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G-ORDERED FUNCTIONS, WITH APPLICATIONS IN STATISTICS. I. THEORY--ETC(U)
SEP 77 J C CONLON, R LEON, F PROSCHAN AFOSR-74-2581
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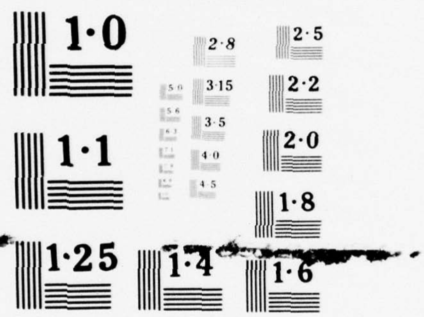
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G-ORDERED FUNCTIONS, WITH APPLICATIONS
IN STATISTICS, I. THEORY

by

J.C. Conlon[1], R. Leon[2], F. Proschan[1], J. Sethuraman[3]

FSU Statistics Report M432
AFOSR Technical Report No. 73
USARO-D Technical Report No. 25

September, 1977

The Florida State University
Department of Statistics
Tallahassee, Florida 32306

[1]Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Grant AFOSR-74-2581C. The United States Government is authorized to reproduce and distribute reprints for governmental purposes.

[2]Research supported by National Institute of Environmental Health Sciences under Grant 1-T32-ES07011 at Florida State University.

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[3]Research supported by the United States Army Research Office, Durham, under Grant No. DAAGG29-76-G-0238.

(See 1473)

ABSTRACT

This is Part I of a two-part paper which generalizes a rearrangement ordering, develops the theory of functions isotonic with respect to the more general ordering, and presents applications of this theory in statistics. Using the theory of reflection groups, we define reflection ordering (a generalization of transposition ordering) and G-ordered functions (a generalization of functions decreasing in transposition). (See Hollander, Proschan, and Sethuraman (Ann. Statist. 5, 1977, 722-733).) Reflection ordering is closely related to G-majorization (a point x G-majorizes a point y if y is an element of the convex hull of the G-orbit of x) and G-ordered functions contain G-monotone functions as special cases (G-monotone increasing functions preserve the G-majorization ordering). We develop many preservation properties for G-ordered functions and we prove a preservation theorem for G-monotone functions under an integral transform. In Part II we present applications in statistics.

1. Introduction and Summary.

In this two-part paper we generalize a rearrangement ordering, develop the theory of functions isotonic with respect to the more general ordering, and present applications of this theory in statistics. Hollander, Proschan, and Sethuraman (1977) define a rearrangement ordering, called transposition ordering, and the corresponding order-preserving functions, called functions decreasing in transposition (DT). Using the theory of reflection groups, we define reflection ordering as a generalization of transposition ordering. Functions which preserve reflection ordering are called G-ordered functions and this class of functions contains the class of DT functions as a special case.

This two-part paper continues the unification of the theory of stochastic comparisons. Earlier work in this area had made use of the majorization ordering (closely related to transposition ordering) and Schur functions (special cases of DT functions).

Majorization is a well-known partial ordering on Euclidean n-space and Schur-convex functions preserve the ordering. Hardy, Littlewood, and Pólya (1952), Bechenbach and Bellman (1961), Mitrinović (1970), and Berge (1963) provide many of the classical results in this area. Various authors have used majorization and Schur functions to obtain inequalities useful in probability and statistics. See, for instance, Marshall and Proschan (1965), Marshall, Olkin, and Proschan (1967), Marshall and Olkin (1974), Proschan and Sethuraman (1977), and Nevius, Proschan, and Sethuraman (1977). Galambos (1971) proves majorization results for vectors of probabilities of Boolean functions of events; Marshall, Walkup, and Wets (1967) study order-preserving functions with applications to majorization and order statistics; and Eaton (1970) uses majorization and Schur functions to establish expectation inequalities for sums of symmetric Bernoulli random variables. In addition, Olkin (1972) and Wong and Yue (1973) establish inequalities for the multinomial distribution based on majorization between parameter vectors. Hollander, Proschan, and Sethuraman (1977) use DT functions to obtain a preservation theorem for Schur functions under an integral transform. They develop many properties of DT functions and obtain from these properties some useful results in probability and statistics.

Using the theory of reflection groups, Eaton and Perlman (1976) introduce a partial ordering on Euclidean n-space, called G-majorization, which contains the majorization ordering as a special case. They define G-monotone increasing functions which preserve the G-majorization ordering. We use G-ordered functions to prove a convolution result for G-monotone decreasing functions and also to establish a preservation theorem for G-monotone functions under an integral transform.

In Section 2 we define reflection ordering for elements of a reflection group G and for elements of V, a linear subspace of Euclidean n-space. A key property of a reflection group is that it can be decomposed into finite reflection groups and orthogonal groups. This simplifies the problem of establishing preservation properties for G-ordered functions.

In Section 3, we define functions on the group G, on a space V, and on V^2 which preserve reflection ordering. We term these functions, G-ordered functions, and we prove many preservation properties for them. The composition theorem for G-ordered functions highlights this section. It is reminiscent of the composition theorem for TP functions found in Karlin (1968) and is of use in further developing the theory of G-ordered functions.

In Section 4 we relate reflection ordering to the G-majorization ordering of Eaton and Perlman (1976) and show that G-monotone functions are a special case of G-ordered functions. Using the properties of G-ordered functions, we establish a preservation theorem for G-monotone functions under an integral transform. We show that much of the theory of G-monotone functions is subsumed under the theory of G-ordered functions.

2. Reflection Groups and Reflection Ordering.

In this section we introduce the notion of reflection ordering for any arbitrary reflection group. We use the notion of a fundamental region in Euclidean n-space with respect to a finite reflection group. (See Benson and Grove (1971), pp. 27-33.) We derive an analogous notion of a fundamental region for the orthogonal group on any subspace of Euclidean n-space. In this case, the region is a closed set. We combine the above-mentioned notions along with a key proposition of Eaton and Perlman (1976) to define a closed fundamental region for any arbitrary reflection group. Each distinct closed fundamental region defines a partial ordering, called reflection ordering, on the elements of the group. We present a short summary of the derivation of reflection groups and fundamental regions for finite reflection groups. Following that is a derivation of a closed fundamental region for the orthogonal group which leads to the definition of reflection ordering on any arbitrary reflection group with respect to a closed fundamental region. We conclude this section with an example of reflection ordering: the well-known "transposition ordering" of Hollander, Proschan, and Sethuraman (1977).

Throughout this section and the rest of this paper R^n denotes Euclidean n-space. Elements of R^n are represented by column vectors and the transpose of a vector z is denoted by z'. The unit ball in R^n is denoted by B_n, i.e.

$B_n = \{x \in R^n : ||x|| = 1\}$, where $||x|| = \sqrt{x'x}$ is the usual Euclidean norm.

<u>Definition 2.1</u>. Suppose $r \in B_n$ and I_n is the $n \times n$ identity matrix. The matrix, $M_r = I_n - 2rr'$, is called the <u>reflection</u> defined by r.

Geometrically, M_r reflects points across the $(n-1)$-dimensional subspace of R^n perpendicular to r. Clearly $M_r = M_{-r} = M_r' = M_r^{-1}$. In particular, we note that $M_r \in O(R^n)$, the group of all $n \times n$ orthogonal matrices.

<u>Definition 2.2</u>. A closed subgroup G of $O(R^n)$ is called a <u>reflection group</u> if there exists a subset Δ_G^* of B_n such that G is the smallest closed subgroup of $O(R^n)$ containing the set of reflections $\{M_r : r \in \Delta_G^*\}$.

We call Δ_G^* a <u>generating system</u> of G. A minimal generating system of G is called a <u>set of fundamental roots</u> of G.

