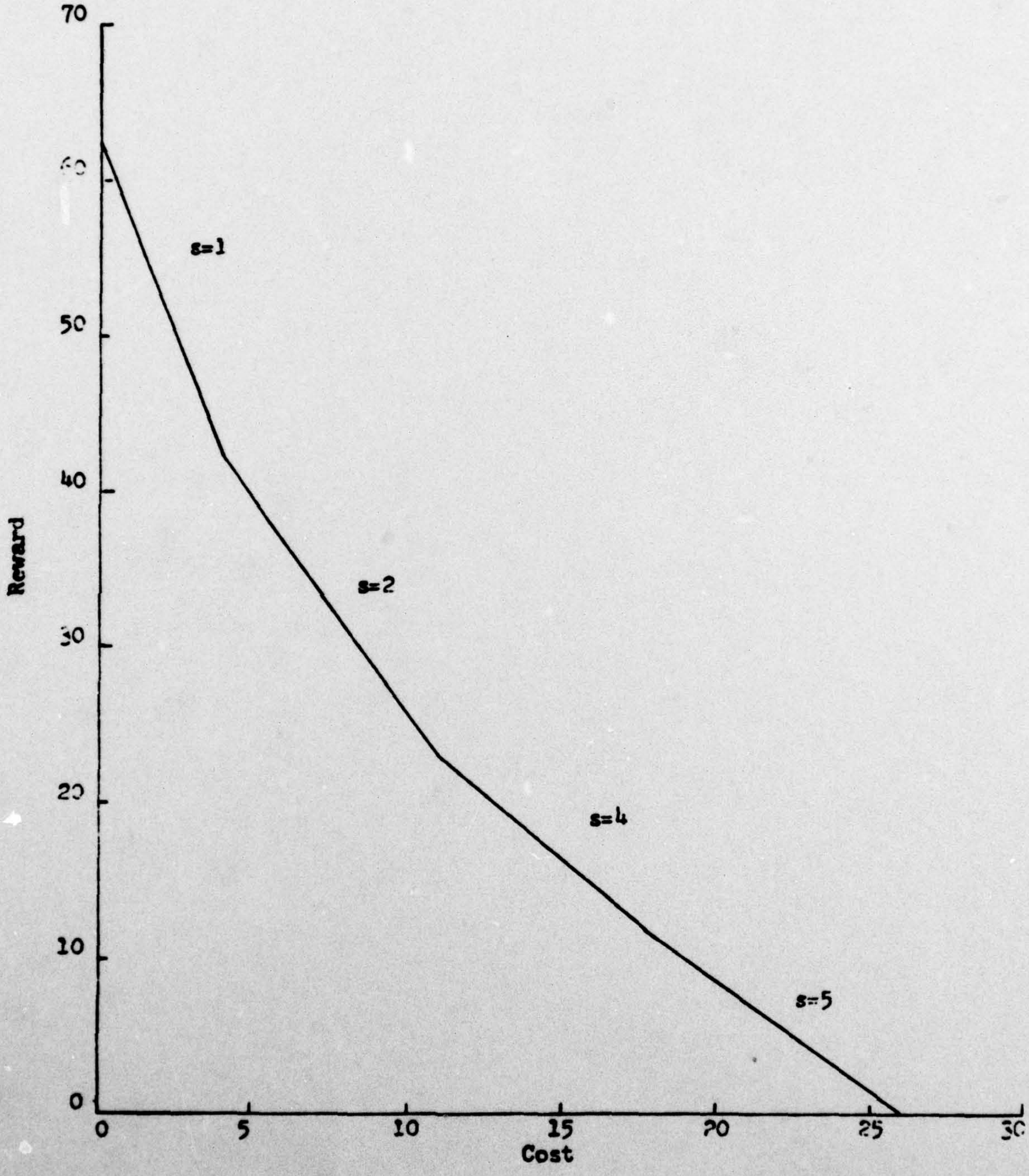


Figure 5-6 Net expected reward as a function of cost

5.3 Rules of Optimality

We shall now develop some rules of optimality with proof of each. The following statement of the problem applies to all rules. The horizon is M transitions. The periodic acquisition of information occurs at intervals k , ℓ , etc., such that $Kk = L\ell = M$. The cost of one acquisition is C , and the cost is linear, i.e., two acquisitions imply $2C$. The decision occurs but once and is governed by

$$g(m) = f^m(1-f) \ , \quad 1 \le m \le M \tag{5.4}$$

Rule 1

$$V_T(M,s=\ell) = \frac{1 - f^M[\alpha(\ell) - C]}{1 - f^\ell} \tag{5.5}$$

where

$$\alpha(\ell) = \sum_{j=1}^{\ell} [v^*(j) - v^*(j)] \ g(j) \tag{5.6}$$

Proof

The expression follows from recognition of three results. The first is that

$$1 + f^\ell + f^{2\ell} + \ldots + f^{(L-1)\ell} = \frac{1 - f^{L\ell}}{1 - f^\ell}$$

The second immediately follows as $L\ell = M$ by assumption. The third result comes from the "memoryless" property of the geometric distribution. Assume, for example, the period of acquisition is ℓ , and we are interested in the k^{th} acquisition. The expected value is then

$$-\bar{G}(k\ell)C + \sum_j [v(j) - v(j)] \ g(k\ell+j) \tag{5.7}$$

However, we may also write (5.7) as

$$-\bar{G}(k\ell)C + \bar{G}(k\ell)\left\{\sum_j [v^*(j) - v^*(j)] \ g(j)\right\} \tag{5.8}$$

This expression holds for $k = 0, 1, 2, \ldots$.

Rule 2

Let $V_T(M,s=\ell^*)$ represent the optimal acquisition policy for some value of C . The graph (see Fig. 5-6) is piecewise linear in C .

Proof: Immediate.

Rule 3

No acquisition policy is dominant for all values of C .

Proof

For dominance some policy, say acquisition at each ℓ^{th} transition, must yield a value of V_T that intersects the V_T axis at a value greater than any other policy, i.e., if $C = 0$,

$$V_T(M,s=\ell) \geq V_T(M,s=k) \;,\; k \neq \ell$$

Also the policy must intersect the C axis at a value greater than any other policy (see Fig. 5-7). For $C = 0$, V_T is obviously maximized for $\ell = 1$ and decreases in ℓ . On the cost axis the intersection occurs at $\alpha(\ell) - C = 0$. By the definition of $\alpha(\ell)$, (5.6), we know that $\alpha(\ell)$ is an increasing function of ℓ . Therefore, no policy is dominant.

Rule 4

Assume $\alpha(\ell) - C_o \geq 0$ for some ℓ . Then the optimal policy is determined by

$$\frac{\alpha(\ell^*) - C_o}{1 - f^{\ell^*}} \geq \frac{\alpha(\ell) - C_o}{1 - f^{\ell}} \;,\; \ell \neq \ell^* \qquad (5.9)$$

If $\alpha(\ell) - C_o \leq 0$ for all ℓ , then $\ell^* = \infty$ (no acquisition).

Proof

This follows directly from Rule 1.

Rule 4 furnishes a fairly tractable determination of the optimal policy for some particular C_o . For instance in the example we might let $C_o = 10$. Then we could construct Table 5-4, confirming what was graphically depicted in Fig. 5-6.

TABLE 5-4

Optimal Policy for the Example Problem (f = 0.8)

$s = 1$	$\alpha(\ell)$	$\frac{\alpha(\ell) - C_o}{1 - f^{\ell}}$
1	12.70	13.50
2	19.63	26.75 Optimal
4	25.09	25.56
5	25.95	23.72
10	26.47	18.45

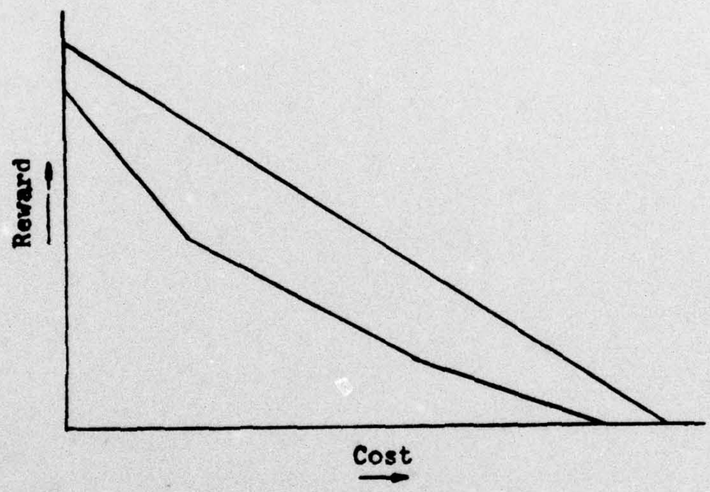


Figure 5-7 A dominant policy

Rule 5

Assume the value of the information perishes by the m^{th} transition counting forward; i.e., $v^*(j) - v^*(j) = 0$, $j \geq m$. Then $\ell^* \leq m$ for all C_o .

Proof

Rule 4 requires

$$\frac{\alpha(\ell^*) - C_o}{1 - f^{\ell^*}} \geq \frac{\alpha(m) - C_o}{1 - f^m} , \quad m \neq \ell^* \qquad (5.10)$$

Assume $\ell^* \geq m$ in contradiction to Rule 5. By the statement of the problem $\alpha(\ell^*) = \alpha(m)$. Therefore, inequality 5.10 reduces to

$$\frac{1}{1 - f^{\ell^*}} \geq \frac{1}{1 - f^m} \qquad (5.11)$$

or

$$f^{\ell^*} \geq f^m \qquad (5.12)$$

which is false for $\ell^* \geq m$. Therefore, ℓ^* must be less than m , which completes the proof.

