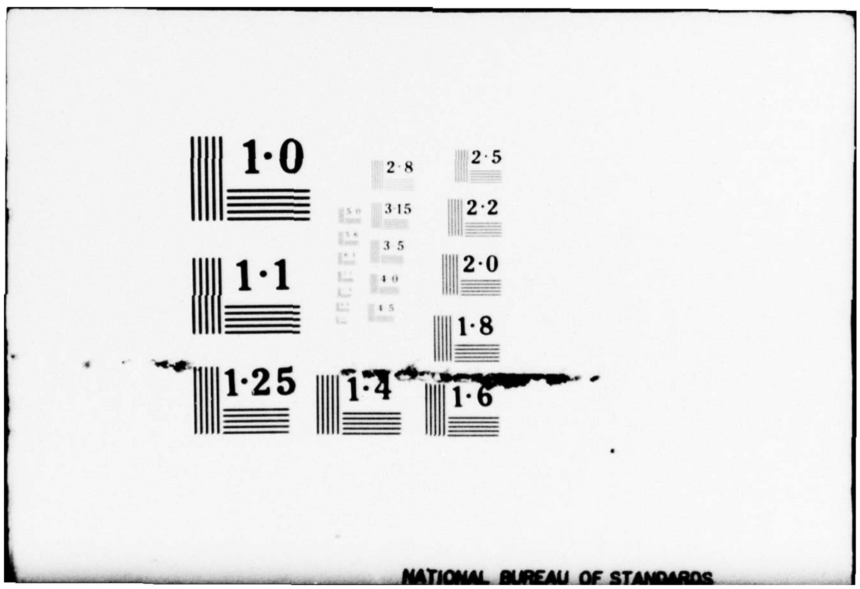
AD-A046 401 ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND REDSTO--ETC F/G 20/11
NONLINEAR ANALYSIS OF ORTHOTROPIC, LAMINATED SHELLS OF REVOLUTI--ETC(U)
AUG 77 C M ELDRIDGE

UNCLASSIFIED DRDMI-TL-77-9 NL

1 OF 2
ADA
046401


1·0
2·8
2·5
3·15
2·2
3·5
2·0
4·0
1·1
4·5
1·8
1·25
1·4
1·6
NATIONAL BUREAU OF STANDARDS

U.S. ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND

TECHNICAL REPORT TL-77-9

NONLINEAR ANALYSIS OF ORTHOTROPIC, LAMINATED SHELLS OF REVOLUTION BY THE FINITE ELEMENT METHOD

Charles M. Eldridge
Ground Equipment and Missile Structures Directorate
~~Technology Laboratory~~

August 1977

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

U.S. ARMY
MIRADCOM

Redstone Arsenal, Alabama 35809

DISPOSITION INSTRUCTIONS

DESTROY THIS REPORT WHEN IT IS NO LONGER NEEDED. DO NOT RETURN IT TO THE ORIGINATOR.

DISCLAIMER

THE FINDINGS IN THIS REPORT ARE NOT TO BE CONSTRUED AS AN OFFICIAL DEPARTMENT OF THE ARMY POSITION UNLESS SO DESIGNATED BY OTHER AUTHORIZED DOCUMENTS.

TRADE NAMES

USE OF TRADE NAMES OR MANUFACTURERS IN THIS REPORT DOES NOT CONSTITUTE AN OFFICIAL INDORSEMENT OR APPROVAL OF THE USE OF SUCH COMMERCIAL HARDWARE OR SOFTWARE.

14

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DRDMI-

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
TL-77-9		9

4. TITLE (and Subtitle)

6 NONLINEAR ANALYSIS OF ORTHOTROPIC, LAMINATED SHELLS OF REVOLUTION BY THE FINITE ELEMENT METHOD.

5. TYPE OF REPORT & PERIOD COVERED

Technical Rep[redacted]t,

6. PERFORMING ORG. REPORT NUMBER

TL-77-9 16

7. AUTHOR(s)

10 Charles M./Eldridge

8. CONTRACT OR GRANT NUMBER(s)

DA Proj No. 1L362303A214
AMCMS Code
no. 632303.2140911.08 17 D9

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Commander, US Army Missile R&D Command
ATTN: DRDMI-TL
Redstone Arsenal, AL 35809

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS

Commander, US Army Missile R&D Command
ATTN: DRDMI-TI
Redstone Arsenal, AL 35809

12. REPORT DATE

11 Aug[redacted]977

13. NUMBER OF PAGES

~~103~~ 12 108p.

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)

Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This report was prepared from the author's Phd dissertation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Structures
Composite material
Orthotropic, laminated shell

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The nonlinear analysis of an orthotropic, laminated shell of revolution with transverse shear deformations is presented. The finite element method employing a curved shell element was the method used to perform the analysis. Each element has two nodes with four degrees of freedom at each node, viz., two translation, one bending rotation and one transverse shear rotation. The classical Kirchhoff-Love assumption for normals to the midsurface was relaxed in favor of the shear deformation mode.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED next page

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

410263

Abstract - Continued

A curved shell element was developed that matches slopes and curvatures as well as displacements at the nodal circles. This significantly reduces the meridional bending moments present at element junctures as compared to the case in which straight line elements (conical frusta) are used to represent a shell of revolution having meridional curvatures.

A local rectilinear coordinate system was established for the element to represent the displacement patterns. The displacements of the curved element were represented in this local system and subsequently transferred to the global system.

A computer program was written in Fortran IV to implement the theory. Several example problems, both linear and nonlinear, were solved and the results compared with solutions from the literature.

The program can be used for analysis of orthotropic shells including fiber composite structures. The angle of wrap and longitudinal and transverse material physical properties are part of the input data.

Two point Gaussian Quadrature Integration was used in the development of the stiffness matrix. The shell thickness and pressure load may vary linearly along the meridian.

An incremental method was used for obtaining the nonlinear effects. The load can be applied in increments. From the first load increment, a linear solution is obtained. The coordinates are then updated and a second solution is found using the second load increment. This procedure is continued until the total load is reached. Excellent agreement with both linear and nonlinear problems from the literature was obtained.

The program requires relatively short execution time and can be run interactively on the CDC 6600 computer.

Some areas of further investigation which are logical and useful extensions of this work were recommended.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

The author hereby expresses his sincere gratitude and deep appreciation to Dr. Ju-chin Huang of the Engineering Science Department, Tennessee Technological University for his guidance, encouragement, and constant surveillance during the formulation of the theory and development of this work. Gratitude is also extended to Dr. D. G. Smith, Dr. S. B. Khleif, Dr. G. R. Buchanan, Dr. D. H. Deason, and Mr. D. K. Buck for their advice and assistance.

Gratitude is expressed to the US Army Missile Research and Development Command for providing the opportunity for this research. A special thanks is given to Mr. John Sofferis of the Command's Computation Center for his advice and assistance with the computer program.

Chapter 1

INTRODUCTION

The continuing search for lighter, stronger, more economical structures, particularly in the aircraft and missile industry, has led to the investigation of various composite materials as a possible applicable type of construction. Many of these materials are orthotropic and multi-layered and have nonlinear physical properties. Tests on actual structures have proven that some of these materials are unlike conventional structural materials in many respects. Although much has been learned about composite material behavior in the past few years, there are still many areas that are unknown and unpredictable. These characteristics pose many difficult problems for the designer and analyst. To obtain the most efficient use of these composites, suitable analysis and design techniques must be developed.

The aircraft and missile industries have many applications for shells of revolution and are continually searching for ways to decrease the weight while maintaining or increasing the strength. The possibility of achieving this by use of composites has created much interest in their application. The need for techniques to analyze these structures was the prime motivation for this present effort. This study investigates an orthotropic, laminated shell of revolution with shear deformation.

The rapid development of the digital computer in the last two decades has contributed significantly to the analyst's ability to treat these complex problems. Numerical methods that would have been impossible to employ prior to the computer are now quite popular and are used extensively throughout the stress analysis community. Both the finite element method and finite difference techniques are used today to solve complex structural problems. Each of these has features which make its application more suitable to a particular type problem.

The finite element method was chosen for this present development for several reasons. Because of the flexibility of their size and shape, finite elements can represent a given body, however complex its shape may be, quite accurately. Structures with holes or discontinuities can be treated with little difficulty. Problems involving variable material properties and geometry such as are encountered with fiber composites do not present any additional difficulty. Geometrical and material non-linearity can be dealt with relatively easily. One of the principal assets of the finite element method is the ease with which boundary conditions can be represented.

The essential feature of the finite element method is that the governing differential equations of equilibrium of the shell are approximated by a set of algebraic equations. This is equivalent to substituting, for the actual structure, an assemblage of discrete elements interconnected at a finite number of nodes. The element stiffness is then evaluated and superimposed to obtain the total stiffness matrix of the entire structure. Finally, the nodal force equilibrium equations are solved simultaneously for the nodal displacements of the complete system [1-5].

In shells of revolution, the structure is divided into a number of short frustums which are connected at their nodal circle. The assemblage is made through equilibrium and compatibility requirements at the nodal circle. Mayer and Harmon [6] employed the conical frustum (singly curved) element in the earlier stage of analysis of shells of revolution. Popov et al. [7] used the bending displacements due to edge loadings from the exact shell theory rather than the usual assumed displacement functions. Their result showed no significant improvement over the simpler assumed functions. Grafton and Strome [8] used conical elements in a true finite element technique. The results are very satisfactory for shells with a straight generated curve. However, there are still some inaccurate moments due to the approximation of a doubly curved shell by a singly curved element. This is mainly caused by the discontinuity of slope at the nodal circle of the substitute structure. To remedy this problem, Jones and Strome [9] developed a doubly curved element which matched both the location and slope of the original shell at nodal circles, thus avoiding unwanted discontinuities of slope at these locations. Khojasteh-Bakht [10] used two local coordinate systems, viz., curvilinear and rectilinear coordinate systems, to formulate his element. The latter proved to be a better approach because it can treat certain constant strain states which the former was unable to accommodate.

Recently, a finite-element technique including transverse shear effect has been attempted. Clough and Felippa [11] have described a simple shear distortion mechanism which can be implemented by expressing the total rotation of a cross section as the sum of the rotation on the middle surface plus a uniform shear strain through the thickness.

Klein [12] has applied the matrix displacement finite element approach to the linear elastic analysis of shells of revolution under axisymmetric loads. The shell is idealized as a series of conical frusta, joined at nodal circles. The external forces are applied at the nodal circles. A comparison with these solutions is made in Chapter 7.

Sharifi [13] used an incremental formulation for the nonlinear finite element analysis of sandwich structures. The nonlinearities considered were due to large displacements. Included in the analysis are axisymmetric shells with axisymmetric loadings and boundary conditions. Curved elements were used in the development.

McNamara [14] investigated nonlinear dynamic problems by using an incremental stiffness finite element analysis. Both geometric and material nonlinearities were considered.

Becker and Brisbane [15] developed the equations for an axisymmetric, orthotropic shell using a straight line element. Their development did not include shear deformations. Shear deformations are important for the analysis of fiber composites.

Nickell and Sato [16] used a curved shell element to analyze an orthotropic, layered shell of revolution. Shear deformations were not included in their analysis.

In this present effort, the finite element method is used with a curved shell element considering a nonlinear laminated, orthotropic shell of revolution and transverse shear deformations.

A polynomial representation of the meridional curve of the shell was chosen which matches the displacements, slope, and curvature at the nodal points. Nonlinear terms are included in the strain-displacement relationships. The stiffness matrix is then derived using these nonlinear relations.

There are four degrees of freedom at a node, viz., two translation, one bending rotation and one transverse shear rotation. The field equations similar to Reissner's theory of thick plates [5] were used as a guideline for formulating the shear deformation degree of freedom. The procedure employed was similar to that of Clough and Felippa [11]. The classical Kirchhoff-Love assumption for normals to the midsurface was relaxed in favor of the assumed shear deformation mode.

A computer program was written implementing the derived equations. The element stiffness matrix was formed by numerical integration. Much of the data is generated internally in the program. The program is limited to ten different materials and 50 nodes, but it can be increased by increasing the dimension statement accordingly. The program is relatively fast. Most of the example problems run required from 3 to 4 seconds execution time on the CDC 6600 computer.

Chapter 2

ELEMENT GEOMETRY

The shell to be considered is axisymmetric; therefore, it is sufficient to define only the shape of its meridional curve. The finite element method will be used for this analysis. The element is shown in Figure 1.

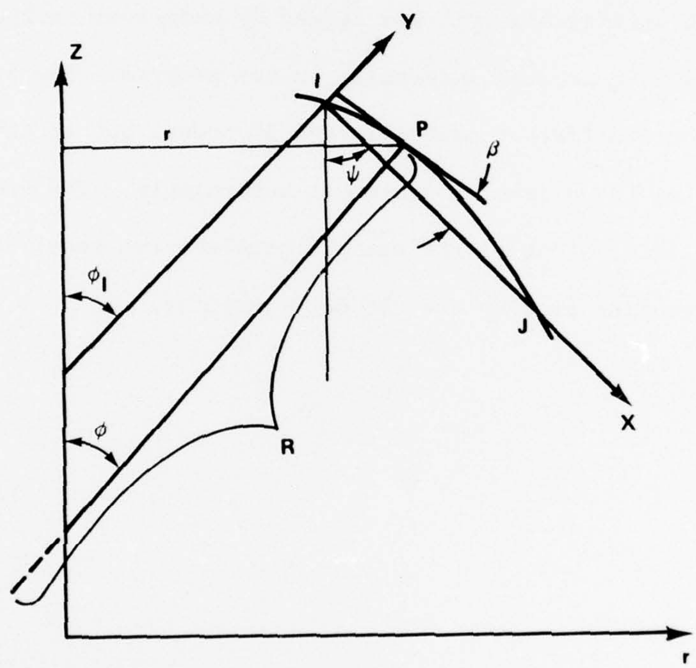


Figure 1. Curved Shell Element.

The element is curved and the two end points of the element are denoted by I and J. For a shell whose reference surface is a surface of revolution, the lines of principal curvature are its meridians and parallel circles. The principal curvilinear coordinates of the reference surface are the angle Φ between the normal to the reference surface and the axis of revolution and the angle θ describing the position of points on the corresponding parallel circle. Since this development is axisymmetric both in geometry and load, it is independent of θ.

A local coordinate system for each curved element is constructed between two adjacent nodal circles by drawing chords between the points. This rectangular Cartesian system which is normalized by the chord length ℓ is denoted by x-y. The global coordinates are represented by r-z. The angles shown in Figure 1 are related by the relation

$$\Phi + \psi + \beta = \frac{\pi}{2} \quad . \tag{1}$$

The angle Φ is the angle between the normal to the reference surface and the axis of revolution. The angle ψ is the angle between the chord of the element and the z-axis. The angle β is defined to be the angle between the chord line (the x-axis) and the tangent to the curved surface at any point.

From Equation (1),

$$\begin{aligned} \sin\beta &= \cos(\Phi + \psi) = \cos\Phi \cos\psi - \sin\Phi \sin\psi \\ \cos\beta &= \sin(\Phi + \psi) = \sin\Phi \cos\psi + \cos\Phi \sin\psi \quad . \end{aligned} \tag{2}$$

To approximate the meridional curve, the following substitute curve is assumed:

$$y = x\left(1 - \frac{x}{\ell}\right)\left(a_1 + a_2 \frac{x}{\ell} + a_3 \frac{x^2}{\ell^2} + a_4 \frac{x^3}{\ell^3}\right) \tag{3}$$

where ℓ = chord length of the element. $(0 < x < \ell)$

Differentiating Equation (3) with respect to x

$$\frac{dy}{dx} = a_1 + \frac{2(a_2 - a_1)}{\ell} x + \frac{3(a_3 - a_2)}{\ell^2} x^2 + \frac{4(a_4 - a_3)}{\ell^3} x^3 - \frac{5a_4}{\ell^4} x^4$$

$$\frac{d^2y}{dx^2} = \frac{2(a_2 - a_1)}{\ell} + \frac{6(a_3 - a_2)}{\ell^2} x + \frac{12(a_4 - a_3)}{\ell^3} x^2 - \frac{20a_4}{\ell^4} x^3 \quad . \tag{4}$$

The constants a_1, a_2, a_3, and a_4 can be determined by evaluating Equations (4) at the end points

$$a_1 = \tan \beta_1$$

$$a_2 = \tan \beta_1 - \frac{\ell}{2R_1} \sec^3 \beta_1$$

$$a_3 = \frac{-\ell}{2R_2} \sec^3 \beta_2 + \frac{\ell}{R_1} \sec^3 \beta_1 - 4 \tan \beta_2 - 5 \tan \beta_1 \tag{5}$$

$$a_4 = \frac{\ell}{2R_2} \sec^3 \beta_2 - \frac{\ell}{2R_1} \sec^3 \beta_1 + 3(\tan \beta_1 + \tan \beta_2) \quad .$$

where the subscripts 1 and 2 on β and R reference these items to the I and J nodes respectively.

The following geometrical relations are given with respect to the element:

$$\Delta r = r_2 - r_1$$

$$\Delta z = z_1 - z_2$$

$$\ell = \sqrt{(\Delta r)^2 + (\Delta z)^2} \tag{6}$$

$$\sin \psi = \frac{\Delta r}{\ell}$$

$$\cos \psi = \frac{\Delta z}{\ell} \quad .$$

After the substitute curve has been established, all the geometric quantities can be written as follows:

$$\tan \beta = \frac{dy}{dx}$$

$$r = r_1 + x \sin \psi + y \cos \psi$$

$$\frac{dr}{dx} = \sin \psi + \tan \beta \cos \psi \qquad (7)$$

$$\frac{d}{dS} = \cos \beta \frac{d}{dx}$$

$$\frac{d\beta}{dx} = \frac{d\beta}{dS}\frac{dS}{dx} = -\frac{1}{R} \sec \beta \quad .$$

Since

$$\frac{dy}{dx} = \tan \beta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan \beta) = \sec^2\beta \frac{d\beta}{dx} \quad ,$$

therefore

$$\frac{d\beta}{dx} = -\frac{1}{R} \sec \beta \quad .$$

The quantity $d\beta/dS$ is negative since β is decreasing as S is increasing. Therefore

$$\frac{d^2y}{dx^2} = -\frac{1}{R} \sec^3\beta \qquad (8)$$

and

$$\cos \phi = \sin \beta \cos \psi + \cos \beta \sin \psi$$
$$\sin \phi = \cos \beta \cos \psi - \sin \beta \sin \psi \quad . \qquad (9)$$

Chapter 3

TRANSFORMATION OF COORDINATE SYSTEMS

The displacement vector of a material point on the midsurface in the local principal curvilinear shell coordinate is denoted by

$$\{f_c\}^T = [u_c, w_c, \chi_c, \gamma_c] \tag{10}$$

where

u_c = the displacement along the meridian.

w_c = the transverse (normal) displacement.

χ_c = the rotation about a meridional tangent.

γ_c = shear deformation.

The displacement components which refer to the local rectilinear coordinates, x-y, are

$$\{f_r\}^T = [u_r, w_r, \chi_r, \gamma_r] \tag{11}$$

and to the global coordinates, r-z, are

$$\{f\}^T = [u, w, \chi, \gamma] \quad . \tag{12}$$

The transformation between these components can be seen as follows:

$$\{f_c\} = [q_c]\,\{f_r\}$$
$$\{f_r\} = [q_r]\,\{f\} \tag{13}$$

where

$$[q_c] = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

and

$$[q_r] = \begin{bmatrix} \sin\psi & -\cos\psi & 0 & 0 \\ \cos\psi & \sin\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{15}$$

Chapter 4

STRAIN-DISPLACEMENT RELATIONS

The general nonlinear strain-displacement relations for large rotation but small strain were derived by Novozhilov [18] and later corrected by Tsao [19]. For shells of revolution with axisymmetric loading, the strain displacement relations can be written as

$$
\begin{aligned}
e_1 &= \frac{du_c}{dS} + \frac{w_c}{R} + \frac{1}{2} \chi_c^2 \\
e_2 &= \frac{1}{r} (u_c \cos \Phi + w_c \sin \Phi) \\
\chi_c &= \frac{dw_c}{dS} - \frac{u_c}{R} \\
\kappa_1 &= - \frac{d}{dS} \left(\frac{dw_c}{dS} - \frac{u_c}{R} + \gamma_1 \right) \\
\kappa_2 &= - \frac{\cos \Phi}{r} \left(\frac{dw_c}{dS} - \frac{u_c}{R} + \gamma_1 \right) \\
\gamma_1 &= -\gamma_c \quad .
\end{aligned}
\tag{16}
$$

The strains defined in Equations (16) are now transformed into the local rectilinear coordinates as follows (recall that $dS = \frac{dx}{\cos \beta}$ and $\beta = \beta_1 - \frac{S}{R}$):

$$
\begin{aligned}
e_1 &= \frac{du_r}{dx} \cos^2 \beta + \frac{dw_r}{dx} \cos \beta \sin \beta + \frac{1}{2} (\chi_c)^2 \\
e_2 &= \frac{1}{r} (u_r \sin \psi + w_r \cos \psi)
\end{aligned}
$$

$$\chi_c = -\sin\beta\cos\beta\,\frac{du_r}{dx} + \cos^2\beta\,\frac{dw_r}{dx} + \gamma_r$$

$$\kappa_1 = -\cos^3\beta\,\frac{d^2w_r}{dx^2} - \frac{2\cos\beta\sin\beta}{R_1}\,\frac{dw_r}{dx} + \sin\beta\cos^2\beta\,\frac{d^2u_r}{dx^2} \qquad (17)$$

$$+ \frac{\sin^2\beta - \cos^2\beta}{R_1}\,\frac{du_r}{dx} - \frac{d\gamma_r}{dx}\cos\beta$$

$$\kappa_2 = -\frac{\cos\phi}{r}\left(\cos^2\beta\,\frac{dw_r}{dx} - \sin\beta\cos\beta\,\frac{du_r}{dx}\right) - \frac{\cos\phi}{r}\,\gamma_r$$

$$\gamma_1 = -\gamma_r \quad .$$

Chapter 5

SHELL DISPLACEMENTS

The shell displacements are written in local rectilinear coordinates. They are represented by four degrees of freedom at a node: two translations, one bending rotation and one transverse shear rotation.

$$u_c = u_r \cos\beta + w_r \sin\beta$$

$$w_c = -u_r \sin\beta + w_r \cos\beta$$

$$\frac{dw_c}{dS} = \left(\frac{dw_c}{dx}\right)\frac{dx}{dS} \tag{18}$$

$$= \frac{dx}{dS}\left(-\frac{du_r}{dx}\sin\beta + \frac{dw_r}{dx}\cos\beta - u_r\cos\beta\frac{d\beta}{dx} - w_r\sin\beta\frac{d\beta}{dx}\right)$$

$$= \cos\beta\left(-\frac{du_r}{dx}\sin\beta + \frac{dw_r}{dx}\cos\beta + \frac{u_r}{r} + \frac{w_r}{r}\tan\beta\right) \quad .$$

As was done by Khojasteh-Bakht [10], the displacement field is assumed to be represented by

$$u_r = \alpha_1 + \alpha_2 x$$

$$w_r = \alpha_3 + \alpha_4 x + \alpha_5 x^2 + \alpha_6 x^3$$

$$\chi_r = \frac{dw_c}{dS} - \frac{u_c}{r}$$

$$\chi_r = \cos\beta\left(-\frac{du_r}{dx}\sin\beta + \frac{dw_r}{dx}\cos\beta + \frac{u_r}{r} + \frac{w_r}{r}\tan\beta\right) \tag{19}$$

$$- \frac{u_r \cos\beta}{r} - \frac{w_r \sin\beta}{r}$$

$$\chi_r = -\frac{du_r}{dx}\sin\beta\cos\beta + \frac{dw_r}{dx}\cos^2\beta + \gamma$$

$$\chi_r = -\alpha_2 \sin\beta\cos\beta + (\alpha_4 + 2\,\alpha_5 x + 3\,\alpha_6 x^2)\cos^2\beta$$

$$\gamma_r = \alpha_7 + \alpha_8 x \quad .$$

In matrix notation this can be written as:

$$\begin{Bmatrix} u_r \\ w_r \\ \chi_r \\ \gamma_r \end{Bmatrix} = \begin{bmatrix} 1 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & x^2 & x^3 & 0 & 0 \\ 0 & -sc & 0 & c^2 & 2xc^2 & 3x^2c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} \tag{20}$$

where

$$s = \sin\beta,\ c = \cos\beta \quad .$$

This can be written symbolically as

$$\{f_r\} = [\Phi]\,\{\alpha\} \tag{21}$$

where $\{\alpha\}$ is the generalized displacements vector for the curved shell element. This displacement function allows rigid body motion without inducing strains.

The shell displacements shown in Equation (20) represent 4 degrees of freedom at a node, two translation, two rotation. The 8 degrees of freedom connected with the nodes of the element are written as the displacement vector

$$\{\delta_r^e\}^T = [u_{r1}, w_{r1}, \chi_{r1}, \gamma_{r1}, u_{r2}, w_{r2}, \chi_{r2}, \gamma_{r2}] \quad . \tag{22}$$

where subscripts 1 and 2 refer to the I and J nodes respectively.