<u>Definition 2.3</u>. The root system of G, denoted Δ_G, is the set $\{r \in B_n : M_r \in G\}$.

For any given $r \in \Delta_G$, partition R^n into the following three subsets:

1. $H_r^+ = \{x \in R^n : r'x > 0\}$,
2. $H_r^- = \{x \in R^n : r'x < 0\}$,
3. $H_r^o = \{x \in R^n : r'x = 0\}$.

Since $M_r x = (I_n - 2rr')\, x = x - 2rr'x$ we note that $M_r x = x$ if and only if $x \in H_r^o$. Thus the set H_r^o is invariant under the transformation defined by the reflection M_r.

We now introduce the notion of a fundamental region for a finite reflection group. From now until we begin the discussion on the decomposition of reflection groups (Proposition 2.7), G will represent a <u>finite</u> reflection group. Define the set $T_G = \{t \in R^n : r't \neq 0$ for each $r \in \Delta_G\}$. Thus T_G is the complement of the set $\{\bigcup_{r\in\Delta_G} H_r^o\}$. When there is no possibility for ambiguity we will drop the subscript G in T_G. For a fixed $t \in T$, define the sets:

1. $\Delta_t^+ = \{r \in \Delta_G : r't > 0\}$,
2. $\Delta_t^- = \{r \in \Delta_G : r't < 0\}$.

Since $r \in \Delta_G$ if and only if $-r \in \Delta_G$, Δ_t^+ and Δ_t^- partition Δ_G into two sets of the same cardinality.

We call Δ_t^+ the set of t-positive roots and we note two useful properties relating to positive roots.

1. For every $g \in G$, $g\Delta_t^+ = \Delta_{gt}^+$.

2. The equality, $\Delta_{gt}^+ = \Delta_t^+$, holds if and only if g is the identity element of G.

For proofs of the above two statements, see Propositions 4.2.2 and 4.2.3 of Benson and Grove (1971).

We partition T into certain regions, termed fundamental regions, by means of the equivalence relation defined below. The equivalence relation is based on the set of positive roots.

<u>Definition 2.4</u>. Suppose $t,s \in T$. Then t is <u>equivalent</u> to s (in symbols, $t \sim s$) if $\Delta_t^+ = \Delta_s^+$.

<u>Definition 2.5</u>. Suppose $t \in T$. The <u>fundamental region</u> F defined by t is the set $\{s \in T : t \sim s\}$.

It is evident that for $t,s \in T$, if $t \sim s$, then t and s define the same fundamental region. For any $t \in T$, gt defines a different fundamental region for each distinct $g \in G$. To see this, note that t is not equivalent to gt for $I_n \neq g \in G$. This is true since $\Delta_{gt}^+ = \Delta_t^+$ if and only if g is the identity element of G as claimed in statement 2 above. Note then that the number of distinct fundamental regions and the number of elements of the group G are equal.

In light of the definition of fundamental regions and the above assertions, one can easily perceive the following properties of any fundamental region F for a finite reflection group G. (See Benson and Grove (1971), p. 27.)

1. F is an open set in R^n.

2. $F \cap gF = \emptyset$ if g is not the identity element of G.

3. $R^n = \cup\{g\overline{F} : g \in G\}$, where $\overline{F}$ is the closure of F in R^n.

Thus the fundamental regions $\{gF : g \in G\}$ are the equivalence classes under the equivalence relation presented in Definition 2.4.

We now present an analogous notion of a closed fundamental region for any arbitrary, not necessarily finite, reflection group. We must first define a notion of a closed fundamental region for $O(V)$, the group of all orthogonal transformations on V, a linear subspace of R^n.

Definition 2.6. Let V be a linear subspace of R^n and suppose that $r \in B_n \cap V$. The closed fundamental region $\overline{F}$ for $O(V)$ defined by r is the set $\{x \in V : x = \alpha r, \alpha > 0\}$.

The region $\overline{F}$ defined above depends intrinsically on the point $r \in B_n \cap V$, but we shall suppress reference to that dependence except where ambiguity may result.

Throughout this section and the rest of this paper V is a linear subspace of R^n. Eaton and Perlman (1976) show that any infinite reflection group acting irreducibly on V is the entire orthogonal group acting on V. A group G is said to act irreducibly on a space V if V contains no proper G-invariant subspace. We make use of the following proposition of Eaton and Perlman (1976) to define reflection ordering for any arbitrary reflection group.

Proposition 2.7. (Eaton and Perlman (1976)). Suppose $G \subseteq O(R^n)$ is a reflection group acting on R^n. Then G is isomorphic with $G_1 \times G_2 \times \dots \times G_k$ acting on $V_1 \oplus V_2 \oplus \dots \oplus V_k$ $(1 \le k \le n)$, where $V_1, V_2, \dots, V_k$ are mutually orthogonal subspaces of R^n with $\sum_{i=1}^{k}$ dimension $(V_i) = n$, and G_i is a reflection group acting irreducibly on V_i.

Definition 2.8. Let G be a reflection group acting on V such that G is isomorphic with $G_1 \times G_2 \times \dots \times G_k$ acting on $V_1 \oplus V_2 \oplus \dots \oplus V_k$. Let $\overline{F}_i$ be a closed fundamental region in V_i for G_i, $i = 1,2,\dots,k$. Then $\overline{F} = \overline{F}_1 \oplus \overline{F}_2 \oplus \dots \oplus \overline{F}_k$ is a closed fundamental region in V for G.

We now begin our discussion of reflection ordering for an arbitrary reflection group with respect to a closed fundamental region. We define reflection ordering for a finite reflection group (Definition 2.10), then for the orthogonal group (Definition 2.12), and finally for any arbitrary reflection group (Definition 2.14) using Proposition 2.7.

Let G be a fixed finite reflection group. In order to define reflection ordering on the group G we present a partition of G. For any fundamental region F, the set $\Delta_F^+ \subseteq \Delta_G$ is the set of F-positive roots; i.e. $\Delta_F^+ = \{r \in \Delta_G : r't > 0$ for all $t \in F\}$. Fix a root $r \in \Delta_G$ and let gF be some fundamental region. Then $r \in \Delta_{gF}^+$ or $r \in \Delta_{gF}^-$. For the given fixed r we partition G into the sets G_r^+ and G_r^-, where $G_r^+ = \{g \in G : r \in \Delta_{gF}^+\}$ and $G_r^- = \{g \in G : r \in \Delta_{gF}^-\}$. Technically, G_r^+ and G_r^- depend on the fundamental region F as well as on the root r. We suppress reference to F unless ambiguity may result.

Definition 2.9. Let G be a finite reflection group acting on V, let F be a fundamental region in V for G, and suppose that $r \in \Delta_F^+$. Then g is r-larger than $M_r g$ (in symbols, $g \overset{r}{\geq} M_r g$) if and only if $g \in G_r^+$.

Note that if $g \in G_r^-$, then g is r^*-larger than $M_{r^*}g$ where $r^* = -r$.

Definition 2.10. Let G be a finite reflection group acting on V, let F be a fundamental region in V for G, and suppose that $g_1,g_2 \in G$. If there exists a sequence $h_o, h_1, \ldots, h_m$ in G satisfying $g_1 = h_o \overset{r_1}{\geq} h_1 \overset{r_2}{\geq} \ldots \overset{r_m}{\geq} h_m = g_2$, where $r_i \in \Delta_F^+$, $i = 1,2,\ldots,m$, then g_1 is <u>$\overline{F}$-larger</u> than g_2 (in symbols, $g_1 \overset{\overline{F}}{\geq} g_2$).

Definition 2.10 presents reflection ordering on the elements of the group G. We now define reflection ordering as a partial ordering on the space V.

The <u>G-orbit</u> of a point $x \in V$ is the set $\{gx : g \in G\}$.

