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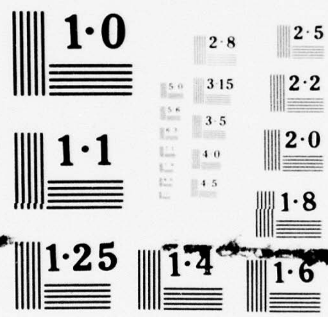

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PERIODIC AND QUASIPERIODIC SOLUTIONS OF $\Delta u + \lambda u + O(u) = 0$

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ABSTRACT

In this paper the boundary value problem $\Delta u+\lambda u+f(u,u_x,u_y)=0$, $u(0,y)=u(1,y)=0$ is studied in the strip $(0,1)\times R$, where f is some C^2-function which, together with its gradient, vanishes at 0, λ is a real parameter. It is shown that, for λ between π^2 and $4\pi^2$, all small solutions are periodic in y. Moreover, singular solutions exist as local H^2-limits of periodic solutions with large periods. For values of λ beyond $4\pi^2$ a formal argument suggests that almost all small solutions are quasiperiodic. The equation is studied as a model for some important but technically cumbersome bifurcation problems in fluid dynamics.

EXPLANATION

A nonlinear elliptic boundary-value problem which models aspects of the technically more cumbersome Taylor and Bénard problems of fluid dynamics is considered. It is shown that all small solutions are periodic in one variable when a parameter (corresponding to a bifurcation parameter) is in a suitable range and indications are given that all small solutions are quasiperiodic for larger parameter values when the nonlinearity is analytic.

AMS(MOS) Subject Classification - 35J60, 76DXX

Key Words - Nonlinear elliptic boundary-value problems, periodic solution, quasiperiodic solution, Taylor and Bénard problems

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§ 1 Introduction

In this study we try to classify all small solutions of the boundary-value problem

$$(1.1)\qquad \begin{aligned} &\Delta u + \lambda u + \tilde{f}(u, D_x u, D_y u) = 0 \\ &u(0,y) = u(1,y) = 0 \end{aligned}$$

in the strip $\Omega = \{(x,y) \in [0,1] \times \mathbb{R}\}$. Here, Δ is the two-dimensional Laplacean, λ a real parameter and $\tilde{f}(u,p,q)$ a real valued C^2-function of its real arguments which, together with its gradient, vanishes for $u = p = q = 0$. The class of solutions considered consists of functions with locally uniform H^s-norm

$$\sup_{\ell \in \mathbb{Z}} \|u\|_{H^s(K_\ell)} < \varepsilon$$

where $0 < s \leq 2$, H^s the usual Sobolev space, and where $K_\ell \equiv [0,1] \times [(\ell-1)\rho, \ell\rho]$ for some $\rho > 0$ denotes a sequence of compacta covering Ω. In view of the physical interpretation given later, our assumption requires small energy input per unit length.

The interest in this question arose from the effort to determine all physically reasonable solutions of the Taylor- and the Bénard problem in fluid dynamics. In both problems a basic (trivial) solution looses its stability to nontrivial solutions which bifurcate at a critical parameter value. In view of the underlying invariance, the set of nontrivial solutions near this bifurcation point is very large [11] but nothing is known about a characterisation of this set. There is an extensive theoretic literature (mathematical or physical) on these problems (see [4] for an excellent survey), however, all approaches assume periodicity in the unbounded variables a priori.

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One might consider this investigation as a first step toward a final answer of those questions, since (1.1) can, to some extent, be regarded as a model for the Navier-Stokes equations - the stationary Burgers equation is a special case of it - .

The conjecture, derived from the linearized equation, that - besides a discrete set of λ-values - all solutions are quasiperiodic is not true without further symmetry restrictions on $\tilde{f}$ as the one dimensional analogon $\ddot{u} + \lambda u + \dot{u}^3 = 0$ already shows. Hence we further impose one of the two following assumptions

$$
(1.2)\qquad
\begin{aligned}
&(a)\quad \tilde{f}(u,p,-q) = \tilde{f}(u,p,q)\\
&(b)\quad \tilde{f}(u,p,-q) = -\tilde{f}(u,p,q) \quad \text{and} \quad \tilde{f}(-u,-p,-q) = \tilde{f}(u,p,q)
\end{aligned}
$$

Given property (1.2) we are able to settle the question in the λ-interval $(\pi^2, 4\pi^2)$ by proving that all small solutions of (1.1) are periodic in y. The question becomes more delicate for $\lambda \in (n^2\pi^2, (n+1)^2\pi^2)$ where quasiperiodic solutions are expected. The method applied for the periodic case breaks down since the invertible part of the linearized operator ceases to be continuously invertible. If the non-linearity $\tilde{f}$ is real analytic then we can show at least, that formally (considering formal power series) all small solutions are quasiperiodic.

Unfortunately, the analogy to the fluid dynamical situations is rather limited for $\lambda \in (\pi^2, 4\pi^2)$ since, for the Taylor- as well as for the Bénard-problem, one expects two independent frequencies ω_1, ω_2, for λ slightly above the critical value. Hence, the analogous model case is $\lambda \in (4\pi^2, 9\pi^2)$ and we conjecture therefore that all small solutions in those fluid-dynamical problems are quasiperiodic. For indications of this fact consult [1],[5],[6],[12].

The exceptional set of values $\lambda = n^2\pi^2$ requires special consideration. We are able to prove for $\lambda = \pi^2$ the following alternative: either, there exist arbitrary small, nontrivial solutions for $\lambda = \pi^2$ or, there exist "singular" solutions near $\lambda = \pi^2$, $u = 0$, whose first Fourier-component is either constant or nonperiodic. The result is natural as analogous ordinary differential equations show. Generalisations to problems where Δ is replaced by a strongly elliptic operator of order 2m are possible, but we avoid them here for technical reasons.

To our knowledge, the problem under consideration has not been studied in the literature. There has been some interest in periodic solutions of nonlinear wave equations (see [8] for a recent list of references) but there, the interest is directed towards the existence of special solutions and their regularity.

We start with an investigation of $\Delta+\lambda$ in a space which is the inductive limit of weighted Sobolev spaces allowing polynomial growth of any order. We show that $\Delta+\lambda$ is onto for every λ, a fact, which justifies the choice of space and which contributes the corner stone for the reduction of (1.1) to a finite dimensional initial value problem - somewhat in the spirit of Liapunov and Schmidt - (sections 2 and 3). In section 4 we prove uniqueness in Theorem 4.4 . Section 5 contains the existence of just enough periodic resp. quasiperiodic solutions. Section 6 is devoted to the existence of singular solutions as envelopes of periodic ones.

We gratefully acknowledge discussions with W. Eckhaus and P. Fife about various topics of this paper.

§ 2 Spaces and their properties

Let be $\Omega := (0,1) \times \mathbb{R}$, $\underline{x} = (x,y) \in \mathbb{R}^2$, and K a nonempty compact subset of Ω. By $H^s_{loc}(\Omega)$ we denote the set of all funktions $u : \Omega \to \mathbb{R}$, with $u \in H^s(K)$ for all $K \subset \Omega$. Here, $H^s(K)$ is the usual real Sobolev space with the norm $(\ \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2,\ |\alpha| = \alpha_1 + \alpha_2)$:

$$\|u\|_{H^s(K)} = \sup_{|\alpha| \leq s} \left(\int_K |D^\alpha u|^2 \, d\underline{x} \right)^{1/2}, \quad s \geq 0$$

$C^0(\overline{\Omega})$ is the Banach space of continuous, real valued functions defined and bounded in $\overline{\Omega}$ equipped with the norm

$$\|u\|_{C^0} = \sup_{\underline{x} \in \overline{\Omega}} |u(\underline{x})| \quad .$$

Furthermore, we need a sequence of Banach spaces X^s_k which are defined as follows :

$$g_k(y) = \max(1, |y|^k), \quad k \in \mathbb{N}_0$$

$$\|u\|_{s,k} = \sup_{|\alpha| \leq s} \left(\int_\Omega g_k^{-2}(y) \, |D^\alpha u(x,y)|^2 \, d\underline{x} \right)^{1/2}$$

$$X^s_k = \{ \ u \in H^s_{loc}(\Omega) \mid \|u\|_{s,k} < \infty \ \}, \quad s \in \mathbb{N}_0$$

Observe that $g_{k+\ell} = g_k g_\ell$ holds.

The spaces X^s_k consist of functions with a given polynomial growth at infinity. Since $X^s_k \subset X^s_{k+1}$ algebraically and topologically, they form a scale of Banach spaces. The boundary conditions $u(0,y) = u(1,y) = 0$ define, via the trace, a subspace of X^s_k for $s > 1/2$ and every k, which is denoted by

$$\overset{o}{X}{}^s_k := \{ \ u \in X^s_k \mid u(0,y) = u(1,y) = 0 \ \} .$$

Finally let X^s be the inductive limit of the Banach spaces X^s_k, $k \in \mathbb{N}_0$, i.e. $X^s = \bigcup_{k=0}^{\infty} X^s_k$ is the locally convex topological (lct) vectorspace equipped with the finest topology which induces on every X^s_k a topology coarser than the given one ([2], p.429). Hence, X^s contains exactly those functions in $H^s_{loc}(\Omega)$ which , for large $|y|$, grow at most like a polynomial, and its lc-topology is such that the injections $X^s_k \to X^s$ are continuous. The spaces $\overset{0}{X}{}^s$ are defined analogously as inductive limits of $\overset{0}{X}{}^s_k$.

A subset M of a vector space is called absolutely convex (ac) if, for every $u,v \in M$, $\lambda u + \mu u \in M$ holds, whenever $|\lambda| + |\mu| \leq 1$, $\lambda,\mu \in \mathbb{R}$. The absolutely convex hull (ach) of M consists of all finite linear combinations

$$\sum_j \mu_j u_j \ , \ \sum_j |\mu_j| \leq 1 \ , \ u_j \in M \quad .$$

Thus, ach(M) is the smallest ac set containing M. The family of those ac subsets $U \subset X^s$, for which $U \cap X^s_k$ is a neighborhood in X^s_k for every k, forms a basis of neighborhoods in X^s. In particular, the sets

$$\text{ach} \left(\bigcup_{k=0}^{\infty} B^k_{\varepsilon_k} \right)$$

form a basis of neighborhoods of 0 *in* X^s *and hence,* generate its topology (cf. [9],p. 79). Here,

$$B^k_{\varepsilon_k} := \{ u \in X^s_k \mid \|u\|_{s,k} < \varepsilon_k \}, \quad \varepsilon_k > 0 \quad .$$

The set M is called bounded if, for every neighborhood U of 0, there exists a $\mu = \mu(U)$, such that $M \subset \nu U$ for all $|\nu| \geq \mu$. If X and Y denote lct-vectorspaces and $L : X \to Y$ is linear and continuous then, L maps bounded sets into bounded sets; such a map is called bounded. A lct-vectorspace X is bornological if and only if every bounded linear map L acting from X into any other lct-vectorspace is continuous ([2],p.177).

As inductive limits of Banach spaces, the X^s are bornological, and even barreled ([9],p.31f). A linear, continuous surjective map $L : X^s \to X^t$ is a *homomorphism*, i.e. maps open sets onto open sets [7]. It is not too hard to prove that X^s is seperated and sequentially complete. Since these properties are not used subsequently we omit the proof.

LEMMA 2.1

The set $M \subset X^s$ is bounded if and only if M is contained in some X_k^s and bounded there.

Proof : If $U \subset X^s$ is a neighborhood of 0 in X^s, so is $U \cap X_k^s$ a neighborhood of 0 in X_k^s. Hence, for $M \subset X_k^s$ bounded, there exists a $\mu > 0$ such that $M \subset \nu(U \cap X_k^s) \subset \nu U$ for all $|\nu| \geq \mu$ Thus M is bounded in X^s.

Now, let $M \subset X^s$ be bounded and assume that M is not contained in some X_k^s or, if so, not bounded there. Then , there is a sequence $u_k \in M$, $u_k \notin B_{k+1}^k$ for all $k \in \mathbb{N}_0$. Hence, we have $\alpha_k \in \mathbb{N}_0^2$ and compacta $K_k \subset \Omega$ such that

$$(2.1) \qquad \int_{K_k} g_k^{-2} \, |D^{\alpha_k} u_k|^2 \, d\underline{x} > k^2 \quad .$$

Choose a sequence (ε_ℓ) with $0 < \varepsilon_\ell < 1$, and $\varepsilon_\ell \leq \min \{ g_{\ell-k}^{-1}(y) \mid 0 \leq k \leq \ell, (x,y) \in \Omega_k \}$. We show that for $u \in U = \operatorname{ach} (\bigcup_{\ell=0}^{\infty} B_{\varepsilon_\ell}^\ell)$ and all k

$$(2.2) \qquad \int_{K_k} g_k^{-2} \, |D^{\alpha_k} u|^2 \, d\underline{x} < 1$$

holds. Hence, $u_k \notin kU$ by (2.1) which yields the contradiction.