The generalized displacements $\{\alpha\}$ are related to the nodal point displacement vector $\{\delta_r^e\}$ by

$$\{\alpha\} = [A_r]\ \{\delta_r^e\} \quad . \tag{23}$$

$\{\alpha\}$ is evaluated as follows:

1) At x = 0

$$\alpha_1 = u_{r1}$$

$$\alpha_3 = w_{r1}$$

$$\alpha_7 = \gamma_{r1}$$

$$-\alpha_2 \sin \beta_1 \cos \beta_1 + \alpha_4 \cos^2\beta_1 = \chi_{r1} \quad .$$

2) At $x = \ell$

$$\alpha_1 + \alpha_2\ell = u_{r2}$$

$$\alpha_3 + \alpha_4\ell + \alpha_5\ell^2 + \alpha_6\ell^3 = w_{r2}$$

$$\alpha_7 + \alpha_8\ell = \gamma_{r2}$$

$$-\alpha_2 \sin \beta_2 \cos \beta_2 + \alpha_4 \cos^2\beta_2 + 2\alpha_5\ell \cos^2\beta_2 + 3\alpha_6\ell^2 \cos^2\beta_2 = \chi_{r2} .$$

Solving this system of equations gives

$$\alpha_1 = u_{r1}$$

$$\alpha_2 = \frac{u_{r2} - u_{r1}}{\ell}$$

$$\alpha_3 = w_{r1}$$

$$\alpha_4 = \frac{\chi_{r1} + \sin\beta_1 \cos\beta_1 \left(\frac{u_{r2} - u_{r1}}{\ell}\right)}{\cos^2\beta_1}$$

$$\alpha_5 = u_{r1} \left(\frac{\tan\beta_2 + 2\tan\beta_1}{\ell^2}\right) - u_{r2} \left(\frac{\tan\beta_2 + 2\tan\beta_1}{\ell^2}\right)$$

$$- \frac{3}{\ell^2} w_{r1} + \frac{3}{\ell^2} w_{r2} - \chi_{r1} \left(\frac{2}{\ell \cos^2\beta_1}\right) - \chi_{r2} \left(\frac{1}{\ell \cos^2\beta_2}\right)$$

$$\alpha_6 = - \frac{u_{r1} (\tan\beta_2 + \tan\beta_1)}{\ell^3} + \frac{u_{r2} (\tan\beta_2 + \tan\beta_1)}{\ell^3}$$

$$+ \frac{2}{\ell^3} w_{r1} - \frac{2}{\ell^3} w_{r2} + \chi_{r1} \left(\frac{1}{\ell^2 \cos^2\beta_1}\right) + \chi_{r2} \left(\frac{1}{\ell^2 \cos^2\beta_2}\right)$$

$$\alpha_7 = \gamma_{r1}$$

$$\alpha_8 = \frac{\gamma_{r2} - \gamma_{r1}}{\ell} .$$

Therefore, from (23)

$$[A_r] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\ell} & 0 & 0 & 0 & \frac{1}{\ell} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1 & 0 & b_1 & 0 & a_1 & 0 & 0 & 0 \\ a_2 & -\frac{3}{\ell^2} & -2b_2 & 0 & -a_2 & \frac{3}{\ell^2} & -b_4 & 0 \\ -a_3 & \frac{2}{\ell^3} & b_3 & 0 & a_3 & -\frac{2}{\ell^3} & b_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\ell} & 0 & 0 & 0 & \frac{1}{\ell} \end{bmatrix} \qquad (24)$$

where

$$a_1 = \frac{\tan \beta_1}{\ell}$$

$$a_2 = \frac{2 \tan \beta_1 + \tan \beta_2}{\ell^2}$$

$$a_3 = \frac{\tan \beta_1 + \tan \beta_2}{\ell^3}$$

$$b_1 = \frac{1}{\cos^2 \beta_1}$$

$$b_2 = \frac{1}{\ell \cos^2 \beta_1}$$

$$b_3 = \frac{1}{\ell^2 \cos^2 \beta_1}$$

$$b_4 = \frac{1}{\ell \cos^2 \beta_2}$$

$$b_5 = \frac{1}{\ell^2 \cos^2 \beta_2} \quad .$$

The transformation of $\{\delta_r\}^e$ to the global coordinate $\{\delta\}^e$ is given by

$$\{\delta_r\}^e = [R] \, \{\delta\}^e \tag{25a}$$

where

$$\{\delta\}^e = \begin{Bmatrix} u_1 \\ w_1 \\ \chi_1 \\ \gamma_1 \\ u_2 \\ w_2 \\ \chi_2 \\ \gamma_2 \end{Bmatrix} \tag{25b}$$

and

$$[R] = \begin{bmatrix} \sin\psi & -\cos\psi & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos\psi & \sin\psi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\psi & -\cos\psi & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\psi & \sin\psi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

Substituting (25) into (23) gives

$$\{\alpha\} = [A_r]\ \{\delta_r\}^e = [A_r]\ [R]\ \{\delta\}^e = [A]\{\delta\}^e \tag{27}$$

where

$$[A] = [A_r]\ [R]$$

$$[A] = \begin{bmatrix} \sin\psi & -\cos\psi & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{-\sin\psi}{\ell} & \dfrac{\cos\psi}{\ell} & 0 & 0 & \dfrac{\sin\psi}{\ell} & \dfrac{-\cos\psi}{\ell} & 0 & 0 \\ \cos\psi & \sin\psi & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1 \sin\psi & a_1 \cos\psi & b_1 & 0 & a_1 \sin\psi & -a_1 \cos\psi & 0 & 0 \\ \begin{matrix} a_2 \sin\psi \\ \dfrac{-3\cos\psi}{\ell^2} \end{matrix} & \begin{matrix} -a_2 \cos\psi \\ \dfrac{-3\sin\psi}{\ell^2} \end{matrix} & -2b_2 & 0 & \begin{matrix} -a_2 \sin\psi \\ \dfrac{+3\cos\psi}{\ell^2} \end{matrix} & \begin{matrix} a_2 \cos\psi \\ \dfrac{+3\sin\psi}{\ell^2} \end{matrix} & -b_4 & 0 \\ \begin{matrix} -a_3 \sin\psi \\ \dfrac{+2\cos\psi}{\ell^3} \end{matrix} & \begin{matrix} a_3 \cos\psi \\ \dfrac{+2\sin\psi}{\ell^3} \end{matrix} & b_3 & 0 & \begin{matrix} a_3 \sin\psi \\ \dfrac{-2\cos\psi}{\ell^3} \end{matrix} & \begin{matrix} -a_3 \cos\psi \\ \dfrac{-2\sin\psi}{\ell^3} \end{matrix} & b_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\ell} & 0 & 0 & 0 & \dfrac{1}{\ell} \end{bmatrix} \tag{28}$$

Chapter 6

STRESS-STRAIN RELATIONS

For an axisymmetric shell of revolution subjected to axisymmetric loadings, the stress resultants and couples can be expressed as

$$\begin{Bmatrix} N_1 \\ N_2 \\ M_1 \\ M_2 \\ Q_1 \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & 0 \\ E_{21} & E_{22} & E_{23} & E_{24} & 0 \\ E_{31} & E_{32} & E_{33} & E_{34} & 0 \\ E_{41} & E_{42} & E_{43} & E_{44} & 0 \\ 0 & 0 & 0 & 0 & E_{55} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ \kappa_1 \\ \kappa_2 \\ -\gamma_1 \end{Bmatrix} . \tag{29}$$

The quantities are related to the principal curvilinear coordinate system with 1 as meridional direction and 2 as circumferential direction. Symbolically this can be written as

$$\{S\} = [E] \{\epsilon\} \tag{30}$$

where [E] is the elasticity matrix. The detail derivation of [E] is given in Appendix A.

Substituting (19) into (17) gives

$$\begin{aligned} \{\epsilon\} &= [\Phi'] \{\alpha\} \\ &= [\Phi'] [A] \{\delta\}^e = [B] \{\delta\}^e \end{aligned} \tag{31}$$

$$[B] = [\Phi'][A]$$

Differentiating the displacement functions with respect to x gives

$$\frac{du_r}{dx} = \alpha_2$$

$$\frac{dw_r}{dx} = \alpha_4 + 2\alpha_5 x + 3\alpha_6 x^2$$

$$\frac{d\gamma_r}{dx} = \alpha_8$$

$$\frac{d^2 w_r}{dx^2} = 2\alpha_5 + 6\alpha_6 x$$

$$\frac{d^2 u_r}{dx^2} = 0 \quad .$$

Substituting these relations into equation (17) gives

$$e_1 = \alpha_2 \cos^2\beta + (\alpha_4 + 2\alpha_5 x + 3\alpha_6 x^2) \cos\beta \sin\beta + \frac{1}{2}(\chi_c)^2$$

$$e_2 = \frac{1}{r}\left[(\alpha_1 + \alpha_2 x) \sin\psi + (\alpha_3 + \alpha_4 x + \alpha_5 x^2 + \alpha_6 x^3) \cos\psi\right]$$

$$\kappa_1 = -\cos^3\beta\,(2\alpha_5 + 6\alpha_6 x) - \frac{2\cos\beta\sin\beta}{R_1}(\alpha_4 + 2\alpha_5 x + 3\alpha_6 x^2)$$

$$+ \frac{\sin^2\beta - \cos^2\beta}{R_1}(\alpha_2) - \alpha_8 \cos\beta$$

$$\kappa_2 = -\frac{\cos\phi}{r}\left[\cos^2\beta\,(\alpha_4 + 2\alpha_5 x + 3\alpha_6 x^2) - \sin\beta\cos\beta\,\alpha_2\right]$$

$$- \frac{\cos\phi}{r}(\alpha_7 + \alpha_8 x)$$

$$\gamma_1 = -(\alpha_7 + \alpha_8 x) \quad .$$

From these equations, the matrix $[\phi']$ may be obtained and may be split into two matrices $[\phi'^{\ell}]$ and $[\phi'^{n}]$ containing linear and nonlinear terms.

$$\{\epsilon\} = [\phi'] \{\alpha\}$$

$$= \left([\phi'^{\ell}] + [\phi'^{n}]\right) \{\alpha\}$$

where $[\phi'^{\ell}]$ is given by

$$[\phi'^{\ell}] = \begin{bmatrix} 0 & \cos^2\beta & 0 & \cos\beta\sin\beta & 2x\cos\beta\sin\beta & 3x^2\sin\beta\cos\beta & 0 & 0 \\ \frac{\sin\phi}{r} & \frac{x\sin\phi}{r} & \frac{\cos\phi}{r} & \frac{x\cos\phi}{r} & \frac{x^2\cos\phi}{r} & \frac{x^3\cos\phi}{r} & 0 & 0 \\ 0 & \frac{\sin^2\beta - \cos^2\beta}{R_1} & 0 & \frac{-2\cos\beta\sin\beta}{R_1} & \begin{matrix}-2\cos^3\beta \\ \frac{-4x\cos\beta\sin\beta}{R_1}\end{matrix} & \begin{matrix}-6x\cos^3\beta \\ \frac{-6x^2\cos\beta\sin\beta}{R_1}\end{matrix} & 0 & -\cos\beta \\ 0 & \frac{\cos\phi\sin\beta\cos\beta}{r} & 0 & \frac{-\cos\phi\cos^2\beta}{r} & \frac{-2x\cos\phi\cos^2\beta}{r} & \frac{-3x^2\cos\phi\cos^2\beta}{r} & \frac{-\cos\phi}{r} & \frac{-x\cos\phi}{r} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -x \end{bmatrix} \quad (32)$$

and the non-zero first row of $[\phi'^{n}]$ is

$$[\phi_1'^{n}] = \left[0 \quad \frac{-\sin\beta\cos\beta}{2}\chi \quad 0 \quad \frac{\cos^2\beta}{2}\chi \quad x\cos^2\beta\chi \quad \frac{3}{2}x^2\cos^2\beta\chi \quad 0 \quad 0\right] \quad (32a)$$

The subscript 1 is used to denote the first row of the matrix.

From Equations (13) and (21)

$$\{f_r\} = [\phi]\{\alpha\} = [\phi][A]\{\delta\}^e$$

$$\{f\} = [q_r]^{-1}\{f_r\} = [q_r]^T\{f_r\} \quad (33)$$

$$\{f\} = [q_r]^T[\phi][A]\{\delta\}^e$$

$$= [N]\{\delta\}^e$$

where

$$[N] = [q_r]^T \; [\Phi] \; [A] \quad . \tag{34}$$

The element stiffness matrix and equivalent nodal force may be obtained from the following formulas:

$$[k^e] = \iint_{A_e} [B]^T \; [E] \; [B] \; dA \tag{35}$$

$$\{F_p^e\} = \iint_{A_e} [N]^T \; \{P\} \; dA \tag{36}$$

where $\{P\}$ is the surface traction vector. The derivation of Equations (35) and (36) is given in Appendix B.

$$[\Phi]^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 0 & -sc & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & c^2 & 0 \\ 0 & x^2 & 2\,x\,c^2 & 0 \\ 0 & x^3 & 3\,x^2\,c^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & x \end{bmatrix} \tag{37}$$

$$[q_r] = \begin{bmatrix} \sin\psi & -\cos\psi & 0 & 0 \\ \cos\psi & \sin\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{38}$$

$$[\phi]^T\ [q_r] = \begin{bmatrix} \sin\psi & -\cos\psi & 0 & 0 \\ x\sin\psi & -x\cos\psi & -sc & 0 \\ \cos\psi & \sin\psi & 0 & 0 \\ x\cos\psi & x\sin\psi & c^2 & 0 \\ x^2\cos\psi & x^2\sin\psi & 2\,x\,c^2 & 0 \\ x^3\cos\psi & x^3\sin\psi & 3\,x^2\,c^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & x \end{bmatrix} \tag{39}$$

$$\{p_r\} = \begin{Bmatrix} p_x \\ p_y \\ 0 \end{Bmatrix} \tag{40}$$

where

$$p_x = p_t \cos\beta - p_n \sin\beta$$

$$p_y = p_t \sin\beta + p_n \cos\beta \quad .$$

$$\{p_r\} = [q_r]\ \{p\} \tag{41}$$

$$\{p\} = [q_r]^T\ \{p_r\} \tag{42}$$

$$= \begin{bmatrix} \sin\psi & \cos\psi & 0 & 0 \\ -\cos\psi & \sin\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \\ 0 \\ 0 \end{Bmatrix} \tag{43}$$

$$= \begin{Bmatrix} p_x \sin\psi + p_y \cos\psi \\ -p_x \cos\psi + p_y \sin\psi \\ 0 \\ 0 \end{Bmatrix} \tag{44}$$

$$[C]^T [q_r] \{p\} = \begin{Bmatrix} p_x \sin^2 \psi + p_y \cos \psi \sin \psi + p_x \cos^2 \psi - p_y \sin \psi \cos \psi \\ p_x x \sin^2 \psi + p_y x \cos \psi \sin \psi + p_x x \cos^2 \psi - p_y x \cos \psi \sin \psi \\ p_x \cos \psi \sin \psi + p_y \cos^2 \psi - p_x \cos \psi \sin \psi + p_y \sin^2 \psi \\ p_x x \cos \psi \sin \psi + p_y x \cos^2 \psi - p_x x \sin \psi \cos \psi + p_y \sin^2 \psi \\ p_x x^2 \sin \psi \cos \psi + p_y x^2 \cos^2 \psi - p_x x^2 \sin \psi \cos \psi + p_y x^2 \sin^2 \psi \\ p_x x^3 \sin \psi \cos \psi + p_y x^3 \cos^2 \psi - p_x x^3 \sin \psi \cos \psi + p_y x^3 \sin^2 \psi \\ 0 \\ 0 \end{Bmatrix} \quad (45)$$

$$= \begin{Bmatrix} p_x \\ p_x\, x \\ p_y \\ p_y\, x \\ p_y\, x^2 \\ p_y\, x^3 \\ 0 \\ 0 \end{Bmatrix} \quad (46)$$

This value may now be substituted into equations (34) and (36) to obtain $\{F_p^e\}$.

Chapter 7

RESULTS AND CONCLUSIONS

NUMERICAL EXAMPLES

To demonstrate the numerical accuracy of the method, some selected problems were solved and compared to known results.

First, a circular, monolithic, thin plate with clamped edges and subjected to a uniformly distributed load was considered. The plate had a radius of 100 inches and a thickness of 1.0 inch. Young's modulus was 1×10^6 psi and Poisson's ratio was 0.3. The plate was divided into five elements.

Using only five elements, the results from this program agree to within 7% of the exact results using large deflection theory shown in [20]. For the particular loading case, elementary theory differs from the exact solution by over 70%. The results are shown in Figure 2.

A comparison with Klein's [12] solution using linear analysis is shown in Figure 3. This was the analysis of a circular, flat plate under axisymmetric pressure loading. It can be seen that as the number of elements increased in the linear solution, it approached the nonlinear solution using only five elements.

A hemispherical shell is shown in Figure 4. A comparison was made with the theoretical values presented in [2]. The values calculated using the ORTHO2 program agreed with the theoretical values within the accuracy with which the curves could be read.

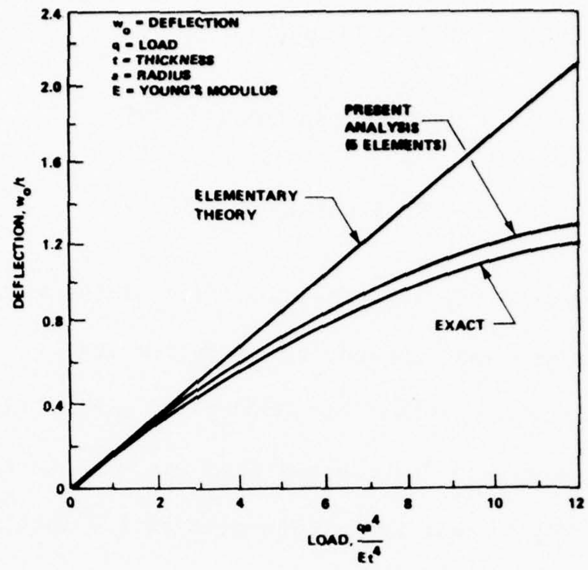


Figure 2. Circular Plate, Large Deflections.

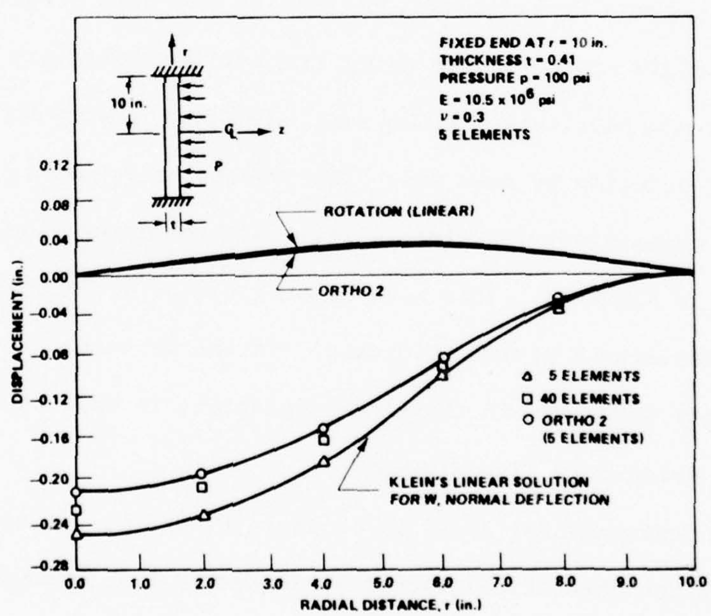


Figure 3. Circular Flat Plate Under Axisymmetric Pressure Loading.

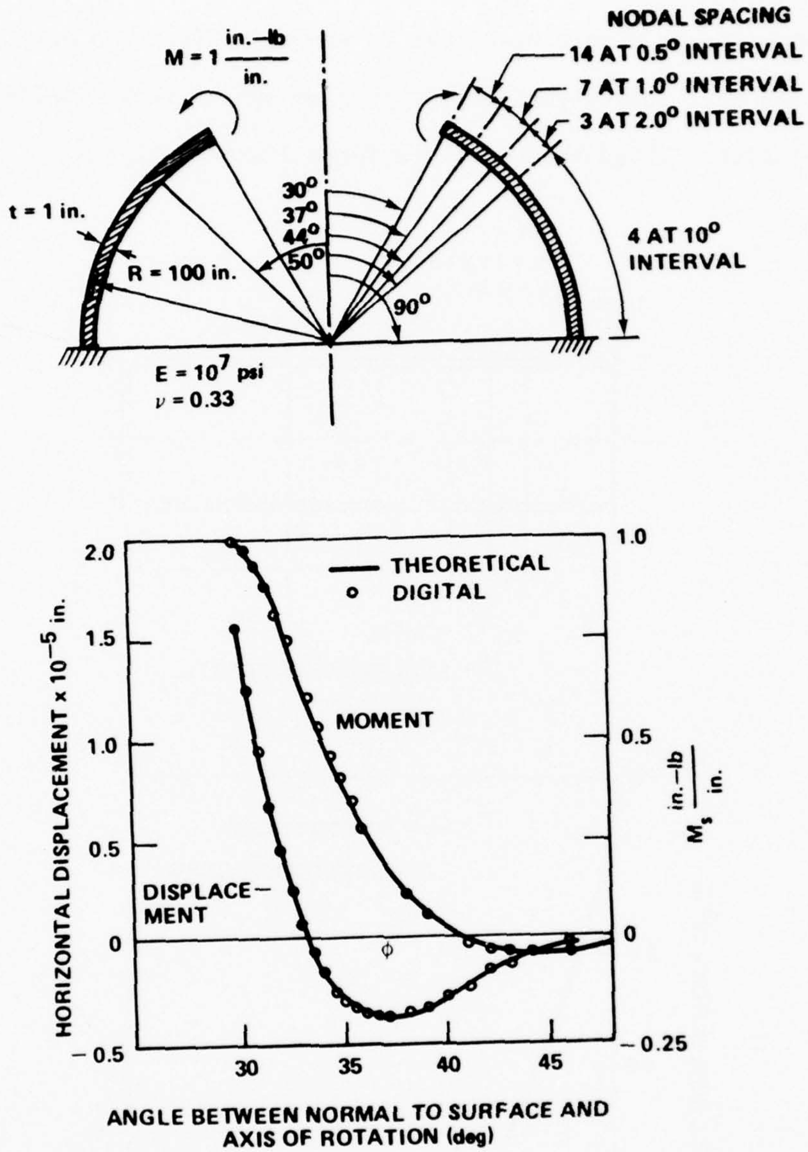
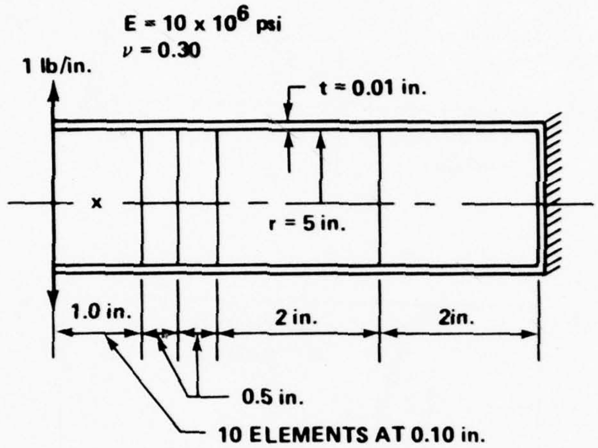


Figure 4. A Hemispherical Shell Solution by Finite Elements (Grafton and Strome, J.A.I.A.A., 1963).

The radial deflection and meridional moment for a cylindrical shell subjected to a unit edge load is shown in Figures 5 and 6. It can be seen that the present ORTHO2 program agrees very closely with the exact solution. These solutions are for a linear analysis.



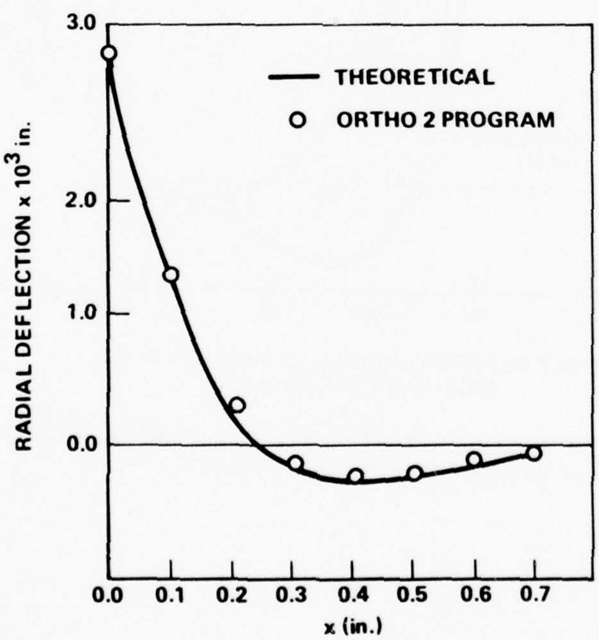


Figure 5. Cylindrical Shell with Unit Edge Load.

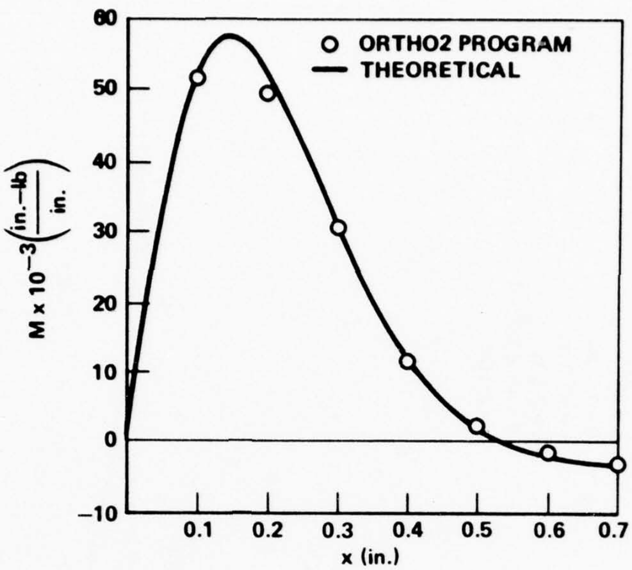


Figure 6. Meridional Moment for the Cylindrical Shell.

