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SOME RESULTS ON LIAPUNOV FUNCTIONS AND GENERATED DYNAMICAL SYST--ETC(U)
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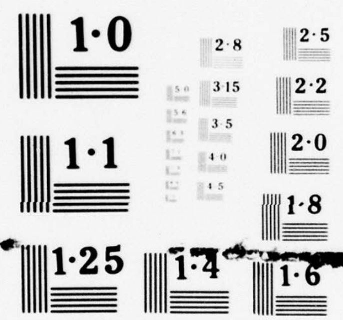

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SOME RESULTS ON LIAPUNOV FUNCTIONS

AND GENERATED DYNAMICAL SYSTEMS*

by

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Accompanying Statement

Given some autonomous evolution equation set in a Banach space $\mathscr{B}$, our present concern lies in setting up a corresponding dynamical system on a metric space $\mathscr{X} \subset \mathscr{B}$, and then applying the Liapunov approach to obtain qualitative information about the behavior of motions. Specifically, the results presented here are related to the following areas of difficulty in application:

a) setting up nonlinear dynamical systems that are not necessarily quasicontractive (Theorem 2.2),

b) locating positive invariant sets, with possibly empty interior, by using a lower semicontinuous Liapunov function V (Proposition 3.3),

c) estimating the derivative $\dot{V}$ along motions for

a a lower semicontinuous function V (Theorem 3.4),

d) using l.s.c. Liapunov functions to assure precompactness of positive orbits (Theorem 3.4 with Proposition 3.5),

e) using l.s.c. Lipaunov functions with the Invariance Principle (Theorem 3.6).

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ABSTRACT

This paper presents several results pertaining to the use of lower semicontinuous Liapunov functions in the analysis of autonomous abstract evolution equations. Such functions can be useful in setting up a nonlinear dynamical system that need not satisfy any exponential estimate, as well as in locating positive invariant sets of the resulting dynamical system. Other results concern the computation of the derivative of a lower semicontinuous Liapunov function, the use of such a function to assure precompactness of positive orbits, and a version of the Invariance Principle that is valid for lower semicontinuous Liapunov functions.

1. Introduction

Given some autonomous evolution equation set in a Banach space $\mathscr{B}$, our present concern lies in setting up a corresponding dynamical system on a metric space $\mathscr{X} \subset \mathscr{B}$, and then applying the Liapunov approach to obtain qualitative information about the behavior of motions. Specifically, the results presented here are related to the following areas of difficulty in applications:

a) setting up nonlinear dynamical systems that are not necessarily quasicontractive (Theorem 2.2),

b) locating positive invariant sets, with possibly empty interior, by using a lower semicontinuous Liapunov function V (Proposition 3.3),

c) estimating the derivative $\dot{V}$ along motions for a lower semicontinuous function V (Theorem 3.4),

d) using l.s.c. Liapunov functions to assure precompactness of positive orbits (Theorem 3.4 with Proposition 3.5),

e) using l.s.c. Liapunov functions with the Invariance Principle (Theorem 3.6).

We take this opportunity to define much of our notation and terminology. The symbols $\mathscr{R}$ and $\mathscr{R}^+$ denote the real line

$(-\infty,\infty)$ and nonnegative real line $[0,\infty)$, respectively, while $\bar{\mathscr{R}}$ represents the extended real line $[-\infty,\infty]$ with $\pm\infty$ considered as points, $-\infty < \alpha < \infty$ for every $\alpha \in \mathscr{R}$.

Definition 1.1: A mapping $u:\mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$, $\mathscr{X}$ a metric space, is a dynamical system on $\mathscr{X}$ if the family $\{u(t,\cdot)\}_{t\geq 0}$ is a $C_\circ$-semigroup of continuous operators on $\mathscr{X}$; equivalently, if $u(0,x) = x$, $u(t,u(\tau,x) = u(t+\tau,x)$, $u(t,\cdot): \mathscr{X} \to \mathscr{X}$ is continuous, and $u(\cdot,x):\mathscr{R}^+ \to \mathscr{X}$ is continuous (right-continuous at $t = 0$) for all $t,\tau \in \mathscr{R}^+$, $x \in \mathscr{X}$.

As every dynamical system is equivalent to a $C_\circ$-semigroup of continuous operators, the theory of $C_\circ$-semigroups provides a means of relating autonomous abstract evolution equations with dynamical systems; for linear dynamical systems, the complete relationship is defined by the Hille-Phillips-Yosida Theorem [10]. For nonlinear $C_\circ$-semigroups, the connection with evolution equations is less well established [4]; most available results are restricted to the "quasicontractive" case [4], a restriction that is often but not always met, even when $\mathscr{X} = \mathscr{R}^n$. In Theorem 2.2 we describe a means of relaxing this restriction by determining a class of positive invariant sets in the process of setting up the dynamical system.

Definition 1.2: For u a dynamical system on a metric space $\mathscr{X}$

and $x \in \mathscr{X}$, the mapping $u(\cdot,x):\mathscr{R}^+ \to \mathscr{X}$ is the <u>motion</u> originating at x, the set $\gamma(x) = \bigcup_{t\geq 0} u(t,x)$ is the <u>positive orbit</u> of the motion, and $\Omega(x) = \bigcap_{\tau>0} (\mathrm{Cl} \bigcup_{t\geq\tau} u(t,x))$ is the (possibly empty) <u>positive limit set</u> of the motion; equivalently, $y \in \Omega(x)$ if there exists a sequence $\{t_n\}_{n=1,2,\ldots}$ such that $t_n \to \infty$ and $u(t_n,x) \to y$ as $n \to \infty$. A set $\mathscr{S} \subset \mathscr{X}$ is <u>positive invariant</u> under u if $x \in \mathscr{S}$ implies that $\gamma(x) \subset \mathscr{S}$; $\mathscr{S}$ is <u>invariant</u> under u if there exists a mapping $v:\mathscr{R} \times \mathscr{S} \to \mathscr{S}$ such that $v(0,x) = x$ and $v(t+s,x) = u(t,v(s,x))$ for all $x \in \mathscr{S}$, $t \in \mathscr{R}^+$, $s \in \mathscr{R}$.

It is apparent that every invariant set is positive invariant, and we note that the closure of a positive invariant set is itself positive invariant. The positive limit set $\Omega(x)$ is directly related to the asymptotic behavior of the motion $u(\cdot,x)$ as $t \to \infty$ if $\gamma(x)$ is precompact. The well known Invariance Principle [5, 11, 14] provides a very useful means of locating $\Omega(x)$ when a suitable Liapunov function is available.

<u>Definition 1.3</u>: Let u be a dynamical system on a metric space $\mathscr{X}$, and let $V:\mathscr{X}\to\bar{\mathscr{R}}$ be lower semicontinuous. V is a <u>l.s.c. Liapunov function</u> for u on a subset $\mathscr{G}$ if $\dot{V}(x) \leq 0$ for every $x \in \mathscr{G}$, where $\dot{V}:\mathscr{X}\to\bar{\mathscr{R}}$ is defined by

$$\dot{V}(x) \equiv \liminf_{t \searrow 0} \frac{1}{t}[V(u(t,x)) - V(x)] \quad \text{if} \quad |V(x)| < \infty,$$

$$\dot{V}(x) \equiv 0 \quad \text{if} \quad V(x) = +\infty, \quad \dot{V}(x) \equiv 1 \quad \text{if} \quad V(x) = -\infty.$$

In applications, the computation of $\dot{V}$ often poses severe difficulties, and few general results are known [22, 23]. Theorem 3.4 provides a very simple and unrestricted means of obtaining a lower bound for $-\dot{V}$.

Liapunov functions are usually defined to be continuous, and continuity of V is essential in most versions of the Invariance Principle [5, 11, 14]; however, with Dafermos [6], we believe that l.s.c. Liapunov functions may be useful in establishing precompactness of positive orbits in certain problems, and such functions appear to be useful for other purposes as well [7]. Here we suggest their usefulness in determining positive invariant sets with possibly empty interior; moreover, by modifying an idea of Ball [1], we obtain in Theorem 3.6 a version of the Invariance Principle that is valid for l.s.c. Liapunov functions and general dynamical systems, thereby extending an earlier result of Dafermos [6, 7].