These rules will be useful in analyzing other distributions.

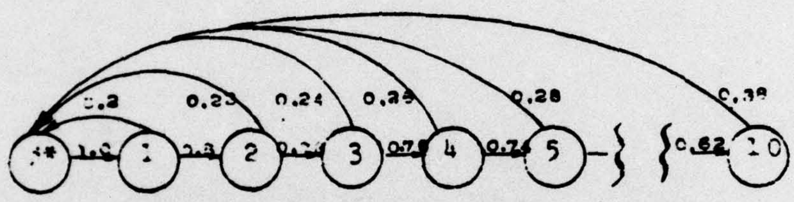
5.4 Other Markovian Distributions

5.4.1 Increasing (Decreasing) Decision Occurrence Rates

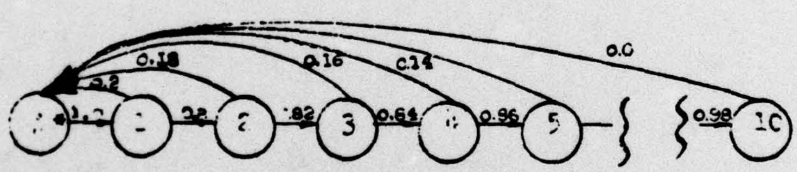
We adapt a concept from Wagner [23] to define an increasing (decreasing) decision occurrence rate. We shall use a special breed of Markov chain where one state which we shall designate j^* is the state, "Decision occurs," and the other states, $i = 0,1,\ldots$, are directly associated with transitions that have no decision occurrence. We let

$$r(i) = p_{ij^*} \qquad (5.13)$$

and define an increasing (decreasing) occurrence rate if $r(i)$ is increasing (decreasing) in i . Roughly this translates as the decision maker feels that as time passes and given that the decision has not occurred the probability of the decision occurring increases (decreases). The distributions of Figs. 5-8a and 5-8b depict increasing and decreasing



(a) Increasing occurrence rate



(b) Decreasing occurrence rate

Fogure 5-8 Increasing and decreasing occurrence rates

occurrence rates, respectively. Figure 5-9 is a graph of V_T for the three cases: increasing, constant, and decreasing occurrence rates.

5.4.2 Optimal Acquisition Policies

We note in Fig. 5-9 that for the increasing (decreasing) occurrence rate the optimal acquisition period for a given C_o is less than or equal (greater than or equal) the period for the "constant" occurrence rate. We now show this is a general result.

Theorem 5.1

Let $\{p_{i,j*}\}$ define a Markov chain as defined in the previous paragraph. If $p_{i,j*}$ is increasing (decreasing) in i, then for a particular cost of information acquisition C_o, the optimal periodic acquisition occurs at a period s where s is less than or equal (greater than or equal) that s determined by Rule 4.

Proof

The proof proceeds by induction on s, the acquisition period. The proof will be for the increasing case as the decreasing case is then obvious. The proof is based on determining the values of C_o for which the decision maker is indifferent between acquisition at each transition and at every second transition. We then show for the increasing case that this C_o is larger than that for the constant occurrence case. This implies that the optimal acquisition period is less. The intersection for the "constant" case is determined from

$$V_T(M,s=1) = V_T(M,s=2) \tag{5.14}$$

Using (5.3) we express the equality as

$$\begin{aligned}&-C_o[\bar{G}(0)+\bar{G}(1)+\ldots+\bar{G}(M-1)] + [g(1)+g(2)+\ldots+g(M)]\,[v^*(1)-v^*(1)] = \\ &-C_o[\bar{G}(0)+\bar{G}(2)+\ldots+\bar{G}(M-2)] + [g(1)+g(3)+\ldots+g(M-1)]\,[v^*(1)-v^*(1)] \\ &\qquad + [g(2)+g(4)+\ldots+g(M)]\,[v^*(2)-v^*(2)]\end{aligned} \tag{5.15}$$

Let the equating value of C_o in (5.15) be $C_o(1)$ and solving for this quantity yields

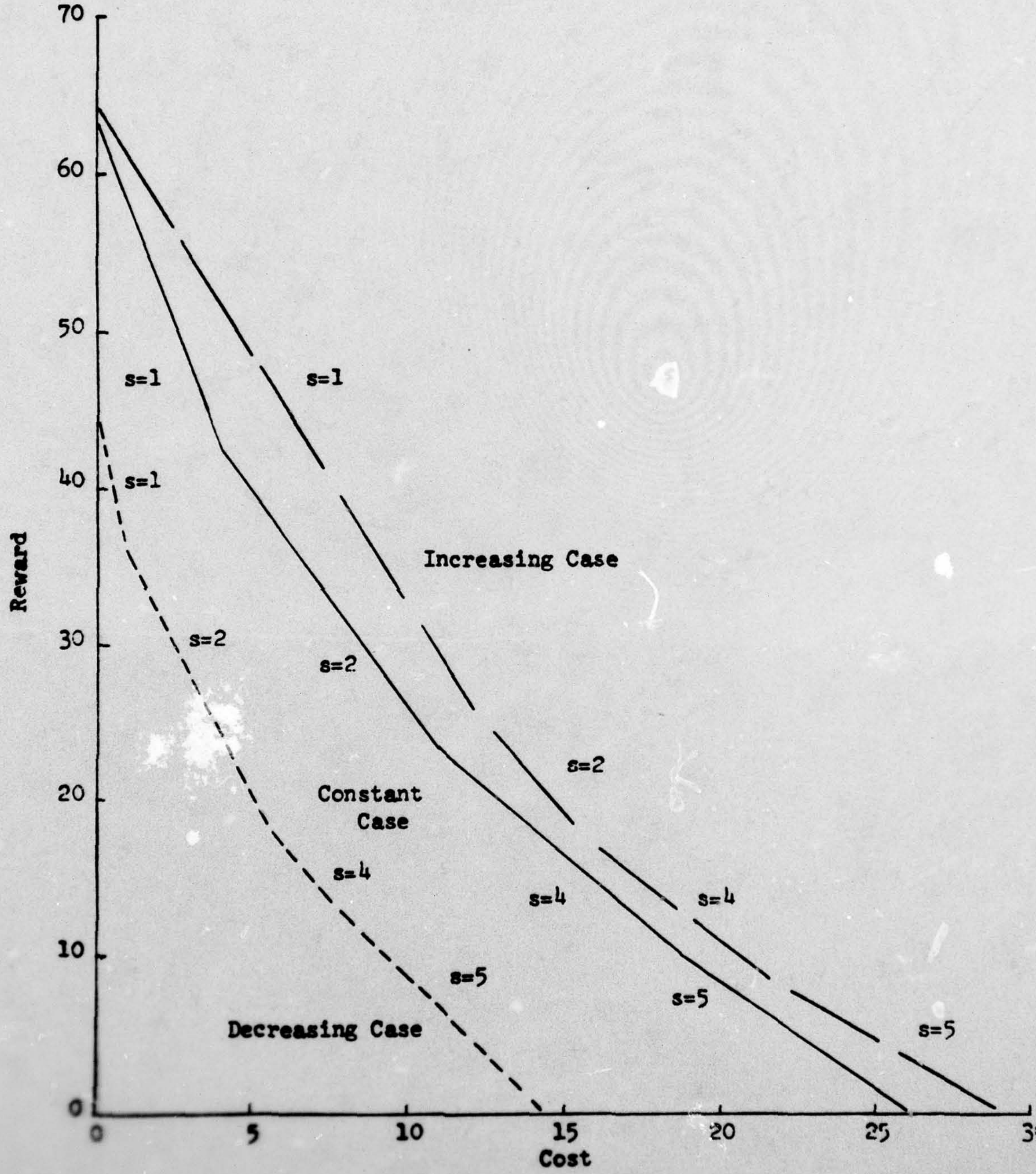


Figure 5-9 Rewards as a function of cost for increasing, constant, and decreasing occurrence rates

$$C_o(1) = \frac{g(2) + g(4) + \ldots + g(M)}{\overline{G}(1) + \overline{G}(3) + \ldots + \overline{G}(M-1)} [\nu^*(1) - \nu^*(2)] \quad (5.16)$$

The numerator of (5.16) is equivalently expressed as

$$\overline{G}(1)p_{1,j^*} + \overline{G}(3)p_{3,j^*} + \ldots + \overline{G}(M-1)p_{M-1,j^*} \quad (5.17)$$

Thus, for (5.16) we may write

$$C_o(1) = \frac{\overline{G}(1)p_{1,j^*} + \overline{G}(3)p_{3,j^*} + \ldots + \overline{G}(M-1)p_{M-1,j^*}}{\overline{G}(1) + \overline{G}(3) + \ldots + \overline{G}(M-1)} [\nu^*(1)-\nu^*(2)]$$

(5.18)

Let $C_o^+(1)$ represent the equating value for a Markov chain with an increasing decision occurrence rate. Then

$$C_o^+(1) = \frac{\overline{G}(1)^+ p^+_{1,j^*} + \overline{G}(3)^* p^+_{3,j^*} + \ldots + \overline{G}(M-1)^+ p^+_{M-1,j^*}}{\overline{G}(1)^+ + \overline{G}(3)^+ + \ldots + \overline{G}(M-1)^+} [\nu^*(1)-\nu^*(2)]$$

(5.19)

One may then compare the right-hand sides of (5.18) and (5.19). Since $p^+_{ij^*} \geq p_{ij^*}$ for all values of $i \leq M$, the conclusion is

$$C_o^+(1) \geq C_o(1) \quad (5.20)$$

This situation is depicted in Fig. 5-10. The implication is that the decision maker would adhere to a "s = 1" policy at a greater cost for the increasing occurrence rate situation. Thus, the optimal acquisition policy for the increasing occurrence rate is $s = 1$ while the optimal policy for the constant rate is $s = 1$ or $s = 2$ (or perhaps even higher). Therefore, we have proven the assertion for $s = 1$.