A spherical shell under uniform normal pressure was analyzed. The shell and its properties are shown in Figure 7. A comparison was made with the exact solution given by [18]. It is readily seen that the present solution agrees with the exact solution very well.

The numerical influence of the shear deformation becomes much clearer when a circular sandwich plate with clamped edges subjected to a distributed load of 14 psi is considered. The plate has a radius of 10 in., the thickness of core layer is 0.75 in., and the thickness of upper and lower facings is 0.028 in. and 0.022 in., respectively. Young's modulus of facings is 10^7 psi, Poisson's ration is 0.3, and the shear modulus of core is 30,000 psi. The plate is also divided into 5, 10, and 20 elements. The results are given in Figure 8. The maximum deflection of the plate is shown to converge to the theoretical value of 0.0415 in. as reported by Plantema [19].

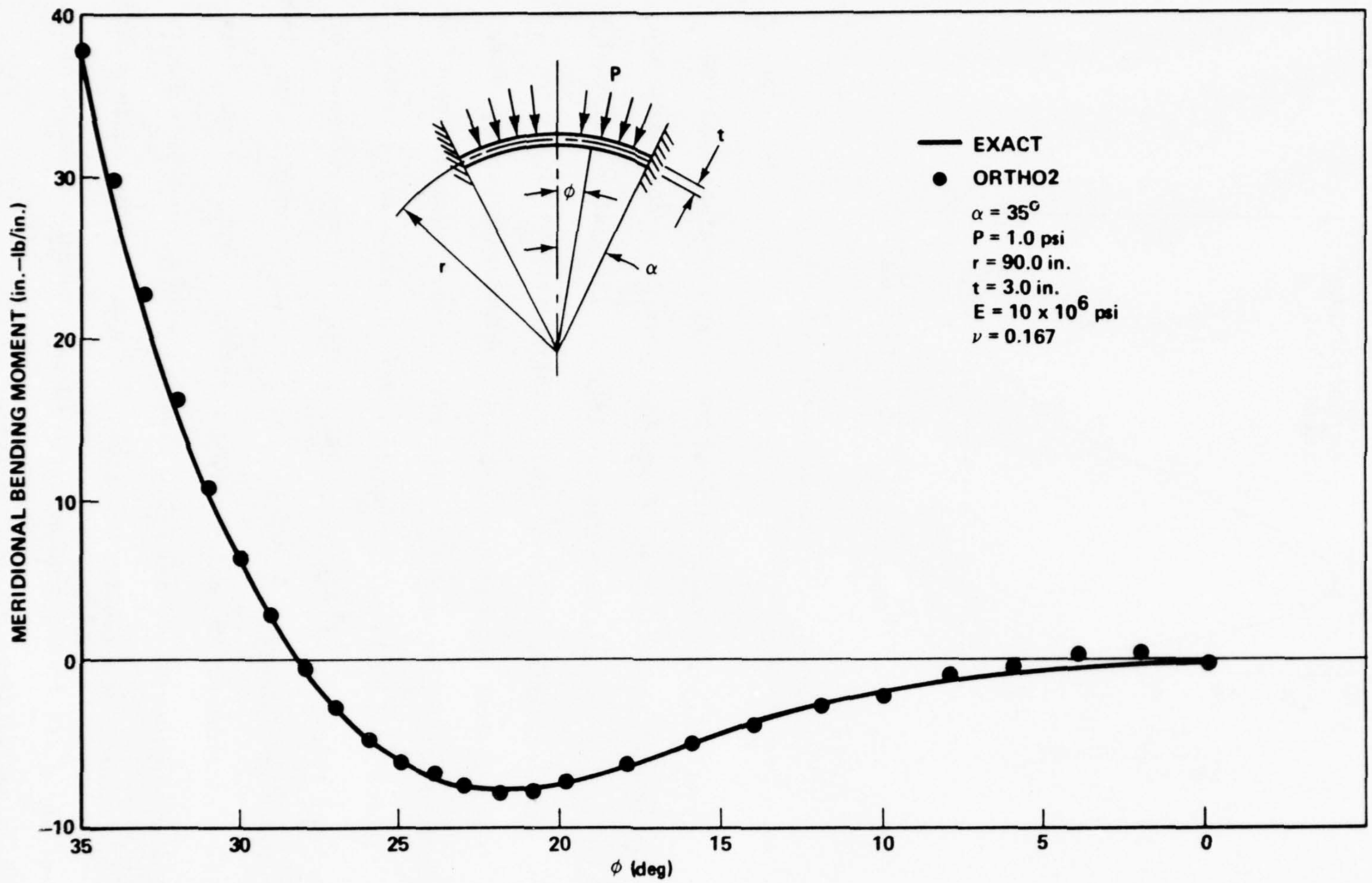


Figure 7. Meridional Bending Moment.

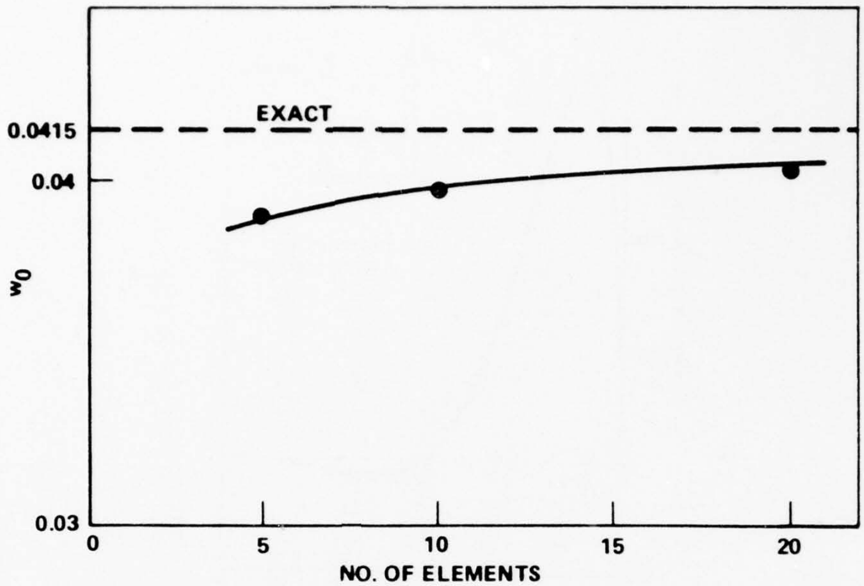


Figure 8. Convergence of Center Deflection of Circular Sandwich Plate.

Nickell [14] obtained a solution for a cylinder loaded with a radial load on one end and a moment (see Figure 9). The results are compared with the present solutions in Figures 10 through 12. The results agree with Nickell's within the limits of accuracy with which the curves can be read.

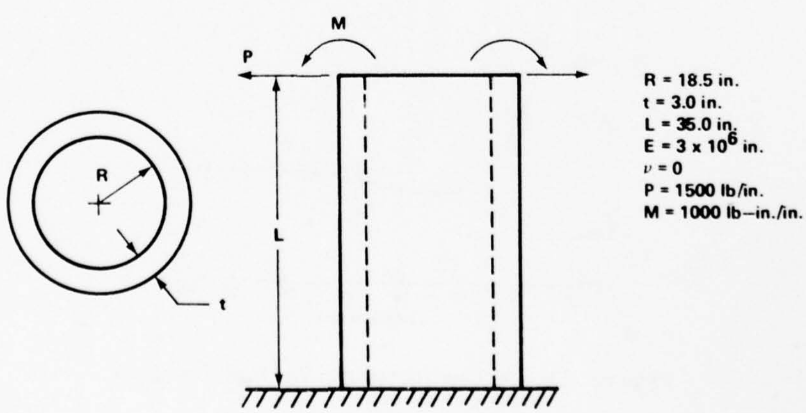


Figure 9. Locally Loaded Cylinder.

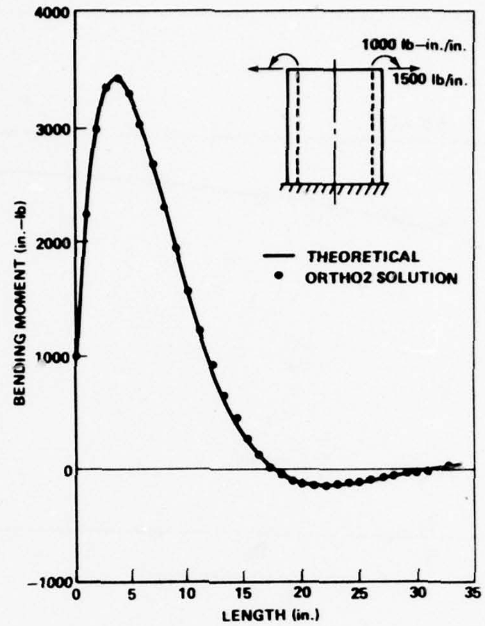


Figure 10. Bending Moment Versus Length.

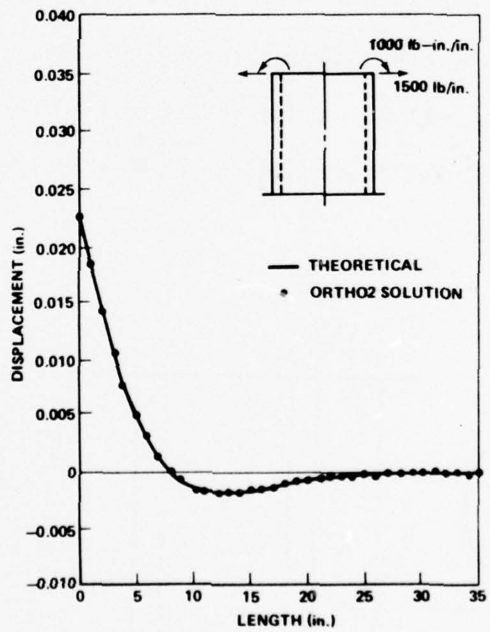


Figure 11. Displacement Versus Length.

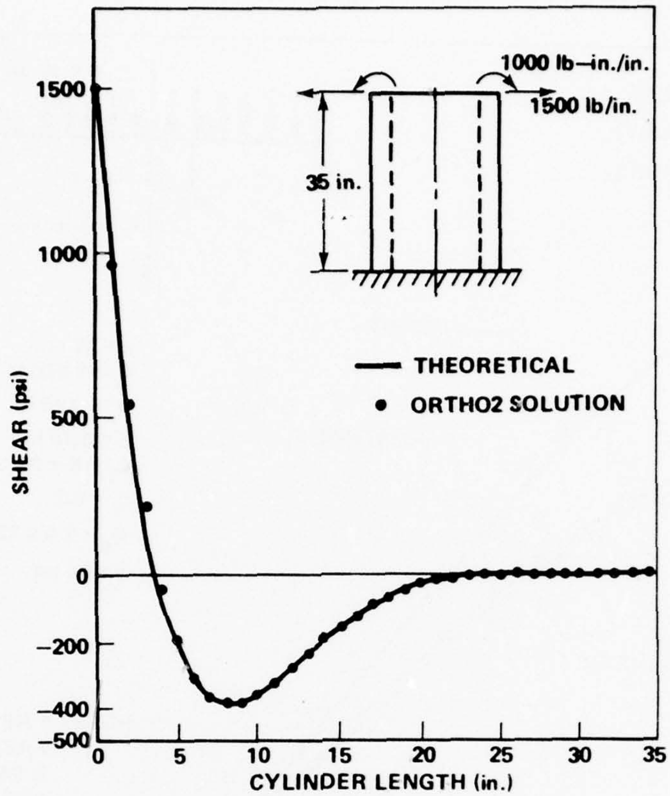


Figure 12. Shear Versus Length.

Sharifi [13] analyzed a clamped circular sandwich plate under a uniform lateral pressure. A comparison with his solution and the linear solution is shown in Figure 13. He used an incremental formulation for a nonlinear finite element analysis of sandwich structures. The nonlinearities considered were due to large displacements, as a result of finite rotations, and plastic deformations of the facings. The ORTHO2 program solution agreed very closely with these results as evidenced in Figure 13. The nonlinear solution differs significantly from the linear.

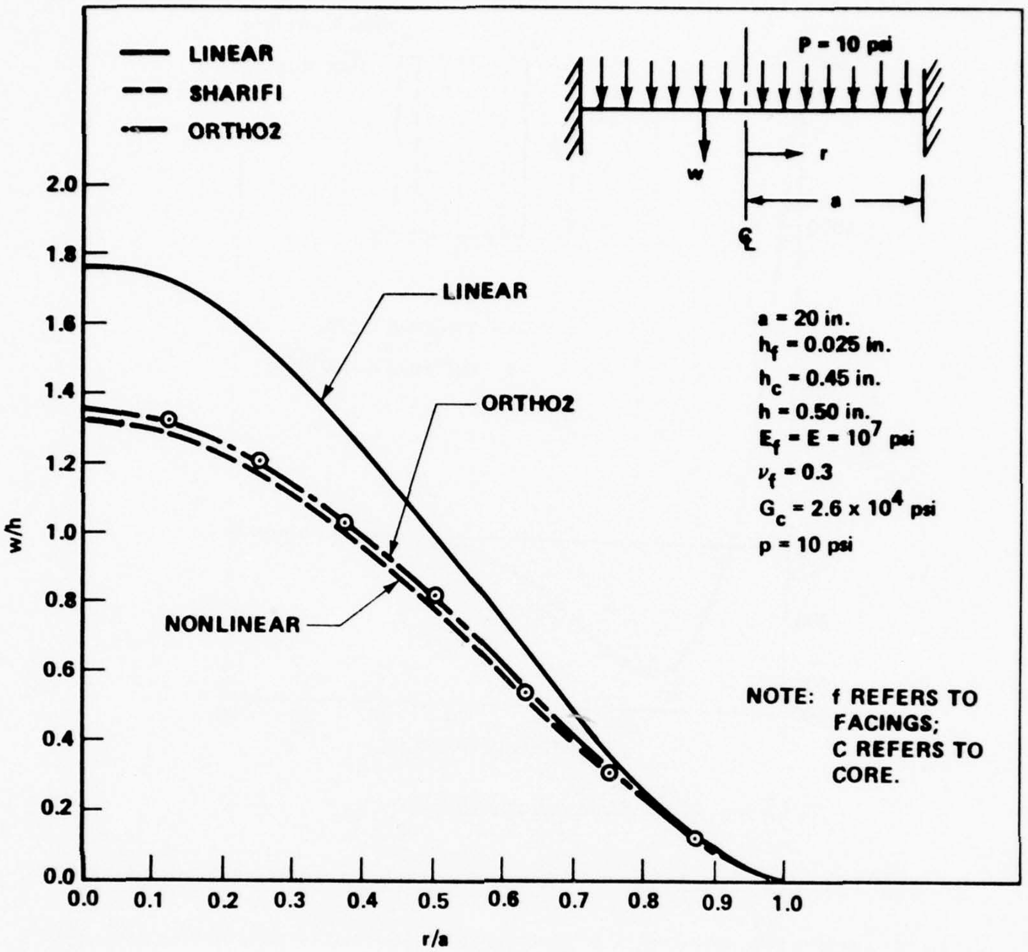


Figure 13. Transverse Deflections in a Clamped Circular Sandwich Plate.

CONCLUSIONS

A method for performing a nonlinear analysis on an orthotropic, laminated shell of revolution has been presented. The shell was assumed to be torsion-free. The classical Kirchhoff-Love assumption for normals to the midsurface was relaxed in favor of the assumed shear deformation. The method is quite general and applicable to any shell geometry possessing axial symmetry.

A finite element method that used the displacement model was selected to analyze the system. The meridional curve of the shell was represented by a series of curved elements having local coordinates. An element was developed that matches slopes and curvatures as well as displacements at its nodal circle.

The geometric approximation of curved shells usually associated with the finite element method is minimized by using the curved element. The use of this curved element significantly reduces the meridional bending moment usually present at the nodal circles when a curved structure is approximated by a straight line (conical) segment. A smaller number of elements can be used in comparison to that of a conical element.

A computer program was developed to solve the derived equations. The program was shown to be a versatile and flexible method of implementing the basic theory. Several problems were solved and the results compared with both linear and nonlinear solutions from the literature. In most cases, the results from this program agreed with the linear solutions within the limits of accuracy with which the curves could be read. Agreement with nonlinear solutions was good and could usually be further improved by taking more elements. The shell thickness and pressure may vary linearly along the meridian. The convergence and accuracy of the method were found to be entirely satisfactory as evidenced by the numerical examples.

The Gaussian Quadrature Integration method was used in the derivation of the stiffness matrix. Several tests were made to determine the most efficient method. As many as eight points were used. After examining the different schemes, it was decided that the two point Gaussian Integration scheme gave the best results.

The accuracy obtained by this method depends directly on the extent to which the assumed displacement patterns are able to reproduce the deformation actually developed within the element. Since the chosen displacement patterns satisfy the requirements of completeness and conformity (continuity of displacement at element boundary) as the size of the element decreases indefinitely, the solution obtained converges to the exact solution.

The finite element method is obviously a powerful tool in the analysis of orthotropic, composite structures. There still remains much work to do in this area. A logical extension of this work is to include stability criteria. Other items which should be considered in the future are: crossover effects, cracking or "crazing" of the matrix material, an appropriate failure criterion, and material properties which are different in tension and compression.

Appendix A

ELASTICITY MATRIX

Individual curved finite elements can, in general, be composed of a number of anisotropic layers of varying thickness along the meridional coordinate. For a single lamina, considering shear deformations, the constitutive relation is given as

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \\ \tau_{L\zeta} \\ \tau_{T\zeta} \end{Bmatrix} = \begin{bmatrix} Q'_{11} & Q'_{12} & 0 & 0 & 0 \\ Q'_{12} & Q'_{22} & 0 & 0 & 0 \\ 0 & 0 & Q'_{44} & 0 & 0 \\ 0 & 0 & 0 & Q'_{55} & 0 \\ 0 & 0 & 0 & 0 & Q'_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \\ \gamma_{L\zeta} \\ \gamma_{T\zeta} \end{Bmatrix} \tag{A-1}$$

where the transverse normal stress σ_ζ has been omitted and the laminae are orthotropic with respect to the principal elastic axes L-T. These axes need not coincide with the axes of the curvilinear coordinate system 1-2, (Figure 14), (1 is the meridional direction) and

$$Q'_{11} = E_L/(1 - \nu_{LT}\,\nu_{TL})$$

$$Q'_{12} = \nu_{LT}\,E_T/\,(1 - \nu_{LT}\,\nu_{TL})$$

$$= \nu_{TL}\,E_L/\,(1 - \nu_{TL}\,\nu_{LT})$$

$$
\begin{aligned}
Q'_{22} &= E_T/(1 - \nu_{LT}\,\nu_{TL}) \\
Q'_{44} &= G_{LT} \\
Q'_{55} &= G_{L\zeta} \\
Q'_{66} &= G_{T\zeta} \quad .
\end{aligned}
\tag{A-2}
$$

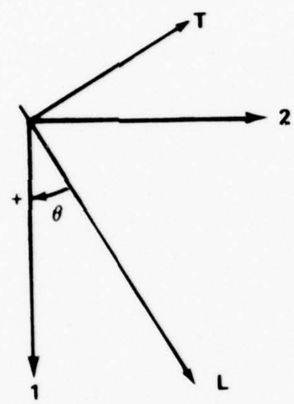


Figure 14. Material Axes.

Equation (A-1) can also be written for the kth layer in the following forms:

$$
\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}_k = \begin{bmatrix} Q'_{11} & Q'_{12} & 0 \\ Q'_{12} & Q'_{22} & 0 \\ 0 & 0 & Q'_{44} \end{bmatrix}_k \begin{Bmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{Bmatrix}
\tag{A-3}
$$

and

$$
\begin{Bmatrix} \tau_{L\zeta} \\ \tau_{T\zeta} \end{Bmatrix}_k = \begin{bmatrix} Q'_{55} & 0 \\ 0 & Q'_{66} \end{bmatrix}_k \begin{Bmatrix} \gamma_{L\zeta} \\ \gamma_{T\zeta} \end{Bmatrix}_k \quad .
$$

To develop a theory for structural laminates with individual layers having their elastic axes oriented at various angles relative to the coordinate axes, the stress-strain Equations (A-3) must be rotated through the positive angle θ so that the transformed stress-strain equations are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{14} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{24} \\ \bar{Q}_{14} & \bar{Q}_{24} & \bar{Q}_{44} \end{bmatrix}_k \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}_k \qquad \text{(A-4)}$$

and

$$\begin{Bmatrix} \tau_{1\zeta} \\ \tau_{2\zeta} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{55} & 0 \\ 0 & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \gamma_{1\zeta} \\ \gamma_{2\zeta} \end{Bmatrix}_k$$

where

$$\begin{aligned}
\bar{Q}_{11} &= Q'_{11} \cos^4 \theta + 2(Q'_{12} + 2\, Q'_{44}) \sin^2 \theta \cos^2 \theta + Q'_{22} \sin^4 \theta \\
\bar{Q}_{12} &= (Q'_{11} + Q'_{22} - 4\, Q'_{44}) \sin^2 \theta \cos^2 \theta + Q'_{12} (\sin^4 \theta + \cos^4 \theta) \\
\bar{Q}_{22} &= Q'_{11} \sin^4 \theta + 2(Q'_{12} + 2\, Q'_{44}) \sin^2 \theta \cos^2 \theta + Q'_{22} \cos^4 \theta \\
\bar{Q}_{14} &= (Q'_{11} + Q'_{12} - 2\, Q'_{44}) \sin \theta \cos^3 \theta + (Q'_{12} - Q'_{22} + 2\, Q'_{44}) \sin^3 \theta \cos \theta \\
\bar{Q}_{24} &= (Q'_{11} - Q'_{12} - 2\, Q'_{44}) \sin^3 \theta \cos \theta + (Q'_{12} - Q'_{22} + 2\, Q'_{44}) \sin \theta \cos^3 \theta \\
\bar{Q}_{44} &= (Q'_{11} + Q'_{22} - 2\, Q'_{12} - 2\, Q'_{44}) \sin^2 \theta \cos^2 \theta + Q'_{44} (\sin^4 \theta \cos^4 \theta) \\
\bar{Q}_{55} &= Q'_{55} \\
\bar{Q}_{66} &= Q'_{66} \quad .
\end{aligned} \qquad \text{(A-5)}$$

Substituting the midsurface strain and curvatures into Equations (A-4) the following expression is obtained:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [\bar{Q}]_k \begin{Bmatrix} e_1 \\ e_2 \\ 2e_{12} \end{Bmatrix} + \zeta\, [\bar{Q}]_k \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ 2\kappa_{12} \end{Bmatrix} \tag{A-6}$$

and

$$\begin{Bmatrix} \tau_{1\zeta} \\ \tau_{2\zeta} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55} & 0 \\ 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \gamma_{1\zeta} \\ \gamma_{2\zeta} \end{Bmatrix}$$

By integrating over the total thickness of the laminate, the generalized stress resultants in terms of midsurface strain and curvature are given as

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ M_1 \\ M_2 \\ M_{12} \\ Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} [C] & [D^*] & 0 \\ [D^*] & [D] & 0 \\ 0 & 0 & [S] \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ 2e_{12} \\ \kappa_1 \\ \kappa_2 \\ 2\kappa_{12} \\ \gamma_{1\zeta} \\ \gamma_{2\zeta} \end{Bmatrix} \tag{A-7}$$

where

$$[C] = \sum_{k=1}^{m} [\overline{Q}^{(k)}] (h_k - h_{k-1})$$

$$[D^*] = \frac{1}{2} \sum_{k=1}^{m} [\overline{Q}^{(k)}] (h_k^2 - h_{k-1}^2) \qquad \text{(A-8)}$$

$$[D] = \frac{1}{3} \sum_{k=1}^{m} [\overline{Q}^{(k)}] (h_k^3 - h_{k-1}^3)$$

$$[S] = \sum_{k=1}^{m} [\overline{Q}^{(k)}] (h_k - h_{k-1})$$

in which h_k and h_{k-1} = the distances, respectively, from the midsurface to the inner and outer surfaces of the k-th layer.

For an axisymmetric shell of revolution subjected to axisymmetric loadings, $N_{12} = M_{12} = Q_2 = e_{12} = \kappa_{12} = \gamma_{2\zeta} = 0$.

Hence,

$$\begin{Bmatrix} N_1 \\ N_2 \\ M_1 \\ M_2 \\ Q_1 \end{Bmatrix} = \begin{bmatrix} [C] & [D^*] & 0 \\ [D^*] & [D] & 0 \\ 0 & 0 & S_{55} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ \kappa_1 \\ \kappa_2 \\ \gamma_{1\zeta} \end{Bmatrix} \qquad \text{(A-9)}$$

or symbolically

$$\{S\} = [E] \{\epsilon\} \quad . \qquad \text{(A-10)}$$

Appendix B

ELEMENT STIFFNESS MATRIX

The element stiffness matrix is found by writing the total potential energy of the axisymmetric shell of revolution and minimizing it for the imposed constraints and loading conditions.

The potential energy for a linear elastic shell of revolution in the absence of thermal and body forces can be formulated as follows:

$$\pi = \iiint_V \frac{1}{2} \{\epsilon\}^T \{\sigma\} \, dV - \iint_{A_1} \{f\}^T \{P\} \, dA \qquad \text{(B-1)}$$

where the vectors $\{\epsilon\}$, $\{\sigma\}$, $\{f\}$, and $\{P\}$ represent the strain, stress, displacement, and equivalent surface traction vectors, respectively.