To prove (2.2) observe, that for $0 \leq \ell \leq k$, $u \in B_{\varepsilon_\ell}^\ell$,

$$\int_{K_k} g_k^{-2} \, |D^{\alpha_k} u|^2 \, d\underline{x} \leq \int_{\Omega} g_\ell^{-2} \, |D^{\alpha_k} u|^2 \, d\underline{x} < \varepsilon_\ell^2 < 1$$

holds and, if $\ell > k$, $u \in B_{\varepsilon_\ell}^\ell$

$$\int_{K_k} g_k^{-2} \, |D^{\alpha_k} u|^2 \, d\underline{x} \leq \frac{1}{\min\limits_{\underline{x} \in \Omega_k} g_{\ell-k}^{-2}} \int_{\Omega_k} g_\ell^{-2} \, |D^{\alpha_k} u|^2 \, d\underline{x}$$

$$\leq \frac{\varepsilon_\ell^2}{\min\limits_{\underline{x} \in \Omega_k} g_{\ell-k}^{-2}} < 1$$

Since U is the ach of $B^{\ell}_{\varepsilon_\ell}$, (2.2) follows readily.

LEMMA 2.2

Let the sequence $(u_\ell) \subset X^s$ be bounded and such that $(u_{\ell|K})$ converges in $H^s(K)$ for every compact $K \subset \Omega$. Then, (u_ℓ) converges in X^s.

Proof : According to Lemma 2.1, $(u_\ell) \subset X^s_k$ holds for some k, and (u_ℓ) bounded there : $\|u_\ell\|_{s,k} \leq C$. We show, there exists an $u \in X^s_k$ such that $\lim\limits_{\ell\to\infty} u_\ell = u$ in X^s_{k+1}. Since X^s_{k+1} lies continuously in X^s, the assertion follows.
Construct an $u \in H^s_{loc}(\Omega)$ such that $u_{\ell|K}$ converges towards $u_{|K}$. Hence, for every $K \subset \Omega$ we have an index ℓ with

$$\sup_{|\alpha|\leq s} \left(\int_K g_k^{-2} \, |D^\alpha(u-u_\ell)|^2 \, d\underline{x} \right)^{1/2} < 1$$

and thus

$$\sup_{|\alpha|\leq s} \left(\int_K g_k^{-2} \, |D^\alpha u|^2 \, d\underline{x} \right)^{1/2} \leq \|u_\ell\|_{s,k} + 1 < C + 1 \quad .$$

This yields $u \in X^s_k$. For arbitrary $\delta > 0$ choose $K \subset \Omega$ so large that $|g_1^{-1}(y)| < \delta(4C+2)^{-1}$ for all $(x,y) \in \Pi \setminus K$.
Hence

$$\begin{aligned} \|u - u_\ell\|_{s,k+1} &\leq \sup_{|\alpha|\leq s} \left(\int_K g_{k+1}^{-2} \, |D^\alpha(u-u_\ell)|^2 \, d\underline{x} \right)^{1/2} \\ &+ \sup_{|\alpha|\leq s} \left(\int_{\Omega\setminus K} g_{k+1}^{-2} \, |D^\alpha(u-u_\ell)|^2 \, d\underline{x} \right)^{1/2} \\ &< \frac{\delta}{2} + \frac{\delta}{2(2C+1)} \|u - u_\ell\|_{s,k} < \delta \end{aligned}$$

for all sufficiently large ℓ; which proves the assertion.

Subsequently, we need a characterisation of X^s_k by Fourier components which is an easy consequence of Parseval's equality. We introduce for $g : \Pi \to \mathbb{R}$ the Banach spaces Y^s_k :

$$(2.3)\qquad \begin{aligned} |u|_{s,k} &= \sup_{0\le\gamma\le s} \int_{\mathbb{R}} g_k^{-2}\, |D^\gamma u|^2\, dy \ , \\ Y_k^s &= \{u \in H^s_{loc}(\mathbb{R}) \mid |u|_{s,k} < \infty\} \end{aligned}$$

LEMMA 2.3

Set $\overset{\circ}{X}{}_k^0 = X_k^0$ and assume $u \in X_k^s$ for $s = 0,1,$or 2. Define

$$u_\nu = \sqrt{2}\int_0^1 u(x,\cdot)\sin\nu\pi x\, dx,$$

then we have $u_\nu \in Y_k^s$ for all $\nu \in \mathbb{N}$ and

$$(2.4)\qquad \sum_1^\infty (\nu\pi)^{2\beta} |D^\gamma u_\nu|^2_{0,k} = \| D_x^\beta D_y^\gamma u \|^2_{0,k}$$

for $\beta,\gamma \in \mathbb{N}_0$, $\beta + \gamma \le s$.

Conversely, let $(u_\nu \mid u_\nu \in Y_k^s)$ be a sequence for which the left side of (2.4) is finite. Then, the functions u_ν are the Fourier components of some u in $\overset{\circ}{X}{}_k^s$ which satisfies (2.4).

Proof : $u_\nu \in Y_k^s$ follows from

$$D^\gamma u_\nu = \sqrt{2}\int_0^1 D_y^\gamma u(x,\cdot)\sin\nu\pi x\, dx$$

and hence

$$|D^\gamma u_\nu|^2_{0,k} \le \|D^\gamma u\|^2_{0,k}$$

Define

$$\varphi_\beta^\nu(x) = \sqrt{2}\begin{cases} \sin\nu\pi x & \text{for } \beta = 0 \\ \cos\nu\pi x & \text{for } \beta = 1 \\ -\sin\nu\pi x & \text{for } \beta = 2 \end{cases}$$

and

$$u_\nu^{\beta,\gamma} = \int_0^1 D_x^\beta D_y^\gamma\, u(x,\cdot)\, \varphi_\beta^\nu(x)\, dx$$

Then

$$u_\nu^{\beta,\gamma} = (\nu\pi)^\beta\, D^\gamma u_\nu$$

holds, and Parseval's idendity implies (2.4).

For the proof of the converse set $u_N^{\beta,\gamma} = \sum_1^N (\nu\pi)^\beta D^\gamma u_\nu \varphi_\beta^\nu$ then $u_N^{\beta,\gamma}$ converges in X_k^0 towards $u^{\beta,\gamma}$, which is the weak $D_x^\beta D_y^\gamma$- derivative of $u = u^{0,0}$. Moreover the traces $u(0,\cdot)$, $u(1,\cdot)$ vanish for $s = 1,2$.

§ 3 Some Estimates

The solution of equation (1.1) requires some knowledge about the action of the twodimensional Laplacean in the spaces X^s. We define $A : \overset{0}{X}{}^2 \to X^0$ by $Au = \Delta u$. The parameter λ is assumed to lie in $[n^2\pi^2, (n+1)^2\pi^2)$ for some fixed $n \in \mathbb{N}_0$.
$L_\nu(\lambda) = D_y^2 + (\lambda - \nu^2\pi^2)$ denotes the operator generated by A on the Fourier components u_ν of u resp. f_ν of f (see Lemma 2.3).

LEMMA 3.1

Let be $f \in X_k^0$, $\nu \in \mathbb{N}$ and $\nu \geqq n+1$. Then there exists a unique solution $u_\nu \in Y_\ell^2$ of $L_\nu(\lambda)u_\nu = f_\nu$. It satisfies the inequality

$$(3.1) \qquad |D^\gamma u_\nu|_{0,\ell} \leqq \frac{c_1(k)}{\nu^{2-\gamma}} |f_\nu|_{0,k} \quad , \quad \gamma = 0,1,2 \quad ,$$

for some constant $c_1(k)$, which is independent of ℓ and ν.

Proof : Since $L_\nu(\lambda)u = 0$ has only exponentially growing solutions, the uniqueness is trivial. By means of the Green's function

$$G_\nu(y,\eta) = \frac{1}{2\omega_\nu} e^{-\omega_\nu(y-\eta)} \quad \text{for } \eta < y$$

$$G_\nu(y,\eta) = G_\nu(\eta,y) \quad , \quad \omega_\nu = \sqrt{\nu^2\pi^2 - \lambda}$$

one obtains the solutions by the formula

$$(3.2) \qquad u_\nu(y) = \int_{\mathbb{R}} G_\nu(y,\eta)\, f_\nu(\eta)\, d\eta$$

Setting $f_\nu = g_k h_\nu$, *one obtains* $h_\nu \in L_2(\mathbb{R})$ *and, using* Cauchy-Schwarz's *inequality*, $\gamma = 0$ or 1 :

$$|D^\gamma u_\nu(y)|^2 \leqq \omega_\nu^{2\gamma} \int_{\mathbb{R}} G(y,\eta) g_k^2(\eta)\, d\eta \int_{\mathbb{R}} G(y,\eta) h_\nu^2(\eta)\, d\eta$$

$$\leqq \omega_\nu^{2\gamma-2}\, c_1(k)\, g_k^2(y) \int_{\mathbb{R}} G(y,\eta) h_\nu^2(\eta)\, d\eta$$

Here, we used an estimate for the first integral which can be established by elementary calculations. Now, Fubini's theorem yields :

$$|D^\gamma u_\nu|^2_{o,\ell} \leq c_1(k)\ \omega_\nu^{2\gamma-2} \int_{\mathbb{R}} h_\nu^2(\eta)\ \left(\int_{\mathbb{R}} G(y,\eta) g^{-2}_{\ell-k}(y)\ dy\right)\ d\eta$$

The inequality $|g^{-2}_{\ell-k}(y)| \leq 1$ for $\ell \geq k$ and all $y \in \mathbb{R}$ implies

$$|D^\gamma u_\nu|^2_{o,\ell} \leq c_1 \omega_\nu^{2\gamma-4} \int_{\mathbb{R}} h_\nu^2(\eta)\ d\eta \quad \text{for } \gamma = 0 \text{ or } 1.$$

The case $\gamma = 2$ follows from the differential equation. Observing $\omega_\nu \sim \nu$ completes the proof.

Define for $s = 0,1,2$ linear projections $P^s : X^s \to X^s$, $Q^s = id - P^s$ by $P^s u = \sum_{\nu=1}^{n} u_\nu(y) \sin \nu\pi x$. P^s and Q^s are continuous and P^2, Q^2 commute with A in the sense that $AP^2 = P^o A$ resp. $AQ^2 = Q^o A$ holds in $\overset{o}{X}{}^2$. Moreover, $Q^s X^s$ is the inductive limit of $Q^s X^s_k$ having all properties mentioned for X^s in § 2.

LEMMA 3.2

Let B_2 denote the restriction of $A+\lambda$ onto $Q^2 X^o$. Then, $B_2 : Q^2\overset{o}{X}{}^2 \to Q^o X^o$ is a topological isomorphism. Furthermore,

(3.3) $\quad \|u\|_{2,k} \leq c_2(k)\ \|B_2 u\|_{o,k}$

holds for $u \in Q^2\overset{o}{X}{}^2$.

The proof that B_2 is bijective is a consequence of Lemma 3.1 which, in addition, shows the continuity of B_2^{-1}. The continuity of B_2 itself is trivial (either direct or by the open mapping principle). Inequality (3.3) follows from (2.4) and (3.1) immediately.

COROLLARY 3.3

Let $f^\rho = f(\cdot,\cdot+\rho)$, $\rho \in \mathbb{R}$, be the ρ-translation of f. Then, for every $f \in X^0$, $(B_2^{-1}f)^\rho = B_2^{-1}f^\rho$.

For the proof use the representation of B_2^{-1} given by (3.2) through its action on the Fourier components. Observe that $G_\nu(y,\eta)$ only depends on the difference $y-\eta$.