Assume the results hold for $s = k$. We shall now prove the theorem for $s = k - 1$, the intersection of the $k - 1$ and k policies. We can express $V_T(M,s=k)$ as

$$V_T(M,s=k) = \sum_{i=0}^{k-1} \sum_{\ell=1}^{k} g(ik+\ell)[\nu^*(\ell) - v^*(\ell)] - \sum_{i=0}^{k-1} \overline{G}(ik)C_o \quad (5.21)$$

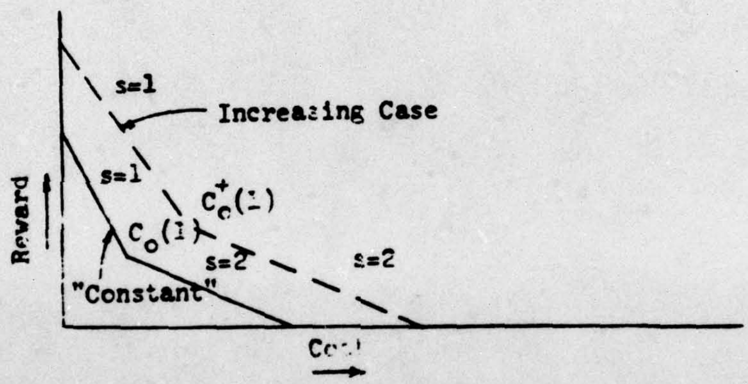


Figure 5-10 s = 1 and s = 2 regions, constant and increasing occurrence cases

Similarly, we write

$$V_T(M,s=k-1) = \sum_{i=0}^{k} \sum_{\ell=1}^{k-1} g[i(k-1)+\ell][\nu^*(\ell)-v^*(\ell)] - \sum_{i=0}^{k} G[i(k-1)]C_o \qquad (5.22)$$

At the intersection the two expressions are equal or

$$\sum_{i=0}^{k-1} \sum_{\ell=1}^{k} g(ik+\ell)[\nu^*(\ell)-v^*(\ell)] - \sum_{i=0}^{k-1} \overline{G}(ik)C_o =$$

$$\sum_{i=0}^{k} \sum_{\ell=1}^{k-1} g[i(k-1)+\ell][\nu^*(\ell)-v^*(\ell)] - \sum_{i=0}^{k} \overline{G}[i(k-1)]C_o \qquad (5.23)$$

The assertion of the theorem is that $C_o(k-1)$ as determined by the solution of (5.5) is less than $C_o^+(k-1)$ as determined by (5.23). If this is true, then one may substitute $C_o(k-1)$ into (5.23), and an equality would no longer exist. Instead the left-hand side would be less than the right-hand side. We may show that

$$C_o(k-1) = \alpha(k-1) - (1-f^{k-1}) [\nu^*(k) - v^*(k)] \qquad (5.24)$$

where $\alpha(k-1)$ was defined by (5.6). We also recognize that

$$\alpha(k-1) = \sum_{j=1}^{k-1} (1-f) f^{j-1}[\nu^*(j) - v^*(j)] \qquad (5.25)$$

Thus, $C_o(k-1)$ is a function of k terms involving $\nu^*(j)$ and $v^*(j)$, $j = 1,2,\ldots,k-1$. We can represent this as

$$C_o(k-1) = \Sigma\zeta(j) [\nu^*(j) - v^*(j)] = \Sigma\nu'(j) \qquad (5.26)$$

Substitution of the expression for $C_o(k-1)$ into (5.23) yields

$$\sum_{i=0}^{k-1} \sum_{\ell=1}^{k} g(ik+\ell)[\nu^*(\ell) - v^*(\ell) - v'(\ell)] \leq$$

$$\sum_{i=0}^{k} \sum_{\ell=1}^{k-1} g[i(k-1)+\ell][\nu^*(\ell) - v^*(\ell) - \nu'(\ell)] \quad (5.27)$$

This completes the proof of the theorem for the increasing rate case. The opposite conclusions hold for the decreasing rate case.

We have seen in this section the utility of the geometric distribution as a benchmark for any singly occurring decision. The next section extends this property to multiple occurring decisions.

5.5 Repetitive Decision Situations

5.5.1 The Model

To this point we have examined decisions that occur but once. However the process may be repetitive. The decision occurrence is described by a probability mechanism. The true state of nature is revealed to the decision maker after each occurrence, and the process continues to the horizon. We could model the process as shown in Fig. 5-11

We may also develop an equivalent Information Value Model as shown in Fig. 5-12.

States 1 through 9 of Fig. 5-12 represent the values of perfect information after one through nine transitions. These values, as initially presented in Table 2-2 are 69.41, 49.17 and so on down to 5.88 at state 9. We also note that state 1 may be entered only through information acquisition. Without the deliberate acquisition the value of information perishes through a minimum of two transitions between decision occurrences.

We offer some rationale for the model. For the deliberate acquisition situation we hypothesize that the decision maker anticipates receipt of the information, and he has his resources mobilized to act on the information after a delay of one period. However, information gained as a result of observing the true state of nature after a decision occurrence has an element of surprise, and two periods are required for reaction.

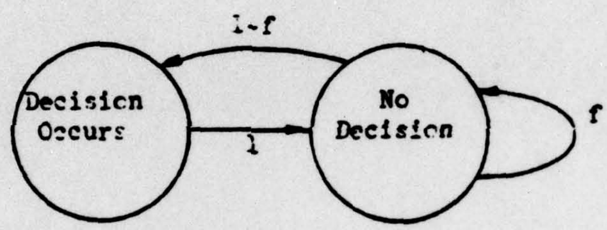


Figure 5-11 The repetitive decision model

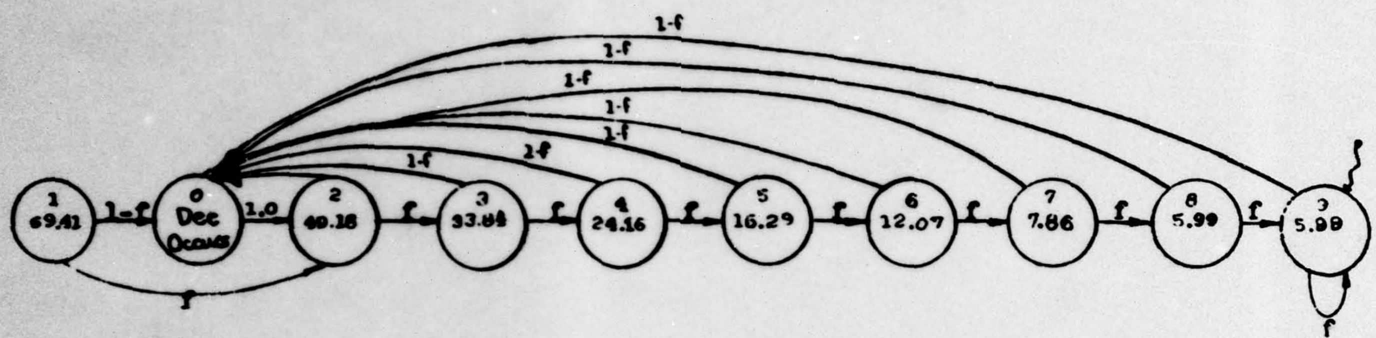


Figure 5-12 Equivalent information value model

The sequence of the action is also pertinent. The decision maker first elects to acquire or not to acquire information. Following this he finds out if the decision occurs or not on that particular transition.

5.5.2 The Value of Information for a Specific Example

Let $f = 0.8$ and the horizon $M = 20$ in the model of Fig. 5-11. We may determine the expected value of a no-information base case. The analysis proceeds by either tagging "special" transitions (see Howard [2]) or by expected state occupancies. The calculation of the occupancies of state 2, as an example, is simplified by collapsing the remaining states as shown in Fig. 5-13.

Let $\omega_{9,2}(m)$ be the expected occupancies of state 2 conditioned on starting in state 9 and m transitions having occurred. Then

$$\omega_{9,2}(m) = \frac{0.2}{1.2}(m) - \frac{0.2}{1.2}\left[\frac{1-(-0.2)^m}{1.2}\right], \quad m \geq 1 \qquad (5.28)$$

From the model we realize that only 0.2 of the occupancies of state 2 result in decision occurrences and the accruing of the 49.17 reward. Also if the horizon is M, only the occupancies in $M-1$ transitions are of interest as one transition is required to go from the information state to the decision occurrence state. We consider states three through nine in a similar manner and arrive at a base case value of 73.22, substantially greater than the base of 5.81 for the singly occurring decision case.

The process of information acquisition is equivalent to starting the model in state 1. For example, if $s = 1$, then the expected value is (20 occupancies of state 1)(0.2)(69.41) = 277.64 . Thus, using the notation of Section 5.2 we have

$$V_T(20,s=1) = (20)(0.2)(69.41) - 73.22 - 20C$$
$$= 204.41 - 20C \qquad (5.29)$$

We develop similar expressions for $s = 5$, 4, and 2 . The graphs of each are plotted in Fig. 5-14 along with comparable graphs from the singly occurring case.