<u>Definition 2.11</u>. Let G and F be as in Definition 2.10. Suppose $x_1, x_2 \in V$ and they also belong to each other's orbit, i.e., $x_2 = gx_1$ for some $g \in G$. Then there exists $x \in \overline{F}$ such that $x_1 = g_1 x$ and $x_2 = g_2 x$ for some $g_1, g_2 \in G$. We say that x_1 is <u>$\overline{F}$-larger</u> than x_2 (in symbols, $x_1 \overset{\overline{F}}{\geq} x_2$) if $g_1 \overset{\overline{F}}{\geq} g_2$.

We now define reflection ordering for the orthogonal group.

<u>Definition 2.12</u>. Suppose $r \in B_n \cap V$ and $g_1, g_2 \in O(V)$. The closed fundamental region $\overline{F}$ defined by r is the set $\{x \in V : x = \alpha r, \alpha > 0\}$. If $x' g_1 x \geq x' g_2 x$ for all $x \in \overline{F}$, then g_1 is <u>$\overline{F}$-larger</u> than g_2 (in symbols, $g_1 \overset{\overline{F}}{\geq} g_2$).

Note that reflection ordering on the elements of $O(V)$ is actually complete. By a simple extension we define reflection ordering on the space V for $O(V)$.

<u>Definition 2.13</u>. Let $\overline{F}$ be as in Definition 2.12 and suppose that $g_1, g_2 \in O(V)$. For any $x \in \overline{F}$ define $x_1 = g_1 x$ and $x_2 = g_2 x$. Then x_1 is <u>$\overline{F}$-larger</u> than x_2 (in symbols, $x_1 \overset{\overline{F}}{\geq} x_2$) if $g_1 \overset{\overline{F}}{\geq} g_2$.

Note that when x_1 and x_2 are elements of the same orbit, the relation $x_1 \overset{\overline{F}}{\geq} x_2$ holds if and only if $u'(x_1 - x_2) \geq 0$ for all $u \in \overline{F}$.

We now define reflection ordering for any arbitrary reflection group.

Definition 2.14. Let G be a reflection group acting on V such that G is isomorphic with $G_1 \times G_2 \times \ldots \times G_k$ acting on $V_1 \oplus V_2 \oplus \ldots \oplus V_k$. Suppose $g_i, h_i \in G_i$, $i = 1,2,\ldots,k$, and define $g = g_1 \oplus g_2 \oplus \ldots \oplus g_k$ and $h = h_1 \oplus h_2 \oplus \ldots \oplus h_k$. If $g_i \overset{\overline{F}_i}{\geq} h_i$, $i = 1,2,\ldots,k$, then we say that g is $\overline{F}$-larger than h (in symbols, $g \overset{\overline{F}}{\geq} h$), where $\overline{F} = \overline{F}_1 \oplus \overline{F}_2 \oplus \ldots \oplus \overline{F}_k$.

We conclude this section with an example of reflection ordering, the well-known "transposition ordering" of Hollander, Proschan, and Sethuraman (1977). Let the group G be P_n, the group of all permutation matrices acting on R^n. A generating system of G, Δ_G^*, is the set $\{r_i : i = 1,2,\ldots,n-1\}$, where $r_i' = (0,\ldots,0,-1/\sqrt{2},1/\sqrt{2},0,\ldots,0)$ with $-1/\sqrt{2}$ and $1/\sqrt{2}$ being the i^{th} and $(i+1)^{st}$ coordinates respectively of r_i. A root system of G, Δ_G, is the set $\{\pm r_{ij} : i = 1,2,\ldots,j-1;\ j = 2,3,\ldots,n\}$, where $r_{ij}' = (0,\ldots,0,-1/\sqrt{2},0,\ldots,0,1/\sqrt{2},0,\ldots,0)$ with $-1/\sqrt{2}$ and $1/\sqrt{2}$ being the i^{th} and j^{th} coordinates respectively of r_{ij}. The G-orbit of a point $x \in R^n$ is the set of points defined by the $n!$ permutations of the coordinates of x.

Let the fundamental region F in R^n be the set $\{x \in R^n : x_1 < x_2 < \ldots < x_n\}$. Since for $i < j$, the j^{th} coordinate of any $x \in F$ is larger than the i^{th} coordinate, Δ_F^+ is the set $\{+ r_{ij} : i = 1,2,\ldots,j-1;\ j = 2,3,\ldots,n\}$. The set $G_{r_{ij}}^+$ for any $r_{ij} \in \Delta_G$ contains any permutation matrix g such that for $x \in F$, the i^{th} coordinate of gx is less than the j^{th} coordinate. This is obvious, since $r_{ij}'gx$ is the j^{th} coordinate of gx less the i^{th} coordinate. For $x \in F$, gx is r_{ij}-larger than $M_{r_{ij}} gx$ means that the i^{th} coordinate of gx is smaller than the j^{th} coordinate. The point $M_{r_{ij}} gx$ is a permutation of the i^{th} and j^{th} coordinates of gx. Consequently it is easy to see that reflection ordering for P_n with respect to the fundamental region $\{x \in R^n : x_1 < x_2 < \ldots < x_n\}$ is actually the transposition ordering of Hollander, Proschan, and Sethuraman (1977).

3. G-ordered Functions.

In this section we define functions, termed G-ordered functions, which are isotonic with respect to reflection ord ring. Functions on the group G, functions on V, and functions on V^2 may have the G-ordered property. Although the G-ordered property is essentially a property of functions on the group G, it is more convenient for theoretical development and practical applications to formulate the G-ordered property for functions on V and V^2.

G-ordered functions contain as a special case functions decreasing in transposition (DT). (See Hollander, Proschan, and Sethuraman (1977).) We establish some basic preservation properties for G-ordered functions. For example, we show that the G-ordered property is preserved under mixtures with respect to a positive measure and under composition with respect to a G-invariant measure. The product of a finite number of nonnegative G-ordered functions is G-ordered. Preservation under composition is particularly useful in further developing the properties of G-ordered functions.

Definition 3.1. Let G be a reflection group acting on V and let $\overline{F}$ be a closed fundamental region in V for G. A function f from G into R^1 is

G-ordered with respect to $\overline{F}$ if $g_1 \overset{\overline{F}}{\geq} g_2$ implies $f(g_1) \geq f(g_2)$, for $g_1, g_2 \in G$.

Definition 3.2. A subset X of R^n is said to be G-invariant if $gX \subseteq X$ for all $g \in G$.

Throughout this section G will be a fixed reflection group acting on V, a linear subspace of R^n, and $\overline{F}$ will be a closed fundamental region in V for G. The sets Λ and X, with or without subscripts, will denote G-invariant subsets of V.

Definition 3.3. A function f from X into R^1 is G-ordered with respect to $\overline{F}$ if for every $x \in \overline{F} \cap X$ and for every pair $g_1, g_2 \in G$ such that $g_1 \overset{\overline{F}}{\geq} g_2$, we have $f(g_1 x) \geq f(g_2 x)$.

Note that if f is G-ordered with respect to $\overline{F}$ on X, then whenever $x_1 \overset{\overline{F}}{\geq} x_2$, we have that $f(x_1) \geq f(x_2)$.

Definition 3.4. A function K from $\Lambda \times X$ into R^1 is G-ordered if the following two conditions hold.

(i). $K(g\lambda, gx) = K(\lambda, x)$ for all $g \in G$.

(ii). For every closed fundamental region $\overline{F}$, whenever $\lambda \in \overline{F} \cap \Lambda$, $x \in \overline{F} \cap X$, and $g_1 \overset{\overline{F}}{\geq} g_2$, then $K(\lambda, g_1) \geq K(\lambda, g_2 x)$.