LEMMA 3.4

Let $\rho \geq 1$ be fixed and define $K_\ell = [0,1] \times [(\ell-1)\rho,\ell\rho]$ for $\ell=1,2,\ldots$ and $K_\ell = [0,1] \times [\ell\rho,(\ell+1)\rho]$ for $\ell=-1,-2,\ldots$, $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$.
Assume

$$\sup_{\ell\in\mathbb{Z}'} \|f\|_{H^0(K_\ell)} < \infty$$

for some $f \in X^0$.
Then there exists a ρ-independent constant γ_1 such that

$$\sup_{\ell\in\mathbb{Z}'} \|B_2 Q^0 f\|_{H^2(K_\ell)} \leq \gamma_1 \sup_\ell \|f\|_{H^0(K_\ell)}$$

is satisfied.

Proof : Observe that $g_2^{-2} \geq \rho^{-2}$ holds in K_1 and use Lemma 3.2 to obtain

$$\|B_2^{-1}Q_0 f^{\ell\rho}\|_{H^2(K_1)} \leq \rho^2\|B_2^{-1}Q_0 f^{\ell\rho}\|_{2,2} \leq c_2(2)\rho^2\|f^{\ell\rho}\|_{0,2}$$

$$= c_2(2)\,\rho^2 \sum_{|k|=1}^{\infty} \|g_2^{-1}f^{\ell\rho}\|_{H^0(K_k)}$$

Since $|g_2^{-1}(y)| \leq \min\,(1,1/(k-1)^2\rho^2)$ for all $y \in K_k$ and in view of

$$\sup_{k\in\mathbb{Z}'} \|f^{\ell\rho}\|_{H^0(K_k)} = \sup_{\ell\in\mathbb{Z}'} \|f^{\ell\rho}\|_{H^0(K_1)}$$

the inequality follows immediately.

It is useful subsequently to introduce

$$f_{|\ell} = f\chi_{K_\ell} \quad , \quad \ell \in \mathbb{Z}'$$

where χ_{K_ℓ} denotes the characteristic function of K_ℓ, defined in Lemma 3.4. Instead of $\|f_{|\ell}\|_{H(K_\ell)}$ we use the less correct but unambigous notation $\|f\|_{H(K_\ell)}$.

LEMMA 3.5

Let be $f \in Q^0 X^0$, $f(x,y) = 0$ for $(x,y) \in S_\rho = [0,1] \times (-\infty,\rho)$. Then , $u \in B_2^{-1} f$ restricted to S_ρ has the form :

$$B_2^{-1} f_{|S_\rho} = \sum_{\nu=n+1}^{\infty} u_\nu(\rho) \sin \nu\pi x \; e^{\omega_\nu y}$$

Furthermore, the following inequality holds :

$$\|B_2^{-1} f\|_{H^2(K_{-1})} \leq e^{-\rho\omega_{n+1}} \|B_2^{-1} f\|_{H^2(K_1)}$$

An analogous result is valid, when S_ρ is replaced by $[0,1] \times (-\rho,\infty)$ and the roles of K_{-1} and K_1 are interchanged.

Proof : The representation of u follows by a straightforward computation. To obtain the inequality use Lemma 2.3 and the following estimates :

$$\begin{aligned}
\|D_x^\beta D_y^\gamma u\|^2_{H^0(K_1)} &= \sum_{\nu=n+1}^{\infty} (\nu\pi)^{2\beta} \omega_\nu^{2\gamma} u_\nu(\rho)^2 \int_{-\infty}^{\rho} e^{2\omega_\nu y} \, dy \\
&\geq \frac{e^{2\omega_{n+1}\rho}}{2} \sum_{\nu=n+1}^{\infty} (\nu\pi)^{2\beta} \omega_\nu^{2\gamma-1} u_\nu(\rho)^2 (1-e^{-2\omega_\nu\rho}) \\
&= e^{2\omega_{n+1}\rho} \|D_x^\beta D_y^\gamma u\|^2_{H^0(K_{-1})} \quad , \qquad \text{qed.}
\end{aligned}$$

Now, we study the action of $\Lambda+\lambda$ on $P^2 Q^2_X$. The operator is still onto but an estimate of the form (3.2) cannot be expected. Allowing a higher order growth we arrive at estimates which are sufficient for the uniqueness proof in the next section.

LEMMA 3.6

Let be $f \in X_k^0$ for some $k \in \mathbb{N}_0$, and assume $\nu \leq n$. Then, $L_\nu(\lambda)u_\nu = f_\nu$ possesses a solution which satisfies

$$|D^\gamma u_\nu|_{0,\ell} \leq c_3(k)\ |f_\nu|_{0,k} \quad , \qquad \gamma = 0,1,2$$

where $\ell > k+\frac{3}{2}$ if $\lambda = n^2\pi^2$ and $\nu = n$, and $\ell > k+1$ otherwise.

Proof : Define φ by $L_\nu(\lambda)\varphi = 0$, $\varphi(0) = 0$, $\varphi'(0) = 1$. Consider first the case $\nu < n$ or $\lambda > n^2\pi^2$. Then φ and its derivatives are bounded and

$$u_\nu(y) = \int_0^y \varphi(y-\eta)\ f_\nu(\eta)\ d\eta$$

is the desired solution. With $f_\nu = g_k h_\nu$, one obtains $h_\nu \in H^0(\mathbb{R})$ and, if $\gamma = 0$ or 1 :

$$|D^\gamma u_\nu|^2_{0,\ell} \leq C(\varphi) \int_{\mathbb{R}} g_\ell^{-2}\Big(\Big(\int_0^{|y|} g_k^2\ d\eta\Big) \int_0^{|y|} h_\nu^2(\eta)\ d\eta\Big)\ dy$$

$$\leq 2C(\varphi)\ |f_\nu|^2_{0,k} \int_{\mathbb{R}} g^{-2}_{\ell-(k+\frac{1}{2})}\ dy$$

$$\leq c_3(k)\ |f_\nu|^2_{0,k} \qquad \text{if } \ell > k+1 \quad .$$

For $\nu = n$ and $\lambda = n^2\pi^2$ one has $\varphi(y) = y$. The modified estimation yields

$$|D^\gamma u_\nu|^2_{0,\ell} \leq C_1\ |f_\nu|^2_{0,k} \int_{\mathbb{R}} g^2_{\ell-(k+1)}\ dy \leq c_3(k)\ |f_\nu|^2_{0,k}$$

if $\ell > k+\frac{3}{2}$. The inequalities for $\gamma = 2$ follow from the differential equation, qed.

The proof shows that u_ν is uniquely determined by the initial condition $u_\nu(0) = u_\nu'(0) = 0$. To put this into the right framework define

$$Z_1 = P^2 \overset{o}{X}{}^2$$

$$\tilde{P} : \overset{o}{X}{}^2 \to \ker(A+\lambda) \quad , \quad \tilde{Q} = \mathrm{id} - \tilde{P}$$

(3.4)
$$\tilde{P}u = \sqrt{2} \sum_{\nu=1}^{n} \sin \nu\pi x \Big(\int_0^1 u(x,o) \sin \nu\pi x \, dx \cos\sqrt{\lambda-\nu^2\pi^2}\, y + \frac{1}{\sqrt{\lambda-\nu^2\pi^2}} \int_0^1 D_y u(x,o) \sin \nu\pi x \, dx \sin\sqrt{\lambda-\nu^2\pi^2}\, y \Big)$$

if $\lambda > n^2\pi^2$; otherwise, for $\lambda = n^2\pi^2$, we write $\tilde{P}_{n-1}$ for the above expression when n is replaced by n-1 and define

(3.4')
$$\tilde{P}u = \tilde{P}_{n-1}u + \sqrt{2} \sin n\pi x \Big(\int_0^1 u(x,o) \sin n\pi x \, dx + y \int_0^1 D_y u(x,o) \sin n\pi x \, dx \Big)$$

Note that the trace $D_y u(x,o)$ is well defined since $u \in \overset{o}{X}{}^2$. It is easily seen that $\tilde{P}$ and $\tilde{Q}$ are continuous projections, $\tilde{P}_{|\ker(A+\lambda)} = \mathrm{id}$ and $\tilde{P}u = 0$ if and only if $u_\nu(x,o) = D_y u_\nu(x,o)$ holds for $0 \leqq \nu \leqq n$. Hence, according to our previous Lemma

(3.5)
$$B_1 = (A+\lambda)_{|\tilde{Q}Z_1} : \tilde{Q}Z_1 \to P^o X^o$$
$$A_1 = (A+\lambda)_{|\tilde{P}Z_1} : \tilde{P}Z_1 \to \{0\}$$

B_1 is a topological isomorphism. Therefore, the decomposition

$$\overset{o}{X}{}^2 = \tilde{P}Z_1 \oplus \tilde{Q}Z_1 \oplus Q^2 \overset{o}{X}{}^2$$

defines a decomposition of $A+\lambda$ in the form

(3.6) $$A + \lambda = A_1 \oplus B \;, \quad B = B_1 \oplus B_2 \;,$$

where B is a top isomorphism between $\tilde{Q}Z_1 \oplus Q^2 \overset{o}{X}{}^2$ and X^o.

THEOREM 3.7

Assume that $n^2\pi^2 \leq \lambda < (n+1)^2\pi^2$ holds for some $n \in \mathbb{N}_0$. Then,

(i) $\dim \ker (A + \lambda) = 2n$

(ii) $A + \lambda = A_1 \oplus B$

where A_1 and B are given through (3.5), $B : \tilde{Q}Z_1 \oplus Q^2\overset{\circ}{X}{}^2 \to X^0$ is a topological isomorphism and the following estimate holds

$$\|u\|_{2,\ell} \leq c(k)\, \|Bu\|_{0,k} \quad \text{if } \ell > k + \frac{3}{2} \quad .$$

The following two Lemmas estimate the influence of the initial conditions $u_\nu(0)$, $u'_\nu(0)$ on the $H^2(K_{\pm 1})$-norm of u, which will be crucial for a continuation argument in the uniqueness proof.

LEMMA 3.8

Adopt the notations of Lemma 3.4 assume $\ker(A+\lambda) \neq \{0\}$. Then, there exists a ρ-independent constant γ_2 such that

$$\|u\|_{H^2(K_{\pm 1})} \leq \gamma_2 \rho^k (\|f\|_{H^0(K_{\pm 1})} + \|\tilde{P}u\|_{2,\ell})$$

holds for $k = \max(2,\ell)$, $\ell \geq 1$, and $f \in P^0X^0$, $(A+\lambda)u = f$.

Proof : Observe that

$$(A+\lambda)v = f, \quad \tilde{P}v = 0, \quad f_{|1} = 0$$

yields $v_{|1} = 0$, since $v \in P^2\overset{\circ}{X}{}^2$ and $v_\nu(0) = v'_\nu(0) = 0$, $1 \leq \nu \leq n$. Hence

$$(B_1^{-1}f)_{|1} = (B_1^{-1}(f_{|1}))_{|1} \quad ,$$

and we conclude from Lemma 3.6 :

$$\|u-\tilde{P}u\|_{H^2(K_1)} = \|B_1^{-1}(f|_1)\|_{H^2(K_1)} \leq C_1\rho^2\|B_1^{-1}f|_1\|_{2,2}$$

$$\leq C_2\rho^2\|f|_1\|_{0,0} = C_2\rho^2\|f\|_{H^0(K_1)}$$

In view of $u \in P^2\overset{o}{X}{}^2$ we have

$$\|\tilde{P}u\|_{H^2(K_1)} \leq C_3\rho^{\ell}\|\tilde{P}u\|_{2,\ell}$$

for every $\ell \geq 1$, C_3 independent of ℓ; whence

$$\|u\|_{H^2(K_1)} \leq C_2\rho^2\|f\|_{H^0(K_1)} + C_3\rho^{\ell}\|Pu\|_{2,\ell}$$

which implies the assertion ($\rho \geq 1$).

LEMMA 3.9

The following inequality is valid for all $u \in P^2\overset{o}{X}{}^2$ and $\ell > \frac{3}{2}$:

$$\|\tilde{P}u\|_{2,\ell} \leq \gamma(\ell)\ \|u\|_{H^2(K_{\pm 1})}$$

Proof : Equation (3.4) immediately implies ($0 \leq \beta+\gamma \leq 2$) :

$$\|D_x^{\beta}D_y^{\gamma}\tilde{P}u\|_{0,\ell}^2 \leq C_1 \sum_{\nu=1}^{n} (u_\nu^2(0)+u_\nu'^2(0)) \int_{\mathbb{R}} (y^2+1)g_\ell^{-2}(y)\,dy$$

From Sobolev's imbedding theorem one obtains $u(\cdot,0)$, $D_y u(\cdot,0) \in L_2(0,1)$ for every $u \in H^2(K_{-1})$. Hence

$$|u_\nu'(0)|^2 \leq \int_0^1 |D_y u(x,0)|^2\,dx \leq C_2\|u\|_{H^2(K_{-1})}^2$$

and a similar inequality for $|u_\nu(0)|^2$. Thus, by Lemma 2.3 the assertion follows.