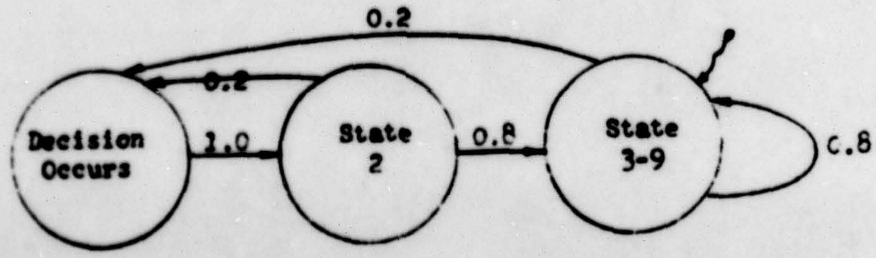


Figure 5-13 The collapsed model

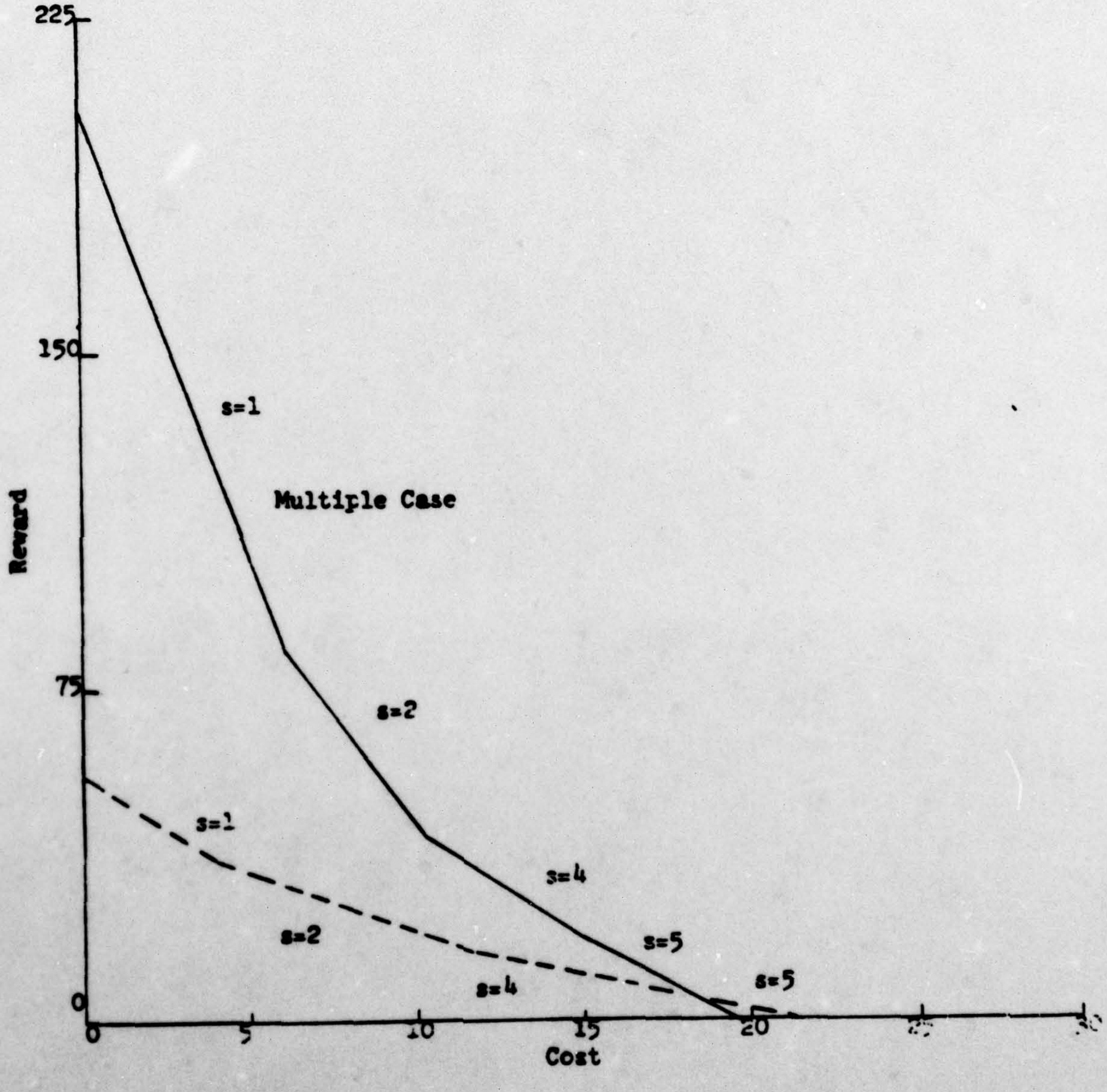


Figure 5-14 Net expected rewards

5.5.3 General Rules of Optimality

We shall now develop rules of optimality to parallel those presented in Section 5.3 for the singly occurring decision. The following statement of the problem applies: The horizon is M transitions. The periodic acquisition of information occurs at intervals k , ℓ , etc., such that $Kk = L\ell = M$. The cost of one acquisition is C , and the cost is linear as previously defined. The probability of the decision occurring, based on the model of Fig. 5-11, is

$$g(m) = \frac{1-f}{2-f} - \frac{1-f}{2-f}(f-1)^m , \quad m \geq 0 \tag{5.30}$$

Rule 1

The expected state occupancy of state j in m transitions conditioned on starting in state 0 , the no information state, is given by

$$\omega_{o,j}(m) = \frac{1-f}{2-f}\Big[m - (j-2)\Big] f^{j-2} - \frac{1-f}{2-f}\left[\frac{1-(f-1)^{m-(j-2)}}{2-f}\right] f^{j-2} \tag{5.31}$$

Proof

The proof follows from Markov state occupancy mechanics.

Rule 2

The expected state occupancy of state j in n transitions conditioned on starting in state 1 as shown in Fig. 5-13 is

$$\omega_{1,j}(m) = 1 , \quad j = 1 ,$$

$$= f^{j-1} + \frac{1-f}{2-f}\Big[m - (j-2)\Big] f^{j-2} - \frac{1-f}{2-f}\left[\frac{1-(f-1)^{m-(j-2)}}{2-f}\right] f^{j-2}, \quad j \neq 1 \tag{5.32}$$

Proof

The proof again follows from Markov mechanics.

Rule 3

The net expected reward conditioned on acquiring information at every ℓ^{th} transition is

$$V_T(M,s=\ell) = \frac{M}{\ell}\Big\{ \sum_j \omega_{\ell,j}(\ell-1)(1-f)[v^*(j)]-C\Big\}$$

$$- \Big\{\sum_j \omega_{0,j}(M-1)(1-f)[v^*(j)]\Big\} \tag{5.33}$$

Proof

The result is derived from Rules 1 and 2 and the definitions of the states used in the model.

Rule 4

Let $V_T(M,s=\ell^*)$ represent the optimal acquisition policy for some value of C. The graph (see Fig. 5-14) is piecewise linear in C.

Proof

Immediate.

There appears to be no readily tractable method for determining the value of ℓ^*, the optimal acquisition period, for some particular value of C. However, we may show that it is less than some value m where m is the transition, counting forward, where the information has perished.

Rule 5

Assume the value of information perishes by the m^{th} transition counting forward; i.e., $v^*(j) - v(j) = 0$, $j \geq m$. Then $\ell^* \leq m$ for all C.

Proof

Assume the existence of two optimal policies k^* and ℓ^* such that $V_T(M,s=k^*) \geq V_T(M,s=j)$ for all $j \geq m$, and $V_T(M,s=\ell^*) \geq V_T(M,s=j)$ for all $j \leq m$. In particular, by assumption, $V_T(M,s=k^*) \geq V_T(M,s=m)$. This implies that

$$\frac{M}{k^*}\left\{\sum_{j=0}^{k^*} [v^*(j) - v^*(j)]\right\} \geq \frac{M}{m}\left\{\sum_{j=0}^{m} [v^*(j) - v^*(j)]\right\} \tag{5.34}$$

However, by the concept of information perishing the bracketed summations are equal which implies

$$\frac{M}{k^*} \geq \frac{M}{m} \tag{5.35}$$

which is false for $k^* \geq m$. Therefore, ℓ^* which is less than or equal to m must be the optimal policy.

Rule 6

No acquisition policy is dominant for all values of C.

Proof

Refer to Fig. 5-14. The intercept on the reward axis obviously decreases with increasing ℓ . However, the intercept on the cost axis increases with increasing ℓ . Thus, no policy dominates.

Rule 7

Let $C(1)$ be the cost for which the decision maker is indifferent between $s = 1$ and $s = 2$. Then

$$C(1) = (1-f)\nu^*(1) - f(1-f)\nu^*(2) \tag{5.36}$$

Proof

Let B = expected reward for the no information case. We can write the equating value for the $s = 1$ and the $s = 2$ case using (5.33) as

$$M[(1-f)\nu^*(1) - C(1)] - B = \frac{M}{2}[(1-f)\nu^*(1) + f(1-f)\nu^*(2) - C] - B \tag{5.37}$$

The results follow from solution of (5.37). This value will be useful in comparison with the singly occurring case.

We had earlier noted the use of the "constant" occurrence case to establish a bound on the optimal acquisition period. This case also serves as a benchmark for the multiple occurring decision although the results are surprisingly different.

Theorem 5.2

Let $C_o(k)$ be the cost of information acquisition for the constant rate, singly occurring decision case for which the decision maker is indifferent between $s = k$ and $s = k + 1$. Let $C(k)$ be similarly defined for the multiple occurring case. Then $C(1) \geq C_o(1)$ but $C(k) \leq C_o(k)$, $k > 1$.