2. Generated Dynamical Systems

If $u: \mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$ is a linear dynamical system, $\mathscr{X}$ a Banach space, it is well known that there exists a closed linear and densely defined operator $A: (\mathscr{D}(A) \subset \mathscr{X}) \to \mathscr{X}$ such that, for every $x_o \in \mathscr{D}(A)$, the motion $u(\cdot, x_o)$ is the unique strong solution of the linear evolution equation

$$\dot{x}(t) = Ax(t) \quad \forall t \in \mathscr{R}^+,$$
$$x(0) = x_o \in \mathscr{D}(A). \tag{1}$$

Furthermore, for all $\lambda \in (0, \lambda_o)$, some $\lambda_o > 0$, $\mathscr{R}(I-\lambda A) = \mathscr{X}$ and $I - \lambda A$ admits a continuous inverse J_λ such that $J_\lambda^n x \to x$ as $\lambda \searrow 0$ for $n=1,2,\ldots,$ and $J_{t/n}^n x \to u(t,x)$ as $n \to \infty$, uniformly on compact t-intervals in $\mathscr{R}^+$, for every $x \in \mathscr{X}$ [13]. Therefore, it is reasonable to say that u is "generated" through the product formula $u(t,x) = \lim_{n \to \infty} J_{t/n}^n x$ [3].

Many analogous results have been obtained for the nonlinear case as well. Crandall and Liggett [4] have shown that if a (possibly multivalued) operator $A: (\mathscr{D}(A) \subset \mathscr{B}) \to \mathscr{B}$, $\mathscr{B}$ a Banach space, is such that $\mathscr{R}(I-\lambda A) \supset \mathrm{Cl}_{\mathscr{B}}\, \mathscr{D}(A)$ for all $\lambda \in (0, \lambda_o)$ and $\omega I - A$ is accretive (in terms of some equivalent norm $||\cdot||_e$) for some $\omega \in \mathscr{R}$, then a dynamical system is generated on $\mathscr{X} = \mathrm{Cl}_{\mathscr{B}}\, \mathscr{D}(A)$ by the same product formula as in the linear case. Moreover, $u: \mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$ is $||\cdot||_e$-quasicontractive, in the sense that $e^{-\omega t}u(t,\cdot) : \mathscr{X} \to \mathscr{X}$ is $||\cdot||_e$-contractive for every $t \in \mathscr{R}^+$. It also has been shown that the motion $u(\cdot,x_o) : \mathscr{R}^+ \to \mathscr{X}$ provides the (unique) strong solution of the evolution equation

$$\begin{aligned} \dot{x}(t) &\in Ax(t) \quad \text{a.e.} \quad t \in \mathscr{R}^+, \\ x(0) &= x_o \in \mathscr{D}(A), \end{aligned} \tag{2}$$

for every x_o such that a strong solution does exist. A number of additional conditions, sufficient for the existence of strong solutions, are also known; e.g., if A is closed and $\mathscr{B}$ is reflexive, or if A is closed and $u(\cdot,x_o)$ is known to be strongly differentiable a.e. on $\mathscr{R}^+$ [4,17]. Although this is a very powerful result, it refers only to dynamical systems that are $||\cdot||_e$-quasicontractive, due to the assumed $||\cdot||_e$-accretiveness of $\omega I - A$. Every linear dynamical system is of this type, but many nonlinear dynamical systems do not possess the quasicontractive property. It appears that major improvements on the results of [4,17] must involve relaxation (probably localization) of the accretiveness condition; for further discussion of this point, see [15].

In our intended applications, it is only some known evolution equation that will be explicitly available for computations, and therefore we are concerned here with a dynamical system that is, in some sense, directly related to a known evolution equation. We wish to make this idea precise, but we do not want to restrict our considerations to quasicontractive dynamical systems. To this end, we note that both the foregoing discussion and recent results on product formulas [3,15] strongly suggest that a dynamical system related to an evolution equation ought to be expressible as a product formula involving the (known) operator A appearing in the evolution equation. This conjecture motivates the following definition.

Definition 2.1: Let $u: \mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$ be a dynamical system on a metric space $\mathscr{X}$. Let there exist a family $\{J_\lambda\}_{\lambda \in [0,\lambda_o)}$, $\lambda_o > 0$, of continuous operators $J_\lambda: \mathscr{X} \to \mathscr{X}$ such that

(i) $J_\lambda^n x \to x$ as $\lambda \searrow 0$, $n = 1,2,\ldots,$ for every $x \in \mathscr{X}$,

(ii) $\lim_{n\to\infty} J_{t/n}^n x = u(t,x)$ exists for all $x \in \mathscr{X}$, $t \in \mathscr{R}^+$,

converging uniformly on compact t-intervals.

Then u is said to be generated by a product formula.

In applications it is usual that $J_\lambda = (I-\lambda A)^{-1}$, where $A: (\mathscr{D}(A) \subset \mathscr{B}) \to \mathscr{B}$ may be multivalued with $\mathscr{B}$ a Banach space, $\mathscr{R}(I-\lambda A) \supset \mathscr{X} \equiv \mathrm{Cl}_{\mathscr{B}}\mathscr{D}(A)$, and $d_{\mathscr{X}}(x,y) \equiv ||x-y||_{\mathscr{B}}$. We shall then say that "A generates u" on the metric space $\mathscr{X}$. As we do not wish to be restricted to the quasicontractive case, we do not insist that A satisfy any (uniform) accretiveness condition. For example, the following proposition shows that a Liapunov approach can be combined with the Crandall-Liggett theory [4] to set up a class of generated dynamical systems that may not be (uniformly) quasiconstractive in terms of any equivalent norm.

<u>Theorem 2.2:</u> For $\mathscr{B}$ a Banach space, consider a (possibly multivalued) $A: (\mathscr{D}(A) \subset \mathscr{B}) \to \mathscr{B}$, a lower semicontinuous $V: \mathscr{B} \to \bar{\mathscr{R}}$ with $V(x) > -\infty$ for every $x \in \mathscr{X}$, and $\alpha < \infty$ such that the set $\mathscr{G}_\alpha = \{x \in \mathscr{B} | V(x) \leq \alpha\} \subset \mathrm{Cl}_{\mathscr{B}} \mathscr{D}(A)$. Let there exist an equivalent norm $||\cdot||_\alpha$ and $\omega, \lambda_o \in \mathscr{R}$, $\lambda_o > 0$, such that for all $\lambda \in (0, \lambda_o)$,

(i) $\mathscr{R}(I-\lambda A) \supset \mathscr{G}_\alpha$,

(ii) $V(x) \leq V(x-\lambda y)$ for all $x \in \mathscr{D}(A)$, $y \in Ax$, such that $x - \lambda y \in \mathscr{G}_\alpha$,

(iii) $||(1+\lambda\omega)(x-\hat{x}) - \lambda(y-\hat{y})||_\alpha \geq ||x - \hat{x}||_\alpha$ for all $x, \hat{x} \in \mathscr{G}_\alpha \cap \mathscr{D}(A)$, $y \in Ax$, $\hat{y} \in A\hat{x}$.

If A_α is the maximal restriction of A to $\mathscr{D}(A_\alpha) \equiv \mathscr{G}_\alpha \cap \mathscr{D}(A)$, then A_α generates a dynamical system u on the (complete) metric space $\mathscr{X} \equiv \mathrm{Cl}_{\mathscr{B}} \mathscr{D}(A_\alpha)$ with $d_{\mathscr{X}}(x,y) \equiv ||x - y||_{\mathscr{B}}$; moreover, V is a l.s.c. Liapunov function for u on $\mathscr{X}$, the estimate $||u(t,x) - u(t,x)||_\alpha \leq e^{\omega t}||x - y||_\alpha$ applies for all $x, y \in \mathscr{X}$, $t \in \mathscr{R}^+$, and $\mathscr{G}_\beta \cap \mathscr{X}$ is positive invariant for each $\beta \leq \alpha$.

If, in addition, $Cl_{\mathscr{B}}\mathscr{D}(A) = \mathscr{B} = \mathscr{R}(I-\lambda A)$ for all sufficiently small $\lambda > 0$, $\bigcup_{\alpha_o<\alpha<\infty} Cl_{\mathscr{B}}\mathscr{D}(A_\alpha) = \mathscr{B}$ for some $\alpha_o \in \mathscr{R}$, and suitable $||\cdot||_\alpha$, $\lambda_o(\alpha) > 0$, and $\omega(\alpha) < \infty$ exist for every finite $\alpha > \alpha_o$, then A generates a dynamical system on $\mathscr{B}$, V is a l.s.c. Liapunov function on $\mathscr{B}$, $Cl_{\mathscr{B}}\mathscr{D}(A_\alpha)$ and $\mathscr{G}_\alpha$ are positive invariant for each finite α, and the estimate $||u(t,x) - u(t,y)||_\alpha \leq e^{\omega(\alpha)t}||x - y||_\alpha$ applies for all $x,y \in Cl_{\mathscr{B}}\mathscr{D}(A_\alpha)$, $t \in \mathscr{R}^+$, for each finite $\alpha > \alpha_o$.