Remark 3.5. Note that condition (i) above can be replaced by:

(i*). $K(M_r\lambda, M_r x) = K(\lambda, x)$ for all r in a set of fundamental roots for G.

The following lemma demonstrates the connections among G-ordered functions on the group G, on X, and on $\Lambda \times X$.

Lemma 3.6. Let $K(g\lambda, gx) = K(\lambda, x)$ for all $g \in G$. Define

(a). $\tilde{K}(x, \lambda) = K(\lambda, x)$ for $\lambda \in \Lambda$, $x \in X$.

(b). $f_\lambda(x) = K(\lambda,x)$ for $\lambda \in \Lambda$, $x \in \mathcal{X}$.

(c). $h_{\lambda,x}(g) = K(\lambda,gx)$ for $\lambda \in \widetilde{\tilde{g}F} \cap \Lambda$, for $x \in \widetilde{\tilde{g}F} \cap \mathcal{X}$, for all $g \in G$, and for some $\tilde{g} \in G$.

Then the following statements are equivalent:

(1). K is G-ordered on $\Lambda \times \mathcal{X}$.

(2). $\tilde{K}$ is G-ordered on $\mathcal{X} \times \Lambda$.

(3). f_λ is G-ordered with respect to $\overline{F}$ on $\mathcal{X}$ for each $\lambda \in \overline{F} \cap \Lambda$.

(4). $h_{\lambda,x}$ is G-ordered with respect to $\widetilde{\tilde{g}F}$ on G for each $\lambda \in \widetilde{\tilde{g}F} \cap \Lambda$ and each $x \in \widetilde{\tilde{g}F} \cap \mathcal{X}$.

The equivalence follows directly from the definitions of G-ordered functions on G, on $\mathcal{X}$, and on $\Lambda \times \mathcal{X}$.||

We now present some preservation properties for G-ordered functions. The proofs of Propositions 3.7, 3.8, and 3.9 below parallel the proofs of corresponding results in Hollander, Proschan, and Sethuraman (1977), so we omit them.

<u>Proposition 3.7</u>. Let K be G-ordered on $\Lambda \times \mathcal{X}$ and let f and h be non-negative G-invariant functions on Λ and $\mathcal{X}$ respectively. Then $f(\lambda)\, K(\lambda,x)\, h(x)$ is G-ordered on $\Lambda \times \mathcal{X}$.

<u>Proposition 3.8</u>. Let $(\Omega,\mathcal{F},\nu)$ be a positive measure space. Suppose that $K_\omega(\lambda,x)$ is G-ordered on $\Lambda \times \mathcal{X}$ for each $\omega \in \Omega$, and suppose that for all $(\lambda,x) \in \Lambda \times \mathcal{X}$, $K_\omega(\lambda,x) \in L^1(\Omega,\mathcal{F},\nu)$. Then $\int_\Omega K_\omega(\lambda,x)d\nu(\omega)$ is G-ordered on $\Lambda \times \mathcal{X}$.

A similar result for mixtures holds for functions G-ordered with respect to $\overline{F}$ on G and on $\mathcal{X}$.

Consider any function $\phi(\lambda,x)$ defined by $\phi(\lambda,x) = c(\lambda)h(x) \exp\left\{\sum_{i=1}^{k} K_i(\lambda,x)\right\}$.

Using Proposition 3.7, Proposition 3.8 with the counting measure, and the fact that increasing functions of G-ordered functions are G-ordered, we may show that ϕ is G-ordered if c and h are G-invariant and K_i is G-ordered, $i = 1,2,\ldots,k$. Note that densities belonging to the multivariate exponential family are special cases of this form.

Note that if K is G-ordered on $\Lambda \times X$, then K is G-ordered on $\Lambda^* \times X^*$, where Λ^* and X^* are any G-invariant subsets of Λ and X respectively. Thus if K, a G-ordered function on $\Lambda \times X$, is the density of a random vector X and u is a G-invariant function on X, then the conditional density of X given $u(X) = u_o$, K_{u_o}, is G-ordered on $\Lambda \times X_o$, where $X_o = \{x \in X : u(x) = u_o\}$.

<u>Proposition 2.9</u>. The product of nonnegative G-ordered functions is G-ordered.

<u>Definition 2.10</u>. A measure μ on X is <u>G-invariant</u> if $\mu(A \cap X) = \mu(gA \cap X)$ for any $g \in G$ and any Borel set A in R^n.

We now present a composition theorem for G-ordered functions on $\Lambda \times X$. We establish first the composition result for G-ordered functions with respect to a G-invariant measure μ for G, a finite reflection group. Then we show the composition result for the orthogonal group on V. Recall that any reflection group is isomorphic with a direct product of groups each of which is either an orthogonal group or a finite reflection group. The composition result for arbitrary reflection groups follows immediately.

<u>Lemma 3.11</u>. Let G be a finite reflection group and suppose that K_1 is G-ordered on $X_1 \times X$ and K_2 is G-ordered on $X \times X_2$. Let $K(x,z) = \int K_1(x,y)K_2(y,z)d\mu(y)$, where the integral is assumed to exist for each $x \in X_1$ and each $z \in X_2$ and μ is a G-invariant measure on X. Then K is G-ordered on $X_1 \times X_2$.

Proof. (i). We show that $K(gx,gz) = K(x,z)$ for all $g \in G$. Suppose $g \in G$. Then

$$K(gx,gz) = \int K_1(gx,y)\, K_2(y,gz)\, d\mu(y)$$
$$= \int K_1(gx,gy)\, K_2(gy,gz)\, d\mu(gy)$$
$$= \int K_1(x,y)\, K_2(y,z)\, d\mu(y)$$
$$= K(x,z), \text{ as desired.}$$

(ii). Suppose $x \in \overline{F} \cap X_1$, $z \in \overline{F} \cap X_2$, and $g_1 \overset{\overline{F}}{\geq} g_2$. We need to show that $K(x,g_1 z) \geq K(x,g_2 z)$. Since G is finite, it suffices to show that $K(x,z) \geq K(x,M_r z)$ for every $r \in \Delta_F^+$. Suppose $r \in \Delta_F^+$. Then

$$K(x,z) - K(x,M_r z) = \int_X K_1(x,y)\, [K_2(y,z) - K_2(y,M_r z)]\, d\mu(y)$$
$$= \int_{H_r^+ \cap X} K_1(x,y)\, [K_2(y,z) - K_2(y,M_r z)]\, d\mu(y) \tag{1}$$
$$+ \int_{H_r^- \cap X} K_1(x,y)\, [K_2(y,z) - K_2(y,M_r z)]\, d\mu(y) \tag{2}$$
$$+ \int_{H_r^o \cap X} K_1(x,y)\, [K_2(y,z) - K_2(y,M_r z)]\, d\mu(y). \tag{3}$$

Since $K_2(y,z) - K_2(y,M_r z) = 0$ for all $y \in H_r^o \cap X$, we drop (3). We use the transformation $y = M_r u$ and invoke the G-invariance property of μ to conclude that (2) is equal to:

$$\int_{H_r^+ \cap X} K_1(x,M_r u)\, [K_2(u,M_r z) - K_2(u,z)]\, d\mu(u). \tag{2*}$$

We now combine (1) and (2*) and factor the integrand to obtain that

$K(x,z) - K(x,M_r z) =$

$$\int_{H_r^+ \cap X} [K_1(x,y) - K_1(x,M_r y)]\, [K_2(y,z) - K_2(y,M_r z)]\, d\mu(y).$$

Both factors of the integrand are nonnegative for $y \in H_r^+ \cap X$, so that $K(x,z) - K(x,M_r z) \geq 0$, as desired.||

<u>Lemma 3.12</u>. Let G be the orthogonal group acting on V and suppose that K_1 is G-ordered on $X_1 \times X$ and K_2 is G-ordered on $X \times X_2$. Define $K(x,z) = \int K_1(x,y)\, K_2(y,z)\, d\mu(y)$, where the integral is assumed to exist for each $x \in X_1$ and each $z \in X_2$ and μ is a G-invariant measure on X. Then K is G-ordered on $X_1 \times X_2$.