Introducing the stress resultant vector

$$\{S\} = t \{\sigma\} \qquad \text{(B-2)}$$

where t is the thickness of the shell, Equation (B-1) may be written as

$$\pi = \iiint_V \frac{1}{2} \{\epsilon\}^T \{S\} \frac{dV}{t} - \iint_{A_1} \{f\}^T \{P\} \, dA \quad . \qquad \text{(B-3)}$$

The first integral is evaluated over the entire volume V of the shell and the second over the portion A_1 of the midsurface of the shell, where the equivalent surface tractions are prescribed. Since the state of displacement throughout the shell is defined element by element, the total potential energy may be considered as the sum of the potential energies of all individual elements, i.e.,

$$\pi = \sum_e \pi^e .$$

The potential energy contribution of element "e" will now be considered. The state of displacement defined for the element in local rectilinear coordinates x-y can be expressed in matrix form in Equation (21) as

$$\{f_r\} = [\phi]\ \{\alpha\} = [\phi][A_r]\ \{\delta_r^e\} \quad . \tag{B-4}$$

Transformation of $\{f_r\}$ into the global coordinate system may be obtained from Equation (13)

$$\{f\} = [q_r]^T\ \{f_r\} = [N]\ \{\delta_r\}^e \tag{B-5}$$

where

$$[N] = [q_r]^T\ [\phi]\ [A_r] \tag{B-6}$$

and the colume vector $\{\delta_r\}^e$ represents the eight discrete parameters (nodal point displacements) of the element as given in Equation (25b). The matrix [N] is a function of spatial coordinates and describes the defined displacement pattern.

Substituting Equation (27) into Equation (31) the following strain-displacement relations are obtained:

$$\{\epsilon\} = [B]\ \{\delta\}^e \tag{B-7}$$

where

$$[B] = [\Phi'] [A] \quad . \tag{B-8}$$

Equation (B-8) is a matrix relating the nodal point displacement vector to the strain vector. The elastic stress-strain relations can be expressed as

$$\{S\} = [E] \{\epsilon\} \tag{B-9}$$

where [E] is a function of the elastic properties of the element. Each element can be assigned different elastic properties. If the relations in Equations (B-9), (B-5), and (B-7) are substituted into (B-3), the potential energy contribution for the element becomes

$$\pi^e = \iiint_{V_e} \frac{1}{2} \{\delta^e\}^T [B]^T [E] [B] \{\delta^e\} \frac{dV}{t} - \iint_{A_{1_e}} \{\delta^e\}^T [N]^T \{P\} \, dA \tag{B-10}$$

where V_e is the volume of the element and A_{1_e} is that part of the mid-surface area of the element which coincides with the midsurface area A_1 of the shell over which the equivalent surface tractions are prescribed.

Since the discrete parameters $\{\delta^e\}$ are not a function of spatial coordinates, the potential energy of the element may be written as

$$\pi^e = \{\delta^e\}^T \left[\iiint_{V_e} \frac{1}{2} [B]^T [E][B] \frac{dV}{t} \right] \{\delta^e\} - \{\delta^e\}^T \iint_{A_{1_e}} [N]^T \{P\} \, dA \quad . \tag{B-11}$$

Since the assumed displacement patterns for each element satisfy various requirements such as completeness and conformity, the best values that can be obtained for the total nodal point displacements of the finite element representation of shells of revolution are those that minimize the total potential energy of the shell under the constraints imposed; i.e., the best value of $\{\delta\}$ are those that satisfy the system of linear equations

$$\frac{\partial \pi}{\partial \{\delta\}} = 0 \tag{B-12}$$

where $\{\delta\}$ is the total nodal displacement vector of the system.

In forming the system of Equations (B-12), it is convenient to have an expression for the spatial derivatives of the potential energy of each element "e" with respect to its own nodal point displacement vector $\{\delta^e\}$, i.e.,

$$\frac{\partial \pi^e}{\partial \{\delta^e\}} = \left[\frac{\partial \pi^e}{\partial u_I} \; \frac{\partial \pi^e}{\partial w_I} \; \frac{\partial \pi^e}{\partial X_I} \; \frac{\partial \pi^e}{\partial \gamma_I} \; \frac{\partial \pi^e}{\partial u_J} \; \frac{\partial \pi^e}{\partial w_J} \; \frac{\partial \pi^e}{\partial X_J} \; \frac{\partial \pi^e}{\partial \gamma_J} \right] \quad . \tag{B-13}$$

By use of Equation (B-10), this expression can be obtained as

$$\frac{\partial \pi^e}{\partial \{\delta^e\}} = \left[\iiint_{V_e} [B]^T \, [E][B] \, \frac{dV}{t} \right] \{\delta^e\} - \left[\iint_{A_{1_e}} [N]^T \, \{P\} \, dA \right] \quad . \tag{B-14}$$

The terms in the first and second brackets are normally defined as the element stiffness matrix $[K^e]$ and the element generalized nodal point force $\{F^e\}$, respectively. Hence,

$$[K^e] = \iiint_{V_e} [B]^T \, [E][B] \, \frac{dV}{t} \tag{B-15}$$

$$\{F^e\} = \iint_{A_{1_e}} [N]^T \, \{P\} \, dA \quad . \tag{B-16}$$

By properly combining the submatrices in Equation (B-14) obtained for each element, the total matrix equation representing Equation (B-12) can be constructed as

$$[K]\{\delta\} = \{F\} \tag{B-17}$$

and then solved for the nodal point displacements. Once the nodal point displacements are obtained, the corresponding stress resultants, stresses, and strains for the defined displacement patterns can be calculated from Equations (B-7) and (B-9).

If Equation (B-8) is substituted into Equation (B-15) and the volume increment for a shell of revolution is taken as

$$dV = 2\pi t \frac{R(x)}{\cos\beta}\, dx \quad , \tag{B-18}$$

then the element stiffness matrix for the axisymmetric shell element takes the form

$$\begin{aligned}[K^e] &= 2\pi \int_o^{\ell} [B]^T [E][B] \frac{R(x)}{\cos\beta}\, dx \\ &= 2\pi [A]^T [G][A] \end{aligned} \tag{B-19}$$

where

$$[G] = \int_o^{\ell} [\phi']^T [E][\phi'] \frac{R(x)}{\cos\beta}\, dx \quad . \tag{B-20}$$

The integration is over the chord length of the meridian cross section of element.

It is assumed that the equivalent surface traction over the mid-surface area A_1 where tractions are prescribed varies linearly between the two nodal circles I and J. That is,

$$\{P_c\}^T = [0 \ (P_n + P_n' \, x) \ \ 0 \ \ 0] \tag{B-21}$$

where $\{P_c\}$ is the surface traction vector expressed in local curvilinear coordinates. Transforming into global coordinates the following is obtained:

$$\{P\} = [q_r]^T \, [q_c]^T \, \{P_c\} \quad . \tag{B-22}$$

Substituting Equations (B-6) and (B-22) into Equation (B-16) the generalized element nodal force vector becomes

$$\{F^e\} = 2\pi \int_o^\ell [A_r]^T \, [\phi]^T \, [q_r][q_r]^T \, [q_c]^T \, \{P_c\} \, \frac{R(x)}{\cos\beta} \, dx \tag{B-23}$$

or

$$\{F^e\} = 2\pi [A_r]^T \int_o^\ell [\phi]^T \, [q_c]^T \, \{P_c\} \, \frac{R(x)}{\cos\beta} \, dx$$

where

$$[\phi]^T \, [q_c]^T \, \{P_c\} = P_n \begin{Bmatrix} -\sin\beta \\ -x\sin\beta \\ \cos\beta \\ x\cos\beta \\ x^2\cos\beta \\ x^3\cos\beta \\ 0 \\ 0 \end{Bmatrix} + P_n' \begin{Bmatrix} -x\sin\beta \\ -x^2\sin\beta \\ x\cos\beta \\ x^2\cos\beta \\ x^3\cos\beta \\ x^4\cos\beta \\ 0 \\ 0 \end{Bmatrix} .$$

Appendix C

EQUILIBRIUM EQUATIONS BY VIRTUAL WORK

The equilibrium equations governing the shell behavior can be derived by using the principle of virtual work.

From Equations (21), (23), and (25a)

$$
\begin{aligned}
\{\bar{f}\} &= [\Phi]\,\{\alpha\} \\
&= [\Phi]\,[\bar{A}]\,\{\bar{u}\} \\
&= [\Phi][\bar{A}][R]\,\{u_g\} \qquad \text{(C-1)} \\
&= [\Phi][A]\,\{u_g\} \\
&= [N]\,\{u_g\}
\end{aligned}
$$

where $[N] = [\Phi][A]$ and $[A] = [\bar{A}][R]$.

Let there be an arbitrary and non-zero virtual nodal displacement $\delta\{u_g\}$ about the deformed position which results in a virtual displacement $\delta\{\bar{f}\}$ and virtual strain $\delta\{\epsilon\}$. The δ prefix denotes a virtual change in the quantity concerned.

By means of the principle of virtual work, the following expression can be written:

$$\delta\{u_g\}^T\{Q\} = \int_{A_m} \delta\{\epsilon\}^T\{\tau\}\,dA_m - \int_{A_m} \delta\{\bar{f}\}^T\{F_s\}\,dA_m \qquad \text{(C-2)}$$

where A_m is the reference surface area of the shell, $\{Q\}$ is the applied nodal force vector and $\{F_s\}$ is the surface traction vector. The stress vector $\{\tau\}$ is given as

$$\{\tau\} = [C]\{\epsilon\}$$
$$= [C][\Phi'][A]\{u_g\} \quad .$$

Substituting Equation (C-1) into (C-2) the following is obtained:

$$\delta\{u_g\}^T \{Q\} + \delta\{u_g\}^T \int_{A_m} [N]^T \{F_s\} \, dA_m = \int_{A_m} \delta\{\epsilon\}^T \{\tau\} \, dA_m$$

or

$$\delta\{u_g\}^T \{P\} = \int_{A_m} \delta\{\epsilon\}^T \{\tau\} \, dA_m \qquad \text{(C-3)}$$

where $\{P\}$ is the equivalent nodal forces of the element defined by the principle of virtual work.

Substituting Equations (23), (25a), and (31) into (C-3) gives:

$$\delta\{u_g\}^T \{P\} = \int_{A_m} \delta\{u_g\}^T [A]^T [\Phi']^T \{\tau\} \, dA_m \quad . \qquad \text{(C-4)}$$

This results in a nonlinear matrix equation for the equivalent nodal forces $\{P\}$, i.e.,

$$\{P\} = \int_{A_m} [A]^T [\Phi']^T [C][\Phi'] [A] \{u_g\} \, dA_m \quad . \qquad \text{(C-5)}$$

Equation (C-5) is now linearized by writing it in the form of an implicit differential.

$$\Delta\{P\} = \int_{A_m} \Delta [A]^T [\Phi']^T [C][\Phi'][A] \{u_g\} dA_m$$

$$+ \int_{A_m} [A]^T \Delta\left([\Phi']^T [C][\Phi']\right) [A] \{u_g\} dA_m \qquad \text{(C-6)}$$

$$+ \int_{A_m} [A]^T [\Phi'][C][\Phi'][A] \Delta \{u_g\} dA_m \quad .$$

It is assumed that a change in the transformation matrix during an increment of load may be neglected. This permits neglecting the first term on the right hand side of equation (C-6). The second term results in the well known initial stress matrix while the third term accounts for the effect of the increment of strain and may be split into two terms separating the linear and nonlinear displacement terms.

$$\int_{A_m} [A]^T [\Phi']^T [C][\Phi'][A] \Delta \{u_g\} dA_m =$$

$$\int_{A_m} [A]^T [\Phi'^{\ell}]^T [C][\Phi'^{\ell}][A] dA_m \Delta \{u_g\} +$$

$$\int_{A_m} [A]^T [\Phi'^{\ell}]^T [C][\Phi'^{n}] [A] dA_m \Delta \{u_g\} +$$

$$\int_{A_m} [A]^T [\Phi'^{n}]^T [C][\Phi'^{\ell}][A] dA_m \Delta \{u_g\} +$$

$$\int_{A_m} [A]^T [\Phi'^{n}]^T [C][\Phi'^{n}][A] dA_m \Delta \{u_g\} =$$

$$\left([K^{(0)}] + [K^{(2)}]\right) \Delta\{u_g\} \quad . \qquad \text{(C-7)}$$

The last three terms have been collected into $[K^{(2)}]$ and will be called the initial displacement matrix. If the area increment of the shell is taken as

$$dA_m = 2\pi \frac{R(x)}{\cos\beta} dx$$

then the element stiffness matrix for the axisymmetric shell element takes the form

$$[K^e] = 2\pi \int_o^{\ell} [B]^T [C][B]\, R(x)\, dx$$

$$= 2\pi\, [A]^T [G][A]$$

where

$$[B] = [\Phi'][A]$$

and

$$[G] = \int_o^{\ell} [\Phi']^T [C][\Phi']\, R(x)\, dx \quad .$$

The integration is over the chord length of the meridian cross section of element.

INITIAL STRESS MATRIX

The second term of Equation (C-6) can be written as:

$$\int_{A_m} [A]^T \triangle\Big([\Phi']^T [C][\Phi']\Big) [A] \{u_g\}\, dA_m =$$

$$2\int_{A_m} [A]^T \Big(\triangle[\Phi']^T\Big) [C][\Phi'][A] \{u_g\} dA_m =$$

$$2\int_{A_m} [A]^T \Big(\triangle[\Phi']^T\Big) \{\tau\}\, dA_m \quad . \tag{C-8}$$

Equation (C-8) can be broken down into a summation of the five stress resultant components, i.e.,

$$2\int_{A_m} [A]^T \triangle[\phi']^T \{\tau\}\, dA_m = \sum_{i=1}^{5} \int_{A_m} \tau_i [A]^T 2\triangle \{\phi_i'^T\}\, dA_m \qquad \text{(C-9)}$$

where i is the index of the stress resultant components and $\{\phi_i'^T\}$ is the transpose of the corresponding i^{th} row of the matrix $[\phi']$.

Since the derivation is based on the current deformed position of the shell element, the increment $\triangle\{\phi_i'^T\}$ can be written as follows:

$$2\triangle\{\phi_i'^T\} = 2\triangle\{\phi_i'^{nT}\} = 2\triangle\{\phi_1'^{nT}\}$$

and

$$2\triangle\{\phi_1'^{nT}\} = 2\left\{\frac{\partial\phi_1'^{nT}}{\partial X}\right\}\triangle X$$

$$= \begin{Bmatrix} 0 \\ -\sin\beta \\ 0 \\ 1 \\ 2x \\ 3x^2 \\ 0 \\ 0 \end{Bmatrix} [0 \quad -\sin\beta \quad 0 \quad 1 \quad 2x \quad 3x^2 \quad 0 \quad 0]\, \triangle\{X\}$$

Appendix D

THEORETICAL BACKGROUND

Some of the basic concepts used in the derivation of the equations presented in the main text are presented in this section.

GEOMETRY OF SHELLS

The geometry of a shell is entirely defined once the midsurface and the thickness at each point are specified. Hence, to describe the shell space, the middle surface or reference surface of the shell must be specified. Let α_1 and α_2 be the curvilinear coordinates for the mid-surface and let them coincide with lines of principal curvature of the surface, and let ζ be a coordinate normal to the midsurface as shown in Figure 15.

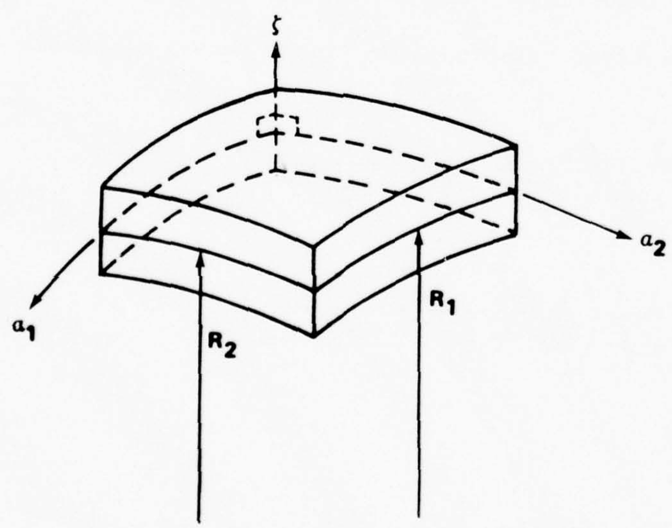


Figure 15. Typical Shell Element.

The position of any point in the midsurface can be defined by the curvilinear coordinates α_1 and α_2. The location of any point in the shell can be related by the three parameters α_1, α_2, and ζ. With this curvilinear coordinate system, a line element in the shell space surrounding the midsurface can be expressed in terms of the differentials of the orthogonal curvilinear coordinates as follows:

$$dS^2 = A_1^2 \left(1 + \frac{\zeta}{R_1}\right)^2 d\alpha_1^2 + A_2^2 \left(1 + \frac{\zeta}{R_2}\right)^2 d\alpha_2^2 + d\zeta^2 \qquad \text{(D-1)}$$

where A_1 and A_2 are the midsurface metrics and R_1 and R_2 are the principal radii of curvature of the surface.

STRAIN-DISPLACEMENT RELATIONSHIPS

The general nonlinear strain-displacement relations for large rotation but small strain were derived by Novozhilov [18] and later corrected by Tsao [19]. Suppressing the nonlinear terms, the following strain-displacement relations for linear theory of shells is obtained:

$$\epsilon_1 = \frac{1}{1 + \frac{\zeta}{R_1}} \left(\frac{1}{A_1} \frac{\partial U}{\partial \alpha_1} + \frac{V}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} + \frac{W}{R_1} \right)$$

$$\epsilon_2 = \frac{1}{1 + \frac{\zeta}{R_2}} \left(\frac{1}{A_2} \frac{\partial V}{\partial \alpha_2} + \frac{U}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} + \frac{W}{R_2} \right) \qquad \text{(D-2)}$$

$$\epsilon_\zeta = \frac{\partial W}{\partial \zeta}$$

$$\gamma_{12} = \frac{1}{1 + \frac{\zeta}{R_1}} \left(\frac{1}{A_1} \frac{\partial V}{\partial \alpha_1} - \frac{U}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \right) + \frac{1}{1 + \frac{\zeta}{R_2}} \left(\frac{1}{A_2} \frac{\partial U}{\partial \alpha_2} - \frac{V}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \right)$$

$$\gamma_{1\zeta} = \frac{\partial U}{\partial \zeta} + \left(\frac{1}{A_1}\frac{\partial W}{\partial \alpha_1} - \frac{U}{R_1}\right) \cdot \frac{1}{\left(1 + \frac{\zeta}{R_1}\right)}$$

$$\gamma_{2\zeta} = \frac{\partial V}{\partial \zeta} + \frac{1}{1 + \frac{\zeta}{R_2}}\left(\frac{1}{A_2}\frac{\partial W}{\partial \alpha_2} - \frac{V}{R_2}\right)$$

where the functions U, V, and W represent the displacement components caused by straining of a material point originally at point $(\alpha_1, \alpha_2, \zeta)$ in the shell in the α_1, α_2, and ζ direction, respectively.

To incorporate the transverse shear deformation, the classical Kirchhoff-Love assumption must be abandoned. The material lines originally straight and normal to the midsurface of the shell remain straight but are no longer normal to the deformed midsurface (Figure 16). This implies that the transverse shear deformation is independent of the coordinate ζ. Hence, the shear rotation can be represented by some average value of the shear strain at midsurface. The displacement components of a point in the shell can be expressed, as a first approximation, by relationships of the form

$$\begin{aligned} U(\alpha_1,\alpha_2,\zeta) &= u(\alpha_1,\alpha_2) + \zeta\ \beta_1(\alpha_1,\alpha_2) \\ V(\alpha_1,\alpha_2,\zeta) &= v(\alpha_1,\alpha_2) + \zeta\ \beta_2(\alpha_1,\alpha_2) \\ W(\alpha_1,\alpha_2,\zeta) &= w(\alpha_1,\alpha_2) \end{aligned} \tag{D-3}$$

where u, v, and w are displacements of the point at midsurface, and β_1 and β_2 are rotations that represent changes of slope of the normal to the midsurface. It should be noted that terms u, v, w, β_1 and β_2 are functions of α_1 and α_2 only.

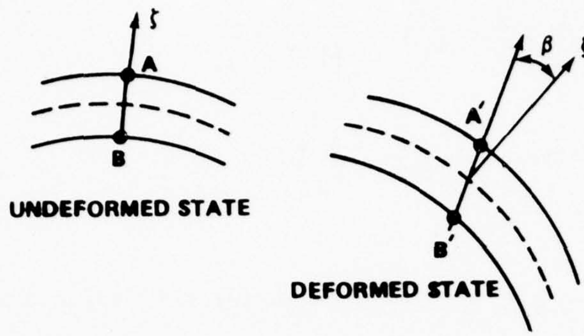


Figure 16. Transverse Shear Deformation.

Substituting Equations (D-3) into the strain-displacement relations Equation (D-2) and suppressing the terms $\frac{\zeta}{R_1}$ yield

$$\epsilon_1 = e_1 + \zeta \kappa_1$$

$$\epsilon_2 = e_2 + \zeta \kappa_2$$

$$\epsilon_\zeta = 0$$

$$\gamma_{12} = 2\, e_{12} + \zeta(2\, \kappa_{12}) \qquad \text{(D-4)}$$

$$\gamma_{1\zeta} = \frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1} + \beta_1$$

$$\gamma_{2\zeta} = \frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2} + \beta_2$$

where

$$e_1 = \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{v}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{w}{R_1}$$

(D-5)

$$e_2 = \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{u}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{w}{R_2}$$

are the extensional strains at the midsurface of the shell,

$$\kappa_1 = \frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}$$

$$\kappa_2 = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} \tag{D-6}$$

are the changes in curvature of the midsurface in the directions of α_1 and α_2, respectively, and

$$2\, e_{12} = \frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{u}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{v}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}$$

$$2\, \kappa_{12} = \frac{1}{A_1}\frac{\partial \beta_2}{\partial \alpha_1} - \frac{\beta_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{1}{A_2}\frac{\partial \beta_1}{\partial \alpha_2} - \frac{\beta_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} \tag{D-7}$$

represent the in-surface shear strain and twist of the midsurface, respectively.

STRESS-STRAIN RELATIONS

Assuming that the in-surface stresses can be represented by a state of generalized plane stress, the stress-strain relations for the shell space and for the orthotropic material can be written as

$$\sigma_1 = \frac{E_1}{1 - \nu_{12}\nu_{21}}\,\epsilon_1 + \frac{\nu_{21}\,E_2}{1 - \nu_{12}\nu_{21}}\,\epsilon_2$$

$$\sigma_2 = \frac{\nu_{12}\,E_1}{1 - \nu_{12}\nu_{21}}\,\epsilon_1 + \frac{E_2}{1 - \nu_{12}\nu_{21}}\,\epsilon_2$$

$$\tau_{12} = G_{12} \gamma_{12}$$

$$\tau_{1\zeta} = G_{1\zeta} \gamma_{1\zeta} \tag{D-8}$$

$$\tau_{2\zeta} = G_{2\zeta} \gamma_{1\zeta}$$

where E_1, E_2, G_{12}, $G_{1\zeta}$, $G_{2\zeta}$, ν_{12}, and ν_{21} are the elastic constants along the three coordinate directions [21].

Since the transverse shear strain has been assumed to be constant across the thickness, the corresponding shear stress is likewise constant and is directly proportional to the shear strain. However, from elementary strength of materials it is known that transverse shear stress is not constant across the thickness of a beam section. Therefore, the average shear strain, which may provide a good approximation to the shear rotation, does not necessarily provide an adequate representation of the shear stress distribution. Hence, a shear stress factor is used in conjunction with the transverse stress-strain Equation (D-8) as suggested by Naghdi [22], that is

$$\tau_{i\zeta} = \frac{5}{6} G_{i\zeta} \gamma_{i\zeta} \quad i = 1, 2 \quad . \tag{D-9}$$

Substituting Equations (D-4) into Equations (D-8) and (D-9) and integrating across the thickness of the shell, the stress resultants and couples are obtained as follows:

$$N_1 = C_{11} e_1 + C_{12} e_2$$

$$N_2 = C_{21} e_1 + C_{22} e_2$$

$$N_{12} = 2G_{12} + e_{12}$$

$$M_1 = D_{11}\,\kappa_1 + D_{12}\,\kappa_2$$

$$M_2 = D_{21}\,\kappa_1 + D_{22}\,\kappa_2 \qquad \text{(D-10)}$$

$$M_{12} = \frac{G_{12}\,t^3}{12}\,(2\,\kappa_{12})$$

$$Q_1 = \frac{5}{6}\,G_{1\zeta}\;t\;\gamma_{1\zeta}$$

$$Q_2 = \frac{5}{6}\,G_{2\zeta}\;t\;\gamma_{2\zeta}$$

where t is the thickness of the shell and

$$C_{11} = \frac{E_1\,t}{1 - \nu_{12}\nu_{21}}$$

$$C_{22} = C_{11}\,\frac{E_2}{E_1}$$

$$D_{11} = \frac{E_1\,t^3}{12(1 - \nu_{12}\nu_{21})} \qquad \text{(D-11)}$$

$$D_{22} = D_{11}\,\frac{E_2}{E_1}$$

$$C_{12} = C_{21} = \nu_{12}\,C_{22} = \nu_{21}\,C_{11}$$

$$D_{12} = D_{21} = \nu_{12}\,D_{22} = \nu_{21}\,D_{11}\quad .$$

SHELLS OF REVOLUTION

The discussion will now be restricted to shells of revolution. The midsurface of the shell is obtained by rotation of a plane curve. This curve is called the meridian and its plane is the meridian plane. The intersection of the surface with planes perpendicular to the axis of rotation are parallel circles and are called parallels. For shells of revolution, the lines of principal curvature are its meridians and parallels [23].