<u>Proof:</u> By condition (iii), $\omega I - A_\alpha$ is $||\cdot||_\alpha$-accretive. Condition (ii) implies that $x \in \mathscr{G}_\alpha$ when $x \in \mathscr{D}(A)$, $y \in Ax$, and $x - \lambda y \in \mathscr{G}_\alpha$; hence, by (i), $\mathscr{R}(I-\lambda A_\alpha) \supset \mathscr{G}_\alpha$. As V is lower semicontinuous, $\mathscr{G}_\alpha$ is a closed subset of $Cl_{\mathscr{B}}\mathscr{D}(A)$; hence, $Cl_{\mathscr{B}}\mathscr{D}(A_\alpha) \subset \mathscr{G}_\alpha \subset \mathscr{R}(I-\lambda A_\alpha)$, and A_α meets all conditions of Theorem I of [4]. It follows that $(I-\lambda A_\alpha)$ has a continuous inverse J_λ (meeting all conditions of Definition 2.1) and the product formula $u(t,x) = \lim_{n\to\infty} (I - \frac{t}{n} A_\alpha)^{-n} x$ leads to a dynamical system $u: \mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$ such that $||u(t,x) - u(t,y)||_\alpha \leq e^{\omega t}||x-y||_\alpha$ for all $x,y \in \mathscr{X}$, $t \in \mathscr{R}^+$ [4]. In order to show that V is a Liapunov function for u on $\mathscr{X}$, we note that (ii) implies that $V(J_\lambda x) \leq V(x)$ for $x \in \mathscr{X}$ and, therefore, $J_\lambda x \in \mathscr{X}$. As V is lower semicontinuous, $V(u(t,x)) \leq \liminf_{n\to\infty} V(J^n_{t/n}x) \leq V(x)$ for $t > 0$ and $x \in \mathscr{X}$, and we conclude that $\dot{V}(x) \leq 0$ for every $x \in \mathscr{X}$. By the same reasoning, we see that $\mathscr{G}_\beta \cap \mathscr{X}$ is positive invariant for each $\beta \leq \alpha$.

Finally, we note that if $\mathrm{Cl}_{\mathscr{B}}\mathscr{D}(A) = \mathscr{B} = \mathscr{R}(I-\lambda A)$ for all sufficiently small $\lambda > 0$, and if suitable $||\cdot||_\alpha$, $\lambda_o(\alpha) > 0$, $\omega(\alpha) < \infty$, exist for every finite $\alpha > \alpha_o$, some $\alpha_o < \infty$, then the above conclusions hold on $\mathrm{Cl}_{\mathscr{B}}\mathscr{D}(A_\alpha)$ for each finite $\alpha > \alpha_o$. If $\bigcup_{\alpha_o<\alpha<\infty} \mathrm{Cl}_{\mathscr{B}}\mathscr{D}(A_\alpha) = \mathscr{B}$, then each $x \in \mathscr{B}$ (resp. $x \in \mathscr{D}(A)$) is in some $\mathrm{Cl}_{\mathscr{B}}\mathscr{D}(A_\alpha) \subset \mathscr{G}_\alpha$ (resp. $\mathscr{D}(A_\alpha)$) for finite $\alpha > \alpha_o$; hence, the remaining conclusions follow and the proof is complete. ∎

If $\mathscr{D}(A)$ is dense and $V(x) \equiv 0$, $\alpha \geq 0$, then Theorem 2.2 and Theorem I of Crandall and Liggett [4] are equivalent; A generates a $||\cdot||_\alpha$-quasicontractive dynamical system on $\mathscr{X} = \mathscr{G}_\alpha = \mathscr{B}$. On the other hand, if $\mathscr{G}_\alpha \neq \mathscr{B}$, Theorem 2.2 provides a constructive method for defining a restriction A_α that, by Theorem I of [2], generates a $||\cdot||_\alpha$-quasicontractive dynamical system on $\mathscr{X} = \mathrm{Cl}_{\mathscr{B}}\mathscr{D}(A_\alpha)$. However, the last part of Theorem 2.2 provides a true extension of Theorem I of [4] if $\omega(\alpha) \to \infty$ as $\alpha \to \infty$; the resulting dynamical system on $\mathscr{B}$ need not be (uniformly) quasicontractive in terms of any equivalent norm. This situation arises, for example, in certain problems in nuclear reactor dynamics which motivated this result; the analyses of [12,21] could be considerably simplified by the use of Theorem 2.2.

There are other uses to which Theorem 2.2 can be put, even for generated dynamical systems that are quasicontractive. For example, consider the nonlinear partial differential equation

$$\frac{\partial}{\partial t} y(\eta,t) = \frac{\partial^2}{\partial \eta^2} y(\eta,t) + f'(y(\eta,t)), \quad t \geq 0, \quad 0 \leq \eta \leq 1, \tag{3}$$

with boundary and initial data

$$\begin{aligned} y(0,t) &= 0 = y(1,t) \quad , \quad t \geq 0 , \\ y(\eta,0) &= x_0(\eta) \quad , \quad 0 \leq \eta \leq 1. \end{aligned} \tag{4}$$

Here $f'(\xi) = \frac{d}{d\xi} f(\xi)$ with $f: \mathscr{R} \to \mathscr{R}$ twice continuously differentiable and $f(0) = 0$, $\sup_{\xi \in \mathscr{R}} f''(\xi) = m < \infty$. This is a slight generalization of a problem considered in a somewhat different context in [2].

In order to place (3),(4) in the form (2), let ∂x denote the generalized derivative of a function $x: [0,1] \to \mathscr{R}$, let $\mathscr{H}_2^n$ denote the space of (equivalence classes of) functions $x \in \mathscr{L}_2$ having n Lebesgue square integrable derivatives $\partial^i x$, $i = 1,\ldots,n$, and let $\mathscr{C}^n$ denote the space of continuous functions having n continuous derivatives. In contrast with [2], we choose to view (3), (4) in the natural topology of $\mathscr{C}^0$; hence, we consider (2) with $\mathscr{B} = \hat{\mathscr{C}}$, where

$$\begin{aligned} \hat{\mathscr{C}} &= \{x \in \mathscr{C}^0 \,|\, x(0) = 0 = x(1)\}, \quad ||x||_{\hat{\mathscr{C}}} = \max_{0 \leq \eta \leq 1} |x(\eta)|, \\ \mathscr{D}(A) &= \{x \in \hat{\mathscr{C}} \,|\, \partial^2 x \in \hat{\mathscr{C}}\} , \\ Ax(\eta) &= \partial^2 x(\eta) + f'(x(\eta)), \quad 0 \leq \eta \leq 1, \quad x \in \mathscr{D}(A). \end{aligned} \tag{5}$$

It is possible to show that $\mathscr{R}(I-\lambda A) = \hat{\mathscr{E}}$ for all sufficiently small $\lambda > 0$, and that $\omega I-A$ is accretive for $\omega = m$; hence, by Theorem I of [4] (equivalently, by our Theorem 2.2 with $V(x) \equiv 0$), it follows that A generates a quasi-contractive dynamical system u on $\hat{\mathscr{E}}$; moreover, Theorem II of [4] implies that $u(\cdot,x_o)$ provides the unique strong solution of (2) for every $x_o \in \mathscr{D}(A)$ such that (2) has a strong solution.