§ 4 Uniqueness

Our aim, to give a complete description of all small solutions of (1.1), requires a proof for the fact that there exists at most one "small" solution with given projection $\tilde{P}u$. The content of this section is a precise formulation of this statement.

We use the notations of the preceding section, in particular the constants $\gamma_1, \gamma_2, \gamma_3$ appearing in various Lemmas. Equation (1.1) is now written in operator notation

$$(4.1) \quad (A + \lambda)u + f(u) = 0 \ , \qquad u \in \overset{0}{X}{}^2$$

where λ satisfies $n^2\pi^2 \leq \lambda \leq (n+1)^2\pi^2$.

4.1 Assumptions for f

$f : H^s_{loc}(\Omega) \to H^0_{loc}(\Omega)$ for some s, $0 \leq s \leq 2$.

$$f_1 = f_{|H^s(K_1)} : H^s(K_1) \to H^0(K_1)$$

is supposed to be differentiable in a neighborhood of 0, and its derivative f_1' should be continuous at 0. Furthermore, $f_1(0) = 0$, $f_1'(0) = 0$, $f(u^\sigma) = f^\sigma(u)$ for all $\sigma \in \mathbb{R}$ is assumed.

Setting

$$p(u,v) = P^0(f(u)-f(v))$$

$$q(u,v) = Q^0(f(u)-f(v)) \quad ,$$

We obtain from 4.1 quite easily, that for every $\delta > 0$ there is a $\eta > 0$ - independent of ℓ - such that

$$(4.2) \quad \begin{aligned} \|p(u,v)\|_{H^0(K_\ell)} &\leq \delta \|u-v\|_{H^s(K_\ell)} \\ \|q(u,v)\|_{H^0(K_\ell)} &\leq \delta \|u-v\|_{H^s(K_\ell)} \end{aligned}$$

for all $\ell \in Z'$ and all $u,v \in H^s_{loc}(\Omega)$ with $\|u\|_{H^s(K_\ell)}, \|v\|_{H^s(K_\ell)} < n$. Since P^0, Q^0 commute with $u \mapsto u^\sigma$ we have $p(u^\sigma, v^\sigma) = p^\sigma(u,v)$ and similarly for q, $\sigma \in \mathbb{R}$.

For $\ell \in \mathbb{N}$, $k = \max(2,\ell)$ define constants $\rho, a, \delta, \varepsilon_m$ satisfying the conditions :

$$
(4.3) \qquad
\begin{aligned}
a &= \frac{1}{2}\min(1,(\gamma_2\gamma_3\rho^k)^{-1}) \\
\rho &\geq \max\left(-\frac{1}{\omega_{n+1}}\ln\frac{a}{2}\ ,\ 1\right) \\
\delta &= \min(a(2\gamma_2\rho^k)^{-1}, a^2(12\gamma_1)^{-1}) \\
\varepsilon_m &= a^m\varepsilon_0\ , \qquad \varepsilon_0 \geq 2n
\end{aligned}
$$

with n taken from (4.2). Observe that the sequence ε_m tends to 0.

4.2 Property $\Pi(m)$

The pair $u,v \in X^s$ is said to have property $\Pi(m)$, $m \in \mathbb{N}_0$, if

$$\sup_\sigma \|u^\sigma\|_{H^s(K_1)} < n\ , \qquad \sup_\sigma \|v^\sigma\|_{H^s(K_1)} < n$$

$$\|u-v\|_{H^2(K_j)} \leq \varepsilon_{m+1-|j|}$$

holds for $j = \pm 1, \ldots, \pm m$.

LEMMA 4.1

Let $u,v \in X^s$ have property $\Pi(m)$ for all $m \in \mathbb{N}_0$. Then, for $\tau = +1$ or $\tau = -1$, one has

$$\|B_2^{-1}\sum_{j=2\tau}^{\tau\infty} q_{|j}(u,v)\|_{H^2(K_\tau)} \leq \frac{1}{6}\varepsilon_{m+1}$$

$$\|B_2^{-1}\sum_{j=-\tau}^{-\tau\infty} q_{|j}(u,v)\|_{H^2(K_\tau)} \leq \frac{1}{6}\varepsilon_{m+1}\ , \qquad m \in \mathbb{N}_0\ .$$

Proof : We proceed by induction. $\Pi(0)$ implies $\|q(u,v)\|_{H^0(K_j)} \leq \varepsilon_0\delta$ for all $j \in Z'$. In view of Lemma 3.4 and (4.3) the inequalities hold for $m = 0$. Let them be satisfied for $m-1$. Since u,v have property $\Pi(m)$ one obtains

$$\|q(u,v)\|_{H^0(K_j)} < \delta\,\varepsilon_{m+1-|j|} \quad \text{for } |j| = 1,2, \ldots ,m .$$

Now, define $\tilde{u} = \sum_{j=3}^{\infty} u_{|j}$ and similarly v. Observe that $\tilde{u}^\rho, \tilde{v}^\rho$ possess property $\Pi(m-1)$, whence

$$\|B_2^{-1}\sum_{j=2}^{\infty} q_{|j}(\tilde{u}^\rho,\tilde{v}^\rho)\|_{H^2(K_1)} \leq \frac{1}{6}\varepsilon_m$$

Since we have $(\sum_{j=2}^{\infty} q_{|j}(\tilde{u}^\rho,\tilde{v}^\rho))_{|k} = 0$ for $k = 1,0,-1,-2, \ldots$, it follows, using Lemma 3.5, by assumption :

$$\|B_2^{-1}\sum_{j=3}^{\infty} q_{|j}(u,v)\|_{H^2(K_1)} = \|B_2^{-1}\sum_{j=2}^{\infty} q^\rho_{|j}(\tilde{u},\tilde{v})\|_{H^2(K_{-1})}$$

$$\leq e^{-\rho\omega_{n+1}}\|B_2^{-1}\sum_{j=2}^{\infty} q_{|j}(\tilde{u}^\rho,\tilde{v}^\rho)\|_{H^2(K_1)} \leq \frac{1}{6}\varepsilon_m\, e^{-\rho\omega_{n+1}}$$

Moreover $\Pi(m-1)$ for u,v and Lemma 3.4 yield :

$$\|B_2^{-1}q_{|2}(u,v)\|_{H^0(K_1)} \leq \gamma_1\delta\varepsilon_{m-1} \quad ,$$

Hence, by (4.3), one obtains :

$$\|B_2^{-1}\sum_{j=2}^{\infty} q_{|j}(u,v)\|_{H^2(K_1)} \leq \gamma_1\delta\varepsilon_{m-1} + \frac{1}{6}\varepsilon_m\, e^{-\rho\omega_{n+1}} \leq \frac{1}{6}\varepsilon_{m+1} .$$

Similarly one shows

$$\|B_2^{-1}\sum_{j=-2}^{-\infty} q_{|j}(u,v)\|_{H^2(K_{-1})} \leq \frac{1}{6}\varepsilon_{m+1} .$$

Let χ_- denote the characteristic function $\chi_{[0,1]\times\mathbb{R}^-}$. Note that $u^\rho_- = u^\rho\chi_-$, $v^\rho_- = v^\rho\chi_-$ have property $\Pi(m)$, implying

$$\|B_2^{-1}\sum_{j=-1}^{-\infty} q_{|j}(u,v)\|_{H^2(K_1)} = \|B_2^{-1}\sum_{j=-2}^{-\infty} q_{|j}(u^\rho_-,v^\rho_-)\|_{H^2(K_{-1})}$$

$$\leq \frac{1}{6}\varepsilon_{m+1} .$$

The last inequality for $\tau = -1$ follows quite similarly, qed.

LEMMA 4.2

If $u,v \in X^s$ have property $\Pi(m)$, for all $m \in \mathbb{N}_0$, then

$$\|B_2^{-1} q(u,v)\|_{H^2(K_{\pm 1})} \leq \frac{1}{2}\, \varepsilon_{m+1}$$

holds for every $m \in \mathbb{N}_0$.

Proof : Property $\Pi(m)$ for u,v and (4.2) imply $\|q_{|1}\|_{H^0(K_1)} \leq \delta\varepsilon_m$. Thus, by Lemma 3.4, one has : $\|B_2^{-1} q_{|1}\|_{H^2(K_1)} \leq \delta\gamma_1\varepsilon_m$. Therefore, in view of Lemma 4.1, we obtain

$$\begin{aligned}
\|B_2^{-1} q\|_{H^2(K_1)} &\leq \|B_2^{-1} q_{|1}\|_{H^2(K_1)} + \|B_2^{-1} \sum_{j=2}^{\infty} q_{|j}\|_{H^2(K_1)} \\
&\quad + \|B_2^{-1} \sum_{j=-1}^{-\infty} q_{|j}\|_{H^2(K_1)} \\
&\leq \delta\gamma_1\varepsilon_m + \frac{1}{3}\,\varepsilon_{m+1} \leq \frac{1}{2}\,\varepsilon_{m+1}
\end{aligned}$$

LEMMA 4.3

Let $m \in \mathbb{N}_0$ be fixed. Assume two solutions $u,v \in \overset{o}{X}{}^2$ of (4.1) have property $\Pi(m)$ and satisfy $\tilde{P}u = \tilde{P}v$ then,

$$\|P^2(u-v)\|_{H^2(K_j)} \leq \frac{\varepsilon_{m+1-|j|}}{2}$$

holds for $|j| = 1,2, \ldots ,m+1$.

Proof : Since $P^0(A+\lambda) = (A+\lambda)P^2$ one obtains

$$(A+\lambda)P^2(u-v) = p(u,v)$$

Now, Lemma 3.8 yields

$$(4.4) \qquad \|P^2(u-v)\|_{H^2(K_1)} \leq \gamma_2\rho^k(\delta\|u-v\|_{H^s(K_1)} + \|\tilde{P}(u-v)\|_{2,\ell})$$

where $k = \max(2,\ell)$, $\ell \geq 1$. Moreover, Lemma 3.9 implies

$$(4.5) \qquad \|\tilde{P}(u-v)\|_{2,\ell} \leq \gamma_3(\ell)\, \|P^2(u-v)\|_{H^2(K_{-1})}.$$

We fix ℓ and show inductively that

$$(4.6) \qquad \|P^2(u^{j\rho}-v^{j\rho})\|_{H^2(K_1)} < \tfrac{1}{2}(\varepsilon_{m+1-j})$$

holds for $0 \leq j \leq m$. For $j = 0$ we obtain from (4.4) and $\tilde{P}u = \tilde{P}v$:

$$\|P^2(u-v)\|_{H^2(K_1)} \leq \gamma_2 \rho^k \delta \varepsilon_m < \frac{1}{2}\, \varepsilon_{m+1}$$

Assume the validity of (4.6) for $j-1 < m$. As we know, $u^{j\rho}, v^{j\rho}$ are solutions of (4.1); they have property $\Pi(m-j)$. Hence, in view of the assumptions, one has :

$$\begin{aligned}
\|P^2(u^{j\rho}-v^{j\rho})\|_{H^2(K_{-1})} &= \|P^2(u^{(j-1)\rho}-v^{(j-1)\rho})\|_{H^2(K_1)} \\
&< \frac{1}{2}\,(\varepsilon_{m+1-(j-1)})
\end{aligned}$$

Combining this inequality with (4.4) and (4.5) yields

$$\begin{aligned}
\|P^2(u-v)\|_{H^2(K_j)} &= \|P^2(u^{j\rho}-v^{j\rho})\|_{H^2(K_1)} \\
&\leq \gamma_2 \rho^k (\delta \varepsilon_{m-j} + \gamma_3(\ell)\, \tfrac{1}{2}(\varepsilon_{m+2-j})) \\
&< \tfrac{1}{2}(\varepsilon_{m+1-j})
\end{aligned}$$

Thus, (4.6) is valid for all n, $0 \leq j \leq m$. A similar argument holds for negative indices j, and the Lemma is proved.