<u>Proof</u>. (i). The proof that $K(gx,gz) = K(x,z)$ for all $g \in G$ is analogous to the proof presented for Lemma 3.11.

(ii). Suppose $z'z = \tilde{z}'\tilde{z}$ and $x'(z - \tilde{z}) \geq 0$. Now $\tilde{z} = gz$ for some $g \in G$; thus we wish to show that $K(x,z) - K(x,\tilde{z}) = K(x,z) - K(x,gz) \geq 0$. Let M_r be the reflection matrix generated by $r = (z - gz)/||z - gz||$ and let H_r^o be the hyperplane perpendicular to r. Note that $M_r \in G$ and that $M_r z = gz$. Write

$$\begin{aligned}
K(x,z) - K(x,gz) &= \int_X K_1(x,y)\, [K_2(y,z) - K_2(y,gz)]\, d\mu(y) \\
&= \int_{H_r^+ \cap X} K_1(x,y)\, [K_2(y,z) - K_2(y,gz)]\, d\mu(y) \\
&+ \int_{H_r^- \cap X} K_1(x,y)\, [K_2(y,z) - K_2(y,gz)]\, d\mu(y) \\
&+ \int_{H_r^o \cap X} K_1(x,y)\, [K_2(y,z) - K_2(y,gz)]\, d\mu(y) \\
&= \int_{H_r^+ \cap X} \{K_1(x,y)\, [K_2(y,z) - K_2(y,gz)] + \\
&\qquad K_1(x,M_r y)\, [K_2(M_r y,z) - K_2(M_r y,gz)]\}\, d\mu(y). \qquad (1)
\end{aligned}$$

In the above we have used the transformation $y = M_r u$, the G-invariance of μ, and the fact that

$$\int_{H_r^o \cap X} K_1(x,y)\,[K_2(y,z) - K_2(y,gz)]\,d\mu(y) = 0.$$

Since $K_2(M_r y,gz) = K_2(M_r y,M_r z) = K_2(y,z)$ and $K_2(y,gz) = K_2(y,M_r z) = K_2(M_r y,z)$, we write (1) as:

$$\int_{H_r^+ \cap X} [K_1(x,y) - K_1(x,M_r y)]\,[K_2(y,z) - K_2(y,gz)]\,d\mu(y). \tag{1*}$$

Now $x'(y - M_r y) = x'rr'(y - M_r y) = ||z - gz||^{-2}\,[x'(z - gz)]\,[(z - gz)'\,(y - M_r y)] \geq 0$ for $y \in H_r^+ \cap X$. Also we have that $y'(z - gz) \geq 0$ for $y \in H_r^+ \cap X$. Consequently both factors of the integrand in (1*) are nonnegative, so that $K(x,z) - K(x,gz) \geq 0$, as desired.||

Theorem 3.13. Let G be an arbitrary reflection group and suppose that K_1 is G-ordered on $X_1 \times X$ and K_2 is G-ordered on $X \times X_2$. Define $K(x,z) = \int K_1(x,y)\,K_2(y,z)\,d\mu(y)$, where the integral is assumed to exist for each $x \in X_1$ and each $z \in X_2$ and μ is a G-invariant measure. Then K is G-ordered on $X_1 \times X_2$.

Proof. (i). The proof that $K(gx,gz) = K(x,z)$ for all $g \in G$ is analogous to the proof presented for Lemma 3.11.

(ii). Suppose G is isomorphic with $G_1 \times G_2 \times \ldots \times G_k$ acting on $V_1 \oplus V_2 \oplus \ldots \oplus V_k$. Then for $j = 1,2$, $X_j = X_j^{(1)} \oplus X_j^{(2)} \oplus \ldots \oplus X_j^{(k)}$, with $X_j^{(i)}$ a G_i-invariant subset of V_i, $i = 1,2,\ldots,k$. Let $\overline{F}$ be a closed fundamental region in V for G. Suppose $x \in \overline{F} \cap X_1$, $z \in \overline{F} \cap X_2$, and $g_1 \overset{\overline{F}}{\geq} g_2$. Define

$z_1 = g_1 z$ and $z_2 = g_2 z$. For $j = 1,2$, write $z_j = z_j^{(1)} + z_j^{(2)} + \ldots + z_j^{(k)}$ with $z_j^{(i)} \in X_j^{(i)}$, $i = 1,2,\ldots,k$. Define the k intermediate points as follows: $\tilde{z}^{(1)} = z_2^{(1)} + z_1^{(2)} + \ldots + z_1^{(k)}$, $\tilde{z}^{(2)} = z_2^{(1)} + z_2^{(2)} + z_1^{(3)} + \ldots + z_1^{(k)}$, $\ldots, \tilde{z}^{(k)} = z_2^{(1)} + z_2^{(2)} + \ldots + z_2^{(k)}$. The closed fundamental region $\overline{F} = \overline{F}_1 \oplus \overline{F}_2 \oplus \ldots \oplus \overline{F}_k$. Now $z_1^{(i)} \overset{\overline{F}_i}{\geq} z_2^{(i)}$, $i = 1,2,\ldots,k$, so that $K(x,\tilde{z}^{(i)}) \geq K(x,\tilde{z}^{(i+1)})$ for $i = 1,2,\ldots,k-1$ as a consequence of Lemma 3.11 if G_i is a finite reflection group or as a consequence of Lemma 3.12 if G_i is the orthogonal group acting on V_i. It follows that $K(x,z_1) \geq K(x,\tilde{z}^{(1)}) \geq \ldots \geq K(x,\tilde{z}^{(k)}) = K(x,z_2)$, as desired.||

The following two corollaries represent preservation results for G-ordered functions on the group G and on the set X. The proofs follow directly from Lemma 3.6 and Theorem 3.13, so we omit them.

Corollary 3.14. Let G be a reflection group and let μ be a uniform measure on G. Let f_1 and f_2 be G-ordered with respect to $\overline{F}$ on G and define $f(g) = \int f_1(g^{-1}g_o)\, f_2(g_o)\, d\mu(g_o)$. Then f is G-ordered with respect to $\overline{F}$ on G.

Corollary 3.15. Let K be G-ordered on $\Lambda \times X$ and let f be G-ordered with respect to $\overline{F}$ on X. Define $h(\lambda) = \int K(\lambda,x)\, f(x)\, d\mu(x)$, where μ is a G-invariant measure on X. Then h is G-ordered with respect to $\overline{F}$ on Λ.

4. G-majorization and G-monotonicity.

G-majorization is a partial ordering on R^n introduced by Eaton and Perlman (1976). G-monotone functions are isotonic with respect to this ordering. In this section we relate reflection ordering to the G-majorization ordering and show that G-monotone functions are special cases of G-ordered functions. We use the properties of G-ordered functions to establish a convolution theorem for G-monotone decreasing functions and also to obtain the preservation of G-monotonicity under the integral transform:

$$h(\lambda) = \int K(\lambda,x)\, f(x)\, d\mu(x).$$

We supply a brief summary of relevant parts of the work of Eaton and Perlman (1976).

The well-known majorization ordering induces a partial ordering on R^n and Schur-convex functions are order preserving with respect to majorization. The G-majorization ordering of Eaton and Perlman (1976) includes majorization as a special case.

<u>Definition 4.1</u>. Let G be a closed subgroup of $O(R^n)$. For $x,y \in R^n$, the point x is said to <u>G-majorize</u> y (in symbols, $x \overset{G}{\geq} y$) if y is an element of the convex hull of the G-orbit of x.