Despite appearances, there is more information to be gained from Theorem 2.2 if a suitable nontrivial function V is used. Extending a function used in [2], we define $V: \hat{\mathscr{E}} \to \bar{\mathscr{R}}$ by

$$
\begin{aligned}
V(x) &= \int_0^1 (\partial x(\eta))^2 d\eta - 2\int_0^1 f(x(\eta))d\eta, \quad x \in \hat{\mathscr{E}} \cap \mathscr{W}_2^1, \\
V(x) &= \infty \quad \text{if} \quad x \in \hat{\mathscr{E}}, \quad x \notin \mathscr{W}_2^1 .
\end{aligned} \tag{6}
$$

In [2] the corresponding dynamical system was set on $\hat{\mathscr{E}} \cap \mathscr{E}^1$ in the topology induced by the natural norm of $\mathscr{E}^1$; in that context V was continuous. Here our dynamical system u is described in the topology of $\mathscr{E}^0$ and $V: \hat{\mathscr{E}} \to \bar{\mathscr{R}}$ is not continuous; however, as will follow from Proposition 3.5 in the next section, $V: \hat{\mathscr{E}} \to \bar{\mathscr{R}}$ is lower semicontinuous. Applying the mean value theorem to $f': \mathscr{R} \to \mathscr{R}$ we find that, for $x \in \mathscr{D}(A)$ and $\lambda > 0$,

$$V(x-\lambda Ax) \geq V(x) + \lambda[2+\lambda(\pi^2-2m)]\int_0^1 [\partial^2 x(\eta)+f'(x(\eta))]^2 d\eta .$$

Given any $\varepsilon > 0$, $\varepsilon < 1$, it follows that there exists $\lambda_o > 0$ such that

$$V(x-\lambda Ax) \geq V(x) + 2\lambda(1-\varepsilon)\int_0^1 [\partial^2 x(\eta)+f'(x(\eta))]^2 d\eta \tag{7}$$

for all $\lambda \in (0,\lambda_o)$, $x \in \mathscr{D}(A)$. Applying Theorem 2.2 we now find that V is a l.s.c. Liapunov function for u on $\bigcup_{\alpha\in\mathscr{R}} \mathscr{G}_\alpha$, each set $\mathscr{G}_\alpha = \{x \in \hat{\mathscr{C}} \mid V(x) \leq \alpha\}$ is positive invariant and closed for $\alpha \in \mathscr{R}$, and $\bigcup_{\alpha\in\mathscr{R}} \mathscr{G}_\alpha = \hat{\mathscr{C}} \cap \mathscr{W}_2^1$ is positive invariant and dense (with empty interior). Noting that $V(x) = \infty$ for $x \in \hat{\mathscr{C}}$, $x \notin \mathscr{W}_2^1$, we see by Definition 1.3 that V is a l.s.c. Liapunov function for u on all of $\hat{\mathscr{C}}$.

From Theorem 2.2 we have obtained the separate conclusions that V is a Liapunov function on $\bigcup_{\alpha\in\mathscr{R}} \mathscr{G}_\alpha = \hat{\mathscr{C}} \cap \mathscr{W}_2^1$ and that $\mathscr{G}_\alpha$ is positive invariant for each $\alpha \in \mathscr{R}$; actually, such conclusions are not independent, whether or not u is known to be generated, and in the following section we point out this fact for general dynamical systems. Other results in the following section are needed in order to continue with our example.

3. Lower Semicontinuous Liapunov Functions

The useful property of a Liapunov function V is that, under relatively weak conditions, its value can be shown to be nonincreasing along motions of the dynamical system; this property leads to many interesting conclusions. In order to obtain conditions sufficient to insure that $V(u(\cdot,x))$ is nonincreasing, we will need the following simple lemma. (Although this result probably is available elsewhere, we have been unable to find it; a similar lemma stated in [14] is not true without a strengthened assumption that we make here.)

Lemma 3.1: Let $f:([0,\beta)\subset\mathscr{R}) \to \bar{\mathscr{R}}$ be defined on $[0,\beta)$, $0 < \beta \le \infty$, with $f(0) < \infty$ and $f(t) > -\infty$ for every $t \in [0,\beta)$, and assume that

(i) f is left lower semicontinuous on $[0,\beta)$; i.e.,

$$\liminf_{t \nearrow t_o} f(t) \geq f(t_o) \quad \text{for every} \quad t_o \in (0,\beta),$$

(ii) the lower right derivative is nonpositive on $[0,\beta)$; i.e.,

$$D_+ f(t) \equiv \liminf_{t \searrow t_o} \frac{f(t)-f(t_o)}{t-t_o} \leq 0 \quad \text{for every} \quad t_o \in [0,\beta).$$

Then f is nonincreasing and differentiable almost everywhere on compact subintervals of $[0,\beta)$; moreover,

$$f(t) \leq f(0) + \int_0^t D_+ f(s)\,ds \qquad \forall t \in [0,\beta).$$

Proof: Choosing some $\varepsilon > 0$, define $f_\varepsilon(t) = f(t) - \varepsilon t$ for $t \in [0,\beta)$. Then f_ε is left lower semicontinuous with $f_\varepsilon(0) = f(0)$ and $f_\varepsilon(t) > -\infty$, $D_+f_\varepsilon(t) \leq -\varepsilon$, for every $t \in [0,\beta)$. We claim that $f_\varepsilon(t) \leq f(0)$ for every $t \in [0,\beta)$; if not, left lower semicontinuity implies the existence of $t_1 \in [0,\beta)$, $t_2 \in (t_1,\beta)$, such that $f_\varepsilon(t) \leq f(0)$ for $t \in [0,t_1]$ and $f_\varepsilon(t) > f(0)$ for $t \in (t_1,t_2)$. However, this leads to the contradiction $D_+f_\varepsilon(t_1) \geq 0$; we conclude that $f_\varepsilon(t) \leq f(0)$ for every $t \in [0,\beta)$ and, as $\varepsilon > 0$ was arbitrary, the same is true for f. Replacing $t = 0$ with $t = \gamma \in (0,\beta)$ and repeating this argument, we find that $f(t) \leq f(\gamma)$ for all $t \in [\gamma,\beta)$ and all $\gamma \in [0,\beta)$; hence, f is nonincreasing and finite-valued on $[0,\beta)$. By a standard result of integration theory (see [16], Section 34.2) it follows that f is a.e. differentiable on compact subsets of $[0,\beta)$ (with derivative equal a.e. to $D_+f(t)$), and that

$$f(t) \leq f(0) + \int_0^t D_+f(s)\,ds$$

for every $t \in [0,\beta)$. The proof is complete. ∎

The following proposition is now obvious.

Proposition 3.2: Let $u: \mathcal{R}^+ \times \mathcal{X} \to \mathcal{X}$ be a dynamical system on a metric space $\mathcal{X}$, and let $V: \mathcal{X} \to \bar{\mathcal{R}}$ be a l.s.c. Liapunov function for u on a set $\mathcal{G} \subset \mathcal{X}$. If $x \in \mathcal{G}$, $V(x) < \infty$, and $u(t,x) \in \mathcal{G}$ for all $t \in [0,\beta)$, $0 < \beta \leq \infty$, then $V(u(\cdot,x))$ is nonincreasing and differentiable a.e. on compact subintervals of $[0,\beta)$; moreover,

$$V(u(t,x)) \leq V(x) + \int_0^t \dot{V}(u(s,x))ds \qquad \forall t \in [0,\beta) \;.$$

Proof: As $u(\cdot,x): \mathcal{R}^+ \to \mathcal{X}$ is continuous and $V: \mathcal{X} \to \bar{\mathcal{R}}$ is lower semicontinuous, we define $f(t) = V(u(t,x))$ for $t \in [0,\beta)$ and note that all conclusions follow from Lemma 3.1. ∎

From Proposition 3.2 it is apparent that $V(u(\cdot,x))$ is nonincreasing on $\mathcal{R}^+$ provided that $\gamma(x)$ is contained within a set $\mathcal{G} \subset \mathcal{X}$ such that V is a Liapunov function on $\mathcal{G}$. If V is not a Liapunov function on all of $\mathcal{X}$, the problem now is to ensure that $\gamma(x)$ is contained in some $\mathcal{G} \subset \mathcal{X}$; this is directly related to the problem of determining positive invariant sets. The following proposition is well known for continuous Liapunov functions; we prove it here for the lower semicontinuous case.