THEOREM 4.4

Let be $n^2\pi^2 \leq \lambda \leq (n+1)^2\pi^2$, $n \in \mathbb{N}_o$. Define $A : \overset{o}{X}{}^2 \to X^0$ by $Au = \Delta u$ and assume 4.1 for f.
Then, there exists a positive ε such that, for every two solutions u,v of equation (4.1), the conditions $\tilde{P}u = \tilde{P}v$ and $\sup\limits_\sigma \|u^\sigma\|_{H^s(K_1)} < \varepsilon$, $\sup\limits_\sigma \|v^\sigma\|_{H^s(K_1)} < \varepsilon$ imply $u = v$.

If $\ker(A+\lambda) = \{0\}$, $\tilde{P}$ is the null operator, and Theorem 4.4 asserts local uniqueness of the trivial solution $u = 0$.

Proof : We show inductively that u,v has property $\Pi(m)$ for all $m \in \mathbb{N}_o$, yielding $\|u-v\|_{H^2(K_j)} = 0$ for $|j| = 1,2, \ldots$ and thus $u = v$.

Set $\varepsilon = \eta \leq \varepsilon_o/2$, then $\Pi(0)$ holds for u,v. Assume, they have property $\Pi(m-1)$. Then, $u^{j\rho}, v^{j\rho}$ have property $\Pi(m-1-|j|)$ for $|j| = 0,1, \ldots ,m-1$. Since $u^{j\rho}, v^{j\rho}$ solve (4.1) one has :

$$Q^2(u^{j\rho}-v^{j\rho}) = B_2^{-1}q(u^{j\rho},v^{j\rho})$$

whence, via Lemma 4.2, one obtains

$$\begin{aligned} \|Q^2(u-v)\|_{H^2(K_{j+1})} &= \|Q^2(u^{j\rho}-v^{j\rho})\|_{H^2(K_1)} \\ &\leq \frac{1}{2}\,\varepsilon_{m-|j|} \quad \text{for } 0 \leq j \leq m-1 \\ \|Q^2(u-v)\|_{H^2(K_{j-1})} &\leq \|Q^2(u^{j\rho}-v^{j\rho})\|_{H^2(K_{-1})} \\ &\leq \frac{1}{2}\,\varepsilon_{m-|j|} \quad \text{for } 0 \geq j \geq -(m-1) \end{aligned}$$

Furthermore, if $\ker(A+\lambda) \neq \{0\}$, Lemma 4.3 implies

$$\|P^2(u-v)\|_{H^2(K_{j\pm1})} \leq \frac{1}{2}\,\varepsilon_{m-|j|}$$

for $j = 0,\pm1, \ldots ,\pm(m-1)$. If $\ker(A+\lambda) = \{0\}$, Q^2 is the idendity on $\overset{o}{X}{}^2$. Hence, in both cases, one obtains

$$\|u-v\|_{H^2(K_j)} < \varepsilon_{m-|j|} \quad \text{for } |j| = 1, \ldots, m$$

Therefore u, v have property $\Pi(m)$ and the Theorem is proved.

COROLLARY 4.5

Assume Theorem 4.4. Then for every $\mu \in \mathbb{R}$ there exist positive numbers $\lambda_0(\mu)$, $\eta_0(\mu)$ such that for every $\lambda \in [\mu-\lambda_0, \mu+\lambda_0]$, $\tilde{u} \in \ker(A+\mu)$, there is at most one solution of (5.1) satisfying $\tilde{P}u = \hat{u}$, $\sup_\sigma \|u^\sigma\|_{H^s(K_1)} \leq \eta_0$.

Proof : Consider in a neighborhood of $(\mu,0)$ the mapping

$$T : (\lambda,u) \mapsto Au + \lambda u + f(u) \quad , \quad T : \mathbb{R} \times \overset{0}{X}{}^2 \to X^0$$

The derivative of T at $(\mu,0)$ is given by $T'(0)(\lambda,u) = Au + \mu u$ which has the kernel $\ker T'(0) = \mathbb{R} \times \ker(A+\mu)$. Moreover, $(\lambda,u) \mapsto (\lambda,\tilde{P}u)$ defines a continuous, linear projection from $\mathbb{R} \times \overset{0}{X}{}^2$ into $\ker T'(0)$. Now the proof proceeds literally as in Theorem 4.4.

§ 5 Existence of Periodic and Quasiperiodic Solutions

While the last theorem proves that, for given $\tilde{u} \in \tilde{P}X^{2}$, there is at most one solution u of (4.1) with $\tilde{P}u = \tilde{u}$, which is uniformly small in $H^{s}_{loc}(\Omega)$, it remains to be shown that there exists such a solution. The nature of this existence problem changes dramatically between the λ - intervals $(\pi^2, 4\pi^2)$ and $(n^2\pi^2, (n+1)^2\pi^2)$, $n \geq 2$.

For the first case we are able to prove, for arbitrary C^p - mappings f satisfying certain symmetry requirements, that all small solutions are periodic in y. Moreover, if a certain non-degeneracy condition is met, "singular" solutions exist which are locally uniform limits of solutions having arbitrarily large irreducible periods (see next section). In the case $\lambda \in (n^2\pi^2, (n+1)^2\pi^2)$, $n \geq 2$, the question of existence is more delicate since problems of small denominatiors are involved. For real analytic f we are able to show that (4.1) can be solved by a formal power series. Convergence and generalisations to C^p - functions f are still open.

We are going to concrete the abstract properties of f, assumed in the preceding section. Let U be some neighborhood of 0 in $\mathbb{R}$ and $\tilde{f} : U \times \mathbb{R}^2 \to \mathbb{R}$ be a C^p-map, $p \geq 2$, satisfying $\tilde{f}(0) = 0$ and $D\tilde{f}(0) = 0$ for its gradient. We consider the equation

$$(5.1) \qquad \begin{aligned} &\Delta u + \lambda u + \tilde{f}(u, D_x u, D_y u) = 0 \\ &u(0,y) = u(1,y) = 0 \end{aligned}$$

for $\lambda \in \mathbb{R}$. Since $H^2_{loc}(\Omega) \subset W^1_{\infty,loc}(\Omega)$, $u, D_x u, D_y u$ are locally L_∞- functions. It follows immediately that $f : u \mapsto \tilde{f}(u, D_x u, D_y u)$ defines a C^p - mapping from the real space $H^2_{loc}(\Omega)$ into $H^0_{loc}(\Omega)$ which, in particular, satisfies assumption 4.1 . Similarly, if f is real analytic for $|u| < \delta$, $|p| < \delta$, $|q| < \delta$ then f, as a mapping from $H^2(K)$ into $H^0(K)$, is analytic in the sense of [4], p.112 for every compact $K \subset \Omega$, if $\|u\|_{H^2(K)} < \delta'$.

The case $\lambda \in (\pi^2, 4\pi^2)$ is considered first. $\tilde{f}$ is assumed to be a C^p-function, $p \geq 2$. Periodic solutions are constructed via the ansatz $u(x,y) = v(x,\omega y)$, where $v(x,z)$ is 2π-periodic in z and where ω varies near $\omega^0 := (\lambda-\pi^2)^{1/2}$. One obtains the boundary-value problem for v :

$$
(5.2) \quad
\begin{aligned}
&(\mathcal{L}(\omega) + \lambda)v + F(\omega,v) = 0 \\
&v(0,z) = v(1,z) = 0, \ v \ \ 2\pi\text{-periodic in } z
\end{aligned}
$$

where the following notations have been used

$$
(5.2a) \quad
\begin{aligned}
&\mathcal{L}(\omega) \equiv D_{xx}^2 + \omega^2 D_{zz}^2 \\
&F(\omega,v) \equiv \tilde{f}(v, D_x v, \omega D_z v)
\end{aligned}
$$

Although only real solutions are of interest we work, for technical reasons, in complex Hilbert-spaces $\mathbb{H}_{\#}^s$, the real subspaces of which are denoted by $H_{\#}^s$. Note that $\mathcal{L}$ and F are defined on real spaces; therefore we have always to assure that they act on real elements.

Set $\Omega_1 = [0,1] \times \mathbb{R}$ and define

$$\mathbb{H}_{\#}^0 \equiv \{v \in H_{loc}^0(\Omega_1) \ / \ v \ 2\pi\text{-periodic in } z\}$$

with the scalar product

$$(v^1, v^2)_0 \equiv \int_{(0,1)\times T_1} v^1 \overline{v^2} \, d\mu \quad ,$$

μ denoting the Lebesgue measure and T_1 the interval $(0,2\pi)$.

Furthermore, we introduce :

$$\overset{o}{\mathbb{H}}{}^2_{\#} \equiv \{ v \in H^2_{loc}(\Omega_1) \;/\; v \in \mathbb{H}^0_{\#} \text{ and } v(0,\cdot) = v(1,\cdot) = 0 \}$$

with the scalar product

$$(v^1,v^2)_2 \equiv \sum_{|\alpha|\leq 2} (D^\alpha v^1, D^\alpha v^2)_0$$

Let V denote a suitable neighbourhood of ω^0 in $\mathbb{R}$. Then, $\mathcal{L}$ and F map $V \times \overset{o}{\mathbb{H}}{}^2_{\#}$ into $\mathbb{H}^0_{\#}$; $\mathcal{L}$ is smooth in ω and linear in v, F is a C^p-mapping satisfying $F(\omega,0) = D_v F(\omega,0) = 0$.

For fixed $\lambda \in (\pi^2, 4\pi^2)$, the kernel of $\mathcal{L}(\omega^0)+\lambda$ is spanned by the $\mathbb{H}^0_{\#}$-orthonormal system

$$(5.3) \qquad \varphi^j(x,z) = \pi^{-1/2} \sin|j|\pi x \; e^{iz_j} \; , \; j = \pm 1$$

where $z_{-j} = -z_j$ per definition. Define P^2 resp. P^0 in $\overset{o}{\mathbb{H}}{}^2_{\#}$ resp. in $\mathbb{H}^0_{\#}$ by

$$(5.4) \qquad P^s v \equiv \sum_{|j|=1} (v,\varphi^j)_0 \varphi^j \quad , \quad s = 0 \text{ or } 2 \quad .$$

P^s is a projection commuting with $\mathcal{L}$ in the following sense

$$P^0 \mathcal{L}(\omega) = \mathcal{L}(\omega) P^2$$

and which, restricted to the real subspaces $\overset{o}{H}{}^2_{\#}$ resp. $H^0_{\#}$, acts as a real projection. Set $Q^s \equiv \mathrm{id} - P^s$ then, for sufficiently small $|\omega - \omega^0|$

$$\mathcal{L}_2(\omega) \equiv (\mathcal{L}(\omega) + \lambda)\big|_{Q^2 \overset{o}{H}{}^2_{\#}}$$

is a topological isomorphism between $Q^2 \overset{o}{H}{}^2_{\#}$ and $Q^0 H^0_{\#}$. Setting

$$c_j \equiv (v,\varphi^j)_0 \; , \quad w \equiv Q^2 v, , \quad c_{-j} = \overline{c_j}$$

($\overline{}$ denoting complex conjugation), one obtains

$$(5.5a)\qquad \mathcal{L}_2(\omega)w = Q^0F(\omega, \sum_{|j|=1} c_j\varphi^j + w)$$

Since $\|F(\omega,v)\|_0 = o(\|v\|_2)$, equation (5.5a) yields, via the implicit funktion theorem, a unique solution $w(\omega,\underline{c}) \in \overset{0}{\mathbb{H}}{}^2_\#$ as a real C^p-funktion of $\omega \in V(\omega^0)$, $\underline{c} \in U(\underline{0})$, where V and U are suitably chosen neighborhoods of ω^0 and $\underline{0} \in \mathbb{C}^2$. Furthermore, w satisfies

$$\|w\|_2 \leqq \gamma|\underline{c}|^2$$

uniformly for $\omega \in V(\omega^0)$. Hence, near $v = 0$, $\omega = \omega^0$, equation (5.2) is equivalent to

$$(5.5b)\qquad (\mathcal{L}(\omega) + \lambda) \sum_{|j|=1} c_j\varphi^j = P^0F(\omega, \sum_{|j|=1} c_j\varphi^j + w(\omega,\underline{c}))$$

For general F, existence of nontrivial solutions of (5.2), and thus of (5.5), cannot be expected without further assumptions Therefore, we require

(5.6)
a) $F(\omega,v)(x,-z) = F(\omega,v(x,-z))$
or
b) $F(\omega,v)(x,-z) = -F(\omega,v(x,-z))$, $F(\omega,-v) = F(\omega,v)$

a.e. in Ω_1

LEMMA 5.1

Assume $c_{-1} = \overline{c_1}$, and define $s_j(\underline{c}) \equiv c_j e^{i\theta_j}$, $j = \pm 1$, $\theta_{-j} = -\theta_j$, then, the following idendities hold :

(i)
$Pv(x,z,s(\underline{c})) = Pv(x,z+\theta_1,\underline{c})$
$w(x,z,s(\underline{c})) = w(x,z+\theta_1,\underline{c})$

(ii)
$Pv(x,z,\overline{\underline{c}}) = Pv(x,-z,\underline{c})$
$w(x,z,\overline{\underline{c}}) = w(x,-z,\underline{c})$, if (5.6a) holds

(iii)
$Pv(x,z,-\overline{\underline{c}}) = -Pv(x,-z,\underline{c})$
$w(x,z,-\overline{\underline{c}}) = -w(x,-z,\underline{c})$, if (5.6b) holds

Here, the ω-dependence of w has been suppressed, P stands for any P^s, and $\bar{\underline{c}}$ denotes the vector $(\overline{c_1}, \overline{c_{-1}})$.