Proof

By Rule 7

$$C(1) = (1-f)\nu^*(1) - f(1-f)\nu^*(2) \tag{5.38}$$

From (5.5) one may show that

$$C_o(1) = (1-f)[\nu^*(1) - \nu^*(2)] \tag{5.39}$$

Comparison of (5.38) and (5.39) proves the assertion regarding $C(1)$ and $C_o(1)$. The remainder of the proof parallels the proof of

Theorem 5.1. For the multiple occurring case we determine $C(k)$ by setting

$$V_T(M,s=k) = V_T(M,s=k+1) \tag{5.40}$$

Then (5.32) may be used to evaluate $V_T(M,s=k)$ and $V_T(M,s=k+1)$ to yield

$$\frac{M}{k}\left[(1-f)\nu^*(1) + \sum_{j=2}^{k} \omega_{1,j}(k-1)\ (1-f)\nu^*(j) - C(k)\right] =$$

$$\frac{M}{k+1}\left[(1-f)\nu^*(1) + \sum_{j=2}^{k+1} \omega_{1,j}(k)\ (1-f)\nu^*(j) - C(k)\right] \tag{5.41}$$

We determine $C_o(k)$ by use of (5.24). The theorem requires that the substitution of (5.24) into (5.41) destroys the equality and results in an inequality with the left-hand side, that is the side with the greatest number of acquisitions, being the lesser side. Substitution of (5.24) into (5.41) confirms this result and proves the theorem.

These results are obviously not as "clean" as those of the singly occurring decision as there is some uncertainty where the cross-over in the bound occurs. However, the ease of determining the values for the constant occurrence, singly occurring case recommends its use for these rough estimates.

5.6 The Case for Periodic Replenishment

We have suggested that the optimal policy in some instances is strictly periodic acquisition of information. An approach, following a development of Barlow and Proschan [9] rigorously supports this contention (under certain limiting conditions).

Suppose that the time between deliberate information acquisitions is described by some distribution function $F(X)$ (and density function $f(X)$). This generates a series of information acquisitions; the time between each acquisition is a random variable, X (see Fig. 5-15a). The X's are identically and independently distributed random variables which we shall designate $\{X_k\}_{k=1}^{\infty}$.

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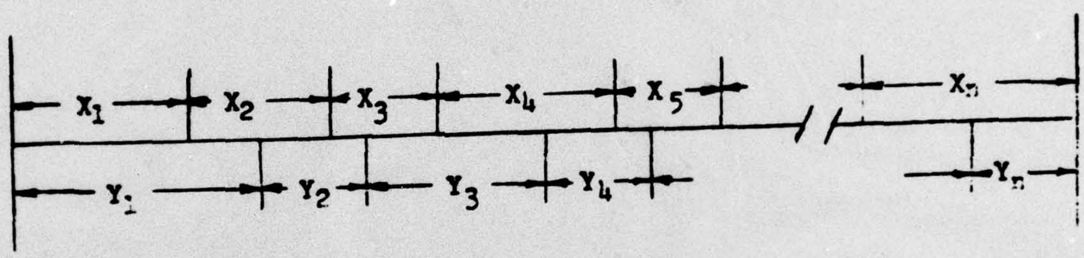

(a)

(b)

Figure 5-15 Relationship of $\{X_k\}$, $\{Y_k\}$, and $\{Z_k\}$

Similarly let the time between decision occurrences, Y , be generated by G(Y) . The set of Y's, designated by $\{Y_k\}_{k=1}^{\infty}$, are also identically and independently distributed (Fig. 5-15a).

Further, define a third set of random variables,

$$Z_k = \min\{X_k, Y_k\}$$

(see Fig. 5-15b). The Z's delineate a series of information replenishments, some of which are by design and some of which occur "free" in that they result from observation of the outcome of a decision.

Let

$N_1(m)$ = number of acquisitions by design by transition m (5.42)

$N_2(m)$ = number of acquisitions by decision occurrence by transition m (5.43)

$N(m) = N_1(m) + N_2(m)$ = total acquisitions by transition m (5.44)

From the definition of Z the

$$Pr\{Z \leq k\} = 1 - \bar{G}(k)\bar{F}(k) \tag{5.45}$$

and one may show that

$$\langle Z|\epsilon\rangle = \sum_{k=0}^{\infty} \bar{G}(k)\bar{F}(k) \tag{5.46}$$

Therefore,

$$\lim_{m\to\infty} \frac{\langle N(m)|\epsilon\rangle}{m} = \frac{1}{\sum_{m=0}^{\infty} \bar{G}(m)\bar{F}(m)} \tag{5.47}$$

Two indicator random variables will be useful in the development. Let

$$V_k = \begin{cases} 1 & \text{if } Z_k = X_k \text{ (replenishment by design)} \\ 0 & \text{otherwise} \end{cases} \tag{5.48}$$

and

$$W_k = \begin{cases} 1 & \text{if } Z_k = Y_k \text{ (replenishment by decision outcome observation)} \\ 0 & \text{otherwise} \end{cases} \tag{5.49}$$

Reflection shows that

$$\langle V|\epsilon\rangle = \Pr\{X \le Y\} = \sum_{m=0}^{\infty} F(m)g(m) \tag{5.50}$$

and

$$\langle W|\epsilon\rangle = \Pr\{Y \le X\} = \sum_{m=0}^{\infty} G(m)f(m) \tag{5.51}$$

These indicator random variables "identify" the method of information replenishment.

We can describe the average reward per transition for an infinite horizon process by two terms: the first is the reward accruing to the decision maker if the decision occurs, and the second is the cost of deliberate information replenishment.

The cost is a constant which we shall label C_1 . We recollect that we receive perfect information from both the deliberate acquisition and from observing the system as the decision occurs. Let the value of this perfect information be C^* and the reward at k transitions later be $C'(k)$. Let

$$C_2(k) = C'(k) - C^* \tag{5.52}$$

so that $C_2(k)$ represents a "cost" of not having perfect information. The average cost per transition is

$$AC = \lim_{m \to \infty} \frac{C_1 \langle N_1(m)|\epsilon\rangle}{m} + \frac{\langle C_2(m) N_2(m)|\epsilon\rangle}{m} \tag{5.53}$$

$$= \frac{C_1 \sum_{m=0}^{\infty} F(m)g(m)}{\sum_{m=0}^{\infty} \bar{F}(m)\bar{G}(m)} + \frac{\sum_{m=0}^{\infty} C_2(m)G(m)f(m)}{\sum_{m=0}^{\infty} \bar{F}(m)\bar{G}(m)} \tag{5.54}$$

We may rewrite the numerator of the first term as

$$C_1 \sum_{m=0}^{\infty} F(m)g(m) = C_1 \sum_{m=0}^{\infty} \bar{G}(m)f(m) \tag{5.55}$$

The denominator of both terms is $\langle Z|\epsilon\rangle$. From basic considerations

$$\langle Z|\epsilon\rangle = \sum_{x=0}^{\infty} \left[\sum_{y=0}^{x} yg(y) + x \sum_{y=x}^{\infty} g(y) \right] f(x) \tag{5.56}$$

This enables one to rewrite (5.54) as

$$AC[F(x)] = \frac{\sum_{x=0}^{\infty} [C_1\bar{G}(x) + C_2(x)G(x)]\ f(x)}{\sum_{x=0}^{\infty} \left[\sum_{y=0}^{x} yg(y) + x \sum_{y=x}^{\infty} g(y) \right] f(x)} \tag{5.57}$$

$$= \frac{\sum_{x=0}^{\infty} [R(x)]\ f(x)}{\sum_{x=0}^{\infty} [S(x)]\ f(x)} \tag{5.58}$$

Since x varies from 0 to ∞ , and assuming neither C_1 nor $C_2(x)$ is infinite, there is some x_o , perhaps infinite (implying no replenishment by design), that minimizes the bracketed quotient of (5.58), i.e.,

$$\frac{[R(x_o)]}{[S(x_o)]} \le \frac{[R(x)]}{[S(x)]}\ , \quad 0 \le x \le \infty \tag{5.59}$$

It follows that

$$\frac{\sum_{x=0}^{\infty} R(x_o)\ f(x_o)}{\sum_{x=0}^{\infty} S(x_o)\ f(x_o)} \le \frac{\sum_{x=0}^{\infty} R(x)\ f(x)}{\sum_{x=0}^{\infty} S(x)\ f(x)} \tag{5.60}$$

or

$$AC[F(x_o)] \le AC[F(x)] \qquad (5.61)$$

In summary:

<u>Theorem 5.3</u>

For the infinite horizon, multi-decision case where the objective function is minimization of the average cost per transition, the optimal policy is strictly periodic replenishment of information.

This, of course, is not an unexpected result for an infinite horizon case. However, we have previously shown in Example 4 that an aperiodic policy may be optimal for a finite horizon case.

5.7 Summary

This chapter has illustrated the concept of information replenishment and optimal policies of replenishment both for the singly occurring decision and multiple decisions. The use of the geometric distribution as a benchmark was discussed, and the final theorem rigorously demonstrated that a periodic acquisition policy was optimal under certain limiting conditions.

CHAPTER 6

RELIABILITY AND MAINTAINABILITY THEORY

6.1 Introduction

In Chapter 5 we were able to capitalize on extant reliability theory for proof of Theorem 5.3. Reliability theory treats the failure of a piece of equipment, i.e., its deterioration from a superior to an inferior state, and the maintainability of equipment, i.e., optimal strategies to restore the equipment to a superior state. This study has treated the deterioration and restoration of information. There is an easy analogy that is readily apparent and that lends support to the greater utilization of the well-established reliability theory. This chapter investigates several possibilities.