Proposition 3.3: Let $u: \mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$ be a dynamical system on a metric space $\mathscr{X}$, and let $V: \mathscr{X} \to \bar{\mathscr{R}}$ be a l.s.c. Liapunov function for u on a disjoint component $\mathscr{G}_\alpha$ of the set $\{x \in \mathscr{X} | V(x) \leq \alpha\}$ for some $\alpha < \infty$. Then, for each $\beta < \alpha$, the set $\mathscr{G}_\beta = \{x \in \mathscr{G}_\alpha | V(x) \leq \beta\}$ is positive invariant and, for every $x \in \mathscr{G}_\beta$, $V(u(\cdot,x))$ is nonincreasing and differentiable almost everywhere on compact t-intervals; moreover,

$$V(u(t,x)) \leq V(x) + \int_0^t \dot{V}(u(s,x))ds \qquad \forall t \in \mathscr{R}^+.$$

Proof: In view of Proposition 3.2, it only remains to be shown that $\mathscr{G}_\beta$ is positive invariant. Let us consider $x \in \mathscr{G}_\beta$, noting that $V(x) \leq \beta$ and, by the continuity of the map $u(\cdot,x)$, $V(u(\cdot,x))$ is lower semicontinuous on $\mathscr{R}^+$. Either $u(t,x)$ remains in $\mathscr{G}_\alpha$ on some finite interval $[0,T]$ or it does not. If not, the lower semicontinuity of $V(u(\cdot,x))$ implies the existence of $\delta > 0$ such that $V(u(t,x)) > \alpha$ for all $t \in (0,\delta)$; therefore, we obtain the contradiction that

$$0 \geq \dot{V}(x) = \liminf_{t \searrow 0} \frac{1}{t} [V(u(t,x)) - V(x)] = + \infty.$$

Hence, $V(u(t,x)) \in \mathscr{G}_\alpha$ for $t \in [0,T]$ for some $T > 0$; applying Proposition 3.2, we find that $V(u(t,x)) \leq V(x) \leq \beta$ $t \in [0,T]$. Repeating this process, we find that either $u(t,x)$

remains in $\mathscr{G}_\beta$ for all $t \in \mathscr{R}^+$ or there exists positive $\tau < \infty$ such that $V(u(t,x)) \leq \beta$ for every $t \in [0,\tau)$ and $V(u(\tau,x)) > \beta$. By the lower semicontinuity of $V(u(\cdot,x))$, the latter case is impossible, and we have shown $\mathscr{G}_\beta$ to be positive invariant. Applying Proposition 3.2, the proof is complete. ∎

Often there are severe difficulties involved in computing $\dot{V}$ when, as is usually the case in applications, the mapping $u:\mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$ is not explicitly known [18,22,23]. For a generated dynamical system, the following result provides a means of obtaining at least a nonnegative lower bound for $-\dot{V}$.

<u>Theorem 3.4</u>: Let $u:\mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$ be a generated dynamical system on a metric space $\mathscr{X}$. Let there exist lower semicontinuous functions $V:\mathscr{X} \to \bar{\mathscr{R}}$ and $U:\mathscr{R}^+ \times \mathscr{X} \to \bar{\mathscr{R}}$, $V(x) > -\infty$ for $x \in \mathscr{X}$, such that

(i) $V(J_\lambda x) \leq V(x) - \lambda U(\lambda, J_\lambda x) \quad \forall x \in \mathscr{X}, \quad \lambda \in (0,\lambda_o)$,

(ii) $\qquad 0 \leq U(0,x) \quad \forall x \in \mathscr{G}_\alpha$, where $\lambda_o > 0$, $\alpha < \infty$, and $\mathscr{G}_\alpha$ is a disjoint component of $\{x \in \mathscr{X} \mid V(x) \leq \alpha\}$. Then $\mathscr{G}_\beta \equiv \{x \in \mathscr{G}_\alpha \mid V(x) \leq \beta\}$ is positive invariant for every $\beta < \alpha$ and V is a l.s.c. Liapunov function on $\mathscr{G}_\alpha$ with $\dot{V}(x) \leq -U(0,x)$ for every $x \in \mathscr{G}_\alpha$.

<u>Proof:</u> For $x \in \mathscr{X}$, $t \in (0,\lambda_o)$, and $n = 1,2,\ldots,$ we see that

$$\begin{aligned}
\frac{1}{t}[V(J^n_{t/n}\, x) - V(x)] &= \frac{1}{t}\sum_{m=1}^{n} [V(J^m_{t/n}\, x) - V(J^{m-1}_{t/n}\, x)] \\
&\leq -\frac{1}{n}\sum_{m=1}^{n} U(\tfrac{t}{n}, J^m_{t/n}\, x) \\
&\leq -\inf_{m} \{U(\tfrac{t}{n}, J^m_{t/n}\, x) \mid m=1,2,\ldots,n\} \\
&\leq -\inf_{m,\tau} \{U(\tfrac{\tau}{m}, J^m_{\tau/m}\, x) \mid 0 < \tau \leq t,\ m=1,2,\ldots n\}\ ,
\end{aligned}$$

where $J^0_\lambda\, x = x$. Since u is generated and V is lower semicontinuous,

$$\begin{aligned}
\frac{1}{t}[V(u(t,x)) - V(x)] &\leq \liminf_{n\to\infty} \frac{1}{t}[V(J^n_{t/n}\, x) - V(x)] \\
&\leq -\inf_{m,\tau} \{U(\tfrac{\tau}{m}, J^m_{\tau/m}\, x) \mid 0 < \tau \leq t,\ m=1,2,\ldots\},
\end{aligned}$$

and it follows that

$$\dot{V}(x) = \liminf_{t \searrow 0} \frac{1}{t}[V(u(t,x)) - V(x)]$$

$$\leq -\limsup_{t \searrow 0} \left[\inf_{m,\tau} \{U(\tfrac{\tau}{m}, J^m_{\tau/m}\, x) \mid 0 < \tau \leq t,\ m=1,2,\ldots\}\right].$$

Denoting the last term by $-f(x)$, $f:\mathscr{X} \to \bar{\mathscr{R}}$, it follows that for each $\varepsilon > 0$ there exists a sequence $\{\tau_k, m_k\}_{,=1,2,\ldots}$, depending on x and ε, such that m_k is a positive integer, $\tau_k > \tau_{k+1} > 0$, $\tau_k \to 0$ as $k \to \infty$, and

$$f(x) + \varepsilon \geq \liminf_{k \to \infty} U(\frac{\tau_k}{m_k}, J^{m_k}_{\tau_{k/m_k}}\, x) .$$

If the sequence $\{m_k\}_{k=1,2,\ldots}$ is bounded, then the lower semicontinuity of U and the fact that $J^m_\lambda\, x \to x$ as $\lambda \searrow 0$, uniformly in $m=1,2,\ldots,n$ for finite n, together imply that $f(x) + \varepsilon \geq U(0,x)$. On the other hand, if the sequence $\{m_k\}_{k=1,2,\ldots}$ is not bounded, then there exists a subsequence $\{t_p, n_p\}_{p=1,2,\ldots}$ of $\{\tau_k, m_k\}_{k=1,2,\ldots}$ such that n_p is a positive integer, $n_p \to \infty$ as $p \to \infty$, $t_p > t_{p+1} > 0$, $t_p \to 0$ as $p \to \infty$, and

$$f(x) + \varepsilon \geq \liminf_{p \to \infty} U(\frac{t_p}{n_p}, J^{n_p}_{t_{p/n_p}}\, x).$$

Then, since $J^n_{t/n}\, x \to u(t,x)$ as $n \to \infty$, uniformly on compact t-intervals, it again follows from the lower semicontinuity of

U that $f(x) + \varepsilon \geq U(0,x)$. As $x \in \mathscr{X}$ and $\varepsilon > 0$ were arbitrary, we obtain $\dot{V}(x) \leq -f(x) \leq -U(0,x)$ for every $x \in \mathscr{X}$.

We now have $\dot{V}(x) \leq -U(0,x) \leq 0$ for every $x \in \mathscr{G}_\alpha$; therefore, V is a l.s.c. Liapunov function on $\mathscr{G}_\alpha$ and the positive invariance of $\mathscr{G}_\beta$, $\beta < \alpha$, follows from Proposition 3.3. The proof is complete. ∎

Remark: If a (possibly multivalued) generator $A:(\mathscr{D}(A) \subset \mathscr{B}) \to \mathscr{B}$ "generates u" in the sense of Section 2, where the Banach space $\mathscr{B} \supset \mathscr{X} \equiv \mathrm{Cl}_{\mathscr{B}}\mathscr{D}(A)$ and $J_\lambda = (I-\lambda A)^{-1}$, then condition (i) is equivalent to

$$\text{(i)}' \quad V(x-\lambda y) - V(x) \geq \lambda U(\lambda,x),$$

for every $x \in \mathscr{D}(A)$, $y \in Ax$, $\lambda \in (0,\lambda_o)$, such that $x - \lambda y \in \mathscr{X}$.