Proof : The proof for Pv follows by inspection. To prove (i) for w one shows that $w_{\theta_1} \equiv w(\cdot, \cdot+\theta_1, \underline{c})$ satisfies (5.5a) if $\underline{c}$ is replaced by $s(\underline{c})$:

$$\mathcal{L}_2(\omega) w_{\theta_1} = (Q^0 F(\omega, Pv+w))_{\theta_1} = Q^0 F(\omega, Pv(s(\underline{c})) + w_{\theta_1})$$

The unique solvability of this equation yields (i). Assertion (ii) is shown quite similarly. To prove (iii) we set $w^- = w(\cdot, -\cdot, \underline{c})$ and obtain

$$\begin{aligned} \mathcal{L}_2(\omega)(-w^-) &= -(Q^0 F(\omega, Pv+w))^- = Q^0 F(\omega, (Pv)^- + w^-) \\ &= Q^0 F(\omega, -Pv(-\bar{\underline{c}}) - (-w^-)) \\ &= Q^0 F(\omega, Pv(-\bar{\underline{c}}) + w^-) \end{aligned}$$

Again, the assertion follows, since (5.5a) is uniquely solvable.

Setting $\tau \equiv \omega^2 - (\omega^0)^2$ and $G(\tau, \underline{c}) \equiv F(\omega, (Pv+w))(\underline{c})$ equation (5.5b) can be written as follows :

$$(5.7)_j \qquad -\tau c_j + (G(\tau, \underline{c}), \varphi^j)_0 = 0 \ , \quad j = \pm 1$$

First, we solve $(5.7)_1$ in the subspace $c_{-1} = c_1 = \overline{c_1}$. Since $G(\tau, \underline{0}) = 0$, we have in this case,

$$N(\tau, \underline{c}) \equiv \begin{cases} \dfrac{(G(\tau, \underline{c}), \varphi^1)_0}{c_1} & \text{if } c_1 = 0 \\ \\ D_{c_1}(G(\tau, c), \varphi^1)_0 & \text{if } c_1 = 0 \end{cases}$$

as a C^{n-1}-function near $\tau = 0$, $\underline{c} = \underline{0}$. Then, $(5.7)_1$ yields, via the implicit function theorem, a unique nontrivial solution $\tau(c_1)$ which, for small $|c_1|$ is a C^{n-1}-curve.

Observe that, in view of Lemma 5.1(i),

$$\left.\frac{(G(\tau,\underline{c}),\varphi^1)_0}{c_1}\right|_{s(\underline{c})} = \left.\frac{(G(\tau,\underline{c}),\varphi^1)_0}{c_1}\right|_{\underline{c}}$$

holds if $c_1 \neq 0$. Hence, by rotation, we obtain, from the solution constructed above, a solution $\tau(\underline{c})$ of $(5.7)_1$ for all small $|\underline{c}|$, $\underline{c} \in \mathbb{C}^2$ satisfying $c_1 = \overline{c_1}$.

It remains to be shown that the solution $\tau(\underline{c})$ of $(5.7)_1$ solves $(5.7)_{-1}$ as well. Property (5.6a) and its implication in Lemma **5.1**(ii) yields :

$$\left.\frac{(G(\tau,\underline{c}),\varphi^1)}{c_1}\right|_{\overline{\underline{c}}} = \frac{(F(\omega,Pv+w),\varphi^1)(\overline{\underline{c}})}{\overline{c_1}}$$

$$= \frac{(F(\omega,Pv+w)(-z),\varphi^1)(\underline{c})}{\overline{c_1}}$$

$$= \frac{(G(\tau,\underline{c}),\overline{\varphi^1})}{\overline{c_1}}$$

The same equality is true if (5.6b) and hence Lemma 5.1(iii) holds. Since we have $\tau(\overline{\underline{c}}) = \tau(\underline{c})$, equation $(5.7)_{-1}$ is satisfied.

THEOREM 5.2

Let be $\lambda \in (\pi^2, 4\pi^2)$ and let $F : V(\omega^0) \times H^2_\# \to H^0_\#$ be a real C^p-map, $p \geq 2$, which satisfies (5.6a) or (5.6b). Then, for every sufficiently small $|\underline{c}|$ $\underline{c} \in \mathbb{C}^2$, $\overline{c_1} = c_{-1}$, there exists a C^{p-1}-solution $\omega(\underline{c})$, $v(\underline{c})$ of (5.2) of the form

$$\omega(\underline{c}) = \omega^0 + O(|\underline{c}|)$$

$$v(\underline{c}) = \sum_{|j|=1} c_j\varphi^j + o(|\underline{c}|)$$

We are now able to complete the characterisation of all small solutions by an existence result which complements the uniqueness theorem 4.4 .

THEOREM 5.3

Let be $\lambda \in (\pi^2, 4\pi^2)$, and let $\tilde{P}$ be defined by (3.4) (n=1). Assume that $\tilde{f} : U \times \mathbb{R}^2 \to \mathbb{R}$ - U some neighborhood of 0 in $\mathbb{R}$ - is a C^p-map, $p \geqq 2$, satisfying $f(0,\underline{0}) = 0$, $Df(0,\underline{0}) = 0$. Suppose further that one of the following two conditions holds :

(a) $\tilde{f}(u,p,-q) = \tilde{f}(u,p,q)$

(b) $\tilde{f}(u,p,-q) = -\tilde{f}(u,p,q)$ and $\tilde{f}(-u,-p,-q) = \tilde{f}(u,p,q)$

Then, for every $\tilde{u} \in \ker(A+\lambda)$ with sufficiently small norm there exists a y-periodic solution u of (5.1) with $\tilde{P}u = \tilde{u}$.

Proof : Note that the assumptions for $\tilde{f}$ guarantee that $F(\omega,v) \equiv \tilde{f}(v, D_x v, \omega D_z v)$ has property (5.6a) or (5.6b) and fulfills the regularity requirements of the preceding theorem. Let $\omega(\underline{c})$, $v(\underline{c})$ be a solution of (5.2). Then $u(x,y) = v(x,\omega y)$ satisfies (5.1). Moreover, $\tilde{P}u$ has the form :

$$\tilde{P}u(x,y) = \frac{2}{\sqrt{\pi}} (\alpha_1 \cos \omega^0 y - \beta_1 \sin \omega^0 y) \sin \pi x + o(|\underline{c}|) ,$$

$$c_1 = \alpha_1 + i\beta_1$$

where the remainder $o(|\underline{c}|)$ is continuously differentiable with respect to α,β. Hence $\tilde{P}u = \tilde{u}$ can be solved using simply the implicit function theorem again, q.e.d. .

Now, we consider the general case $\lambda \in (n^2\pi^2, (n+1)^2\pi^2)$ with $n \in \mathbb{N}$ satisfying $n \geq 2$. Setting $\omega_j^0 := (\lambda - j^2\pi^2)^{1/2}$ we construct quasiperiodic solutions via the ansatz $u(x,y) := v(x, \omega_1 y, \ldots, \omega_n y)$ where $\underline{\omega} \equiv (\omega_1, \ldots, \omega_n)$ is close to $\underline{\omega}^0$ in $\mathbb{R}^n$, and where v is 2π-periodic in every z_j ("v is 2π-periodic in $\underline{z}$"). With the notations

$$\mathcal{L}(\underline{\omega}) \equiv D^2_{xx} + \sum_{j,k=1}^{n} \omega_j \omega_k D^2_{z_j z_k}$$

$$F(\underline{\omega}, v) \equiv f(v, D_x v, \underline{\omega} \cdot \nabla_z v)$$

we obtain the equation

$$(5.8) \qquad \begin{aligned} &(\mathcal{L}(\underline{\omega}) + \lambda)v + F(\underline{\omega}, v) = 0 \\ &v(0,\underline{z}) = v(1,\underline{z}) = 0 \quad , \quad v \ 2\pi\text{-periodic in } \underline{z}. \end{aligned}$$

The formal similarity with (5.2) is not accidental; rather the preceding analysis carries over to the present case if $\omega, z, \Omega_1, \tau_1$, and $|j| = 1$ are replaced by $\underline{\omega}, \underline{z}$, $\Omega_n \equiv [0,1] \times \mathbb{R}^n$, $\tau_n \equiv (0, 2\pi)^n$, and $|j| = 1, \ldots, n$. The functions

$$(5.9) \qquad \varphi^j(x, \underline{z}) \equiv 2^{-\frac{n-1}{2}} \pi^{-\frac{n}{2}} \sin|j|\pi x \cdot e^{iz_j} \ , \quad |j| = 1, \ldots, n$$

form a $\mathbb{H}^0_{\#}$-orthonormal basis of the $2n$-dimensional kernel of $\mathcal{L}(\underline{\omega}^0) + \lambda$, a space which, quite obviously, consists of quasiperiodic solutions only. The projections P^s, Q^s as well as $\mathcal{L}_2(\underline{\omega})$ are defined similarly to the case $n = 1$. The fundamental difference here lies in the fact that $\mathcal{L}_2(\underline{\omega})$, though invertible, has no continuous inverse. Hence, the analogue of (5.5a)

$$(5.10) \qquad \mathcal{L}_2(\underline{\omega}) w = Q^0 F(\underline{\omega}, \sum_{|j|=1}^{n} c_j \varphi^j + w)$$

cannot be solved as simply as before.

Let us assume that $\tilde{f}$ is a real analytic function of its arguments. Then, since F then is a real analytic mapping from $H^2_{\#}$ into $H^0_{\#}$, one can solve (5.10) by a formal power series

$$w = \sum_{\substack{|\alpha|\geq 0 \\ |\beta|\geq 2}} \underline{\tau}^{\alpha}\, \underline{c}^{\beta}\, W_{\alpha\beta}$$

- $\alpha,\beta \in \mathbb{N}_0^n$, $|\alpha|,|\beta|$ their lengths, $\underline{\tau} = (\tau_1, \ldots, \tau_n)$, $\tau_j \equiv \omega_j^2 - \omega_j^{o2}$. Observe that, in view of the injectivity of $\mathcal{L}_2(\underline{\omega}^0)$, the coefficients $W_{\alpha\beta}$ can be determined recursively. The proof of convergence by standard arguments fails however since, with increasing β, the inverse of $\mathcal{L}_2(\underline{\omega}^0)$ grows indefinitely in norm.

While the proof of unique solvability of (5.10) seems to be a formidable task - and we are not yet able to accomplish it - the solution of the system of bifurcation equations analogous to (5.5b) is not. To show this, let us *assume that (5.10) has a unique real analytic solution* for sufficiently small $|\underline{c}|$ and $|\underline{\tau}|$, $\underline{c}_j = \overline{c_j}$. The following Lemma is an immediate consequence of this assumption.

<u>LEMMA 5.4</u>

Assume $c_{-j} = \overline{c_j}$ for $j = 1, \ldots, n$ and denote by $c^{\ell} \in \mathbb{C}^{2n}$ a vector with $c_{\ell} = \overline{c_{\ell}} = 0$. Then, the solution $w(\underline{\omega},\underline{c})(x,\underline{z})$ of (5.10) is independent of z_{ℓ}.