6.2 Definitions

Several definitions from the theory will be useful.

1. Reliability. The classical definition is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered [1].
2. Failure. The complement of reliability.
3. Failure distribution, $[F(t)]$. The distribution function that describes the failure of an item of interest, i.e.,

   $F(t) = Pr\{\text{item has failed or is in a failed state at time } t \mid e\}$

4. Failure rate function, $r(t) = f(t)/\bar{F}(t)$. (This is a widely used concept and is also known as the force of mortality, the Mills ratio, the intensity function, and the hazard rate.)
5. Increasing failure rate (IFR).

   (a) A continuous failure distribution is IFR if

   $$\frac{d}{dt}[r(t)] \geq 0$$

   (b) IFR Markov chains. We have previously used in Section 5.4 a definition of an IFR Markov chain that is due to Wagner [23]. Assume a chain with states 0, 1, 2, ..., N such that the greater value of the state the greater the deterioration of the item. Then the chain is IFR if

$$Pr\{s(n+1) \subset \beta | s(n)=i, \epsilon\}$$

is non-decreasing in i for all sets β of the form

$$\beta = \{k, k+1, k+2, \ldots, N\}$$

for any $k = 0, 1, 2, \ldots, N$. Equivalently, the chain is IFR if

$$r_k(i) = \sum_{j=k}^{N} p_{ij}$$

is non-decreasing in i for all k, $k = 0, 1, 2, \ldots, N$. These equivalent definitions connote the greater the value of the state the greater the probability of further deterioration.

6.3 Results

6.3.1 Control Limit Rules

Theorem 6.1

Let states $\{0,1,2,\ldots,N\}$ be states of a Markov chain where the higher numbered states represent progressively greater deterioration of an item. Let C represent the cost if the item is replaced before it becomes inoperative and $C + A$, $A \geq 0$, the cost for replacement after the item becomes inoperative. Then the optimal policy is to replace the item if and only if the item is in state i, $i+1$, $i+2$, ..., N for some i. (This is a control limit rule where state i represents the control limit.)

Proof

Derman [10].

The calculation of i is not a trivial procedure. Ross [6] presents a linear programming algorithm to determine i. The thrust of the theorem is sufficiently analogous to the concepts of this thesis to merit investigation of the possibility of using a control limit rule to determine optimal policies for information replenishment.

6.3.2 Bounds on the Optimal Replacement Period

A discussion by Barlow and Proschan [9] suggests a method of bounding the optimal replacement period for the infinite horizon case.

We let

$$L(x) = \frac{R(x)}{S(x)} \tag{6.1}$$

where $R(x)$ and $S(x)$ are defined by (5.58).

For some x, say x^*, to be optimal is equivalent to

$$L(x_1) \geq L(x^*) \leq L(x_2) \ , \quad x_1 \leq x^* \leq x_2 \tag{6.2}$$

From the basic definition the left-hand inequality is equivalent to

$$\frac{C_1 \bar{G}(x_1) + C_2(x_1)\, G(x_1)}{\sum_{k=0}^{x_1} \bar{G}(k)} \leq \frac{C_1 \bar{G}(x^*) + C_2(x^*)\, G(x^*)}{\sum_{k=0}^{x^*} \bar{G}(k)} \tag{6.3}$$

This inequality reduces to

$$\sum_{k=0}^{x_1} \bar{G}(k) \left\{ \frac{g(x^*)}{\bar{G}(x^*)} + \frac{G(x_1)[C_2(x^*) - C_2(x_1)]}{\bar{G}(x^*)[C_1 - C_2(x^*)]} \right\} - \bar{G}(x_1) \geq \frac{C_1}{C_1 - C_2(x_1)} \tag{6.4}$$

or

$$H(x_1) \geq \frac{C_1}{C_1 - C_2(x_1)} \tag{6.5}$$

The right-hand inequality in (6.2) reduces to

$$H(x_2) \leq \frac{C_1}{C_1 - C_2(x_2)} \tag{6.6}$$

We are considering only discrete values of x. However, the continuous version of the right-hand side of (6.6) is shown in Fig. 6-1. In addition, if the distribution $G(x)$ is IFR as previously defined, then $H(x)$ is increasing in x. This limits the optimal region to $x^* \leq x_2'$, where $C_1 - C_2(x_2') = 0$. This establishes an upper bound for x^*.

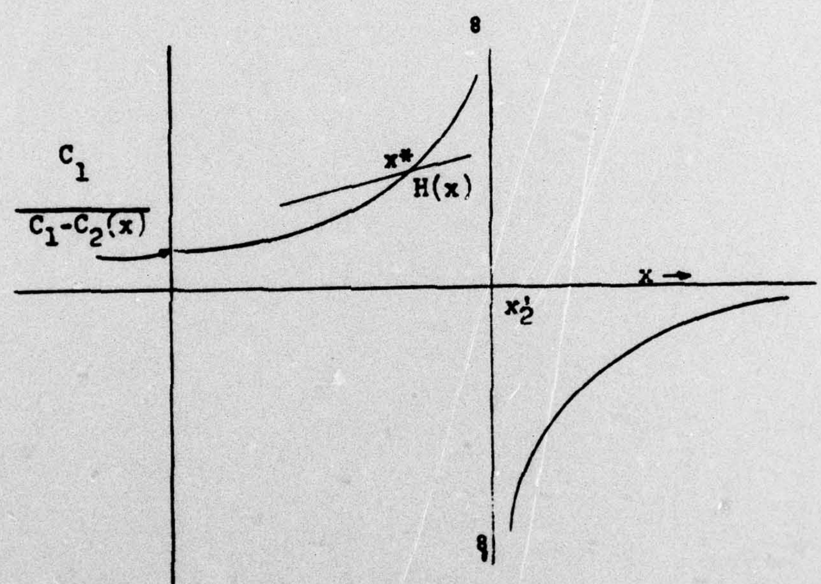

Figure 6-1 Optimal region for x^*

From (6.1) and (5.58) we may establish that

$$L(0) = \infty$$

and

$$L(x) \to C_2/\mu$$

as $x \to \infty$, where C_2 is defined as

$$C_2 = \max_k \{C'(k) - C^*\}$$

from (5.52), and

$$\mu = \sum_{k=0}^{\infty} \overline{G}(k)$$

We rewrite $L(x)$ as

$$L(x) = \frac{C_1 + [C_1 - C_2(x)]\, G(x)}{\sum_{k=0}^{x} \overline{G}(k)} \quad \frac{C_1}{x} \tag{6.7}$$

There is no assurance that $L(x)$ ever crosses the line $L(x) = C_2/\mu$, but if it does, it crosses to the right of the intersection of that line and C_1/x. This intersection is at the point where $x = (C_1/C_2)\mu$. Therefore, this is the lower bound for x^*.

6.4 Summary

This chapter has been intentionally brief. The purpose has been to simply suggest the possibilities of utilizing the existing definitions and results of reliability theory to develop analogous results for perishing information. The theory allowed the establishment of bounds on the optimal acquisition period and appeared to hold some promise for establishing a control limit rule. The theory of reliability is extensive, and the results here are only a fairly cursory survey of possibly applicable approaches.

CHAPTER 7

CONCLUSIONS AND EXTENSIONS

This is the point for reflection and projection. Where are we now, and where do we have to go?

The goals of the first half of the study were straightforward:

1. To describe the phenomena of information perishing.
2. To develop operational definitions.
3. To prove the inevitability of information perishing.
4. To determine several parameters to describe the process.
5. To consider ancillary areas such as the effect of risk aversion, discounting of rewards, and contingency decision making.

These goals have been met.

The goals of the second half of the thesis were:

1. To describe information replenishment.
2. To develop optimal acquisition policies for singly occurring decisions.
3. To develop optimal acquisition policies for multiple occurring decision.
4. To suggest parallels from reliability theory.

These goals have also been met with the reservation that bounds were established for the optimal acquisition policies rather than precise determination of the period between replenishments.

The accomplishment of these goals contribute to the rationalization of the information process. However, the study suggests several needed extensions. These can be grouped into two main divisions: theoretical and applied.

The theoretical extensions are:

1. The development, in the main, centered on Markovian information models. A few results hold for any probability distribution. The generality of many results is limited by the Markov assumption. Consideration of other dynamic probability models would be worthwhile.

2. The bounds on the optimal period between information acquisitions are useful. However, an algorithm to determine an exact value of the optimal period would enhance these results.
3. The foray into the thicket of reliability theory was limited. Much of the theory, as previously noted, concerns parameter estimation and is of little apparent use. However, the results concerning inspection and equipment replacement may have significant application to information economics. This is perhaps the extension of most immediate potential.

The applicatory extensions fall into two sub-groups:

1. Descriptive

   Several questions would merit study at the personal and organization level. Do people and organizations recognize the phenomena of information perishing? If so, how do they cope with this deterioration? Is there a rationale for allocation of resources for information acquisition? Is it related to information perishing?

2. Normative.

   A body of theory should lead to a set of optimal policies to guide individual and organizational decision makers in the allocation of resources for information collection, analysis, and use. This thesis, along with the other Decision Analysis research efforts cited in Chapter 2, form a basis for development of such a normative theory.

The enormity and complexity of completing such a theory is apparent. However, the national intelligence budget today is in the billions, and further billions are spent in information acquisition at corporate and individual levels. The saving of even a small percentage of this huge sum would merit a major and dedicated research effort.