Remark: Since V is lower semicontinuous, each $\mathscr{G}_\beta$ is closed for $\beta \leq \alpha$, although possibly empty; moreover, we note that $\mathscr{G}_\beta$ may be nonempty with empty interior.

As a somewhat trivial application of Theorem 3.2, let a dynamical system $u: \mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$, $\mathscr{X}$ a Banach space, be generated by a (possibly multivalued) operator $A: (\mathscr{D}(A) \subset \mathscr{X}) \to \mathscr{X}$ such that $\omega I - A$ is accretive for some $\omega \leq 0$ and $0 \in \mathscr{R}(A)$. Then defining $V(x) = ||x-x_e||$ with x_e such that $0 \in Ax_e$, application of Theorem 3.2 immediately yields $\dot{V}(x) \leq \omega V(x)$. Of course, as the theory of $C_\circ$-semigroups shows that u must admit the estimate $||u(t,x) - x_e|| \leq e^{\omega t}||x-x_e||$ [4], this estimate can be used in Definition 1.2 to achieve the same conclusion.

As a more interesting application, consider the example which was begun in Section 2. With $\hat{\mathscr{C}}$, A, and V defined by (5) and (6), we set $\mathscr{X} = \hat{\mathscr{C}}$ in Theorem 3.4 and note from (7) that the function $U: \mathscr{R}^+ \times \hat{\mathscr{C}} \to \bar{\mathscr{R}}$, given by

$$
\begin{aligned}
U(\lambda,x) &= \lambda(1-\varepsilon) \int_0^1 [\partial^2 x(\eta) + f^1(x(\eta))]^2 d\eta, \quad x \in \hat{\mathscr{C}} \cap \mathscr{W}_2^2 , \\
U(\lambda,x) &= \infty \quad \text{if} \quad x \in \hat{\mathscr{C}}, \quad x \notin \mathscr{W}_2^2 ,
\end{aligned}
\tag{8}
$$

satisfies conditions (i) and (ii) of Theorem 3.2 for $0 < \lambda < \lambda_0(\varepsilon)$. We shall show later that both $U: \mathscr{R}^+ \times \hat{\mathscr{C}} \to \bar{\mathscr{R}}$ and $V: \hat{\mathscr{C}} \to \bar{\mathscr{R}}$ are lower semicontinuous; hence, recalling that $\varepsilon > 0$ can be chosen arbitrarily small, Theorem 3.4 implies that $\dot{V}(x) \leq - W(x)$ for all $x \in \bigcup_{\alpha\in\mathscr{R}} \mathscr{G}_\alpha = \hat{\mathscr{C}} \cap \mathscr{W}_2^1$, where

$$W(x) = \lambda\int_0^1 [\partial^2 x(\eta) + f'x(\eta)]^2 d\eta, \quad x \in \hat{\mathscr{E}} \cap \mathscr{W}_2^2,$$
$$W(x) = \infty \quad \text{if} \quad x \in \hat{\mathscr{E}}, \quad x \notin \mathscr{W}_2^2. \tag{9}$$

The remaining conclusions of Theorem 3.4 were previously obtained in Section 2 for this example. A new conclusion can be reached by noting the result $\dot{V}(x) \leq - W(x)$ for $x \in \hat{\mathscr{E}} \cap \mathscr{W}_2^1$ and applying Proposition 3.3, from which it follows that $u(t,x)$ cannot remain in $\hat{\mathscr{E}} \cap (\mathscr{W}_2^1 - \mathscr{W}_2^2)$ on any open time interval in $\mathscr{R}^+$. We shall return to this example later.

Theorem 3.4 provides both an estimate for $\dot{V}$ and a family $\{\mathscr{G}_\beta\}_{\beta<\alpha}$ of positive invariant sets. We note that if some $\mathscr{G}_\beta$, $\beta < \alpha$, is bounded (or precompact), then the positive orbit $\gamma(x)$ is bounded (or precompact) for every $x \in \mathscr{G}_\beta$; this suggests the possibility of using Theorem 3.4 to assure precompactness of positive orbits, an essential requirement for useful application of the Invariance Principle [8,11,18,20]. Note that if there exists a smaller metric space $\mathscr{Y} \subset \mathscr{X}$ such that some $\mathscr{G}_\beta$, $\beta < \alpha$, is a $\mathscr{Y}$-bounded subset of $\mathscr{Y}$ and the injection $\mathscr{Y} \to \mathscr{X}$ is compact, then $\mathscr{G}_\beta$ is $\mathscr{X}$-precompact and $\gamma(x)$ is precompact for every $x \in \mathscr{G}_\beta$. This idea is related to the previous approaches of Hale [11] and Slemrod [18] for assuring precompactness of positive orbits, but it is much simpler in that we do not make the assumption of [11] and [18] that u, restricted to $\mathscr{R}^+ \times \mathscr{Y}$, is also a dynamical system on $\mathscr{Y}$. This advantage may be partially offset in applications by the practical problem of assuring that $V: \mathscr{X} \to \mathscr{R}$ is lower semicontinuous on $\mathscr{X}$ [19].

We now provide a sufficient condition for lower semicontinuity that is related to our comments on orbital precompactness. We note that if $V: \mathscr{X} \to \mathscr{R}$ meets the conditions of Theorem 3.4 as well as those of Proposition 3.5, with $d_{\mathscr{X}}(x,y) \equiv ||x-y||_{\mathscr{B}_o}$, then $\mathscr{G}_\alpha \equiv \mathscr{S}_\alpha$ is $\mathscr{X}$-precompact.

Proposition 3.5: Let $F:(\mathscr{D}(F) \subset \mathscr{B}_o) \to \bar{\mathscr{R}}$, $\mathscr{B}_o$ a Banach space, and let $\mathscr{S}_\alpha$ denote the set $\{x \in \mathscr{D}(F) | F(x) \leq \alpha\}$, $\alpha \in \mathscr{R}$. Let $\mathscr{B}_1$ be a reflexive Banach space such that

(i) $\mathscr{B}_1 \subset \mathscr{B}_o$ and the injection $\mathscr{B}_1 \to \mathscr{B}_o$ is compact,

(ii) for every (finite) $\alpha \in \mathscr{R}$, $\mathscr{S}_\alpha \subset \mathscr{B}_1$ and $\mathscr{S}_\alpha$ is both $\mathscr{B}_1$-bounded and $\mathscr{B}_1$-weakly closed.

Then $F:(\mathscr{D}(F) \subset \mathscr{B}_o) \to \bar{\mathscr{R}}$ is lower-semicontinuous on $\mathscr{D}(F)$ and $\mathscr{S}_\alpha$ is precompact in $\mathscr{B}_o$ for every $\alpha \in \mathscr{R}$. If $\mathscr{D}(F)$ is closed in $\mathscr{B}_o$, $\mathscr{S}_\alpha$ is compact in $\mathscr{B}_o$.

Proof: Suppose that F is not lower semicontinuous; then there exists $x_o \in \mathscr{D}(F)$, $\varepsilon > 0$, and a sequence $\{x_n\}_{n=1,2,\ldots} \subset \mathscr{D}(F)$ such that $x_n \xrightarrow{\mathscr{B}_o} x_o$ as $n \to \infty$ and $\liminf F(x_n) = \delta \leq F(x_o) - 2\varepsilon$, $F(x_o) < \infty$. By choosing a subsequence, if necessary, we may assume that $F(x_n) \leq \delta + \varepsilon$ for every n; hence, $\{x_n\}_{n=1,2,...} \subset \mathscr{S}_{\delta+\varepsilon} \subset \mathscr{B}_1$. Since $\mathscr{B}_1$ is reflexive and $\mathscr{S}_{\delta+\varepsilon}$ is $\mathscr{B}_1$-bounded, we may assume (by choosing another subsequence, if necessary) that $\{x_n\}_{n=1,2,...}$ is also $\mathscr{B}_1$-weakly convergent to some $y_o \in \mathscr{B}_1$ [9]; in fact, $y_o \in \mathscr{S}_{\delta+\varepsilon}$ since $\mathscr{S}_{\delta+\varepsilon}$ is weakly closed. As the injection $\mathscr{B}_1 \to \mathscr{B}_o$ is a compact linear operator, it maps $\mathscr{B}_1$ -weakly convergent sequences into $\mathscr{B}_o$ -strongly convergent sequences [9]; hence, $x_o = y_o \in \mathscr{S}_{\delta+\varepsilon}$. This contradicts our assumption that $x_o \notin \mathscr{S}_{\delta+2\varepsilon}$ and implies that $F: (\mathscr{D}(F) \subset \mathscr{B}_o) \to \overline{\mathscr{R}}$ is lower semicontinuous.