The bifurcation equations, corresponding to (5.5b), are as follows :

$$(5.11)_{\ell} \qquad -\tau_{\ell} c_{\ell} + (G(\underline{\tau},\underline{c}),\varphi^{\ell})_0 = 0\ , \quad \tau_{-\ell} = \tau_{\ell}\ , \quad |\ell| = 1, \ldots, n\ .$$

where

$$G(\underline{\tau},\underline{c}) \equiv F(\underline{\omega}, \sum_{|j|=1}^{n} c_j\varphi^j + w(\underline{\omega},\underline{c}))$$

Observe that Lemma 5.1 still holds for $|j| = 1, \ldots ,n$. As in (5.7), we solve $(5.11)_\ell$ first for $\ell = 1, \ldots ,n$ and $c_{-j} = c_j = \overline{c_j}$, $j = 1, \ldots ,n$. $G(\underline{\tau},\underline{c}^\ell)$ is independent of z_ℓ according to the preceding Lemma. Hence $(G(\underline{\tau},\underline{c}),\varphi^\ell)_0 = 0$ if $c_\ell = 0$. Thus

$$N_\ell(\underline{\tau},\underline{c}) \equiv \begin{cases} \dfrac{(G(\underline{\tau},\underline{c}),\varphi^\ell)_0}{c_\ell} & \text{if } c_\ell \neq 0 \\[2ex] D_{c_\ell}(G(\underline{\tau},\underline{c}),\varphi^\ell)_0 & \text{if } c_\ell = 0 \end{cases}$$

is an analytic map near $\underline{\tau} = \underline{c} = \underline{0}$ and $(5.11)_\ell, \ell = 1, \ldots ,n$, yields locally a unique analytic solution $\underline{\tau}(\underline{c})$.
Again it follows, via Lemma 5.1, that

$$\left.\frac{(G(\underline{\tau},\underline{c}),\varphi^\ell)_0}{c_1}\right|_{s(\underline{c})} = \left.\frac{(G(\underline{\tau},\underline{c}),\varphi^1)_0}{c_1}\right|_{\underline{c}}$$

for every $c_1 \neq 0$. Therefore, the system $(5.11)_\ell$, $\ell = 1, \ldots ,n$, has a unique analytic solution $\underline{\tau}(\underline{c})$ in some neighborhood of $\underline{0} \in \mathbb{C}^{2n}$ satisfying $c_{-j} = \overline{c_j}$, $j = 1, \ldots ,n$. The proof, that $\tau_{-\ell} = \tau_\ell$ fulfil as well the equations $(5.11)_\ell$, $\ell = -1, \ldots ,-n$, proceeds literally as in the periodic case.

PROPOSITION 5.5

Let be $\lambda \in (n^2\pi^2,(n+1)^2\pi^2$, $2 \leq n \in \mathbb{N}$ and let $\tilde{P}$ be defined by (3.4). Assume that $f : U \times \mathbb{R}^2 \to \mathbb{R}$ be real analytic near the origin, satisfying the conditions of Theorem 5.3 . Furthermore, suppose that equation (5.10) has a unique real analytic solution near $\underline{c} = \underline{0}$ and $\underline{\omega} = \underline{\omega}^0$. Then there exists, for every $\tilde{u} \in \ker(\Lambda+\lambda)$ with sufficiently small norm, a solution u of (5.1) with $\tilde{P}u = \tilde{u}$, which is quasiperiodic in y.

§ 6 Existence of Singular Solutions

In this section we consider the set of solutions near $u = 0$, $\lambda = \pi^2$. Either there exist nontrivial solutions of (5.1) for $\lambda = \pi^2$ or there are solutions for $\lambda > \pi^2$, which are local H^2-limits of periodic solutions with infinitely growing irreducible periods.

We restrict λ to the interval $(\pi^2, 4\pi^2)$. According to the preceding section the set of solutions with small $H^s(K_\ell)$-norm consists of periodic functions exclusively which form a two-dimensional manifold over ker $(A+\lambda)$. Let be $\omega_0 = (\lambda-\pi^2)^{1/2}$, $u(x,y) = v(x,\omega y)$ and assume $v(x,z)$ to be 2π-periodic in z. Define the real Hilbert-space H^s_e for arbitrary real $s \geq 0$ as follows :

$$H^s_e(\Omega_1) = \{v \in H^s_\#(\Omega_1) \mid v(x,-z) = v(x,z)\}$$

Similarly $\overset{o}{H}{}^s_e(\Omega_1)$, $s \geq 1$, denotes the corresponding subspace of $\overset{o}{H}{}^s_\#$.

6.1 HYPOTHESIS

Assume that $\tilde{f}$ in (5.1) has the properties given there and that $\tilde{f}(u,p,-q) = \tilde{f}(u,p,q)$ holds. In addition we suppose that the mapping $f : u \mapsto f(u,D_x u,D_y u)$ from $H^s_{loc}(\Omega)$ into $H^0_{loc}(\Omega)$ is continuously differentiable near $u = 0$ for some real $s < 2$.

Under this hypothesis $F(\omega,v) = f(v,D_x v,\omega D_z v)$ is continuously differentiable near $v = 0$ and satisfies (5.7a) as well as $F(\omega,0) = 0$, $D_v F(\omega,0) = 0$. Observe that every polynomial in u with $\tilde{f}(0) = D\tilde{f}(0) = 0$ satisfies the assumption 6.1 for $\tilde{f}$.

Note that $\mathcal{L}(\omega)$ - defined in (5.2a) - has a continuous inverse $\mathcal{L}^{-1}(\omega) : H^0_e \to \overset{o}{H}{}^2_e$ which is compact as a mapping from H^0_ℓ into H^s_ℓ. Hence, $\mathcal{L}^{-1}(\omega)F(\omega,\cdot)$ defines a completely continuous operator in $\overset{o}{H}{}^s_e$ which satisfies $\|\mathcal{L}^{-1}(\omega)F(\omega,v)\|_s = o(\|v\|_s)$ uniformly for ω in compact subsets of $\mathbb{R}^+$. Equation (5.2) can be written as follows

$$(6.1) \qquad v = -\lambda\mathcal{L}^{-1}(\omega)v - \mathcal{L}^{-1}(\omega)F(\omega,v)$$

We intend to apply the global result of Rabinowitz [10] for fixed λ and variable ω. Although our ω-dependence is somewhat more general than in [10], the global existence still carries over since, at $\omega = \omega_1^o = (\lambda-\pi^2)^{1/2}$, a simple eigenvalue of $-\lambda\mathcal{L}^{-1}(\omega)$ in $\overset{o}{H}{}^s_e$ crosses at the point 1 the unique circle with nonvanishing velocity.

Denote by C_λ^+ resp. C_λ^- the connected components constructed in [10], Theorem 1.40, which meet the point $(\omega_1^{o2},0)$ in $\mathbb{R}^+ \times \overset{o}{H}{}^s_e$. Then each of them has one of the following properties :

(i) $\quad C_\lambda^\pm$ is unbounded

(ii) $\quad C_\lambda^\pm$ meets another point $(\tilde{\omega}^2,0)$

(iii) $\quad C_\lambda^\pm$ meets the subspace $\{0\} \times \overset{o}{H}{}^s_e$

Subsequently we will show that alternative (iii) holds. In view of equation (6.1) we may consider $C_\lambda^\pm$ as subsets of $\mathbb{R}^+ \times \overset{o}{H}{}^2_e$ having the same properties as in $\mathbb{R}^+ \times \overset{o}{H}{}^s_e$.

Define

$$S^+ \equiv \{v \in \overset{o}{H}{}^2_e \mid v_1 = \sqrt{2}\int_0^1 v(x,\cdot)\sin\pi x\,dx \text{ has exactly two simple zeroes in } [0,2\pi) \text{ and } v_1(0) > 0\}$$

similarly S^- with $v_1(0) < 0$. Since we consider functions which are even in z and since, for $v \in \overset{o}{H}{}^2_e$, v_1 is continuously differentiable in $\mathbb{R}$, the sets $S^\pm$ are open in $\overset{o}{H}{}^2_e$. Moreover, near $(\omega_o^2,0)$, $C_\lambda^+ \setminus \{\omega_o^2,0\}$ is a subset of $\mathbb{R}^+ \times S^\mp$ and similarly for C_λ^-. Observe that $v \in S^+$ and v periodic in z implies that v has the irreducible period 2π, i.e. 2π is the largest possible period. If $u(x,y) = v(x,\omega y)$ then u has the irreducible period $\frac{2\pi}{\omega}$.

identity on X^-. Hence, in both cases, one obtains

Define

$$\Gamma_\lambda \equiv \{u \in \overset{\circ}{H}{}^2_{loc} \mid u(x,y) = v(x, y), \ (\lambda^2, v) \in C^{\pm}_\lambda\}$$

and

$$\Gamma_\lambda \equiv \Gamma^+_\lambda \cup \Gamma^-_\lambda$$

It is easily seen that Γ_λ as well as $\Gamma^{\pm}_\lambda$ are connected subsets of $\overset{\circ}{H}{}^2_{loc}(\Omega)$. Subsequently we use the following notation :

$$E(u) = \sup_{\sigma \in \mathbb{R}} \|u^\sigma\|_{H^2(K_1)}$$

We have two alternatives which are treated seperately :

I An $\varepsilon_0 > 0$ can be found such that for every $\varepsilon \in (0, \varepsilon_0]$ there exists a sequence (λ_n, u_n) of solutions of (5.1) with $\lambda_n \to \pi^2 + 0$, $u_n \in \Gamma_{\lambda_n}$ and $E(u_n) = \varepsilon$.

II For every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\sup_{\substack{u \in \Gamma_\lambda \\ 0 < |\lambda - \pi^2| < \delta(\varepsilon)}} E(u) < \varepsilon$$

holds.

THEOREM 6.2

Suppose hypothesis 6.1 and I hold. Then, for $\lambda = \pi^2$, and for all ε in $[0, \varepsilon_0]$ there exists a solution u_ε of (5.1) with $E(u_\varepsilon) = \varepsilon$.

Proof : Choose the sequence (λ_n, u_n) from I. We may assume that $E(u_n) \leq \|u_n\|_{H^2(K_1)}$ (u^σ_n is again a solution). Since $\overset{\circ}{H}{}^2(K_j)$ lies compactly in $H^s(K_j)$, $s < 2$, we can select a subsequence u_{n_i} converging towards u in $H^s_{loc}(\Omega)$. Therefore, u_{n_i} and $f(u_{n_i})$ converge in $H^0_{loc}(\Omega)$ and are bounded in $H^0(K_j)$ uniformly in j. Hence, by Lemma 2.2, they converge in X^0_1. Using the equation

$$(6.2) \qquad u_{n_i} = -\lambda_{n_i} A^{-1} u_{n_i} - A^{-1} f(u_{n_i})$$

where A, defined in § 3 , is a topological isomorphism from X^{0} into $\overset{0}{X}{}^{2}$, we conclude that (u_{n_i}) converges in $\overset{0}{X}{}^{2}$. Thus $u \in \overset{0}{X}{}^{2}$ is a solution of (5.1) which, in view of $E(u_n) \leq \| u_n \|_{H^2(K_1)}$ satisfies $E(u) = \epsilon$.

Now we turn to case II. Let be $J \subset \mathbb{R}^+$ be compact set; then there are positive constants C_1, C_2 - independent of u and ω - such that

$$(6.3) \qquad C_1 \| v \|_2 \leq E(u) \leq C_2 \| v \|_2$$

holds for $v(x, \omega y) = u(x,y)$, $\omega \in J$.

LEMMA 6.3

Assume II and $\lambda - \pi^2$ sufficiently small. Then

$$C_\lambda^{\pm} \setminus \{(\omega_0^2, 0)\} \subset \mathbb{R}^+ \times S_\lambda^{\pm}$$

Proof : Take $\lambda - \pi^2 < \min(\delta(\eta_0), \lambda_0)$ where η_0, λ_0 are from Corollary 4.5.Then, using II , we obtain :

$$(6.4) \qquad \sup_{u \in \Gamma_\lambda} E(u) < \eta_0$$

Furthermore, near $(\omega_0^2, 0)$ we have $(\omega^2, v) \in \mathbb{R}^+ \times S^+$ if $v \neq 0$ and $(\omega^2, v) \in C_\lambda^+$.

If $C_\lambda^+ \setminus \{(\omega_0^2, 0)\} \not\subset \mathbb{R}^+ \times S_\lambda^+$ we show that C_λ^+ contains a point $(\bar{\omega}^2, 0)$, with $\bar{\omega}^2 \neq \omega_0^2$. Indeed, one would have a point $(\bar{\omega}^2, v) \in C_\lambda^+$ with $v \in \partial S_\lambda^+$ (boundary of S_λ^+). Since v is even in z, v_1 and thus u_1 has a double zero in $[0, 2\pi)$. We may assume $u_1(0) = u_1'(0) = 0$. Corollary 4.5 yields $u = 0$ and hence $v = 0$.