A set of convected normal coordinates Φ, θ, and ζ are used to describe the shells of revolution, where Φ is the angle between the normal to the midsurface of the shell and the axis of revolution, θ is the angle describing the position of points of the corresponding parallel as shown in Figure 17. The radius of curvature of the meridian is R_1. The second radius of curvature R_2 will always be the length of the intercept of the normal to the midsurface between the axis of the shell, i.e., $\overline{AP}$. This is because the normal from two adjacent points P and P' on the same parallel will always intersect on the axis of the shell.

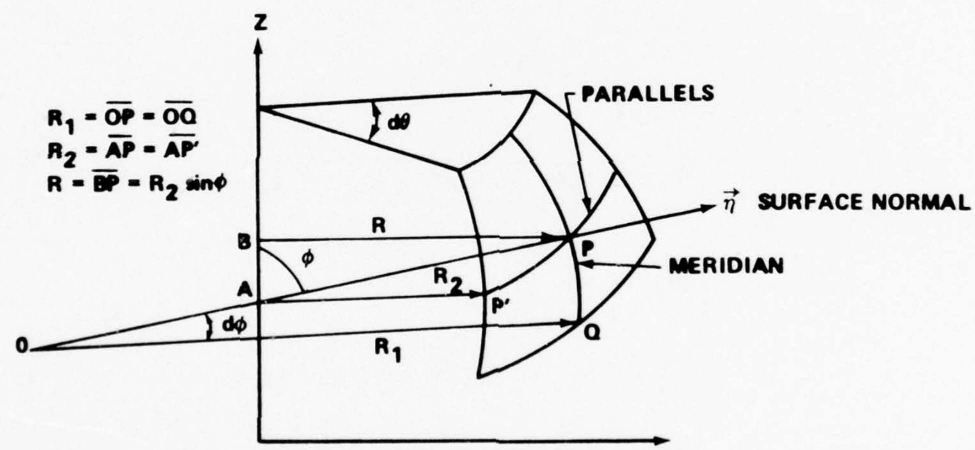


Figure 17. Shell Geometry.

The arc length of a line element in the shell space is given as

$$dS^2 = R_1^2 \, d\Phi^2 + R_2^2 \sin^2\Phi \, d\theta^2 \quad . \qquad \text{(D-12)}$$

Associating α_1 with Φ and α_2 with θ and comparing Equation (D-12) with Equation (D-1), the following is obtained:

$$A_1 = R_1$$

$$A_2 = R = R_2 \sin\phi \quad . \tag{D-13}$$

Note that the term $\frac{\zeta}{R_i}$ in Equation (D-1) is small compared to unity for thin shells.

From Figure 18, by inspection, the following is obtained:

$$\frac{dR}{ds_1} = \cos\phi \quad ,$$

or

$$\frac{dR}{d\phi} = R_1 \cos\phi \quad . \tag{D-14}$$

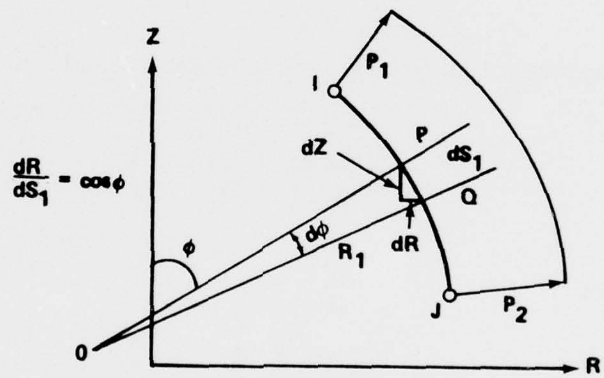


Figure 18. Shell Meridian.

AXISYMMETRIC LOADINGS

For shells of revolution with axisymmetric loadings, all geometric quantities are independent of θ. Consequently, all of the shell variables are independent of θ and, starting with the relationships between the strains and displacements Equations (D-4) through (D-7), the following is obtained:

$$e_1 = \frac{1}{R_1}\frac{du}{d\phi} + \frac{w}{R_1}$$

$$e_2 = \frac{1}{R}\,(u\cos\phi + w\sin\phi)$$

$$\kappa_1 = \frac{1}{R_1}\frac{d\beta_1}{d\phi} \tag{D-15}$$

$$\kappa_2 = \frac{\beta_1}{R}\cos\phi$$

$$\gamma_{1\zeta} = \frac{1}{R_1}\frac{dw}{d\phi} - \frac{u}{R_1} + \beta_1 \quad .$$

Setting $\gamma_{1\zeta}$ equal to some average shear strain as

$$\gamma_{1\zeta} = -\gamma_1 \quad , \tag{D-16}$$

and substituting Equation (D-16) into the last Equation of (D-15) yields

$$\beta_1 = -\left(\frac{1}{R_1}\frac{dw}{d\phi} - \frac{u}{R_1} + \gamma_1\right) \quad . \tag{D-17}$$

Furthermore, let

$$\frac{1}{R_1}\frac{d}{d\phi} = \frac{d}{dS} \tag{D-18}$$

where S is measured along the meridional direction of the midsurface.

Substituting Equations (D-18) and (D-17) into Equation (D-15) the following is obtained:

$$e_1 = \frac{du}{dS} + \frac{w}{R_1}$$

$$e_2 = \frac{1}{R}\,(u\cos\phi + w\sin\phi)$$

$$\kappa_1 = -\frac{d}{dS}\left(\frac{dw}{dS_1} - \frac{u}{R_1} + \gamma_1\right)$$

$$\kappa_2 = -\frac{\cos\phi}{R}\left(\frac{dw}{dS_1} - \frac{u}{R_1} + \gamma_1\right) \qquad \text{(D-19)}$$

$$\gamma_{1\zeta} = -\gamma_1 \quad .$$

The stress resultants and couples reduce from Equation (D-10) to the following set:

$$N_1 = C_{11}\, e_1 + C_{12}\, e_2$$

$$N_2 = C_{21}\, e_1 + C_{22}\, e_2$$

$$M_1 = D_{11}\, \kappa_1 + D_{12}\, \kappa_2 \qquad \text{(D-20)}$$

$$M_2 = D_{21}\, \kappa_1 + D_{22}\, \kappa_2$$

$$Q_1 = \frac{5}{6}\, G_{1\zeta}\, t\, \gamma_{1\zeta} \quad .$$

Appendix E

COMPUTER PROGRAM ORTHO2

The foregoing derivation was implemented in a finite element computer program. For convenience in reference, this program will be referred to as ORTHO2. The element stiffness matrix was formed by numerical integration. The nodal point coordinate and element connection array are generated automatically by the program. A shell of revolution is first divided into as many segments as necessary. Because each segment may be considered as a separate unit, different material properties as well as thickness and pressure can be ascribed to different segments. Each segment in turn may be subdivided into any number of shell elements. Normal pressure and thickness of the shell must be axisymmetric, but may be varied linearly along the meridional direction. The matrix equations are solved by the Cholesky decomposition process which stores only nonzero elements and therefore results in a significant saving of computing time. The program can be used to solve problems of thin, thick, and sandwich shells of revolution as well as multilayered, orthotropic shells such as a fiber reinforced composite. The program is limited to ten different materials and 50 nodes, but can be increased by increasing the dimension statement accordingly. This program requires about 32K core storage and three scratch files. A total of nine sets of input data is needed. The flow chart for ORTHO2 is shown in Figure 19.

An iteration procedure is used in the program. For the nonlinear effect, the load is applied in increments and the coordinates are updated.

This method was compared with that of using the relations in Equation (32a) and was found to give practically the same results. Much of the data is generated internally in the program.

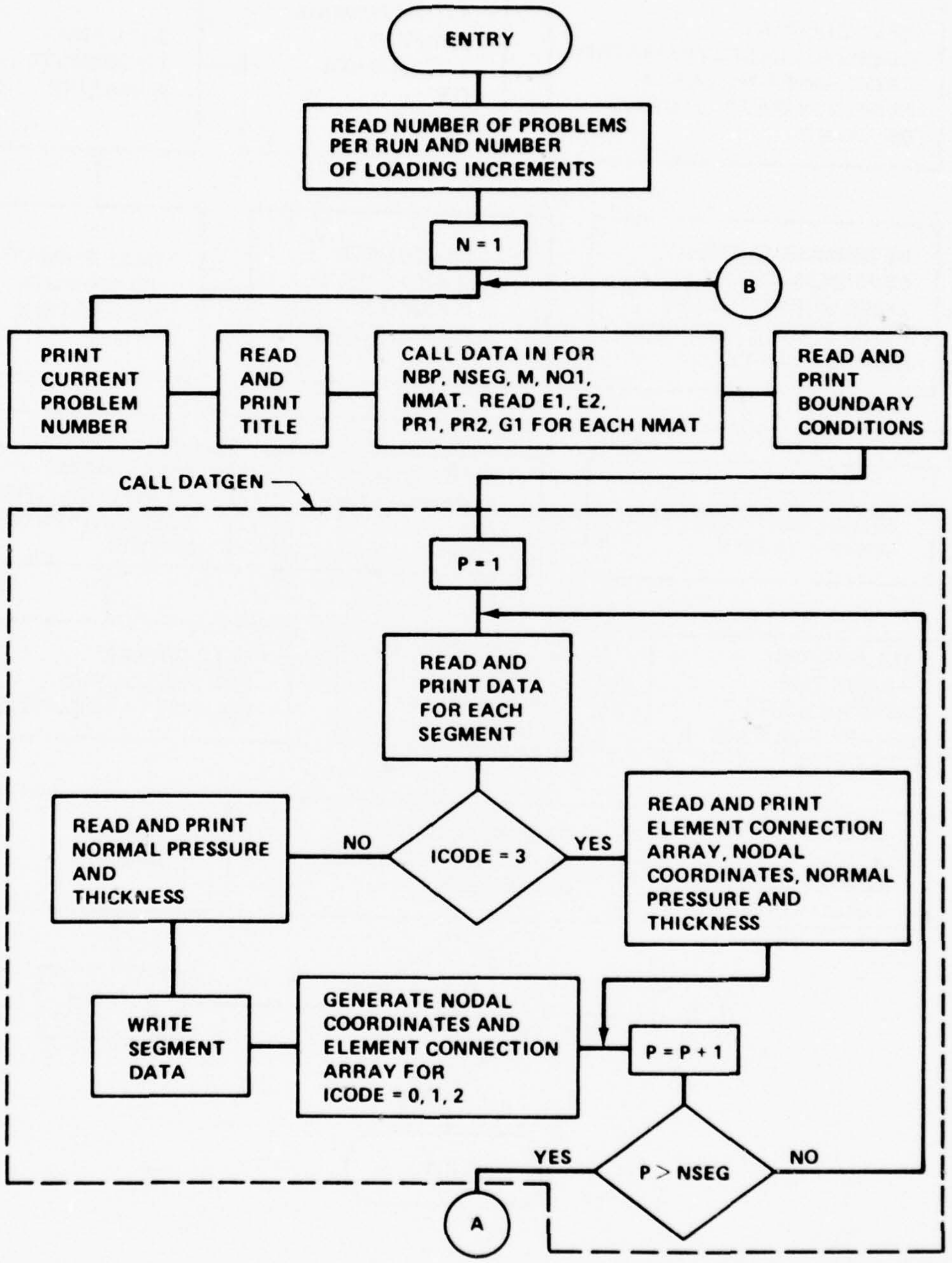


Figure 19. Flow Chart for ORTHO2.

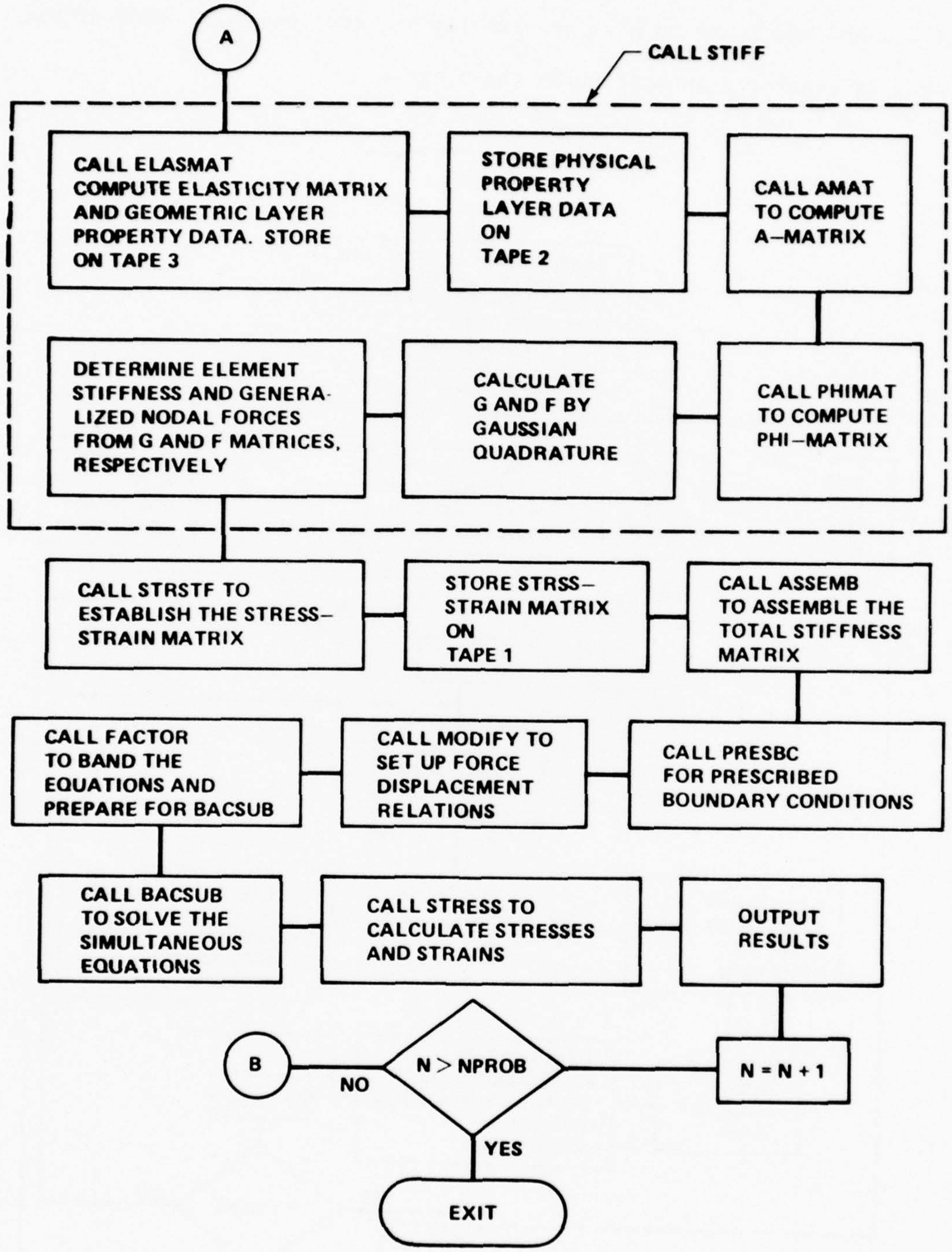


Figure 19. (Concluded).

DATA INPUT INSTRUCTIONS

1. Problem card; Load increment card (2I5):

Col.	1-5	Number of problems per run	(NPROB)
	6-10	Number of load increments	(NINCR)

2. Title card (9A8):

Col.	1-72	Title to be printed with output	(TITLE)

3. Control card (5I5):

Col.	1-5	Number of boundary points	(NBP)
	6-10	Number of segments	(NSEG)
	11-15	Number of elements	(M)
	16-20	Number of nodes	(NQ1)
	21-25	Number of materials	(NMAT)

4. Material cards (7F10.0) one for each material:

Col.	1-10*	Young's modulus in meridional direction	(E1)
	11-20*	Young's modulus in circumferential direction	(E2)
	21-30	Poisson's ratio in meridional direction	(PR1)
	31-40	Poisson's ratio in circumferential direction	(PR2)
	41-50	Enter 0. for thin shell 1. for thick shell with $G = E/2(1+\nu)$ or shear modulus for thick or sandwich shell	(G1)

5. Boundary cards (5I5, 5X, 4F10.2) one for each boundary point:

Col.	1-5	Boundary node number	(I)
	6-10	r-direction 0 free 1 fixed	(ID1)
	11-15	z-direction 0 free 1 fixed	(ID2)
	16-20	Normal rotation 0 free 1 fixed	(ID3)
	21-25	Shear rotation 0 free 1 fixed	(ID4)
	31-40	Prescribed r displacement	(UP)
	41-50	Prescribed z displacement	(WP)
	51-60	Prescribed rotation (angular displacement)	(THP)
	61-70	Skewed boundary (angle)	(AL)

6. Segment cards (1X, 2I2, I5, 5 F10.2, 4I5) one for each segment:

Col. 2-3	0 Conical segment (straight line)	(ICODE)
	1 Spherical segment	
	2 Elliptical segment	
	3 Arbitrary curved segment	
4-5	Number of layers	(NLAYER)
6-10	0 NLAYER is the same as the previous segment	(LAYID)
	1 New NLAYER for the segment New layer data are required	
11-20	R-coordinate of the first node of the segment	(R1)
21-30	Z-coordinate of the first node of the segment	(Z1)
31-40	Total length of the segment if ICODE = 0	(A1)
	Total subtend angle of the segment if ICODE = 1	
	The difference in the R-coordinate of the first and last node of the segment if ICODE = 2	
	Blank if ICODE = 3	
41-50	Angle of slope between the straight line segment and the r-axis if ICODE = 0	(A2)
	Radius of the spherical segment if ICODE = 1	
	Major radius of the elliptic segment if ICODE = 2	
	Blank if ICODE = 3	
51-60	Leave blank if ICODE = 0	(A3)
	Phase angle Φ between the normal to the shell surface and the axis of revolution if ICODE = 1 (to the first node of the segment)	
	Minor radius of the elliptic segment if ICODE = 2	
	Blank if ICODE = 3	
61-65	First element number of the segment	(M1)
66-70	Last element number of the segment	(M2)
71-75	First node number of the segment	(N1)
76-80	Last node number of the segment	(N2)

7a. Pressure loading and thickness cards (8F10.2) one for each segment except ICODE = 3. (Replace 7a by 7b and 7c when ICODE = 3):

Col.	1-10	Normal pressure at the first node of the segment	(P1)
	11-20	Normal pressure at the last node of the segment	(P2)
	21-30	Thickness of the shell or thickness of the core layer of a sandwich shell at the first node of the segment	(T1)
	31-40	Thickness of the shell or thickness of the core layer of a sandwich shell at the last node of the segment.	(T2)

Replace 7a by the following set if ICODE = 3.

7b. Element cards (5I10) one for each element:

Col.	1-10	Element number	(I)
	11-20	Node I } element connection	(J)
	21-30	Node J } element connection	(K)
	31-40	Number of layers	(MLAYER)
	41-50	Layer idenfication code	(LID)

7c. Coordinate cards (I5, 5X, 2F10.2, 2F5.1, 4F10.2) one for each node:

Col.	1-5	Node number	(N3)
	11-20	R-coordinate of the node	(R)
	21-30	Z-coordinate of the node	(Z)
	31-35	Angle between normal to the shell surface and the axis of symmetry	(PHA)
	36-40	Meridian curvature of the shell	(KAPP)
	41-50	Normal pressure at the node	(PP)
	51-60	Thickness of the shell or thickness or the core layer of a sandwich shell	(TT)

8. Concentrated load cards (2I5, F10.2)** one for each load component plus one EOD card:

Col.	1-5	Node number	(I)
	6-10	1 for r-component of the load 2 for z-component of the load 3 for moment loading	(NC1)
	11-20	Magnitude of the loading. Positive if the direction of loading coincides with the positive direction of the coordinates	(V1)

9. Layer data (I5, 5X, 3F10.4) new set of data is required if there is a change in the number of layers:

Col.	1-5	Material type number	(MT1)
	11-20	Distance from reference surface to top of layer at node I	(HI1)
	21-30	Distance from reference surface to top of layer at node J	(HJ1)
	31-40	Wrap angle	(ANGLE1)

---

*All Young's moduli must be scaled down by a factor of 10^6

**The last card of the set must contain a number greater than the total node number (End of Data Card).

A pressurized hemisphere-cylinder shell was chosen as an example for data preparation for the computer program. The shell as shown in Figure 20 is divided into four segments. There are two boundary points. With the boundary condition shown, there will be a boundary release at node No. 1 in the z-direction and at node No. 50 in the r-direction. The total subtend angles for segments 1 and 2 are 80° and 10° respectively (lines 7 and 9 of Figure 21). Note that the angle of slope between the straight line segments and the r-axis for segments 3 and 4 is -90° since the node numbers are increasing downward (see line 13 of Figure 21). A set of sample data cards for this structure is shown in Figure 21.

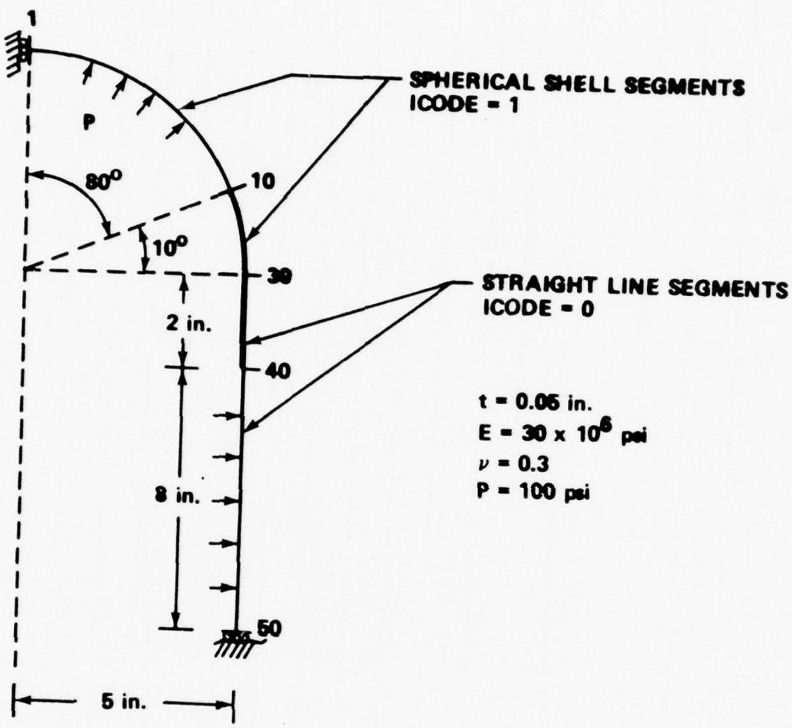


Figure 20. Hemisphere-Cylindrical Shell.