As $\mathscr{S}_\alpha$ is $\mathscr{B}_1$-bounded and the injection $\mathscr{B}_1 \to \mathscr{B}_o$ is compact, $\mathscr{S}_\alpha$ is $\mathscr{B}_o$-precompact. If $\mathscr{D}(F)$ is closed in $\mathscr{B}_o$, lower semicontinuity of F implies that $\mathscr{S}_\alpha$ is closed in $\mathscr{B}_o$; hence, $\mathscr{S}_\alpha$ is compact and the proof is complete. ∎

Remark: As $\mathscr{B}_1$ is locally convex, a closed convex set in $\mathscr{B}_1$ must be weakly closed [9]; hence, we can replace (ii) by the simpler but more restrictive condition

(ii)' for every (finite) $\alpha \in \mathscr{R}$, $\mathscr{S}_\alpha$ is a convex $\mathscr{B}_1$-bounded and $\mathscr{B}_1$-closed subset of $\mathscr{B}_1$.

As an application of Proposition 3.5, consider the function

V of (6) defined on the Banach space $\hat{\mathscr{C}}$ defined by (5). We note that $V = F + H$, where $H: \hat{\mathscr{C}} \to \mathscr{R}$ is continuous and $F: \hat{\mathscr{C}} \to \bar{\mathscr{R}}$ is defined by

$$F(x) = \int_0^1 (\partial x(\eta))^2 d\eta, \quad x \in \hat{\mathscr{C}} \cap \mathscr{W}_2^1 ,$$

$$F(x) = \infty \quad \text{if} \quad x \in \hat{\mathscr{C}}, \quad x \notin \mathscr{W}_2^1 .$$

Defining $\mathscr{S}_\alpha = \{x \in \hat{\mathscr{C}} \mid F(x) \leq \alpha\}$, $\alpha \in \mathscr{R}$, and defining a Hilbert space $\mathscr{B}_1$ to be the set $\hat{\mathscr{C}} \cap \mathscr{W}_2^1$ equipped with the natural norm of $\mathscr{W}_2^1$, we see that the injection $\mathscr{B}_1 \to \hat{\mathscr{C}}$ is compact, $\mathscr{B}_1$ is reflexive, and $\mathscr{S}_\alpha$ is a closed convex bounded subset of $\mathscr{B}_1$ for every $\alpha \in \mathscr{R}$. Hence, by Proposition 3.5 with condition (ii)', $F: \hat{\mathscr{C}} \to \bar{\mathscr{R}}$ is lower semicontinuous and $\mathscr{S}_\alpha$ is compact in $\hat{\mathscr{C}}$. It follows that, as claimed, $V: \hat{\mathscr{C}} \to \bar{\mathscr{R}}$ also is lower semicontinuous. Moreover, if $f: \mathscr{R} \to \mathscr{R}$ is such that there exist $\alpha, \beta \in \mathscr{R}$ for which $\{x \in \hat{\mathscr{C}} \mid V(x) \leq \beta\} \subset \{x \in \hat{\mathscr{C}} \mid F(x) \leq \alpha\}$, then $\mathscr{G}_\beta \subset \mathscr{S}_\alpha$ and $\mathscr{G}_\beta$ is compact; consequently, the positive orbit $\gamma(x)$ is precompact for every $x \in \mathscr{G}_\beta$.

In a similar manner we can show that $W: \hat{\mathscr{C}} \to \bar{\mathscr{R}}$, given by (9), is lower semicontinuous. We note that $W = \tilde{F}\tilde{H}$, where $\tilde{H}: \hat{\mathscr{C}} \to \hat{\mathscr{C}}$ is continuous and $\tilde{F}: \hat{\mathscr{C}} \to \bar{\mathscr{R}}$ is defined by

$$\tilde{F}(x) = 2\int_0^1 (\partial^2 x(\eta))^2 d\eta \ , \quad x \in \hat{\mathscr{C}} \cap \mathscr{W}_2^2$$

$$\tilde{F}(x) = \infty \text{ if } x \in \hat{\mathscr{C}} \ , \quad x \notin \mathscr{W}_2^2 \ .$$

Defining a Hilbert space $\mathscr{B}_1$ to be the set $\hat{\mathscr{C}} \cap \mathscr{W}_2^2$ equipped with the natural norm of $\mathscr{W}_2^2$, the argument made above again applies, and $\tilde{F}: \hat{\mathscr{C}} \to \overline{\mathscr{R}}$ is lower semicontinuous; it follows that $W: \hat{\mathscr{C}} \to \overline{\mathscr{R}}$ is lower semicontinuous. Moreover, since $U = (1-\varepsilon)W$ and $\varepsilon < 1$, it follows that $U: \mathscr{R}^+ \times \hat{\mathscr{C}} \to \overline{\mathscr{R}}$ given by (8) is also lower semicontinuous, as was previously claimed.

In combination with a result such as Proposition 3.5, Theorem 3.4 provides a means of assuring precompactness of positive orbits; hence, the Invariance Principle is made available for the study of positive limit sets and asymptotic behavior of motions. The usual form of the Invariance Principle is given by (a) of Theorem 3.6 below, and its proof is well known [5,11,14]. This form of the Invariance Principle requires a continuous Liapunov function, which seems unfortunate since a l.s.c. Liapunov function, capable of showing precompactness of positive orbits, will not be continuous unless $\mathscr{X}$ is a compact metric space; hence, a second (continuous) Liapunov function must be found, and Liapunov functions often are very difficult to find. This disadvantage led Dafermos [6,7] to an extension of the Invariance Principle which employs finite-valued l.s.c.

Liapunov functions; unfortunately, his extension applies to only a very special class of dynamical systems, wherein every motion is known to be stable a-priori. A more generally useful extension seems to be provided by our result (b) of Theorem 3.6 below. Related to an idea used by Ball [1] in yet another extension of the Invariance Principle, result (b) of Theorem 3.6 extends the Invariance Principle to l.s.c. Liapunov functions and general dynamical systems. Rather than requiring knowledge of $\dot{V}$ as in (a), result (b) requires only an explicitly known lower semicontinuous and nonnegative lower bound for $-\dot{V}$, such as the estimate $U(0,x)$ provided by Theorem 3.4.

Theorem 3.6: Let $u:\mathscr{R}^+ \times \mathscr{X} \to \mathscr{X}$ be a dynamical system on a metric space $\mathscr{X}$, and let $V:\mathscr{X} \to \bar{\mathscr{R}}$ be a l.s.c. Liapunov function for u on a set $\mathscr{G} \subset \mathscr{X}$ such that $\dot{V}(x) \leq -W(x) \leq 0$ for all $x \in \mathscr{G}$, where $W:(\bar{\mathscr{G}} \subset \mathscr{X}) \to \bar{\mathscr{R}}$ is lower semicontinuous on $\bar{\mathscr{G}} = \mathrm{Cl}_{\mathscr{X}}\mathscr{G}$ and $V(y) > -\infty$ for all $y \in \bar{\mathscr{G}}$. If $\gamma(x) \subset \mathscr{G}$, then $\Omega(x) \subset \mathscr{M}^+$, where $\mathscr{M}^+$ is the largest positive invariant subset of

(a) $\mathscr{M}_1 = \{z \in \bar{\mathscr{G}} | \dot{V}(z) = 0\}$ if V is continuous (in fact, $\Omega(x) \subset \mathscr{M}^+ \cap V^{-1}(\beta)$ for some $\beta \in \mathscr{R}$),

or

(b) $\mathscr{M}_2 = \{z \in \bar{\mathscr{G}} | W(z) = 0\}$ if V is only lower semicontinuous.

If, in addition, $\mathscr{X}$ is complete and $\gamma(x)$ is precompact, then $u(t,x) \xrightarrow{\mathscr{X}} \mathscr{M}$ as $t \to \infty$, where $\mathscr{M}$ is the largest invariant subset of $\mathscr{M}_1$ if V is continuous, or of $\mathscr{M}_2$ if V is only lower semicontinuous.