<u>Definition 4.2</u>. A function f from X, a subset of R^n, into R^1 is <u>G-monotone increasing (decreasing)</u> if $x \overset{G}{\geq} y$ implies $f(x) \geq (\leq)\, f(y)$.

When G is P_n, the permutation group, G-majorization coincides with the familiar majorization ordering. (See Eaton and Perlman (1976).) Consequently the class of G-monotone functions coincides with the class of Schur functions when G is the permutation group.

When G is a finite group, there exists a polygonal path from a point x to any point in the convex hull of the G-orbit of x. This is a generalization of the famous path lemma for majorization of Hardy, Littlewood, and Pólya (1952, p. 47). Stated formally:

Lemma 4.3. (Eaton and Perlman (1976)). Let G be a finite reflection group. Suppose $x \overset{G}{\geq} y$, $x \neq y$. Then there exists a sequence of points $z_o, z_1, \ldots, z_m$ such that $z_o = y$, $z_m = x$, and

$$z_{j-1} = [\lambda_j I_n + (1 - \lambda_j) M_{r_j}] z_j, \; 1 \leq j \leq m,$$

where $r_j \in \Delta_G$, $0 \leq \lambda_j < 1$, and I_n is the $n \times n$ identity matrix.

Note that $z_j \overset{G}{\geq} z_{j-1}$ for $j = 1,2,\ldots,m$.

Before we show the relationship between G-ordered functions and G-monotone functions we establish some technical lemmas. We will use results for finite reflection groups and orthogonal groups to obtain results for arbitrary reflection groups. Proofs are omitted where uninstructive.

Lemma 4.4. Let G be a reflection group and suppose that $r \in \Delta_G$. Let $u_1, u_2, \ldots, u_n$ be an orthonormal basis for R^n such that $u_1 = r$. Suppose $x,y \in R^n$ and $u_i'x = u_i'y$, $i = 2,3,\ldots,n$. Then $x \overset{G}{\geq} y$ if and only if $|r'x| \geq |r'y|$.

Lemma 4.5. Let G be a finite reflection group and let $\overline{F}$ be a closed fundamental region in R^n for G. Then for $r \in \Delta_F^+$, we have that $g \overset{\overline{F}}{\geq} M_r g$ if and only if $r'gx \geq 0$ for all $x \in \overline{F}$.

Lemma 4.6. Let G be a finite reflection group and let $\overline{F}$ be a closed fundamental region in R^n for G. Then for $r \in \Delta_F^+$, the relation, $g \overset{\overline{F}}{\geq} M_r g$, holds if and only if $\lambda + gx \overset{G}{\geq} \lambda + M_r gx$ $(\lambda - M_r gx \overset{G}{\geq} \lambda - gx)$ for all λ such that $r'\lambda \geq 0$.

Proof. Without loss of generality assume that $g = I_n$. We show that $\lambda + x \overset{G}{\geq} \lambda + M_r x$ if and only if $(r'\lambda)(r'x) \geq 0$. Let $u_1, u_2, \ldots, u_n$ be an orthonormal basis for R^n such that $u_1 = r$. Now

$$\lambda + x = ((r'\lambda + r'x)\, r + \sum_{i=2}^{n} (u_i'\lambda + u_i'x)u_i)$$

and

$$\lambda + M_r x = ((r'\lambda - r'x)\, r + \sum_{i=2}^{n} (u_i'\lambda + u_i'x)u_i).$$

Thus $\lambda + x \overset{G}{\geq} \lambda + M_r x$ if and only if $|r'\lambda + r'x| \geq |r'\lambda - r'x|$ by Lemma 4.4. But $|r'\lambda + r'x| \geq |r'\lambda - r'x|$ if and only if $(r'\lambda)(r'x) \geq 0$.

As a consequence of Lemma 4.5, $I_n \overset{F}{\geq} M_r I_n$ if and only if $r'x \geq 0$. Under the assumption that $r'\lambda \geq 0$, $\lambda + x \overset{G}{\geq} \lambda + M_r x$ if and only if $r'x \geq 0$. Thus we conclude that $g \overset{\overline{F}}{\geq} M_r g$ if and only if $\lambda + gx \overset{G}{\geq} \lambda + M_r gx$ for all λ such that $r'\lambda \geq 0$.

The proof that $g \overset{\overline{F}}{\geq} M_r g$ if and only if $\lambda - M_r gx \overset{G}{\geq} \lambda - gx$ is analogous.||

Lemma 4.7. Let G be an orthogonal group and let $\overline{F}$ be a closed fundamental region in R^n for G. Then for $g_1, g_2 \in G$, the relation, $g_1 \overset{\overline{F}}{\geq} g_2$, holds if and only if $(\lambda + g_1 x)'(\lambda + g_2 x) \geq (\lambda + g_2 x)'(\lambda + g_2 x)$ $[(\lambda - g_1 x)'(\lambda - g_1 x) \leq (\lambda - g_2 x)'(\lambda - g_2 x)]$ for every λ, $x \in \overline{F}$.

Lemma 4.8. Let G be a reflection group acting on V, let $\overline{F}$ be a closed fundamental region in V for G, and suppose that $r \in \Delta_F^+$. Then $g \overset{\overline{F}}{\geq} M_r g$ if and only if $\lambda + gx \overset{G}{\geq} \lambda + M_r gx$ for all $\lambda, x \in \overline{F}$.

Lemma 4.9. Suppose $r \in B_n$ and $z \in R^n$. Then for any α, $0 \le \alpha < 1$, there exist points $\lambda_\alpha, x_\alpha \in R^n$ such that $z = \lambda_\alpha + x_\alpha$, $(r'\lambda_\alpha)(r'x_\alpha) \ge 0$, and $(\alpha I_n + (1 - \alpha)M_r)(\lambda_\alpha + x_\alpha) = \lambda_\alpha + M_r x_\alpha$.

Theorem 4.10. Let G be a reflection group acting on V. Let $K(\lambda,x)$ be of the form $f(\lambda + x)$ $(f(\lambda - x))$. Then $K(\lambda,x)$ is G-ordered on V^2 if and only if $f(\lambda + x)$ $(f(\lambda - x))$ is G-monotone increasing (decreasing) on V.

Proof. We show that $K(\lambda,x)$ is G-ordered if and only if $f(\lambda + x)$ is G-monotone increasing. The proof that $K(\lambda,x)$ is G-ordered if and only if $f(\lambda - x)$ is G-monotone decreasing is analogous.

(i). For all $g \in G$, $K(g\lambda,gx) = f(g\lambda + gx) = f(g(\lambda + x))$. Thus $K(g\lambda,gx) = K(\lambda,x)$ if and only if $f(g(\lambda + x)) = f(\lambda + x)$ for all $g \in G$.