If $(\omega_0^2,0)$ is the only point in C_λ^+ with $v = 0$, the assertion is proved. Since, for some neighborhood U of $(\omega_0^2,0)$ $C_\lambda^+ \cap U$ and $C_\lambda^- \cap U$ are disjoint, C_λ^+ cannot leave $\mathbb{R}^+ \times \dot{S}^+$ near $(\omega_0^2,0)$ and hence everywhere. Up to here, the proof for C_λ^- is the same.

It remains to show that $(\omega_0^2,0)$ is the only point in $C_\lambda = C_\lambda^+ \cap C_\lambda^-$ with $v = 0$. Assume the contrary: Then, there exists a sequence $(\omega_n^2,v_n) \in C_\lambda \cap (\mathbb{R}^+ \times S)$, $S = S^+ \cup S^-$, such that $\|v_n\|_2 \to 0$; $\omega_n^2 \to \tilde{\omega}^2$ with $\tilde{\omega}^2 \neq \omega_0^2$. Therefore $v_{n,1}(0) \to 0$, $v_{n,1}'(0) = 0$ holds which implies $u_{n,1}(0) \to 0$, $u_{n,1}'(0) = 0$. In view of (3.4) and Theorem 4.4 we conclude that u_n must coincide with the solutions constructed in Theorem 5.5 for large n. But those solutions have period near $\frac{2\pi}{\omega_0}$ whereas u_n has period near $\frac{2\pi}{\tilde{\omega}}$. Since $v_n \in S$, the periods are irreducible and the contradiction follows.

Subsequently, the constants η_0 and λ_0 are taken from Corollary 4.5 for $\mu = \pi^2$.

LEMMA 6.4

Suppose that case II holds. Then, for sufficiently small $\lambda - \pi^2$, the projection of $C_\lambda^\pm \subset \mathbb{R}^+ \times \overset{\circ}{H}{}_e^2$ into $\mathbb{R}^+$ forms an interval $(0,b]$, $b > 0$.

Proof : We show first that C_λ^+ is contained in some interval $(0,b_0]$. Otherwise, since $\omega_0^2 < 3\pi^2$ for $\pi^2 < \lambda < 4\pi^2$, there is a sequence $C_{\lambda_n}^+$, $\lambda_n \to \pi^2$, such that $C_{\lambda_n}^+$ intersects $\{3\pi^2\} \times \overset{\circ}{H}{}_e^2$ for every n. Hence, we have a sequence $u_n \in \Gamma_{\lambda_n}$ with period $2\pi/\sqrt{3}\pi$ in y satisfying - in view of II - $E(u_n) \to 0$. Dividing (6.2) by $E(u_n)$ yields :

$$\frac{u_n}{E(u_n)} = - \lambda_n \Lambda^{-1} \left(\frac{u_n}{E(u_n)} + \frac{f(u_n)}{E(u_n)} \right)$$

As in the proof of Theorem 6.2 one obtains a subsequence $u_i = u_{n_i}/E(u_{n_i})$ converging towards some u in X_1^0. Furthermore, $f(u_{n_i})/E(u_{n_i}) \to 0$ in $H_{loc}^0(\Omega)$, hence in X_1^0. The above equation implies $u_i \to u$ in $\overset{0}{X}{}^2$ and thus $(\Lambda + \pi^2)u = 0$. Since all u_i have the irreducible period $2\pi/\sqrt{3}\pi$ and since $E(u_i) = 1$ holds, u possesses the same properties. However, there is no such solution in ker $(\Lambda+\pi^2)$. Therefore, the projection of C_λ^+ into $\mathbb{R}^+$ is contained in some interval $(0,b_0]$.

Now, let be $\lambda-\pi^2 < \min(\delta(\eta_0),\lambda_0)$. In view of II we have $\sup\limits_{u\in\Gamma_\lambda} E(u) < \eta_0$ which, by (6.3) yields:

$$\sup_{\substack{(\omega^2,v)\in C_\lambda^+ \\ \omega^2\in[a,b_0]}} \|v\|_2 \leq \frac{1}{C_1}\eta_0$$

for every $a \in (0,b_0)$, where the constant C_1 only depends on a. Therefore, C_λ^+ possesses property P_3. Since C_λ^+ is connected and closed, the same is true for its projection on $\mathbb{R}^+$ which proves the assertion for C_λ^+. An identical argument holds for C_λ^-.

We call a nontrivial solution of (5.1) singular, if it is the $H^2_{loc}(\Omega)$-limit of functions in Γ_λ but does not belong to Γ_λ. Note that such a singular solution u is nonperiodic in y or u_1 is constant.

THEOREM 6.5

Let II and hypothesis 6.1 be valid and assume $\lambda-\pi^2 > 0$ be sufficiently small. Then, there is an interval $J \equiv (\alpha,\beta) \subset \mathbb{R}$ containing 0 such that, for every $a \in J$, there exists an unique solution u of (5.1) with $E(u) < \eta_0$ and $u_1(0) = a$, $u_1'(0) = 0$. Among these solutions, there are those of arbitrary large period.

Furthermore, there exist two singular solutions u^j satisfying $u_1^1(0) = \alpha$, $u_1^2(0) = \beta$, $u^{j\prime}(0) = 0$, $E(u) \leq \eta_0$. The map $a \mapsto u$ from $[\alpha,\beta]$ into $\overset{0}{X}{}^2$ is continuous.

Proof : According to II we have (6.4). The linear map $\phi: u \to u_1(0)$ from $H^2_{loc}(\Omega)$ into $\mathbb{R}$ is continuous and injective by Corollary 4.5. Hence, ϕ^{-1} is continuous. Since Γ_λ is connected and bounded in $H^2_{loc}(\Omega)$, $J \equiv \phi\Gamma_\lambda$ is an interval with $0 \in J$. We show : J is open.

Take a curve s_1 in C_λ^+ connecting $(\omega_1^2, v^1) \in C_\lambda^+$ and $(\omega_0^2, 0)$ and let $\sigma_1 \equiv \{ u \in \Gamma_\lambda / u(x,y) = v(x,\omega y), (\omega^2, v) \in s_1 \}$ be the trace in Γ_λ. Then $\Phi\sigma_1$ is an interval $[0,\beta_1]$. If $(\omega_2^2, v^2) \notin s_1$, we obtain, in view of Corollary 4.5, for the corresponding interval $[0,\beta_2]$, $\beta_2 > \beta_1$. Therefore $\Phi(u^1) \in \overset{\circ}{J}$ ($u^j(x,y) = v^j(x,\omega^j y)$) and, since a similar argument holds for C_λ^-, J is open.

In view of Lemma 6.4., to every $(\omega_1^2, v^1) \in C_\lambda^+$ there exists a $(\omega_2^2, v^2) \in C_\lambda^+$, $(\omega_2^2, v^2) \notin s_1$ and with $0 < \omega_2^2 < \omega_1^2$. Thus we obtain a sequence $u_n \in \Gamma_{\lambda_n}^+$ with $u_{n,1}(0) \to \alpha$ having period $2\pi/\omega_n$ which increase indefinitely. We conclude - as in the proof of Theorem 6.2 - that a subsequence u_{n_i} converges towards a solution u^1 of (5.1) in $H^2_{loc}(\Omega)$ satisfying

$$(6.5a) \quad u_1^1(0) = \alpha, \; u_1^{1\prime}(0) = 0, \; E(u^1) \leqq \eta_0 .$$

Similarly one constructs a solution u^2 with

$$(6.5b) \quad u_1^2(0) = \beta, \; u_1^{2\prime}(0) = 0, \; E(u^2) \leqq \eta_0 .$$

The continuity of the map

$$a \mapsto \begin{cases} \Phi^{-1} a & \text{for } a \in J \\ u^j & \text{on the boundary of } J \end{cases}$$

acting from $[\alpha,\beta]$ into $\overset{\circ}{X}{}^2$, follows in $\overset{\circ}{J}$ from the continuity of Φ^{-1}. At the boundary we argue as follows : Every sequence u_n satisfying $u_{n,1}(0) \to \alpha$ (or β), converges towards u^1 (or u^2) in $H^2_{loc}(\Omega)$ since - by Corollary 4.5 - the u^j's are uniquely determined through (6.5).

To prove that the solutions u^j are singular, i.e. $u^j \notin \Gamma_\lambda$, we show that u_1^j is either constant or not periodic. Note that the orbits $\ell_u = \{(u_1(y), u_1'(y)) / y \in \mathbb{R}\}$ of periodic functions $u \in \Gamma_\lambda$ in $\mathbb{R}^2$ are simply closed curves enclosing the origin. Since, according to Corollary 4.5, the ℓ_u's do not intersect and since

Γ_λ is connected in $H^2_{loc}(\Omega)$, the set $U = \bigcup_{u \in \Gamma_\lambda} \ell_u$ forms an open neighborhood of 0. The orbits of u^j, $j=1,2$, must belong to the boundary ∂U. Hence, if u_1^j is not constant but periodic, it possesses at least two simple zeros per period. Since u_1^j is the local C^1-limit of functions with arbitrary large irreducible period which have exactly two simple zeros per period, a contradiction follows.

If hypothesis 6.1 b) is valid a similar result holds in the space of odd functions :

$$H^s_\sigma(\Omega_1) \equiv \{v \in H^s_\#(\Omega_1) / \ v(x,-z) = -v(x,z)\ \}$$

Choose for $S^\pm$ the set of those $v \in \overset{o}{H}{}^2_\sigma$ which have exactly two simple zeroes in $[0,2\pi)$ and satisfy $v_1'(0) > 0$ (< 0). In this case one obtains solutions u of (5.1) with $u_1(0) = 0$, $u_1'(0) = a$ for all a in the closure cl J of J.

COROLLARY 6.6

Let $\tilde{f}$ satisfy the hypothesis 6.1 a) or 6.1 b). Moreover, assume II to hold and take $\lambda-\pi^2 > 0$ sufficiently small. Then, there exists an open neighborhood U of 0 in ker $(A+\lambda)$ and a continuous map Ψ : cl $U \to \overset{o}{X}{}^2$ such that $u = \Psi(\tilde{u})$, $\tilde{u} \in$ cl U is a solution of (5.1) with $E(u) \leq \eta_0$. For $\tilde{u} \in U$, $\Psi(\tilde{u})$ is periodic, for $\tilde{u} \in \partial U$, $\Psi(\tilde{u})$ is singular.

Proof : According to Theorem 6.5 and the following remark a component $\Gamma_\lambda \subset H^2_{loc}(\Omega)$ of periodic solutions u exist in each case, satisfying $E(u) < \eta_0$. Their orbits in the (u_1,u_1')-plane form an open neighborhood of 0 in $\mathbb{R}^2$. Since u^σ is also a solution for all $\sigma \in \mathbb{R}$ we obtain solutions u of (5.1) satisfying $u_1(0) = a$, $u_1'(0) = b$, $E(u) < \eta_0$ for arbitrary $(a,b) \in U$.

Now, consider the case $(a,b) \in \partial U$. As in the proof of Theorem 6.5 we conclude, for every sequence $(a_n,b_n) \in U$ converging toward (a,b), that the corresponding periodic solutions u_n converge in $H^2_{loc}(\Omega)$ towards a solution u of (5.1) which, by Corollary 4.5, is uniquely determined, and which satisfies $u_1(0) = a, u_1'(0) = b$. Hence, the mapping $(a,b) \mapsto u$ from $\mathbb{R}^2$ into $H^2_{loc}(\Omega)$ is injective and continuous. The orbit of u in the (u_1, u_1')-plane lies in ∂U; hence u is singular, qed.

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In this paper the boundary value problems $\Delta u+\lambda u+f(u,u_x,u_y)=0$, $u(0,y)=u(1,y)=0$ is studied in the strip $(0,1)\times R$, where f is some C^2-function which, together with its gradient, vanishes at 0, λ is a real parameter. It is shown that, for λ between π^2 and $4\pi^2$, all small solutions are periodic in y. Moreover, singular solutions exist as local H^2-limits of periodic solutions with large periods. For values of λ beyond $4\pi^2$ a formal argument suggests that almost all small solutions are quasiperiodic. The equation is studied as a model for some important but technically cumbersome bifurcation problems in fluid dynamics.

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