Col. No.	1	3	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80
Card 1			1	10														
2		HEMISPHERE-CYLINDER SHELL																
3			2	4	49	50	1											
4			30.		30.		.3		.3									
5			1	1	0	1	1											
6			50	0	1	1	1											
7		1	1	1	0.		15.		80.		5.		0.		1	9	1	10
8			100.		100.		.05		.05									
9		1	1						10.		5.		80.		10	29	10	30
10			100.		100.		.05		.05									
11		0	1						2.		-90.				30	39	30	40
12			100.		100.		.05		.05									
13		0	1						8.		-90.				40	49	40	50
14			100.		100.		.05		.05									
15			1000															
16			1		.025		.025											

Figure 21. Sample Data Cards.

```
      PROGRAM MAIN (INPUT=65,OUTPUT=65,TAPE5=INPUT,TAPE6=OUTPUT,TAPE1,TA
     1PE2,TAPE3)
C
C
C       VERSION 7
C   FINITE ELEMENT ANALYSIS OF AXISYMMETRIC LAMINATED SHELLS OF
C   REVOLUTION USING CURVED ELEMENT
C
C   GLOBAL COORDINATES * R-Z
C
CCCCCCC
      REAL    L,KAPPA1,KAPP,KAPPAI,KAPPAJ
      REAL    NULC,NUCL
      COMMON           EL(10),EC(10),NUCL(10),NULC(10),GL(10)
     *,FSAVE(1000),XK(250),F1(1000)
        COMMON        CSPHI,SNPHI,CSPHJ,SNPHJ,KAPPAI,KAPPAJ,SNPSI,CSPSI
     1    ,SNBTI,SNBTJ,CSBTI,CSBTJ
      COMMON         S(8,8),GE(8),AA(8,8),G(8,8),GAA(8,8)
     1,LL1(1000),R(250),Z(250),L(1000, 8),F(1000)
     2,J(250),K(250), AL(250),PHA(250),KAPP(250)
     3 ,IPP(250),C(5,5),SS(10,8),SE(10,8)
     4,MLAYER(250),PP(250),TT(250)
     5,LID(250),LP(1000)
      COMMON ISF3(35)
      DIMENSION TITLE(9),L(4)
      READ(5,10) NPROB,NINCR
   10 FORMAT(2I5)
      DO 930 NPR=1,NPROB
      WRITE(6,8801)NPR
 8801 FORMAT (1H0* PROBLEM NO. *,I5,//)
      READ(5,1111) TITLE
      WRITE(6,1112) TITLE
 1111 FORMAT(9A8)
 1112 FORMAT(1X,9A8,//,5X,*INPUT DATA**,//)
      FNINCR=NINCR
      FAC=1./FNINCR
      REWIND 3
      MAXDIF=0
      PI=3.141592653529
      R(1)=0.0
```

```
      Z(1)=C.C
      CALL DATAIN(NBP,NSEG,M,NQ1)
      N=4*NQ1
      DO 18C II=1,N
      LL1(II)=C
      DP(II)=C.
      F1(II)=C.
      FSAVE(II)=C.
  180 F(II)=C.
      DO 181 KK=1,NQ1
      XK(KK)=C.
  181  AL(KK)=C.
      DO 182 LL=1,NBP
      READ(5,196) I,ID1,ID2,ID3,ID4,UP,WP,THP, AL(I)
      WRITE(6,196) I,ID1,ID2,ID3,ID4,UP,WP,THP, AL(I)
      II=4*(I-1)
      LL1(II+1)=ID1
      LL1(II+2)=ID2
      LL1(II+3)=ID3
      LL1(II+4)=ID4
      DP(II+1)=UP
      DP(II+2)=WP
      DP(II+3)=THP
  182 CONTINUE
  196 FORMAT(5I5,5X,4F1C.2)
      CALL DATGEN(NSEG,M)
      DO 5C III=1,M
      MMM=IABS(K(III)-J(III))
      IF(MMM.GT.MAXDIF) MAXDIF=MMM
   50 CONTINUE
      NW=4*(MAXDIF+1)-1
      NN=NW+1
C
C
C CONCENTRATED LOADINGS
C I,NODE NO.,NC1,DIRECTION NC.,  1 FOR R-DIR. AND 2 FOR Z-DIR.
C V1. VALUE OF LOADING, POSITIVE IF COINCIDE WITH COOR. DIRECTION
C LAST DATA CARD MUST HAVE A NUMBER GREATER THAN THE NUMBER OF NODES
C
C
```

```
      WRITE(6,9C19)                                                     ORTHO 82
 9C19 FORMAT(//,* NODAL FORCES *,/,* NCDE  CIR. NO.        LOAD*)       ORTHO 83
  552 READ(5,553)I,NC1,V1                                               ORTHO 84
      WRITE(6,553) I,NC1, V1                                            ORTHO 85
  553 FORMAT(2I5,F1C.2)                                                 ORTHO 86
      IF(I.GT.NQ1)GO TC 56C                                             ORTHO 87
      K1=4*I-4+NC1                                                      ORTHO 88
      F1(K1)=F1(K1)+V1*2.*PI *R(I)                                      ORTHO 89
      IF(R(I).EQ.C.C) F1(K1)=F1(K1)+V1                                  ORTHO 90
      GO TO 552                                                         ORTHO 91
  560 CONTINUE                                                          ORTHO 92
      WRITE(6,38)                                                       ORTHO 93
   38 FORMAT(/,1X,*ELEMENT NUMBER     J          K        MLAYER*)      ORTHO 94
      DO 3CC1 N1=1,M                                                    ORTHO 95
      WRITE(6,361) N1,J(N1),K(N1),MLAYER(N1)                            ORTHO 96
 3001 CONTINUE                                                          ORTHO 97
  361 FORMAT(5I1C)                                                      ORTHO 98
      WRITE(6,39)                                                       ORTHO 99
   39 FORMAT(/,1CX,*ELEMENT LAYER MATERIAL    HI          HJ          ANGLE*) ORTHO1C0
      DO 2CCC INCR=1,NINCR                                              ORTHO1C1
      DO 13C II=1,N                                                     ORTHO1C2
      F(II)=C.                                                          ORTHO1C3
      DO 13C JJ=1,NN                                                    ORTHO1C4
  130 U(II,JJ)=C.                                                       ORTHO1C5
      REWIND 1                                                          ORTHO1C6
      REWIND 2                                                          ORTHO1C7
      DO 43C II=1,M                                                     ORTHO1C8
      I=II                                                              ORTHO1C9
      TI=TT(J(I))                                                       ORTHO110
      TJ=TT(K(I))                                                       ORTHO111
      RI=R(J(I))                                                        ORTHO112
      RJ=R(K(I))                                                        ORTHO113
      ZI=Z(J(I))                                                        ORTHO114
      ZJ=Z(K(I))                                                        ORTHO115
      A=ZI-ZJ                                                           ORTHO116
      B =RJ-RI                                                          ORTHO117
      L=C.                                                              ORTHO118
      T3=A**2+B**2                                                      ORTHO119
      IF(T3.GT.C.) L=SQRT(T3)                                           ORTHO120
      IF(RI.EQ.C)RI=1./1C.**16                                          ORTHO121
```

BEST AVAILABLE COPY

```
      IF(RJ.EQ.C)RJ=1./1C.**16                                           ORTHO122
      FNJ=XK(I1)                                                         ORTHO123
      CALL  STIFF(I1,PI,RI,A,B ,L,ZI,RJ,ZJ,INCR,FNJ)                     ORTHO124
      CALL  STRSTF(L,RI,RJ,TI,TJ)                                        ORTHO125
      WRITE(1)I,S,SS,RI,RJ,T,SE                                          ORTHO126
      CALL  ASSEMB(I1)                                                   ORTHO127
  430 CONTINUE                                                           ORTHO128
      DO 44C I2=1,N                                                      ORTHO129
  440 F(I2)=(F(I2)+F1(I2))*FAC                                           ORTHO130
      CALL  PRESBC(N,NN,DP,L,F,LL1)                                      ORTHO131
      DO 1CCC I1=1,NQ1                                                   ORTHO132
      IF( AL(I1).EQ.C.) GC TC 1CCO                                       ORTHO133
      K4=4*(I1-1)+1                                                      ORTHO134
      AL1= AL(I1)*PI/18C.                                                ORTHO135
      ALF = TAN(AL1)                                                     ORTHO136
      CALL  MODIFY(F,L,NN,K4,ALF)                                        ORTHO137
 1000 CONTINUE                                                           ORTHO138
      CALL  FACTOR(N,NW,L,LL1)                                           ORTHO139
      CALL  BACSUB(N,NW,L,LL1,F)                                         ORTHO140
      DO 81C I1=1,N                                                      ORTHO141
  810 FSAVE(I1)=FSAVE(I1)+F(I1)                                          ORTHO142
      CALL  STRESS(M,INCR,NINCR)                                         ORTHO143
      IF(INCR.EQ.NINCR) WRITE(6,800)                                     ORTHO144
  800 FORMAT(///,5CX,*NODAL DISPLACEMENT*,//,                            ORTHO145
     1* NODE*,12X,*R-DIRECTICN*,7X,*Z-DIRECTICN*,7X,                     ORTHO146
     2*R UPDATED     *,7X,*Z UPDATED*,5X,                                ORTHO147
     2*NORMAL ROTATION*,5X,*SHEAR ROTATICN*/)                            ORTHO148
      WRITE(6,9CC) INCR                                                  ORTHO149
  900 FORMAT(1X,7HINCR = ,I3)                                            ORTHO150
       DO 927 I1=1,NQ1                                                   ORTHO151
      K1=4*I1-4                                                          ORTHO152
      DO 928 JJ=1,4                                                      ORTHO153
  928 Q(JJ)=FSAVE(K1+JJ)                                                 ORTHO154
      UC=PP(I1)                                                          ORTHO155
      WC=TT(I1)                                                          ORTHO156
      R(I1)=R(I1)+F(K1+1)                                                ORTHO157
      Z(I1)=Z(I1)+F(K1+2)                                                ORTHO158
      PHA(I1)=PHA(I1)-F(K1+3)*180./PI                                    ORTHO159
       IF(INCR.EQ.NINCR)                                                 ORTHO160
     1WRITE(6,929) I1,Q(1),Q(2),R(I1),Z(I1),Q(3),Q(4)                    ORTHO161
```

```
  927 CONTINUE                                                                              ORTH0162
 2000 CONTINUE                                                                              ORTH0163
  929 FORMAT(I5,8X,1P6E16.7)                                                                ORTH0164
  930 CONTINUE                                                                              ORTH0165
 9999 CALL EXIT                                                                             ORTH0166
      END                                                                                   ORTH0167
      SUBROUTINE DATAIN(NBP,NSEG,M,NQ1)                                                     ORTH0168
      REAL    NULC,NUCL                                                                     ORTH0169
      COMMON              EL(10),EC(10),NUCL(10),NULC(10),GL(10)                            ORTH0170
     *,FSAVE(1000),XK(250),F1(1000)                                                         ORTH0171
C                                                                                           ORTH0172
C                                                                                           ORTH0173
C NBP, NUMBER OF TOTAL BOUNDARY POINTS                                                      ORTH0174
C NSEG, NUMBER OF TOTAL SEGMENTS                                                            ORTH0175
C M,NUMBER OF TOTAL ELEMENTS                                                                ORTH0176
C NQ1, NUMBER OF TOTAL NODES                                                                ORTH0177
C NMAT, NUMBER OF TOTAL MATERIALS                                                           ORTH0178
C                                                                                           ORTH0179
C                                                                                           ORTH0180
      READ(5,10) NBP,NSEG,M,NQ1,NMAT                                                        ORTH0181
   10 FORMAT(5I5)                                                                           ORTH0182
      WRITE(6,11) NBP,NSEG,M,NQ1,NMAT                                                       ORTH0183
   11 FORMAT(1X,*NUMBER OF BOUNDARY POINTS=*  I5,/,                                         ORTH0184
     1          *NUMBER OF SEGMENTS          =* I5,//,                                      ORTH0185
     2          *NUMBER OF ELEMENTS          =*I5,//,                                       ORTH0186
     3          *NUMBER OF NODES             =*I5,//,                                       ORTH0187
     4          *NUMBER OF MATERIALS         =*I5,///)                                      ORTH0188
C                                                                                           ORTH0189
C                                                                                           ORTH0190
C ONE SET OF MATERIAL DATA FOR EACH MATERIAL                                                ORTH0191
C E1,E2 YOUNG-S MODULUS                                                                     ORTH0192
C  PR1,PR2    POISSON*S RATIOS                                                              ORTH0193
C G1,SHEAR MODULUS                                                                          ORTH0194
C G1=1,SHEAR MODULUS IS SET EQUAL TO E/2(1+V)                                               ORTH0195
C G1=0.,TRANSVERSE SHEAR EFFECT IS SUPPRESSED                                               ORTH0196
C                                                                                           ORTH0197
C                                                                                           ORTH0198
      WRITE(6,12)                                                                           ORTH0199
   12 FORMAT(1X,*MATERIAL    EL            EC            NUCL          NULC                 ORTH0200
     1         GL *)                                                                        ORTH0201
```

```
      DO 30 III=1,NMAT                                                  ORTHO202
      READ(5,21) E1,E2,PR1,PR2,G1                                       ORTHO203
   21 FORMAT(7F10.0)                                                    ORTHO204
      E1=E1*10.**6                                                      ORTHO205
      E2=E2*10.**6                                                      ORTHO206
      IF(G1.EQ.1.0) G1=E1/(2.*(1.+PR1))                                 ORTHO207
      WRITE(6,22) III,E1,E2,PR1,PR2,G1                                  ORTHO208
   22 FORMAT(1X,I4,7E14.3)                                              ORTHO209
      IF(G1.EQ.0.0) G1=10.**50                                          ORTHO210
      EL(III)=E1                                                        ORTHO211
      EC(III)=E2                                                        ORTHO212
      NULC(III)=PR1                                                     ORTHO213
      NUCL(III)=PR2                                                     ORTHO214
      GL(III)=G1                                                        ORTHO215
   30 CONTINUE                                                          ORTHO216
      IF(G1.GT.10.**30) WRITE(6,20)                                     ORTHO217
   20 FORMAT(1X,//,* TRANVERSE SHEAR EFFECT IS NEGLECTED FOR THIS PROB.*ORTHO218
     1,//)                                                              ORTHO219
      RETURN                                                            ORTHO220
      END                                                               ORTHO221
      SUBROUTINE DATGEN(NSEG,M)                                         ORTHO222
      REAL    KAPP                                                      ORTHO223
      COMMON ISF11 (2300)                                               ORTHO224
      COMMON ISFC(12)                                                   ORTHO225
      COMMON       S(8,8),GE(8),AA(8,8),G(8,8),GAA(8,8)                 ORTHO226
     1,LL1(1000),R(250),Z(250),L(1000, 8),F(1000)                       ORTHO227
     2,J(250),K(250), AL(250),PHA(250),KAPP(250)                        ORTHO228
     3 ,IPP(250),C(5,5),SS(10,8),SE(10,8)                               ORTHO229
     4,MLAYER(250),PP(250),TT(250)                                      ORTHO230
     5,LID(250),DP(1000)                                                ORTHO231
      PI=3.141592653529                                                 ORTHO232
      DO 2001 I1=1,M                                                    ORTHO233
 2001 LID(I1)=0                                                         ORTHO234
C                                                                       ORTHO235
C                                                                       ORTHO236
C  ICODE=0, STRAIGHT LINE SEGMENT,A1=TOTAL LENGHTH, A2=SLOPE ANGLE      ORTHO237
C  ICODE=1, CIRCULAR SEGMENT, A1=TOTAL SUBTEND ANGLE, A2=RADIUS, A3=PHASEORTHO238
C           ANGLE.                                                      ORTHO239
C  ICODE=2 ELLIPTIC SEGMENT  A1=HORIZONTAL PROJECTION OF THE SEGMENT    ORTHO240
C  LENGTH, A2=MAJOR AXIS, A2=MINOR AXIS                                 ORTHO241
```

```
C ICODE=3 , ARBITRARY CURVED SEGMENT                                   ORTHO242
C NLAYER= NUMBER OF LAYERS                                             ORTHO243
C LAYID=0 NLAYER IS THE SAME AS THE PREVIOUS SEGMENT                   ORTHO244
C LAYID=1 NEW NLAYER FOR THE SEGMENT  NEW LAYER DATA IS REQUIRED       ORTHO245
C R1,Z1, COORDINATES OF THE FIRST NODE OF THE FIRST SEGMENT            ORTHO246
C N1,N2, THE FIRST AND LAST NODE NO. OF SEGMENT                        ORTHO247
C M1,M2  THE FIRST AND LAST  ELEMENT NO. OF SEGMENT                    ORTHO248
C                                                                      ORTHO249
C                                                                      ORTHO250
      DO 1260 IJ=1,NSEG                                                ORTHO251
      READ(5,190) ICODE,NLAYER,LAYID,R1,Z1,A1,A2,A3,M1,M2,N1,N2        ORTHO252
      WRITE(6,100)ICODE,NLAYER,LAYID,R1,Z1,A1,A2,A3,M1,M2,N1,N2        ORTHO253
  190 FORMAT(1X,2I2,I5,5F10.2,4I5)                                     ORTHO254
  100 FORMAT(//,3I5,5F10.2,4I5)                                        ORTHO255
      IF(R1.EQ.0.0.AND.Z1.EQ.0.0) GO TO 4000                           ORTHO256
      GO TO 4001                                                       ORTHO257
 4000 CONTINUE                                                         ORTHO258
      R1 = R(N1)                                                       ORTHO259
      Z1 = Z(N1)                                                       ORTHO260
 4001 CONTINUE                                                         ORTHO261
      IP=ICODE + 1                                                     ORTHO262
      IF(IP.NE.4) GO TO 371                                            ORTHO263
      WRITE(6,31) IJ                                                   ORTHO264
      DO 373 IK=M1,M2                                                  ORTHO265
      IPP(IK)=IP                                                       ORTHO266
      READ(5,361) I,J(I),K(I),MLAYER(I),LID(I)                         ORTHO267
  373 CONTINUE                                                         ORTHO268
  361 FORMAT(5I10)                                                     ORTHO269
      MEL=M2-M1+1                                                      ORTHO270
      NP1=MEL+1                                                        ORTHO271
      DO 80 IJK=1,NP1                                                  ORTHO272
      READ(5,194)N3,R(N3),Z(N3),PHA(N3),KAPP(N3),PP(N3),TT(N3)         ORTHO273
      WRITE(6,199) N3,R(N3),Z(N3),AL(N3),PP(N3),TT(N3),PHA(N3)         ORTHO274
   80 CONTINUE                                                         ORTHO275
  194 FORMAT(    I5,5X,2F10.2,2F5.1,4F10.2)                            ORTHO276
      GO TO 1260                                                       ORTHO277
C                                                                      ORTHO278
C                                                                      ORTHO279
C P1=NORMAL PRESSURE AT NODE I                                         ORTHO280
CP2==NORMAL PRESSURE AT NODE J                                         ORTHO281
```

```
C T1=TOTAL  THICKNESS AT NCDE I                                        ORTHO282
C T2=TOTAL  THICKNESS AT NCDE J                                        ORTHO283
C                                                                      ORTHO284
C                                                                      ORTHO285
  371 CONTINUE                                                         ORTHO286
      READ(5,19 ) P1,P2,T1,T2                                          ORTHO287
      WRITE(6,19) P1,P2,T1,T2                                          ORTHO288
   19 FORMAT(8F10.2)                                                   ORTHO289
      DELP=(P2-P1)/(N2-N1)                                             ORTHO290
      DELT=(T2-T1)/(N2-N1)                                             ORTHO291
      DELX=A1/(N2-N1)                                                  ORTHO292
      SLOPE=A2*PI/180.                                                 ORTHO293
      WRITE(6,31) IJ                                                   ORTHO294
   31 FORMAT(/,1X,* SEGMENT NO.*I5,/,* NODE          P          Z      ORTHO295
     1   AL          PP          TT          PHA*)                     ORTHO296
      DO 375 IK=M1,M2                                                  ORTHO297
      IPP(IK)=IP                                                       ORTHO298
      I=IK                                                             ORTHO299
      J(I)=IK                                                          ORTHO300
      K(I)=IK+1                                                        ORTHO301
      MLAYER(I)=NLAYER                                                 ORTHO302
      IF(IK.EQ.M1) LID(I)=LAYID                                        ORTHO303
  375 CONTINUE                                                         ORTHO304
  376 CONTINUE                                                         ORTHO305
      DO 260 I1=N1,N2                                                  ORTHO306
      IF(IJ.NE.1.AND.I1.EQ.N1) GO TO 192                               ORTHO307
      I=I1-N1+1                                                        ORTHO308
      PP(I1)=P1+DELP*(I-1)                                             ORTHO309
      TT(I1)=T1+DELT*(I-1)                                             ORTHO310
      GO TO (263,261,262,260),IP                                       ORTHO311
  263 CONTINUE                                                         ORTHO312
      DR=DELX*COS(SLOPE)*(I-1)                                         ORTHO313
      DZ=DELX*SIN(SLOPE)*(I-1)                                         ORTHO314
      R(I1)=R1+DR                                                      ORTHO315
      Z(I1)=Z1+DZ                                                      ORTHO316
      PHA(I1)=-A2                                                      ORTHO317
      KAPP(I1)=0.                                                      ORTHO318
      GO TO 192                                                        ORTHO319
  261 CONTINUE                                                         ORTHO320
      RAD=A2                                                           ORTHO321
```

```
      ANG1=A3*PI/18C.
      ANG=DELX*(PI/18C.)*(I-1)+A3*PI/180.
C     DZ=RAD*SIN(ANG)-RAD*SIN(ANG1)
C     DR=RAD*COS(ANG)-RAD*COS(ANG1)
      DR=RAD*SIN(ANG)-RAD*SIN(ANG1)
      DZ=RAD*COS(ANG)-RAD*COS(ANG1)
      R(I1)=R1+DR
      Z(I1)=Z1+DZ
      PHA(I1) = A3+DELX*(I-1)
      KAPP(I1)=1./RAD
      GO TO 192
  262 CONTINUE
      DR=DELX*(I-1)
      R(I1)=R1+DR
      Z(I1)=((A2*A2*A3*A3-A3*A3*R(I1)*R(I1))/(A2*A2))**0.5
      RR=(A2**4*R(I1)**2+A3**4*Z(I1)**2)**0.5
      RR1=RR**3/(A2**4*A3**4)
      RR2=RR/(A3**2)
      SNPHI=R(I1)/RR2
      PHA(I1)=ASIN(SNPHI)*18C./PI
      KAPP(I1)=1./RR1
  192 WRITE(6,199) I1,R(I1),Z(I1),AL(I1),PP(I1),TT(I1),PHA(I1)
  199 FORMAT(1X,I4,5X,7F1C.3)
  260 CONTINUE
 1260 CONTINUE
      RETURN
      END
      SUBROUTINE ASSEMB(I)
      REAL    KAPP
      COMMON ISF11 (23CC)
      COMMON ISFC(12)
      COMMON      S(8,8),GE(8),AA(8,8),G(8,8),GAA(8,8)
     1,LL1(1CCC),R(250),Z(250),L(1000, 8),F(1000)
     2,J(25C),K(25C), AL(250),PHA(250),KAPF(250)
     3 ,IPP(25C),C(5,5),SS(10,8),SE(10,8)
     4,MLAYER(25C),PP(25C),TT(250)
     5,LID(25C),DP(1CCC)
      DIMENSION KK(8)
      KK(4)=4.*J(I)
      KK(8)=4.*K(I)
```

```
      KK(3)=KK(4)-1                                                     ORTH0362
      KK(2)=KK(3)-1                                                     ORTH0363
      KK(1)=KK(2)-1                                                     ORTH0364
      KK(7)=KK(8)-1                                                     ORTH0365
      KK(6)=KK(7)-1                                                     ORTH0366
      KK(5)=KK(6)-1                                                     ORTH0367
      DO 400 M=1,8                                                      ORTH0368
      II=KK(M)                                                          ORTH0369
      DO 400 N=1,8                                                      ORTH0370
      IF(KK(N).LT.II) GO TO 400                                         ORTH0371
      JJ=KK(N)-II+1                                                     ORTH0372
      U(II,JJ)=U(II,JJ)+S(M,N)                                          ORTH0373
  400 CONTINUE                                                          ORTH0374
      RETURN                                                            ORTH0375
      END                                                               ORTH0376
      SUBROUTINE MODIFY(F,U,NN,K4,ALF)                                  ORTH0377
      DIMENSION F(1)    ,U(1000,8)                                      ORTH0378
      F(K4)=F(K4)+ALF*F(K4+1)                                           ORTH0379
      F(K4+1)=0.                                                        ORTH0380
      U(K4,1)=U(K4,1)+ALF*(ALF*(U(K4+1,1)+1.)+2.*U(K4,2))               ORTH0381
      U(K4,2)=-ALF                                                      ORTH0382
      U(K4+1,1)=1.                                                      ORTH0383
      DO 29 N1=3,NN                                                     ORTH0384
   29 U(K4,N1)=U(K4,N1)+ALF*U(K4+1,N1-1)                                ORTH0385
      DO 28 N1=2,NN                                                     ORTH0386
   28 U(K4+1,N1   )=0.                                                  ORTH0387
      K5=K4-1                                                           ORTH0388
      DO 34 N1=1,K5                                                     ORTH0389
      N11=N1+1                                                          ORTH0390
      N12=N1+2                                                          ORTH0391
      IF (N11.GT.NN) GO TO 34                                           ORTH0392
      IF (N12.GT.NN) GO TO 34                                           ORTH0393
      U(K4-N1,N1+1)=U(K4-N1,N1+1)+ALF*U(K4-N1,N1+2)                     ORTH0394
      U(K4-N1,N1+2)=0.                                                  ORTH0395
   34 CONTINUE                                                          ORTH0396
      RETURN                                                            ORTH0397
      END                                                               ORTH0398
      SUBROUTINE PRESBC(N,NN,DP,U,F,LL1)                                ORTH0399
      DIMENSION DP(1),F(1),U(1000,8),LL1(1)                             ORTH0400
      DO 200 N1=1,N                                                     ORTH0401
```

```
      IF(LL1(N1).NE.1) GC TO 200                                        ORTHO4C2
      DO 1CC M1=2,NN                                                    ORTHO4C3
      K1=N1-M1+1                                                        ORTHO4C4
      IF(K1.LE.C) GO TC 5C                                              ORTHO4C5
      F(K1)=F(K1)-L(K1,M1)*DP(N1)                                       ORTHO4C6
      U(K1,M1)=C.                                                       ORTHO4C7
   50 K1=N1+M1-1                                                        ORTHO408
      IF(K1.GT.N) GC TC 1CC                                             ORTHO4C9
      F(K1)=F(K1)-L(N1,M1)*DP(N1)                                       ORTHO410
      U(N1,M1)=C.                                                       ORTHO411
  10C CONTINUE                                                          ORTHO412
      U(N1,1)=1.                                                        ORTHO413
      F(N1)=DP(N1)                                                      ORTHO414
  200 CONTINUE                                                          ORTHO415
      RETURN                                                            ORTHO416
      END                                                               ORTHO417
      SUBROUTINE FACTOR(N,NBAND,L,LL1)                                  ORTHO418
      DIMENSION L(1CCC,8),LL1(1)                                        ORTHO419
      DO 16C I=1,N                                                      ORTHO420
      IF(LL1(I).EQ.1) GC TC 16C                                         ORTHO421
      S1=0.                                                             ORTHO422
      NW1= MAXC(1,I-NBAND)                                              ORTHO423
      KK=I-1                                                            ORTHO424
      IF(KK.LT.NW1) GO TO 51                                            ORTHO425
      DO 5C K=NW1,KK                                                    ORTHO426
C     IF(LL1(K).EQ.1) GO TC 5C                                          ORTHO427
   40 S1=S1+L(K,I-K+1)**2*L(K,1)                                        ORTHO428
   50 CONTINUE                                                          ORTHO429
   51 U(I,1)=U(I,1)-S1                                                  ORTHO430
      NW2= MINC(N,I+NBAND)                                              ORTHO431
      JJ=I+1                                                            ORTHO432
      IF(JJ.GT.NW2)GO TO 160                                            ORTHO433
      DO 15C J=JJ,NW2                                                   ORTHO434
C     IF(LL1(J).EQ.1) GC TC 15C                                         ORTHO435
   90 S2=0.                                                             ORTHO436
      NW1= MAXC(1,J-NBAND)                                              ORTHO437
      K1=I-1                                                            ORTHO438
      IF(K1.LT.NW1) GO TC 141                                           ORTHO439
      DO 14C K=NW1,K1                                                   ORTHO440
C     IF(LL1(K).EQ.1) GC TC 140                                         ORTHO441
```

```
  130 S2=S2+U(K,I-K+1)*U(K,1)*U(K,J-K+1)                              ORTHO442
  140 CONTINUE                                                          ORTHO443
  141 U(I,J-I+1)=(U(I,J-I+1)-S2)/U(I,1)                                 ORTHO444
  150 CONTINUE                                                          ORTHO445
  160 CONTINUE                                                          ORTHO446
      RETURN                                                            ORTHO447
      END                                                               ORTHO448
      SUBROUTINE BACSUB(N,NBAND,U,LL1,F)                                ORTHO449
      DIMENSION U(1000,8),LL1(1),F(1)                                   ORTHO450
      DO 60 I=1,N                                                       ORTHO451
      IF(LL1(I).EQ.1) GO TO 60                                          ORTHO452
      S1=0.                                                             ORTHO453
      NW1= MAX0(1,I-NBAND)                                              ORTHO454
      KK=I-1                                                            ORTHO455
      IF(KK.LT.NW1)GO TO 51                                             ORTHO456
      DO 50 K1=NW1,KK                                                   ORTHO457
C     IF(LL1(K1).EQ.1) GO TO 50                                         ORTHO458
   40 S1=S1+U(K1,I-K1+1)*F(K1)                                          ORTHO459
   50 CONTINUE                                                          ORTHO460
   51 F(I)=F(I)-S1                                                      ORTHO461
   60 CONTINUE                                                          ORTHO462
C     BACKWARD SUBSTITUTION                                             ORTHO463
      DO 110 I1=1,N                                                     ORTHO464
      I=N+1-I1                                                          ORTHO465
C     IF(LL1(I).EQ.1) GO TO 110                                         ORTHO466
      S1=0.                                                             ORTHO467
      NW2= MIN0(N,I+NBAND)                                              ORTHO468
      J1=I+1                                                            ORTHO469
      IF(J1.GT.NW2)GO TO 101                                            ORTHO470
      DO 100 JJ= J1,NW2                                                 ORTHO471
C     IF(LL1(JJ).EQ.1) GO TO 100                                        ORTHO472
      S1=S1+U(I,1)*U(I,JJ-I+1)*F(JJ)                                    ORTHO473
  100 CONTINUE                                                          ORTHO474
  101 CONTINUE                                                          ORTHO475
      F(I)=(F(I)-S1)/U(I,1)                                             ORTHO476
  110 CONTINUE                                                          ORTHO477
      RETURN                                                            ORTHO478
      END                                                               ORTHO479
      SUBROUTINE STIFF(I,PI,RI,A,B,L,ZI,RJ,ZJ,INCR,FNJ)                 ORTHO480
      REAL   L,KAPPAI,KAPP,KAPPAI,KAPPAJ                                ORTHO481
```