APPENDIX A

PROOF OF THEOREMS 2.3 and 2.4

A.1 Purpose

The purpose of this appendix is to prove Theorems 2.3 and 2.4, important proofs but of such length that the reader is distracted from the logic of Chapter 2.

A.2 Proof of Theorem 2.3

Theorem 2.3 consists of showing that $\rho(n) \le |\lambda_1|$ for the constant reward, invariant transition matrix case.

Proof

The proof is by induction on n .

$$\rho(n) = \begin{cases} \dfrac{\Delta(n-1)}{\Delta(n)} & \text{for } \Delta(n) \neq 0 \\ 0 & \text{for } \Delta(n) = 0 \end{cases} \qquad (A.1)$$

where

$$\Delta(n) = \nu^*(n) - v^*(n) \qquad (A.2)$$

If $\rho(n) = 0$, i.e., $\Delta(n-1)$ or $\Delta(n) = 0$, then $\rho(n) \le |\lambda_1|$ is true trivially. Therefore, we shall assume this is not true. For the first step of the induction we prove $\rho(1) \le |\lambda_1|$. By definition

$$p(1) = \frac{\Delta(0)}{\Delta(1)} \qquad (A.3)$$

Let $M-1$ be the transition (counting forward) with 1 to go and M be the transition with 0 to go. From (2.22) we may deduce that the expected reward at transition M conditioned on some starting state, say state 1, is

$$\langle v(M) \,|\, s(0)=1, \delta(M)=\delta^*, c\rangle = \max_k \Big\{ \underbrace{{}_0q_{11} \;\; {}_0q_{12} \;\cdots\; {}_0q_{1N}} + \lambda_1^M \underbrace{{}_1q_{11} \;\; {}_1q_{12} \;\cdots\; {}_1q_{1N}} \;\cdots + \lambda_{N-1}^M \underbrace{{}_{N-1}q_{11} \;\; {}_{N-1}q_{12} \;\cdots\; {}_{N-1}q_{1N}} \Big\} [R] \qquad (A.4)$$

where ${}_0q_{11}$, ${}_0q_{12}$, ..., ${}_0q_{1N}$ are elements of $[Q_o]$ and

$$[R] = \{r_\ell^{(k)}\} , \quad k = 1, 2, \ldots, k-1 \tag{A.5}$$

Therefore, for $\Delta(0)$ we write

$$\Delta(0) = \sum_{i=1}^{N} \pi_i \left\{ \max_k \sum_{j=0}^{N-1} \lambda_j^M \sum_{\ell=1}^{N} {}_jq_{i\ell}\, r_\ell^{(k)} \right\} - \sum_{i=1}^{N} \pi_i(\infty) r_i^{(0)} \tag{A.6}$$

Similarly,

$$\begin{aligned} \Delta(1) = &\sum_{i=1}^{N} \pi_i \left\{ \max_k \sum_{j=0}^{N-1} \lambda_j^{M-1} \sum_{\ell=1}^{N} {}_jq_{i\ell}\, r_\ell^{(k)} \right\} - \sum_{i=1}^{N} \pi_i(\infty) r_i^{(0)} \\ &+ \sum_{i=1}^{N} \pi_i \left\{ \max_k \sum_{j=0}^{N-1} \lambda_j^{M} \sum_{\ell=1}^{N} {}_jq_{i\ell}\, r_\ell^{(k)} \right\} - \sum_{i=1}^{N} \pi_i(\infty) r_i^{(0)} \end{aligned} \tag{A.7}$$

From (A.1), (A.3), (A.6), and (A.7) we see that

$$\rho(1) = \frac{\Delta(0)}{\Delta(0) + \alpha\Delta(0)} \tag{A.8}$$

where $\alpha \geq 1$ by Theorem 2.2. Therefore,

$$\rho(1) = \frac{1}{1+\alpha} \leq \frac{1}{2} \tag{A.9}$$

So if $|\lambda_1| \geq 1/2$ the theorem holds.
Assume now that $0 \leq |\lambda_1| \leq 1/2$. Then one must show that

$$\rho(1) = \frac{\sum_i \pi_i \left\{ \max_k \sum_j \lambda_j^M \sum_\ell {}_jq_{i\ell} r_\ell^{(k)} \right\} - \sum_i \pi_i r_i^{(0)}}{\sum_i \pi_i \left\{ \max_k \sum_j \lambda^{M-1} \sum_\ell {}_jq_{i\ell} r_\ell^{(k)} \right\} - \sum_i \pi_i r_i^{(0)} + \sum_i \pi_i \left\{ \max_k \sum_j \lambda_j^M \sum_\ell {}_jq_{i\ell} r_\ell^{(k)} \right\} - \sum_i \pi_i r_i^{(0)}}$$

$$\leq |\lambda_1| \tag{A.10}$$

We may prove inequality (A.10) is valid by the method of contradiction. Assume (A.10) is false and that $\rho(1) \geq |\lambda_1|$. This implies

$$\sum_i \pi_i \Big[\max_k \sum_j \lambda_j^M \sum_\ell {}_jq_{i\ell} r_\ell^{(k)}\Big] - \sum_i \pi_i r_i^{(0)}$$

$$- |\lambda_1| \Big\{ \sum_i \pi_i \Big[\max_k \sum_j \lambda_j^{M-1} \sum_\ell {}_jq_{i\ell} r_\ell^{(k)} + \max_k \sum_j \lambda_j^M \sum_\ell {}_jq_{i\ell} r_\ell^{(k)}\Big]$$

$$- 2 \sum \pi_i r_i^{(0)} \Big\} \geq 0 \qquad \text{(A.11)}$$

Let

$$\max_k \sum_j \lambda_j^M \sum_\ell {}_jq_{i\ell} r_\ell^{(k)} = \sum_j \lambda_j^M \sum_\ell {}_jq_{i\ell} r_\ell^{**} \qquad \text{(A.12)}$$

and

$$\max_k \sum_j \lambda_j^{M-1} \sum_\ell {}_jq_{i\ell} r_\ell^{(k)} = \sum_j \lambda_j^{M-1} \sum_\ell {}_jq_{i\ell} r_\ell^{*} \qquad \text{(A.13)}$$

The two "constant" terms in (A.11) are negative, i.e.,

$$- \sum_i \pi_i r_i^{(0)} + 2|\lambda_1| \sum_i \pi_i r_i^{(0)} \leq 0 \qquad \text{(A.14)}$$

as $|\lambda_1| \leq 1/2$ by assumption. Therefore, for (A.10) to be positive, as assumed,

$$\sum_i \pi_i \Big\{ \sum_j \lambda_j^M \sum_\ell {}_jq_{i\ell} r_k^{**} - |\lambda_1| \Big[\sum_j \lambda_j^{M-1} \sum_\ell {}_jq_{i\ell} r_k^{*} + \sum_j \lambda_j^M \sum_\ell {}_jq_{i\ell} r_\ell^{**} \Big] \Big\} \qquad \text{(A.15)}$$

must be greater than zero. Let expression (A.15) = Γ , and by assumption $\Gamma \geq 0$. As

$$\sum_i \pi_i \sum_j \lambda_j^{M-1} \sum_\ell {}_jq_{i\ell} r_k^{**} \leq \sum_i \pi_i \sum_j \lambda_j^{M-1} \sum_\ell {}_jq_{i\ell} r_k^{*} \qquad \text{(A.16)}$$

by the optimality of the decisions, then

$$0 \leq \Gamma \leq \sum_i \pi_i \Big\{ \sum_j \Big\{ \Big[\lambda_j^M - |\lambda_1| \lambda_j^{M-1} - |\lambda_1| \lambda_j^M \Big] \Big\} \sum_\ell {}_jq_{i\ell} r_k^{**} \Big\} \qquad \text{(A.17)}$$

or

$$0 \le \Gamma \le \sum_{i} \pi_i \left\{ \sum_{j} \left\{ \left[\lambda_j - |\lambda_1| - |\lambda_j| \lambda_j \right] \lambda_j^{M-1} \right\} \sum_{\ell} {}_j q_{i\ell} r_k^{**} \right\} \tag{A.18}$$

and

$$0 \le \Gamma \le \sum_{i} \pi_i \left\{ \sum_{j} \left[\lambda_j - |\lambda_1| - |\lambda_1| \lambda_j \right] \lambda_j^{M-1} \sum_{\ell} {}_j q_{i\ell} r_k^{*} \right\} \tag{A.19}$$

If the bracketed expression containing the "$|\lambda_1|$" terms in (A.19) is positive, the right side of the inequality is positive; conversely, if this term is negative, the right side is negative. If $\lambda_j \ge 0$, then

$$[\lambda_j - |\lambda_1| - |\lambda_1|\lambda_j] \le 0 \ , \quad \text{as} \quad |\lambda_1| \ge \lambda_j \ , \quad \forall_j \tag{A.20}$$