(iia). Let f be G-monotone increasing. Suppose that $\lambda,x \in \overline{F}$ and $r \in \Delta_F^+$. Then $I_n \overset{\overline{F}}{\ge} M_r I_n$, which implies that $\lambda + x \overset{G}{\ge} \lambda + M_r x$ by Lemma 4.8. Thus

$$K(\lambda,x) - K(\lambda,M_r x) = f(\lambda + x) - f(\lambda + M_r x) \ge 0.$$

(iib). Let K be G-ordered and suppose that $z_1 \overset{G}{\ge} z_2$. Suppose G is isomorphic with $G_1 \times G_2 \times \ldots \times G_k$ acting on $V_1 \oplus V_2 \oplus \ldots \oplus V_k$. Write $z_1 = z_1^{(1)} + z_1^{(2)} + \ldots + z_1^{(k)}$ and $z_2 = z_2^{(1)} + z_2^{(2)} + \ldots z_2^{(k)}$, so that $z_1^{(i)} \overset{G_i}{\ge} z_2^{(i)}$, $i = 1,2,\ldots,k$. Let J be a subset of $\{1,2,\ldots,k\}$ such that for all $i \in J$, G_i is a finite reflection group. Denote the subset of $\{1,2,\ldots,k\}$ for which G_i is an orthogonal group by J^c. For each $i \in J$ assume $z_2^{(i)} = (\alpha^{(i)}I + (1 - \alpha^{(i)})M_{r_i})\, z_1^{(i)}$, where $0 \le \alpha^{(i)} < 1$. By Lemma 4.9, there exist $\lambda_{\alpha(i)}$ and $x_{\alpha(i)}$ such that $z_1^{(i)} = \lambda_{\alpha(i)} + x_{\alpha(i)}$, $z_2^{(i)} = \lambda_{\alpha(i)} + M_{r_i} x_{\alpha(i)}$ and $(r_i'\lambda_{\alpha(i)})(r_i'x_{\alpha(i)}) \geq 0$. For each $i \in J^c$, write $z_1^{(i)} = \lambda^{(i)} + x_1^{(i)}$ and $z_2^{(i)} = \lambda^{(i)} + x_2^{(i)}$, where $\lambda^{(i)} = \dfrac{z_1^{(i)} + z_2^{(i)}}{2}$, $x_1^{(i)} = \dfrac{z_1^{(i)} - z_2^{(i)}}{2}$, and $x_2^{(i)} = \dfrac{z_2^{(i)} - z_1^{(i)}}{2}$. Now

$$\begin{aligned} f(z_1) - f(z_2) &= f(z_1^{(1)} + z_1^{(2)} + \dots + z_1^{(k)}) \\ &\quad - f(z_2^{(1)} + z_2^{(2)} + \dots + z_2^{(k)}) \\ &= f((\tilde{\lambda}^{(1)} + \tilde{x}_1^{(1)}) + (\tilde{\lambda}^{(2)} + \tilde{x}_1^{(2)}) + \dots + (\tilde{\lambda}^{(k)} + x_1^{(k)})) \\ &\quad - f((\tilde{\lambda}^{(1)} + \tilde{x}_2^{(1)}) + (\tilde{\lambda}^{(2)} + \tilde{x}_2^{(2)}) + \dots + (\tilde{\lambda}^{(k)} + \tilde{x}_2^{(k)})) \\ &= K(\tilde{\lambda}^{(1)} + \tilde{\lambda}^{(2)} + \dots + \tilde{\lambda}^{(k)}, \tilde{x}_1^{(1)} + \tilde{x}_1^{(2)} + \dots + x_1^{(k)}) \\ &\quad - K(\tilde{\lambda}^{(1)} + \tilde{\lambda}^{(2)} + \dots + \tilde{\lambda}^{(k)}, \tilde{x}_2^{(1)} + \tilde{x}_2^{(2)} + \dots + x_2^{(k)}) \\ &\geq 0, \end{aligned}$$

where $\tilde{\lambda}^{(i)} = \begin{cases} \lambda_{\alpha(i)} & i \in J \\ \lambda^{(i)} & i \in J^c \end{cases}$, $\tilde{x}_1^{(i)} = \begin{cases} x_{\alpha(i)} & i \in J \\ x_1^{(i)} & i \in J^c \end{cases}$, and $\tilde{x}_2^{(i)} = \begin{cases} M_{r_i} x_{\alpha(i)} & i \in J \\ x_2^{(i)} & i \in J^c \end{cases}$.

The inequality holds since $(r_i'\tilde{\lambda}^{(i)})(r_i'\tilde{x}_1^{(i)}) \geq 0$ for each $i \in J$ and $\lambda^{(i)\prime}\tilde{x}_1^{(i)} \geq \lambda^{(i)\prime}\tilde{x}_2^{(i)}$ for each $i \in J^c$. ||

<u>Definition 4.11</u>. A measure μ on V is said to be <u>translation invariant</u> if $\mu(A \cap V) = \mu((A + x) \cap V)$ for all Borel sets A in R^n and all $x \in V$.

Corollary 4.12. The convolution of G-monotone decreasing functions on R^n with respect to a translation invariant measure is G-monotone decreasing for G any reflection group acting on R^n.

Proof. Let f_1 and f_2 be G-monotone decreasing on R^n and define $h(x) = \int f_1(x - y)\, f_2(y)\, d\mu(y)$. Then

$$h(x - z) = \int f_1(x - z - y)\, f_2(y)\, d\mu(y)$$
$$= \int f_1(x - u)\, f_2(u - z)\, d\mu(u).$$

By Theorem 4.11, $f_1(x - u)$ and $f_2(u - z)$ are G-ordered on R^{2n}. By Theorem 3.13, $h(x - z)$ is G-ordered on R^{2n}. Thus we apply Theorem 4.11 again to conclude that h is G-monotone decreasing on R^n.||

Remark 4.13. For Corollary 4.12 it is not necessary that the functions be G-monotone decreasing on R^n. Suppose X is a subset of R^n such that the set $\mathcal{U} \overset{\text{def}}{=} \{u \in R^n : u = x + y;\ x,y \in X\}$ is G-invariant, then the convolution of G-monotone decreasing functions on X is G-monotone decreasing. This condition is satisfied if X forms a semigroup under addition, for then $\mathcal{U} \equiv X$.

Definition 4.14. Suppose Λ and X form semigroups under addition. A function K on $\Lambda \times X$ is said to have the G-ordered generalized semigroup property with respect to a translation invariant measure μ, if for $\lambda_1, \lambda_2 \in \Lambda$ and $x \in X$, there exist G-ordered functions K_1 and K_2 on $\Lambda \times X$ such that

$$K(\lambda_1 + \lambda_2, x) = \int K_1(\lambda_1, x - y)\, K_2(\lambda_2, y)\, d\mu(y).$$

We now state and prove the main preservation theorem for G-monotone functions under an integral transform.

Theorem 4.15. Let Λ, X be as in Definition 4.14 and let a function K on $\Lambda \times X$ have the G-ordered generalized semigroup property with respect to a G-invariant and translation invariant measure μ. Then $h(\lambda) = \int K(\lambda,x)\, f(x)\, d\mu(x)$ is G-monotone increasing (decreasing) on Λ if f is G-monotone increasing (decreasing) on X.

Proof. We show that f is G-monotone increasing implies that h is G-monotone increasing. We show that $h(\lambda + \lambda^*)$ is G-ordered on Λ^2 and conclude that h is G-monotone increasing on Λ using Theorem 4.10. Write

$$\begin{aligned} h(\lambda + \lambda^*) &= \int_X K(\lambda + \lambda^*, x)\, f(x)\, d\mu(x) \\ &= \int_X \int_X K_1(\lambda, x - y)\, K_2(\lambda^*, y)\, d\mu(y)\, f(x)\, d\mu(x) \\ &= \int_X K_2(\lambda^*, y) \int_X K_1(\lambda, x - y)\, f(x)\, d\mu(x)\, d\mu(y) \\ &= \int_X K_2(\lambda^*, y) \int_{X_y} K_1(\lambda, x)\, f(y + z)\, d\mu(z)\, d\mu(y), \end{aligned}$$

where $X_y = \{u \in R^n : u = x - y;\ x, y \in X\}$. Since X forms a semigroup under addition, $X_y \supseteq X$ for all $y \in X$. On the set $X_y - X$, $K_1(\lambda, \cdot)$ is zero; hence we replace X_y by X for the region of integration of the inside integral. Thus

$$h(\lambda + \lambda^*) = \int_X K_2(\lambda^*, y) \int_X K_1(\lambda, z)\, f(y + z)\, d\mu(z)\, d\mu(y).$$

We apply Theorem 3.13 to conclude that $\int_X K_1(\lambda, z)\, f(y + z)\, d\mu(z)$ is G-ordered on $\Lambda \times X$. We apply Theorem 3.13 again to conclude that $h(\lambda + \lambda^*)$ is G-ordered on Λ^2. Thus h is G-monotone increasing on Λ.