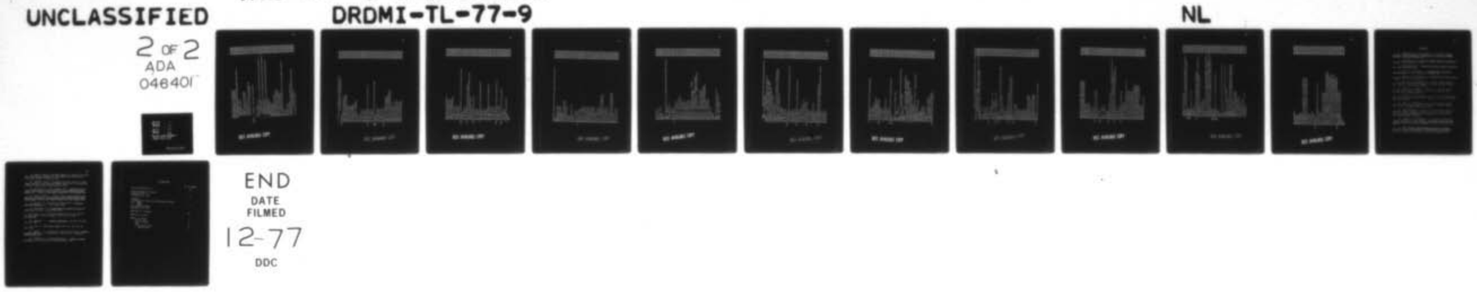
```
      REAL    NULC,NUCL                                                 ORTH0482
      DIMENSION MT1(10),HI1(10),HJ1(10),ANGLE1(10)                      ORTH0483
      COMMON           EL(10),EC(10),NUCL(10),NULC(10),GL(10)           ORTH0484
     *,FSAVE(1000),XK(250),F1(1000)                                     ORTH0485
        COMMON      CSPHI,SNPHI,CSPHJ,SNPHJ,KAPPAI,KAPPAJ,SNPSI,CSPSI   ORTH0486
     1    ,SNBTI,SNBTJ,CSBTI,CSBTJ                                      ORTH0487
      COMMON        S(8,8),GE(8),AA(8,8),G(8,8),GAA(8,8)                ORTH0488
     1,LL1(1000),R(250),Z(250),U(1000, 8),F(1000)                       ORTH0489
     2,J(250),K(250),ALP(250),PHA(250),KAPF(250)                        ORTH0490
     3 ,IPP(250),C(5,5),SS(10,8),SE(10,8)                               ORTH0491
     4,MLAYER(250),PP(250),TT(250)                                      ORTH0492
     5,LID(250),DP(1000)                                                ORTH0493
      COMMON /SET3/ AU(2,2),AL(2,2),DSTARU(2,2),DSTARL(2,2),            ORTH0494
     1DSTAR1(2,2),DU(2,2),DL(2,2),D1(2,2),D2(2,2),GBARU,GBARL           ORTH0495
      DIMENSION XI(2),AI(2),PHI(5,8),          DD(5,8),FP(8),FPA(8)     ORTH0496
     1,PHIPI(8,8),PHIPA(8,8),SI(8,8)                                    ORTH0497
      DATA XI/-.577302692,.577302692/                                   ORTH0498
      DATA AI/1.,1./                                                    ORTH0499
C*****  INITIALIZATION                                                  ORTH0500
        DO 1 N1=1,8                                                     ORTH0501
      FP(N1)=0.                                                         ORTH0502
      FPA(N1)=0.                                                        ORTH0503
      DO 1 N2=1,8                                                       ORTH0504
      G(N1,N2)=0.                                                       ORTH0505
    1 GAA(N1,N2)=0.                                                     ORTH0506
      DO 2 N1=1,5                                                       ORTH0507
      DO 2 N2=1,8                                                       ORTH0508
    2 DD(N1,N2)=0.                                                      ORTH0509
      DO 3 N1=1,5                                                       ORTH0510
      DO 3 N3=1,5                                                       ORTH0511
    3  C(N1,N3)=0.                                                      ORTH0512
      DELP=PP(K(I))-PP(J(I))                                            ORTH0513
      P=PP(J(I))                                                        ORTH0514
      TI=TT(J(I))                                                       ORTH0515
      TJ=TT(K(I))                                                       ORTH0516
      T=(TI+TJ)/2.                                                      ORTH0517
     KP1=J(I)                                                           ORTH0518
     K1=4*KP1-4                                                         ORTH0519
        SNPSI=B/L                                                       ORTH0520
        CSPSI=A/L                                                       ORTH0521
```

```
      IF(INCR.NE.1) GO TC 613                                           ORTHO522
      IP=IPP(I)                                                         ORTHO523
        GO TO (2C1C,3C1C,3C1C),IF                                       ORTHO524
 2010   CSPHI=SNPSI                                                     ORTHO525
        SNPHI=CSPSI                                                     ORTHO526
        CSPHJ=SNPSI                                                     ORTHO527
        SNPHJ=CSPSI                                                     ORTHO528
        KAPPAI=C.                                                       ORTHO529
        KAPPAJ=C.                                                       ORTHO530
        GO TO 41CC                                                      ORTHO531
 3C10 CONTINUE                                                          ORTHO532
      PHII=PHA(J(I))*PI/18C.                                            ORTHO533
      PHIJ=PHA(K(I))*PI/18C.                                            ORTHO534
      CSPHI=COS(PHII)                                                   ORTHO535
      SNPHI=SIN(PHII)                                                   ORTHO536
      CSPHJ= COS(PHIJ)                                                  ORTHO537
      SNPHJ=SIN(PHIJ)                                                   ORTHO538
        KAPPAI=KAPP(J(I))                                               ORTHO539
          KAPPAJ=KAPP(K(I))                                             ORTHO540
 41C0 CONTINUE                                                          ORTHO541
      NLAYER=MLAYER(I)                                                  ORTHO542
      LID1=LID(I)                                                       ORTHO543
      HNI=-T/2.                                                         ORTHO544
      HNJ=-T/2.                                                         ORTHO545
      DO 6C5 N1=1,2                                                     ORTHO546
      DO 6C5 N2=1,2                                                     ORTHO547
      AO(N1,N2)=C.                                                      ORTHO548
      AL(N1,N2)=C.                                                      ORTHO549
      DSTARO(N1,N2)=C.                                                  ORTHO550
      DSTARL(N1,N2)=C.                                                  ORTHO551
      DSTAR1(N1,N2)=C.                                                  ORTHO552
      DO(N1,N2)=C.                                                      ORTHO553
      DL(N1,N2)=C.                                                      ORTHO554
      D1(N1,N2)=C.                                                      ORTHO555
      D2(N1,N2)=C.                                                      ORTHO556
  605 CONTINUE                                                          ORTHO557
      GBARU=C.                                                          ORTHO558
      GBARL=C.                                                          ORTHO559
C                                                                       ORTHO560
C                                                                       ORTHO561
```

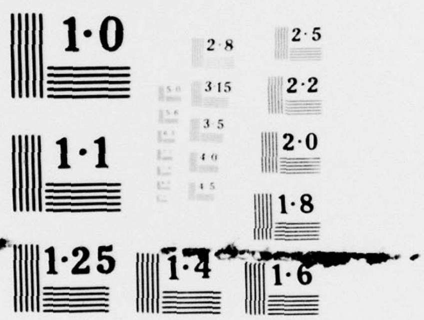
```
C LAYER DATA                                                              ORTH0562
C NEW SET OF DATA IS REQUIRED IF THERE IS A CHANGE IN NUMBER OF LAYER     ORTH0563
C MT1=MATERIAL TYPE NUMBER                                                ORTH0564
C HI1=DISTANCE FROM REFERENCE SURFACE TO TOP OF LAYER AT NODE I           ORTH0565
C HJ1=DISTANCE FROM REFERENCE SURFACE TO TOP OF LAYER AT NODE J           ORTH0566
C ANGLE1=WRAP ANGLE                                                       ORTH0567
C                                                                         ORTH0568
C                                                                         ORTH0569
      DO 610 IJK=1,NLAYER                                                 ORTH0570
      IF(LID1.NE.1) GO TO 604                                             ORTH0571
      READ(5,611) MT1(IJK),HI1(IJK),HJ1(IJK),ANGLE1(IJK)                  ORTH0572
      WRITE(6,612) I,IJK,MT1(IJK),HI1(IJK),HJ1(IJK),ANGLE1(IJK)           ORTH0573
  604 CONTINUE                                                            ORTH0574
      MT=MT1(IJK)                                                         ORTH0575
      HI=HI1(IJK)                                                         ORTH0576
      HJ=HJ1(IJK)                                                         ORTH0577
      ANGLE=ANGLE1(IJK)                                                   ORTH0578
      CALL ELASMAT(MT,T,HI,HNI,ANGLE,IJK,HJ,HNJ,TI,TJ)                    ORTH0579
      HNI=HI                                                              ORTH0580
      HNJ=HJ                                                              ORTH0581
  610 CONTINUE                                                            ORTH0582
  611 FORMAT (I5,5X,3F10.4)                                               ORTH0583
  612 FORMAT(1X,3I10,3F10.4)                                              ORTH0584
      GSTARO=TI/GBARO                                                     ORTH0585
      GSTARL=TJ/GBARL                                                     ORTH0586
      WRITE(2) AO,AL,DSTARO,DSTARL,DSTAR1,CC,DL,D1,C2,GSTARO,GSTARL       ORTH0587
      GO TO 614                                                           ORTH0588
  613 CONTINUE                                                            ORTH0589
      READ (2) AO,AL,DSTARO,DSTARL,DSTAR1,CC,DL,D1,C2,GSTARO,GSTARL       ORTH0590
      PHII=PHA(J(I))*PI/180.                                              ORTH0591
      PHIJ=PHA(K(I))*PI/180.                                              ORTH0592
      THETA1=(PHIJ-PHII)/2.                                               ORTH0593
      KAPPAI=2.*SIN(THETA1)/L                                             ORTH0594
      KAPPAJ=KAPPAI                                                       ORTH0595
      CSPHI=COS(PHII)                                                     ORTH0596
      SNPHI=SIN(PHII)                                                     ORTH0597
      CSPHJ=COS(PHIJ)                                                     ORTH0598
      SNPHJ=SIN(PHIJ)                                                     ORTH0599
  614 CONTINUE                                                            ORTH0600
      SNBTI=CSPHI*CSPSI-SNPHI*SNPSI                                       ORTH0601
```

AD-A046 401 ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND REDSTO--ETC F/G 20/11
NONLINEAR ANALYSIS OF ORTHOTROPIC, LAMINATED SHELLS OF REVOLUTI--ETC(U)
AUG 77 C M ELDRIDGE
UNCLASSIFIED DRDMI-TL-77-9 NL