Therefore, for $\lambda_j \ge 0$, the assumption that $\Gamma \ge 0$ is false. Similarly, if $\lambda_j \le 0$, then

$$-|\lambda_j| + |\lambda_1|\ |\lambda_j| - |\lambda_1| \le 0 \ , \quad \text{as} \quad |\lambda_j| \le |\lambda_1| \le \frac{1}{2} \tag{A.21}$$

by assumption. So once again the assumption that Γ is positive is false. Therefore, we conclude

$$\Gamma \le 0 \tag{A.22}$$

which is contrary to inequality (A.17). It follows that the assumption that $\rho(1) \ge |\lambda_1|$ is false, and

$$\rho(1) \le |\lambda_1| \tag{A.23}$$

We continue the induction by assuming

$$\frac{\Delta(n-2)}{\Delta(n-1)} \le |\lambda_1| \tag{A.24}$$

and prove

$$\frac{\Delta(n-1)}{\Delta(n)} \le |\lambda_1| \tag{A.25}$$

We let the

n transition to go → L - 1 transition counting forward

n-1 transition to go → L transition counting forward

n-2 transition to go → L + 1 transition counting forward

Let r_ℓ^* and r_ℓ^{**} denote optimal decisions at L - 1 and L, respectively. Then we must show that

$$\frac{\Delta(n-1)}{\Delta(n)} = \frac{\sum_i \pi_i \sum_j \lambda_j^L \sum_\ell {}_j q_{i\ell} r_\ell^{**} - \sum_i \pi_i r_i^{(0)} + \Delta(n-2)}{\sum_i \pi_i \sum_j \lambda_j^{L-1} \sum_\ell {}_j q_{i\ell} r_\ell^{*} - \sum_i \pi_i r_i^{(0)} + \Delta(n-1)} \le |\lambda_1| \qquad \text{(A.26)}$$

Assume the contrary of the proof, or that

$$\frac{\Delta(n-1)}{\Delta(n)} \ge |\lambda_1|$$

Therefore, cross-multiplying and transposing in (A.26) yields

$$\sum_i \pi_i \Big[\sum_j \lambda_j^L \sum_\ell {}_j q_{i\ell} r_\ell^{**} - |\lambda_1| \sum_j \lambda_j^{L-1} \sum_\ell {}_j q_{i\ell} r_\ell^{*} \Big] + \Big[-\sum_i \pi_i r_i^{(0)} + |\lambda_1| \sum_i \pi_i r_i^{(0)} \Big] \qquad \text{(A.27a)}$$

$$+ \Delta(n-2) - |\lambda_1| \, \Delta(n-1) \ge 0 \qquad \text{(A.27b)}$$

Now either (A.27a) or (A.27b) or both must be positive to satisfy the inequality. As $|\lambda_1| \le 1$, the second bracketed expression of (A.27a) is less than zero. Therefore, for (A.27a) to be greater than zero, the first bracketed term must be positive. However, by reasoning similar to that used in the proof of (A.16) through (A.19) and noting the optimality of the decisions we see that

$$\sum_i \pi_i \Big[\sum_j \lambda_j^L \sum_\ell {}_j q_{i\ell} r_\ell^{**} - |\lambda_1| \sum_j \lambda_j^{L-1} \sum_\ell {}_j q_{i\ell} r_\ell^{**} \Big] \ge \sum_i \pi_i \Big[\sum_j \lambda_j^L \sum_\ell {}_j q_{i\ell} r_\ell^{**} - |\lambda_1| \sum_j \lambda_j^{L-1} \sum_\ell {}_j q_{i\ell} r_\ell^{*} \Big]$$

(A.28)

After rewriting the left side of (A.28) one may show

$$0 \ge \sum_i \pi_i \Big\{ \sum_j \lambda_j^{L-1} [\lambda_j - |\lambda_1|] \sum_\ell {}_j q_{i\ell} r_\ell^{**} \Big\} \qquad \text{(A.29)}$$

as

$$[\lambda_j - |\lambda_1|] \leq 0 \tag{A.30}$$

for all λ_j . Thus (A.27a) is negative and (A.27b) must be positive for our assumption to hold. For

$$\Delta(n-2) - |\lambda_1| \; \Delta(n-1) \geq 0 \tag{A.31}$$

then

$$\frac{\Delta(n-2)}{\Delta(n-1)} \geq |\lambda_1| \tag{A.32}$$

which violates the induction hypothesis. We have now shown that both (A.27a) and (A.27b) are negative, contrary to our assumptions.

We are in a position to state an important result.

Theorem 2.3

For the n-state Markov process with k reward decisions and an invariant transition matrix $\rho(n) \leq |\lambda_1|$, the absolute value of the largest transient eigenvalue.

A.2 Proof of Theorem 2.4

We had limited the previous proof to decision situations where the decision was limited to a choice of state, and the transition matrix was invariant. However, we may also extend the result to the situation where the decision maker may elect not only the reward structure but also the transition matrix. The notation is the same as for the previous theorem. Our method of proof is to develop an expression that is equivalent to (A.4). From this it follows that the remaining proof is identical.

The theorem is

Theorem 2.4

For a n-state Markov process let $k \subset K$ represent an index set of reward decisions and $l \subset L$ represent an index set of transition matrix decisions. Then

$$\rho(n) \leq |\lambda_1|'$$

where $|\lambda_1|' = \sup\{\lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_1^{(L)}\}$, $\lambda_1^{(l)}$ being the greatest absolute value of the transient eigenvalues of $[P^{(l)}]$.

Proof

We assume the existence of some stationary optimal policy based on the no-information case with an associated $\underline{\pi}$ and reward $r_i^{(0)}$]. We also assume in the proof of

$$\rho(1) = \frac{\Delta(0)}{\Delta(1)} < |\lambda_1|' \tag{A.33}$$

that we may determine, based on an optimal policy from $m = 0$ to $m = M-2$, the value of $\underline{\pi_i(M-2)}$, where i reflects the initial starting state. We further assume the decision maker elects matrix $[P']$ at transition $M-2$ and $[P'']$ at transition $M-1$ so that

$$\underline{\pi_i(M-1)} = \underline{\pi_i(M-2)}\,[P'] \tag{A.34}$$

and

$$\underline{\pi_i(M)} = \underline{\pi_i(M-1)}\,[P''] \tag{A.35}$$

$$= \underline{\pi_i(M-2)}\,[P']\,[P''] \tag{A.36}$$

where $[P']$ and $[P'']$ may or may not be the same transition matrices. We further assume $[P']$ has the largest (in absolute value) eigenvalue.

We now use N differential matrices to express P' so that (A.36) becomes

$$\underline{\pi_i(M)} = \underline{\pi_i(M-2)}\left\{[Q_o][P''] + \lambda_o[Q_1][P''] + \ldots + \lambda_{N-1}[Q_{N-1}][P'']\right\} \tag{A.37}$$

or

$$\underline{\pi_i(M)} = \underline{\pi_i(M-2)}\left\{[W_o] + \lambda_1[W_1] + \ldots + \lambda_{N-1}[W_{N-1}]\right\} \tag{A.38}$$

Expression (A.38) allows us to write the expected reward at transition M conditioned on $s(0) = i$ as

$$\langle v(M)\,|\,s(0)=i, \delta(M)=\delta^*, e\rangle = \max_k \underline{\pi_i(M)}\,[R] \tag{A.39}$$

where $[R] = \{r_\ell^{(k)}\}$, or finally

$$\langle v(M) | s(0)=i, \delta(M)=\delta^*, \epsilon\rangle = \max_k \Big\{ \lfloor {}_0\omega_{i1}\ {}_0\omega_{i2} \cdots {}_0\omega_{iN} \rfloor$$

$$+ \lambda_1 \lfloor {}_1\omega_{i1}\ {}_1\omega_{i2} \cdots {}_1\omega_{iN} \rfloor$$

$$+ \cdots + \lambda_{N-1} \lfloor {}_{N-1}\omega_{i1}\ {}_{N-1}\omega_{i2} \cdots {}_{N-1}\omega_{iN} \rfloor \Big\} [R] \tag{A.40}$$

$$= \max_k \sum_{j=0}^{N-1} \lambda_j \sum_{\ell=1}^{N} {}_j\omega_{i\ell} r_\ell^{(k)} \tag{A.41}$$

However, with the exception of the power of λ_j , (A.41) is exactly the same form as (A.4), the initial step in proving Theorem 2.3. Therefore, we infer the remainder of the proof follows and that Theorem 2.4 is valid.

This completes the proof of both theorems.

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

A dissertation submitted to the Department of Engineering-Economic Systems and the Committee on Graduate Studies of Stanford University in partial fulfillment of the Requirements for the Degree of Doctor of Philosophy, August 1976.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

DECISION ANALYSIS; DYNAMICS OF INFORMATION; INFORMATION REPLENISHMENT; VALUE OF INFORMATION; INFORMATION PERISHING; MILITARY INTELLIGENCE; MARKOVIAN DECISION PROCESSES

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The dissertation treats the dynamics of a decision maker's value of information. There are two main parts, a section on the depreciation (perishing) of information and a section on the appreciation (replenishment) of information.

(Continued)

Block 20 (continued)

A notion, widely held by decision analysts but tenuously defined, is that the value of any specific information diminishes over time. This concept, termed information perishing, is rigorously defined and illustrated by the use of a Markov model in the first section of the study.

The main assertions of the section are:

1. Information perishing is inevitable (not only for the Markov model of information but for any state of information described by a probability distribution).

2. For the Markov model, the absolute value of the largest transient eigenvalue is an upper bound for the rate of information perishing.

3. The rate of perishing is a decreasing function of time.

A short transition section alters the basic decision model to allow an element of uncertainty for the exact timing of the decision. Basically, the new model of the decision process recognizes that many decisions in real life are "triggered" by events which may be described by some stochastic process. Without this uncertainty, the decision-maker could simply discount the value of information because of perishing and would reduce his problem to a static case. However, the uncertainty in timing forces consideration of optimal policies of information replenishment, the second main area of the thesis.

The major results of this section are:

1. Rules of optimality are developed for singly and multiple occurring decisions.

2. The optimality of periodic replenishment (under certain limiting conditions) is established.

3. The suggestion that some of the research results of reliability and maintainability theory may be applied to information replenishment strategy.

The thesis closes with the customary delineation of areas of further application and research.