Proof: It is well known that $\Omega(x)$ is closed and positive invariant [11,14]. If $\mathscr{X}$ is complete and $\gamma(x)$ is precompact, then $\Omega(x)$ is nonempty, compact, connected, and invariant; moreover, $u(t,x) \to \Omega(x)$ as $t \to \infty$. Assuming that $\gamma(x) \subset \mathscr{G}$, we have $\Omega(x) \subset \mathrm{Cl}_{\mathscr{X}}\gamma(x) \subset \bar{\mathscr{G}}$. If $\gamma(x)$ is not precompact $\Omega(x)$ may be empty, in which case the theorem is obviously true but vacuous; hence, we will assume that $\Omega(x)$ is nonempty. There now are several cases to be considered.

If $V(u(t,x)) \equiv \infty$ for all $t \in \mathscr{R}^+$, Definition 1.3 implies that $W(u(t,x)) \equiv 0$ on $\mathscr{R}^+$; hence, by the lower semicontinuity and nonnegativity of W, $W(z) = 0$ for every $z \in \Omega(x)$ and result (b) applies.

If $V(0) = \infty$ but $V(u(t^*,x)) < \infty$ for some $t^* > 0$, we may replace x by $x^* = u(t^*,x)$ and note that $\Omega(x^*) = \Omega(x)$, $V(x^*) < \infty$; hence, the proof of (b) for this case can be embedded in the proof for the following case.

If $V(x) < \infty$, Proposition 3.2 shows that $V(u(\cdot,x)):\mathscr{R}^+ \to \mathscr{X}$ is nonincreasing as well as finite-valued on $\mathscr{R}^+$. This implies that $V(u(t,x)) \to \beta < \infty$ as $t \to \infty$, where $\beta = \inf_{t \in \mathscr{R}^+} V(u(t,x))$ [16]; as $\mathrm{Cl}\gamma(x) \subset \overline{\mathscr{G}}$, $\Omega(x)$ is nonempty, and $V(y) > -\infty$ for every $y \in \overline{\mathscr{G}}$, lower semicontinuity of V implies $\beta > -\infty$. If $V: \mathscr{X} \to \overline{\mathscr{R}}$ is continuous, it follows from the definition of $\Omega(x)$ (Definition 1.2) that $V(z) = \beta$ for every $z \in \Omega(x)$; furthermore, as $\Omega(x)$ is positive invariant, $\dot{V}(z) = 0$ for every $z \in \Omega(x)$ and the well known result (a) follows. On the other hand, if V is only lower semicontinuous, we note as in the proof of Proposition 3.2 that $V(u(\cdot,x)):\mathscr{R}^+ \to \mathscr{R}$ is differentiable a.e. and

$$V(u(t,x)) - V(x) \leq \int_0^t \dot{V}(u(s,x))ds, \quad t \in \mathscr{R}^+.$$

Therefore, considering any sequence $\{t_n\}_{n=1,2,\ldots} \subset \mathscr{R}^+$ such that $t_n \to \infty$ and $u(t_n,x) \to z \in \Omega(x)$ as $n \to \infty$, the uniqueness of the limit β implies that for any $T > 0$,

$$0 \geq \int_0^T \dot{V}(u(s+t_n,x))ds \geq [V(u(t_n+T,x)) - V(u(t_n,x))] \to 0 \quad \text{as} \quad n \to \infty.$$

As W is nonnegative and lower semicontinuous, we apply Fatou's Lemma [16] to obtain

$$\begin{aligned}\int_0^T W(u(s,z))ds &\leq \int_0^T [\liminf_{n \to \infty} W(u(s,u(t_n,x))]ds \\ &\leq \liminf_{n \to \infty} \int_0^T W(u(s+t_n,x)ds \\ &\leq -\lim_{n\to\infty} \int_0^T \dot{V}(u(s+t_n,x))ds = 0 \quad .\end{aligned}$$

Therefore, $W(u(s,z)) = 0$ a.e. $s \in [0,T]$, and the lower semicontinuity of W now implies that $0 = W(u(0,z)) = W(z)$; hence, $\Omega(x) \subset \{z \in \overline{\mathscr{G}} | W(z) = 0\}$ and the proof of (b) is complete. ∎

For an application of Theorem 3.6 we return to the example begun in Section 2, with $\hat{\mathscr{C}}$,A,V,W defined by (5), (6), (9). As V is only lower semicontinuous on $\hat{\mathscr{C}}$ and W is a lower bound for $-\dot{V}$ only on the (positive invariant) set $\mathscr{G} = \hat{\mathscr{C}} \cap \mathscr{W}_2^1$, we note that $\overline{\mathscr{G}} = \hat{\mathscr{C}}$ and

$$\mathscr{M}_2 = \{x \in \hat{\mathscr{C}} \mid \partial^2 x(\eta) + f'(x(\eta)) = 0 \text{ a.e. } \eta \in [0,1]\}$$

$$= \{x \in \mathscr{D}(A) | \ Ax = 0\} .$$

Hence, $\mathscr{M}_2$ consists solely of equilibria of the dynamical system $u: \mathscr{R}^+ \times \hat{\mathscr{C}} \to \hat{\mathscr{C}}$; as equilibria are invariant, $\mathscr{M} = \mathscr{M}_2$. It follows that, for $x \in \hat{\mathscr{C}} \cap \mathscr{W}_2^1$, $\Omega(x)$ consists solely of equilibria. If $f: \mathscr{R} \to \mathscr{R}$ is such that

$$\mathscr{G}_\alpha \subset \{x \in \hat{\mathscr{C}} \cap \mathscr{W}_2^1 | \int_0^1 (\partial x(\eta))^2 d\eta \leq \beta\}$$

for some (finite) $\alpha, \beta \in \mathscr{R}$, where $\mathscr{G}_\alpha = \{x \in \hat{\mathscr{C}} | V(x) \leq \alpha\}$, then our earlier results imply that $\mathscr{G}_\alpha$ is compact; hence, $\gamma(x)$ is precompact for every $x \in \mathscr{G}_\alpha$ and $u(t,x) \to \mathscr{M} \cap \mathscr{G}_\alpha$ (strongly in $\hat{\mathscr{C}}$) as $t \to \infty$, for every $x \in \mathscr{G}_\alpha$. If equilibria are isolated in $\hat{\mathscr{C}}$, it then follows from connectedness of the positive limit set $\Omega(x)$ that $\Omega(x)$ consists of exactly one equilibrium for $x \in \mathscr{G}_\alpha$. The approach used here might have simplified considerably the analysis of [2], which was performed (in a different space) under stronger assumptions on $f: \mathscr{R} \to \mathscr{R}$.

We see that result (b) of Theorem 3.6 is not as strong as result (a). Specifically, if V is not continuous, V may not be constant on the positive limit set $\Omega(x)$ of a motion $u(\cdot,x)$; in fact, we may not even have $\dot{V}$ zero on $\Omega(x)$ unless $-\dot{V}$ is lower semicontinuous on $\bar{\mathscr{G}}$. In contrast, the extension provided by Dafermos [6,7] for finite-valued l.s.c. Liapunov functions yields $\dot{V}$ zero on $\Omega(x)$ without assuming $-\dot{V}: \mathscr{X} \to \bar{\mathscr{R}}$ to be lower semicontinuous on $\bar{\mathscr{G}}$, provided that all motions of u are known to be stable a-priori; however, this is a very strong proviso which does not hold in general for our example. It would be extremely difficult to compare our extension with that of Ball (Theorems 2.2 and 2.3 of [1]), wherein the assumptions on V are of a very different nature; we ask the reader interested in such a comparison to consult that paper [1]. Here we only mention that, in our example, the function V defined by (6) violates an assumption of both Ball [1] and Dafermos [6,7], as it is not finite-valued everywhere on $\hat{\mathscr{C}}$.

We emphasize that Theorem 3.6, like Propositions 3.2 and 3.3, applies to a general dynamical system (Definition 1.1);

specifically, Theorem 3.6 does not assume the dynamical system to be generated in the sense of Definition 2.1. At present, it seems to be an open question as to whether or not all dynamical systems are generated in the sense of Definition 2.1.

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This paper presents several results pertaining to the use of lower semicontinuous Liapunov functions in the analysis of autonomous ab- evolution equations. Such functions can be useful in setting onlinear dynamical system that need not satisfy any exponentia e, as well as in locating positive invariant sets of the re- dynamical system. Other results concern the computation of vative of a lower semicontinuous Liapunov function, the use a function to assure precompactness of positive orbits, and

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20. ABSTRACT (continued)

a version of the Invariance Principle that is valid for lower semicontinuous Liapunov functions.