To show f G-monotone decreasing implies h G-monotone decreasing, we need only consider $-f$ which is G-monotone increasing and deduce that $-h$ is G-monotone increasing.||

Theorem 4.15 is an extension and generalization of a similar preservation theorem under an integral transform (Theorem 3.7) of Hollander, Proschan, and Sethuraman (1977). It yields their theorem as a special case when G is the permutation group, $K_1 = K_2 = K$, and the coordinates of points in Λ and $\mathcal{X}$ are positive real numbers or positive integers.

Definition 4.16. Let $\Lambda, \mathcal{X}$ be as in Definition 4.14. A function K on $\Lambda \times \mathcal{X}$ is said to have the G-ordered conditional generalized semigroup property with respect to a translation invariant measure μ, if there exists a σ-finite measure space $(\Omega, \mathcal{F}, \nu)$ and functions $K_\omega(\lambda,x)$, $\omega \in \Omega$, such that:

(i). $K(\lambda,x) = \int_\Omega K_\omega(\lambda,x)\, d\nu(\omega)$,

(ii). For each $\omega \in \Omega$, K_ω has the G-ordered generalized semigroup property with respect to μ.

Corollary 4.17. The conclusion Theorem 4.15 holds if $K(\lambda,x)$ now has the G-ordered conditional generalized semigroup property with respect to μ.

Proof. Let $h_\omega(\lambda) \overset{\text{def}}{=} \int_{\mathcal{X}} K_\omega(\lambda,x)\, f(x)\, d\mu(x)$. Then by Theorem 4.15, $h_\omega(\lambda)$ is G-monotone increasing (decreasing) on Λ for each $\omega \in \Omega$. Now

$$
\begin{aligned}
h(\lambda) &= \int_{\mathcal{X}} K(\lambda,x)\, f(x)\, d\mu(x) \\
&= \int_{\mathcal{X}} \int_{\Omega} K_\omega(\lambda,x)\, d\nu(\omega)\, f(x)\, d\mu(x) \\
&= \int_{\Omega} \int_{\mathcal{X}} K_\omega(\lambda,x)\, f(x)\, d\mu(x)\, d\nu(\omega) \\
&= \int_{\Omega} h_\omega(\lambda)\, d\nu(\omega).
\end{aligned}
$$

We apply the mixture result, Proposition 3.8, and Theorem 4.10 to conclude that $h(\lambda)$ is G-monotone increasing (decreasing).||

We present further extensions of Theorem 4.15 in Corollaries 4.18, 4.19, 4.20, and 4.21. The proofs are fairly routine, so we omit them.

Corollary 4.18. Let $\phi(\lambda,x)$ have the G-ordered generalized semigroup property on $\Lambda \times X$ with respect to a G-invariant and translation invariant measure μ. Let ℓ_1 and ℓ_2 be linear, G-invariant functions on Λ and X respectively. Define $h(\lambda) = \int \phi(\lambda,x)\, K(\ell_1(\lambda),\ell_2(x))\, f(x)\, d\mu(x)$, where K is a function on $\Lambda \times X$ only through ℓ_1 and ℓ_2. Then f G-monotone increasing (decreasing) on X implies h G-monotone increasing (decreasing) on Λ.

Corollary 4.19. Let ϕ and K be as in Corollary 4.18. Let T be a linear transformation from V into V such that $\lambda \overset{G}{\geq} \tilde{\lambda}$ if and only if $T\lambda \overset{G}{\geq} T\tilde{\lambda}$. Define $\tilde{K}(T\lambda,Tx) = \phi(\lambda,x)\, K(\ell_1(\lambda),\ell_2(x))$ and $h(T\lambda) = \int \tilde{K}(T\lambda,x)\, f(x)\, d\mu(x)$. Then f G-monotone increasing (decreasing) implies h G-monotone increasing (decreasing) on $T\Lambda$.

Corollary 4.20. Let ϕ and ψ have the G-ordered generalized semigroup property on $X_1 \times X_2$. Define $K(x,z) = \int \phi(x,y)\, \psi(z,y)\, d\mu_2(y)$ and $h(x) = \int K(x,z)\, f(z)\, d\mu_1(z)$ where μ_1 and μ_2 are G-invariant measures on X_1 and X_2 respectively. Then f G-monotone increasing (decreasing) on X_1 implies h G-monotone increasing (decreasing) on X_1.

Note that the conclusion of Corollary 4.20 holds as long as the preservation of G-monotonicity under an integral transform holds with ϕ and ψ as kernels of the transform.

Corollary 4.21. Let the random vector X have density $K(\lambda,x)$ possessing the G-ordered generalized semigroup property on $\Lambda \times X$. Suppose $Y = f(X)X$, where f, a function from X into R^1, is such that $f(x) = f(y)$ for all y in the convex hull of the G-orbit of x, and let $\phi(\lambda,y)$ be the density of Y. Define $h(\lambda) = \int \phi(\lambda,y)\, \ell(y)\, d\mu(y)$, where μ is a G-invariant and translation invariant measure on X. Then ℓ G-monotone increasing (decreasing) on X implies h G-monotone increasing (decreasing) on Λ.

Again we should remark that the conclusion of Corollary 4.21 holds as long as the preservation of G-monotonicity under an integral transform holds with K as the kernel of the transform.

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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1. REPORT NUMBER USARO-D No. 25 AFOSR No. 73	2. GOVT. ACCESSION NO. AFOSR-TR- 77 - 1254	3. RECIPIENT'S CATALOG NUMBER
4. TITLE G-ordered Functions, with Applications in Statistics, I. Theory.		5. TYPE OF REPORT Technical Report 6. PERFORMING ORGANIZATION REPORT NUMBER FSU Statistical Report M432
7. AUTHOR(s) J.C. Conlon, R. Leon, F. Proschan, J. Sethuraman		8. CONTRACT OR GRANT NUMBER(s) NIH T32 ES07011 AFOSR-74-2581C ARO-D No. DAAG29-76-G-0238
9. PERFORMING ORGANIZATION NAME AND ADDRESS Florida State University Department of Statistics Tallahassee, Florida 32306		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research Bolling Air Force Base, D.C. 20332		12. REPORT DATE September, 1977 13. NUMBER OF PAGES 29
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) U.S. Army Research Office - Durham P.O. Box 12211 Research Triangle Park, NC 27709		15. SECURITY CLASS (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS

G-ordered Functions, Reflection ordering, G-majorization, G-monotonicity, Fundamental region

20. ABSTRACT

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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 19 18 AFOSR-TR-77-1254
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER 9 Technical rept.,
4. TITLE (and Subtitle) 6 G-ORDERED FUNCTIONS, WITH APPLICATIONS IN STATISTICS. I. THEORY.
5. TYPE OF REPORT & PERIOD COVERED Interim
6. PERFORMING ORG. REPORT NUMBER FSU St Rpt M432
7. AUTHOR(s) 10 J. C./Conlon, R./Leon, F./Proschan, J./Sethuraman
8. CONTRACT OR GRANT NUMBER(s) 15 AFOSR-74-2581, DAAG29-76-G-0238
9. PERFORMING ORGANIZATION NAME AND ADDRESS Florida State University Department of Statistics Tallahassee, FL 32306
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 16 17 A5 61102F 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, DC 20332
12. REPORT DATE 11 Sep[redacted]77
13. NUMBER OF PAGES 28 12 34p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
15. SECURITY CLASS. (of this report) UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) 14 FSU-Statistics-M432

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
18. SUPPLEMENTARY NOTES
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UNCLASSIFIED

400 277

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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