2 OF 2
ADA
046401



END
DATE
FILMED
12-77
DDC



NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

```
      SNBTJ=CSPHJ*CSPSI-SNPHJ*SNPSI                                     ORTH0602
      CSBTI=SNPHI*CSPSI+CSPHI*SNPSI                                     ORTH0603
      CSBTJ=SNPHJ*CSPSI+CSPHJ*SNPSI                                     ORTH0604
      CALL AMAT(AA,SNPSI,CSPSI,L,SNBTI,CSBTI,SNBTJ,CSBTJ)               ORTH0605
        ETAPI=SNBTI/CSBTI                                               ORTH0606
        ETAPJ=SNBTJ/CSBTJ                                               ORTH0607
        B1=ETAPI                                                        ORTH0608
        ETAPPI=-KAPPAI/(CSBTI**3)                                       ORTH0609
        ETAPPJ=-KAPPAJ/(CSBTJ**3)                                       ORTH0610
        B2=B1+ETAPPI*L/2.                                               ORTH0611
        B3=ETAPPJ*L/2.-4.*ETAPJ-3.*B1-2.*B2                             ORTH0612
        B4=3.*ETAPJ+2.*B1+B2-L*ETAPPJ/2.                                ORTH0613
C                                                                       ORTH0614
C*****    GAUSSIAN QUADRATURE INTEGRATION                               ORTH0615
C                                                                       ORTH0616
      DO 4 NI=1,2                                                       ORTH0617
      Y=L*(1.+XI(NI))/2.                                                ORTH0618
        ETA=B1*Y+(B2-B1)*Y**2/L+(B3-B2)*Y**3/L**2+(B4-B3)*Y**4/L**3     ORTH0619
     1     -B4*Y**5/L**4                                                ORTH0620
        ETAP=B1+2.*(B2-B1)*Y/L+3.*(B3-B2)*Y*Y/L/L+4.*(B4-B3)*Y**3/L**3  ORTH0621
     1  -5.*B4*Y**4/L**4                                                ORTH0622
        ETAPP=2.*(B2-B1)/L+6.*(B3-B2)*Y/L**2+12.*(B4-B3)*Y*Y/L**3       ORTH0623
     1  -20.*B4*Y**3/L**4                                               ORTH0624
        BETA=ATAN(ETAP)                                                 ORTH0625
      CSBT=COS(BETA)                                                    ORTH0626
      SNBT=SIN(BETA)                                                    ORTH0627
      KAPPA1=-ETAPP*(CSBT**3)                                           ORTH0628
      CSPHIX=SNBT*CSPSI+CSBT*SNPSI                                      ORTH0629
      SNPHIX=CSBT*CSPSI-SNBT*SNPSI                                      ORTH0630
      RR=RI+(Y*SNPSI+ETA*CSPSI)                                         ORTH0631
      XMUL=AI(NI)*RR/CSBT*0.5*L                                         ORTH0632
      CALL PHIMAT(PHI,CSBT,SNBT,SNPSI,CSPSI,RR,KAPPA1,Y,CSPHIX)         ORTH0633
      X=1.-Y/L                                                          ORTH0634
      ZZ=Y/L                                                            ORTH0635
      DO 300 NI=1,2                                                     ORTH0636
      DO 300 NJ=1,2                                                     ORTH0637
       C(NI,NJ)=X*A0(NI,NJ)+ZZ*AL(NI,NJ)                                ORTH0638
       C(NI,NJ+2)=X*X*DSTAR0(NI,NJ)+ZZ*ZZ*CSTARL(NI,NJ)                 ORTH0639
     1           +X*ZZ*DSTAR1(NI,NJ)                                    ORTH0640
       C(NI+2,NJ)= C(NI,NJ+2)                                           ORTH0641
```

```
      C(NI+2,NJ+2)=X*X*X*DD(NI,NJ)+ZZ*ZZ*ZZ*DL(NI,NJ)                    ORTH0642
     1             +X*X*ZZ*D1(NI,NJ)+X*ZZ*ZZ*C2(NI,NJ)                    ORTH0643
  300 CONTINUE                                                            ORTH0644
      C(5,5)=GSTAR0*X+GSTARL*ZZ                                           ORTH0645
      ADD1=-XMUL*(P+DELP*Y/L)*SNBT                                        ORTH0646
      ADD2= XMUL*(P+DELP*Y/L)*CSBT                                        ORTH0647
      FP(1)=FP(1)+ADD1                                                    ORTH0648
      FP(2)=FP(2)+Y*ADD1                                                  ORTH0649
      FP(3)=FP(3)+ADD2                                                    ORTH0650
      FP(4)=FP(4)+Y*ADD2                                                  ORTH0651
      FP(5)=FP(5)+Y *ADD2*Y                                               ORTH0652
      FP(6)=FP(6)+Y *ADD2*Y*Y                                             ORTH0653
      FP(7)=0.                                                            ORTH0654
      FP(8)=0.                                                            ORTH0655
      DO 400 II=1,5                                                       ORTH0656
      DO 400 JJ=1,8                                                       ORTH0657
      DD(II,JJ)=0.                                                        ORTH0658
      DO 400 MM=1,5                                                       ORTH0659
  400 DD(II,JJ)=DD(II,JJ)+C(II,MM)*PHI(MM,JJ)                             ORTH0660
      DO 450 II=1,8                                                       ORTH0661
      DO 450 JJ=1,8                                                       ORTH0662
      DO 430 MM=1,5                                                       ORTH0663
  430 G(II,JJ)=G(II,JJ)+PHI(MM,II)*DD(MM,JJ)*XMUL                         ORTH0664
  450 CONTINUE                                                            ORTH0665
      CSBT2=CSBT*CSBT                                                     ORTH0666
      CSBT4=CSBT2**2                                                      ORTH0667
      DO 1000 II=1,8                                                      ORTH0668
      DO 1000 JJ=1,8                                                      ORTH0669
 1000 PHIPI(II,JJ)=0.                                                     ORTH0670
      PHIPI(2,2)=SNBT*SNBT*CSBT2                                          ORTH0671
      PHIPI(2,4)=-SNBT*CSBT2*CSBT                                         ORTH0672
      PHIPI(2,5)=-2.*SNBT*Y*CSBT2*CSBT                                    ORTH0673
      PHIPI(2,6)=-3.*SNBT*Y*Y*CSBT2*CSBT                                  ORTH0674
      PHIPI(4,2)=PHIPI(2,4)                                               ORTH0675
      PHIPI(4,4)=CSBT4                                                    ORTH0676
      PHIPI(4,5)=2.*Y*CSBT4                                               ORTH0677
      PHIPI(4,6)=3.*Y*Y*CSBT4                                             ORTH0678
      PHIPI(5,2)=PHIPI(2,5)                                               ORTH0679
      PHIPI(5,4)=PHIPI(4,5)                                               ORTH0680
      PHIPI(5,5)=4.*Y*Y*CSBT4                                             ORTH0681
```

```
      PHIPI(5,6)=6.*Y**3*CSBT4                                          ORTHO682
      PHIPI(6,2)=PHIPI(2,6)                                             ORTHO683
      PHIPI(6,4)=PHIPI(4,6)                                             ORTHO684
      PHIPI(6,5)=PHIPI(5,6)                                             ORTHO685
      PHIPI(6,6)=9.*Y**4*CSBT4                                          ORTHO686
      DO 11CC II=1,8                                                    ORTHO687
      DO 11CC JJ=1,8                                                    ORTHO688
      PHIPA(II,JJ)=C.                                                   ORTHO689
      DO 11CC MM=1,8                                                    ORTHO690
 1100 PHIPA(II,JJ)=PHIPA(II,JJ)+PHIPI(II,MM)*AA(MM,JJ)*XMUL             ORTHO691
    4 CONTINUE                                                          ORTHO692
      DO 5CC II=1,8                                                     ORTHO693
      DO 5CC MM=1,8                                                     ORTHO694
  500 FPA(II)=FPA(II)+AA(MM,II)*FP(MM)*2.*PI                            ORTHO695
      WRITE(1) FPA(1),FPA(2),FPA(3),FPA(5),FPA(6),FPA(7)                ORTHO696
     1,CSPHI,CSPHJ,SNPHI,SNPHJ                                          ORTHO697
      DO 55C II=1,8                                                     ORTHO698
  550 F(K1+II)=F(K1+II)+FPA(II)                                         ORTHO699
      DO 6 N1=1,8                                                       ORTHO7C0
      DO 6 N2=1,8                                                       ORTHO7C1
      DO 6 N3=1,8                                                       ORTHO702
    6 GAA(N1,N2)=GAA(N1,N2)+G(N1,N3)*AA(N3,N2)                          ORTHO703
      DO 7 N1=1,8                                                       ORTHO704
      DO 7 N2=1,8                                                       ORTHO7C5
      S(N1,N2)=C.                                                       ORTHO7C6
      DO 7 N3=1,8                                                       ORTHO707
    7 S(N1,N2)=S(N1,N2)+AA(N3,N1)*GAA(N3,N2)                            ORTHO708
      DO 12CC II=1,8                                                    ORTHO7C9
      DO 12CC JJ=1,8                                                    ORTHO710
      SI(II,JJ)=C.                                                      ORTHO711
      DO 12CC MM=1,8                                                    ORTHO712
 1200 SI(II,JJ)=SI(II,JJ)+AA(MM,II)*PHIPA(MM,JJ)                        ORTHO713
      DO 13CC II=1,8                                                    ORTHO714
      DO 13CC JJ=1,8                                                    ORTHO715
 1300  S(II,JJ)= S(II,JJ)+SI(II,JJ)*FNJ                                 ORTHO716
      DO 8 N1=1,8                                                       ORTHO717
      DO 8 N2=1,8                                                       ORTHO718
    8 S(N1,N2)=S(N1,N2)*2.*PI                                           ORTHO719
      RETURN                                                            ORTHO720
      END                                                               ORTHO721
```

```
      SUBROUTINE AMAT(AA,SNPSI,CSPSI,L,SNBTI,CSBTI,SNBTJ,CSBTJ)          ORTH0722
      REAL L                                                            ORTH0723
      DIMENSION AA(8,8)                                                 ORTH0724
      TNBTI=SNBTI/CSBTI                                                 ORTH0725
      TNBTJ=SNBTJ/CSBTJ                                                 ORTH0726
      A1=TNBTI/L                                                        ORTH0727
      A2=(2.*TNBTI+TNBTJ)/L/L                                           ORTH0728
      A3=(TNBTI+TNBTJ)/L/L/L                                            ORTH0729
      B1=1./CSBTI/CSBTI                                                 ORTH0730
      B2=B1/L                                                           ORTH0731
      B3=B2/L                                                           ORTH0732
      B4=1./CSBTJ/CSBTJ/L                                               ORTH0733
      B5=B4/L                                                           ORTH0734
      DO 10 N1=1,8                                                      ORTH0735
      DO 10 N2=1,8                                                      ORTH0736
   10 AA(N1,N2)=0.                                                      ORTH0737
      AA(1,1)= SNPSI                                                    ORTH0738
      AA(1,2)= -CSPSI                                                   ORTH0739
      AA(2,1)= -SNPSI/L                                                 ORTH0740
      AA(2,2)= CSPSI/L                                                  ORTH0741
      AA(2,5)=-AA(2,1)                                                  ORTH0742
      AA(2,6)=-AA(2,2)                                                  ORTH0743
      AA(3,1)=-AA(1,2)                                                  ORTH0744
      AA(3,2)=AA(1,1)                                                   ORTH0745
      AA(4,1)=-A1*SNPSI                                                 ORTH0746
      AA(4,2)=A1*CSPSI                                                  ORTH0747
      AA(4,3)=B1                                                        ORTH0748
      AA(4,5)=-AA(4,1)                                                  ORTH0749
      AA(4,6)=-AA(4,2)                                                  ORTH0750
      AA(5,1)=A2*SNPSI-3.*CSPSI/L/L                                     ORTH0751
      AA(5,2)=-A2*CSPSI-3.*SNPSI/L/L                                    ORTH0752
      AA(5,3)=-2.*B2                                                    ORTH0753
      AA(5,5)=-AA(5,1)                                                  ORTH0754
      AA(5,6)=-AA(5,2)                                                  ORTH0755
      AA(5,7)=-B4                                                       ORTH0756
      AA(6,1)=-A3*SNPSI+2.*CSPSI/L/L/L                                  ORTH0757
      AA(6,2)=A3*CSPSI+2.*SNPSI/L/L/L                                   ORTH0758
      AA(6,3)=B3                                                        ORTH0759
      AA(6,5)=-AA(6,1)                                                  ORTH0760
      AA(6,6)=-AA(6,2)                                                  ORTH0761
```

```
      AA(6,7)=B5                                                        ORTHO762
      AA(7,4)= 1.                                                       ORTHO763
      AA(8,4)= -1./L                                                    ORTHO764
      AA(8,8)= 1./L                                                     ORTHO765
      RETURN                                                            ORTHO766
      END                                                               ORTHO767
      SUBROUTINE PHIMAT(PHI,CSBT,SNBT,SNPSI,CSPSI,RR,KAPPA1,Y,CSPHIX)   ORTHO768
      REAL    KAPPA1                                                    ORTHO769
      DIMENSION PHI(5,8)                                                ORTHO770
      DO 1C N1=1,5                                                      ORTHO771
      DO 1C N2=1,8                                                      ORTHO772
   10 PHI(N1,N2)=C.                                                     ORTHO773
      PHI(1,2)= CSBT*CSBT                                               ORTHO774
      PHI(1,4)= SNBT*CSBT                                               ORTHO775
      PHI(1,5)= PHI(1,4)*2.*Y                                           ORTHO776
      PHI(1,6)= PHI(1,4)*3.*Y*Y                                         ORTHO777
      PHI(2,1)= SNPSI/RR                                                ORTHO778
      PHI(2,2)= PHI(2,1)*Y                                              ORTHO779
      PHI(2,3)= CSPSI/RR                                                ORTHO780
      PHI(2,4)= PHI(2,3)*Y                                              ORTHO781
      PHI(2,5)= PHI(2,4)*Y                                              ORTHO782
      PHI(2,6)= PHI(2,5)*Y                                              ORTHO783
      PHI(3,2)= (SNBT*SNBT-CSBT*CSBT)   *KAPPA1                         ORTHO784
      PHI(3,4)=-2.*CSBT*SNBT   *KAPPA1                                  ORTHO785
      PHI(3,5)= -4.*CSBT*SNBT*Y   *KAPPA1-2.*CSBT**3                    ORTHO786
      PHI(3,6)= -6.*CSBT*SNBT*Y*Y   *KAPPA1-6.*Y*CSBT**3                ORTHO787
      PHI(3,8)= -CSBT                                                   ORTHO788
      PHI(4,2)= -SNBT*CSBT *CSPHIX/RR                                   ORTHO789
      PHI(4,4)= -CSPHIX/RR*CSBT*CSBT                                    ORTHO790
      PHI(4,5)= -CSPHIX/RR*2.*Y*CSBT*CSBT                               ORTHO791
      PHI(4,6)= -CSPHIX/RR*3.*Y*Y*CSBT*CSBT                             ORTHO792
      PHI(4,7)= -CSPHIX/RR                                              ORTHO793
      PHI(4,8)= -CSPHIX/RR*Y                                            ORTHO794
      PHI(5,7)= -1.                                                     ORTHO795
      PHI(5,8)= -Y                                                      ORTHO796
      RETURN                                                            ORTHO797
      END                                                               ORTHO798
      SUBROUTINE STRSTF(L,KI,RJ,TI,TJ)                                  ORTHO799
      REAL    L,KAPP,KAPPAI,KAPPAJ                                      ORTHO800
      COMMON ISF11 (23CC)                                               ORTHO801
```

```
      COMMON        CSPHI,SNPHI,CSPHJ,SNPHJ,KAPPAI,KAPPAJ,SNPSI,CSPSI  ORTHO802
     1    ,SNBTI,SNBTJ,CSBTI,CSBTJ                                     ORTHO803
      COMMON       S(8,8),GE(8),AA(8,8),G(8,8),GAA(8,8)                ORTHO804
     1,LL1(1000),R(250),Z(250),U(1000, 8),F(1000)                      ORTHO805
     2,J(250),K(250),ALP(250),PHA(250),KAPF(250)                       ORTHO806
     3 ,IPP(250),C(5,5),SS(10,8),SE(10,8)                              ORTHO807
     4,MLAYER(250),PP(250),TT(250)                                     ORTHO808
     5,LID(250),DP(1000)                                               ORTHO809
      COMMON /SET3/ A0(2,2),AL(2,2),DSTAR0(2,2),DSTARL(2,2),           ORTHO810
     1DSTAR1(2,2),D0(2,2),DL(2,2),D1(2,2),D2(2,2),GBAR0,GBARL          ORTHO811
      DIMENSION PHI(5,8),PHIA(5,8)                                     ORTHO812
      GSTAR0=TI/GBAR0                                                  ORTHO813
      GSTARL=TJ/GBARL                                                  ORTHO814
      DO 250 NI=1,5                                                    ORTHO815
      DO 250 NJ=1,5                                                    ORTHO816
  250 C(NI,NJ)=0.                                                      ORTHO817
      CALL PHIMAT(PHI,CSBTI,SNBTI,SNPSI,CSPSI,RI,KAPPAI,0.,CSPHI)      ORTHO818
      DO 30 N1=1,5                                                     ORTHO819
      DO 30 N2=1,8                                                     ORTHO820
      PHIA(N1,N2)=0.                                                   ORTHO821
      DO 30 N3=1,8                                                     ORTHO822
   30 PHIA(N1,N2)=PHIA(N1,N2)+PHI(N1,N3)*AA(N3,N2)                     ORTHO823
      DO 300 NI=1,2                                                    ORTHO824
      DO 300 NJ=1,2                                                    ORTHO825
      C(NI,NJ)=A0(NI,NJ)                                               ORTHO826
      C(NI,NJ+2)=DSTAR0(NI,NJ)                                         ORTHO827
      C(NI+2,NJ)=C(NI,NJ+2)                                            ORTHO828
  300 C(NI+2,NJ+2)=D0(NI,NJ)                                           ORTHO829
      C(5,5)=GSTAR0                                                    ORTHO830
      DO 40 N1=1,5                                                     ORTHO831
      DO 40 N2=1,8                                                     ORTHO832
      SE(N1,N2)=PHIA(N1,N2)                                            ORTHO833
      SS(N1,N2)=0.                                                     ORTHO834
      DO 40 N3=1,5                                                     ORTHO835
   40 SS(N1,N2)=SS(N1,N2)+C(N1,N3)*PHIA(N3,N2)                         ORTHO836
      CALL PHIMAT(PHI,CSBTJ,SNBTJ,SNPSI,CSPSI,RJ,KAPPAJ,L ,CSPHJ)      ORTHO837
      DO 45 N1=1,5                                                     ORTHO838
      DO 45 N2=1,8                                                     ORTHO839
      PHIA(N1,N2)=0.                                                   ORTHO840
      DO 45 N3=1,8                                                     ORTHO841
```

```
 45 PHIA(N1,N2)=PHIA(N1,N2)+PHI(N1,N3)*AA(N3,N2)                     ORTHO842
    DO 400 NI=1,2                                                    ORTHO843
    DO 400 NJ=1,2                                                    ORTHO844
    C(NI,NJ)=AL(NI,NJ)                                               ORTHO845
    C(NI,NJ+2)=DSTARL(NI,NJ)                                         ORTHO846
    C(NI+2,NJ)=C(NI,NJ+2)                                            ORTHO847
400 C(NI+2,NJ+2)=DD(NI,NJ)                                           ORTHO848
    C(5,5)=GSTARL                                                    ORTHO849
    DO 50 N1=1,5                                                     ORTHO850
    DO 50 N2=1,6                                                     ORTHO851
    SE(N1+5,N2)=PHIA(N1,N2)                                          ORTHO852
    SS(N1+5,N2)=0.                                                   ORTHO853
    DO 50 N3=1,5                                                     ORTHO854
 50 SS(N1+5,N2)=SS(N1+5,N2)+C(N1,N3)*PHIA(N3,N2)                     ORTHO855
200 RETURN                                                           ORTHO856
    END                                                              ORTHO857
    SUBROUTINE STRESS(M,INCR,NINCR)                                  ORTHO858
    REAL    KAPP                                                     ORTHO859
    REAL    NULC,NUCL                                                ORTHO860
    COMMON           EL(10),EC(10),NUCL(10),NULC(10),GL(10)          ORTHO861
   *,FSAVE(1000),XK(250),F1(1000)                                    ORTHO862
    COMMON ISFC(12)                                                  ORTHO863
    COMMON        S(8,8),GE(8),AA(8,8),G(8,8),GAA(8,8)               ORTHO864
   1,LL1(1000),R(250),Z(250),L(1000, 8),F(1000)                      ORTHO865
   2,J(250),K(250), AL(250),PHA(250),KAPF(250)                       ORTHO866
   3 ,IPP(250),C(5,5),SS(10,8),SE(10,8)                              ORTHO867
   4,MLAYER(250),PP(250),TT(250)                                     ORTHO868
   5,LID(250),DP(1000)                                               ORTHO869
    DIMENSION EPS(4),SIG(4),TAU13(2) ,GAMMA(2),CC(2,2)               ORTHO870
    DIMENSION FD(10)                                                 ORTHO871
    DIMENSION FE( 8),FF(10)                                          ORTHO872
    FINCR=INCR                                                       ORTHO873
    FNINCR=NINCR                                                     ORTHO874
    FAC1=FINCR/FNINCR                                                ORTHO875
    PI=3.141592654                                                   ORTHO876
    REWIND 1                                                         ORTHO877
    IF(INCR.EQ.NINCR) REWIND 2                                       ORTHO878
    IF(INCR.EQ.NINCR) REWIND 3                                       ORTHO879
    IF(INCR.EQ.NINCR) WRITE(6,570)                                   ORTHO880
570 FORMAT(1H1,50X,*OUTPUT DATA*,//,                                 ORTHO881
```

```
     *          5X,*EL. NO.*,3X,*I/J*,2X,*COORDINATES*,25X,*FORCE  RESUORTHO882
     *LTANTS*,25X,*MOMENT RESULTANTS*                                   ORTHO883
     *        /21X,*R*,7X,*Z*, 9X,*LONG. RESULT.*,5X,*CIRC. RESULT.*,4X,*ORTHO884
     * SHEAR RESULT.*,6X,*LONG. MOMENT*,5X,*CIRC. MOMENT*)              ORTHO885
      DO 780 I1=1,M                                                     ORTHO886
      I=I1                                                              ORTHO887
      K1=4*(J(I)-1)                                                     ORTHO888
      K2=4*(K(I)-1)                                                     ORTHO889
C     DO 597 I2=1,8                                                     ORTHO890
C 597 GE(I2)=0.                                                         ORTHO891
      DO 610 JJ=1,4                                                     ORTHO892
C     IF(LL1(K1+JJ).EQ.1)GO TO 606                                      ORTHO893
      GE(JJ)=    F(K1+JJ)                                               ORTHO894
C 606 IF(LL1(K2+JJ).EQ.1)GO TO 610                                      ORTHO895
      GE(JJ+4)=    F(K2+JJ)                                             ORTHO896
  610 CONTINUE                                                          ORTHO897
      READ (1) FRJ,FZJ,FMJ,FRK,FZK,FMK,CSPHI,CSPHJ,SNPHI,SNPHJ          ORTHO898
      READ (1)I,S,SS,RI,RJ,T,SE                                         ORTHO899
      DO 612 N1=1,8                                                     ORTHO900
      FE(N1)=0.                                                         ORTHO901
      DO 611 N2=1,8                                                     ORTHO902
  611 FE(N1)=FE(N1)+S(N1,N2)*GE(N2)                                     ORTHO903
      IF(N1.GT.0.AND.N1.LT.5)FE(N1)=FE(N1)/2./PI/RI                     ORTHO904
      IF(N1.GT.4.AND.N1.LT.9)FE(N1)=FE(N1)/2./PI/RJ                     ORTHO905
  612 CONTINUE                                                          ORTHO906
      DEI=2.*PI*RI/FAC1                                                 ORTHO907
      DEJ=2.*PI*RJ/FAC1                                                 ORTHO908
      FRJ=FRJ/DEI                                                       ORTHO909
      FZJ=FZJ/DEI                                                       ORTHO910
      FMJ=FMJ/DEI                                                       ORTHO911
      FRK=FRK/DEJ                                                       ORTHO912
      FZK=FZK/DEJ                                                       ORTHO913
      FMK=FMK/DEJ                                                       ORTHO914
      UC=GE(1)*CSPHI-GE(2)*SNPHI                                        ORTHO915
      WC=GE(1)*SNPHI+GE(2)*CSPHI                                        ORTHO916
      IF(I.LT.M) GO TO 400                                              ORTHO917
      UCK=GE(5)*CSPHJ-GE(6)*SNPHJ                                       ORTHO918
      WCK=GE(5)*SNPHJ+GE(6)*CSPHJ                                       ORTHO919
  400 CONTINUE                                                          ORTHO920
      FL1=(FE(1)-FRJ)*CSPHI-(FE(2)-FZJ)*SNPHI                           ORTHO921
```

```
      FQ1=(FE(1)-FRJ)*SNPHI+(FE(2)-FZJ)*CSPHI                           ORTHO922
      FL2=(FE(5)-FRK)*CSPHJ-(FE(6)-FZK)*SNPHJ                           ORTHO923
      FQ2=(FE(5)-FRK)*SNPHJ+(FE(6)-FZK)*CSPHJ                           ORTHO924
      FE(3)=FE(3)-FMJ                                                   ORTHO925
      FE(7)=FE(7)-FMK                                                   ORTHO926
      XK(I1)=XK(I1)+FL2                                                 ORTHO927
      DO 615 N1=1,10                                                    ORTHO928
      FF(N1)=0.                                                         ORTHO929
      DO 615 N2=1,8                                                     ORTHO930
  615 FF(N1)=FF(N1)+SS(N1,N2)*GE(N2)                                    ORTHO931
      IF(R(J(I)).EQ.0.0) GO TO 805                                      ORTHO932
      GO TO 810                                                         ORTHO933
  805 FQ1 = 0.0                                                         ORTHO934
      FE(3) = FE(7)                                                     ORTHO935
      FF(4) = FF(9)                                                     ORTHO936
      FF(2) = FF(7)                                                     ORTHO937
      FL1 = FL2                                                         ORTHO938
  810 CONTINUE                                                          ORTHO939
      IF(INCR.NE.NINCR) GO TO 780                                       ORTHO940
      WRITE(6,613)I,J(I),R(J(I)),Z(J(I)),  FL1,FF(2),FQ1,FE(3),FF(4),   ORTHO941
     1K(I),R(K(I)),Z(K(I)),  FL2,FF(7),FQ2,FE(7),FF(9)                  ORTHO942
  613 FORMAT(1H016,I10,0P2F8.3,1P5E18.7/I17,0P2F8.3,1P5E18.7)           ORTHO943
      DO 617 N1=1,10                                                    ORTHO944
      FD(N1)=0.                                                         ORTHO945
      DO 617 N2=1,8                                                     ORTHO946
  617 FD(N1)=FD(N1)+SE(N1,N2)*GE(N2)                                    ORTHO947
      NLAYER=MLAYER(I)                                                  ORTHO948
      DO 800 L1=1,NLAYER                                                ORTHO949
      READ (3) MT,ANGLE,HI,HNI,HJ,HNJ,CC,TI,TJ                          ORTHO950
      EPS(1)=FD(1)+FD(3)*(HI+HNI)/2.                                    ORTHO951
      EPS(2)=FD(2)+FD(4)*(HI+HNI)/2.                                    ORTHO952
      EPS(3)=FD(6)+FD(8)*(HJ+HNJ)/2.                                    ORTHO953
      EPS(4)=FD(7)+FD(9)*(HJ+HNJ)/2.                                    ORTHO954
      SIG(1)=CC(1,1)*EPS(1)+CC(1,2)*EPS(2)                              ORTHO955
      SIG(2)=CC(2,1)*EPS(1)+CC(2,2)*EPS(2)                              ORTHO956
      SIG(3)=CC(1,1)*EPS(3)+CC(1,2)*EPS(4)                              ORTHO957
      SIG(4)=CC(2,1)*EPS(3)+CC(2,2)*EPS(4)                              ORTHO958
      TAU13(1)=FQ1/TI                                                   ORTHO959
      TAU13(2)=FQ2/TJ                                                   ORTHO960
      GAMMA(1)=TAU13(1)/GL(MT)                                          ORTHO961
```

```
      GAMMA(2)=TAU13(2)/GL(MT)
      WRITE(2) I ,HI,HNI,L1,J(I),EPS(1),EPS(2),GAMMA(1),SIG(1),SIG(2),
     1TAU13(1),K(I),EPS(3),EPS(4),GAMMA(2),SIG(3),SIG(4),TAU13(2)
  800 CONTINUE
  780 CONTINUE
      IF(INCR.NE.NINCR) GO TO 814
      REWIND 2
      WRITE(6,571)
  571 FORMAT(////,50X,*FIBER STRAINS AND STRESESS*,/,
     11X,*ELEMENT*3X*HI*5X*HNI*3X*LAYER*2X*I/J*5X*EPS(1)*10X*EPS(2)*
     210X*GAMMA*10X*SIG(1)*10X*SIG(2)*10X*TAU13*)
      DO 782 II=1,M
      NLAYER=MLAYER(II)
      DO 781 JJ=1,NLAYER
      READ(2)  I ,HI,HNI,L1,J(I),EPS(1),EPS(2),GAMMA(1),SIG(1),SIG(2),
     1TAU13(1),K(I),EPS(3),EPS(4),GAMMA(2),SIG(3),SIG(4),TAU13(2)
      WRITE(6,813) I,HI,HNI,L1,J(I),EPS(1),EPS(2),GAMMA(1),SIG(1),SIG(2)
     1,TAU13(1),K(I),EPS(3),EPS(4),GAMMA(2),SIG(3),SIG(4),TAU13(2)
  781 CONTINUE
  782 CONTINUE
  813 FORMAT(1H016.2F8.3,2I5  ,1P6E16.7/I33,1P6E16.7)
  814 CONTINUE
      RETURN
      END
      SUBROUTINE ELASMAT(MT,T,HI,HNI,ANGLE1,IJK,HJ,HNJ,TI,TJ)
      REAL    NULC,NUCL
      COMMON            EL(10),EC(10),NUCL(10),NULC(10),GL(10)
     *,FSAVE(1000),XK(250),F1(1000)
      COMMON ISFCM(14461)
      COMMON /SET3/ A0(2,2),AL(2,2),DSTAR0(2,2),DSTARL(2,2),
     1DSTAR1(2,2),D0(2,2),DL(2,2),D1(2,2),D2(2,2),GBAR0,GBARL
      DIMENSION C(2,2)
      ANGLE=ANGLE1*3.1415926/180.
      DEMON=1.-NULC(MT)*NUCL(MT)
      C(1,1)=EL(MT)/DEMON
      C(1,2)=NUCL(MT)*C(1,1)
      C(2,1)=C(1,2)
      C(2,2)=EC(MT)/DEMON
      IF(ANGLE.EQ.0.) GO TO 615
      CS=COS(ANGLE)
```

```
      SN=SIN(ANGLE)                                              ORTH1002
      CS2=CS*CS                                                  ORTH1003
      SN2=SN*SN                                                  ORTH1004
      CS4=CS2*CS2                                                ORTH1005
      SN4=SN2*SN2                                                ORTH1006
      C11=CS4*C(1,1)+2.*SN2*CS2* C(1,2)+SN4*C(2,2)               ORTH1007
      C22=SN4*C(1,1)+2.*SN2*CS2* C(1,2)+CS4*C(2,2)               ORTH1008
      C12=SN2*CS2*(C(1,1)+C(2,2))+C(1,2)*(SN4+CS4)               ORTH1009
      C(1,1)=C11                                                 ORTH1010
      C(1,2)=C12                                                 ORTH1011
      C(2,1)=C(1,2)                                              ORTH1012
      C(2,2)=C22                                                 ORTH1013
  615 CONTINUE                                                   ORTH1014
      WRITE(3) MT,ANGLE,HI,HNI,HJ,HNJ,C  ,TI,TJ                  ORTH1015
      DO 620 I=1,2                                               ORTH1016
      DO 620 J=1,2                                               ORTH1017
      AO(I,J)=AO(I,J)+C(I,J)*(HI-HNI)                            ORTH1018
      AL(I,J)=AL(I,J)+C(I,J)*(HJ-HNJ)                            ORTH1019
      DSTARO(I,J)=DSTARO(I,J)+0.5*C(I,J)*(HI*HI-HNI*HNI)         ORTH1020
      DSTARL(I,J)=DSTARL(I,J)+0.5*C(I,J)*(HJ*HJ-HNJ*HNJ)         ORTH1021
      DSTAR1(I,J)=DSTAR1(I,J)+C(I,J)*(HI*HJ-HNI*HNJ)             ORTH1022
      DO(I,J)=DO(I,J)+C(I,J)*(HI*HI*HI-HNI*HNI*HNI)/3.           ORTH1023
      DL(I,J)=DL(I,J)+C(I,J)*(HJ*HJ*HJ-HNJ*HNJ*HNJ)/3.           ORTH1024
      D1(I,J)=D1(I,J)+C(I,J)*(HI*HI*HJ-HNI*HNI*HNJ)              ORTH1025
  620 D2(I,J)=D2(I,J)+C(I,J)*(HI*HJ*HJ-HNI*HNJ*HNJ)              ORTH1026
      GBARO=GBARO+(HI-HNI)/(TI*GL(MT))                           ORTH1027
      GBARL=GBARL+(HJ-HNJ)/(TJ*GL(MT))                           ORTH1028
      RETURN                                                     ORTH1029
      END                                                        ORTH1030
```

REFERENCES

1. Zienkiewicz, O. C. and Holister, G. S. (editors), Stress Analysis: Recent Developments in Numerical and Experimental Methods, New York, John Wiley and Sons, 1965.

2. Zienkiewicz, O. C., The Finite Element Methods in Engineering Science, London, McGraw-Hill, 1971.

3. Przemieniecki, J. C., The Theory of Matrix Structural Analysis, New York, McGraw-Hill, 1968.

4. Desai, C. S. and Abel, J. F., Introduction to the Finite Element Method, New York, van Nostrand Reinhold, 1972.

5. Martin, H. C. and Carey, G. F., Introduction to Finite Element Analysis, New York, McGraw-Hill, 1973.

6. Meyer, R. R. and Harmon, M. B., "Conical Segment Method for Analyzing Open Crown Shells of Revolution for Edge Loadings," AIAA J., Vol. 1, No. 4, April 1963, pp. 886-891.

7. Popov, E. P., Penzien, J. and Lu, Z. A., "Finite Element Solution for Axisymmetric Shells," J. of EM Proc. ASCE, Vol. 90, No. EM5, October 1964, pp. 119-145.

8. Grafton, P. E. and Strome, D. R., "Analysis of Axisymmetrical Shells by the Direct Stiffness Method," AIAA J., Vol. 1, No. 10, October 1963, pp. 2342-2347.

9. Jones, R. E. and Strome, D. R., "Direct Stiffness Method Analysis of Shells of Revolution Utilizing Curved Elements," AIAA J., Vol. 4, No. 9, September 1966, pp. 1519-1525.

10. Khojasteh-Bakht, J., "Analysis of Elastic-Plastic Shells of Revolution Under Axisymmetric Loading by the Finite Element Method," Ph.D. Dissertation, Department of Civil Engineering, University of California, Berkeley, California.

11. Clough, R. W. and Felippa, C. A., "A Refined Quadrilateral Element for Analysis of Plate Bending," Proceedings Second Conference on Matrix Methods in Structural Mechanics, Wright-Patterson AFB, Ohio, 1968.

12. Klein, Stanley, Matrix Methods in Structural Mechanics, Report No. AFFDL-TR-66-80, Air Force Flight Dynamics Laboratory and Air Force Institute of Technology Air University, November 1966.

13. Sharifi, Parviz, "Nonlinear Analysis of Sandwich Structures," Ph.D. Dissertation, Department of Civil Engineering, University of California, Berkeley, California, 1970.

14. McNamara, John F., "Incremental Stiffness Method for Finite Element Analysis of the Nonlinear Dynamic Problem," Ph.D. Dissertation, Division of Engineering, Brown University, 1972.

15. Becker, Eric B., and Brisbane, John J., Application of the Finite Element Method to Stress Analysis of Solid Propellant Rocket Grains, Vol. 1, Rohm and Haas Company, Report No. S-76, November 1965.

16. Nickell, Robert E., and Sato, Tomio, Finite Element Stress Analysis of Orthotropic, Layered Shells of Revolution Using a Curved Shell Element, Rohm and Haas Company, Report No. S-264, October 1970.

17. Reissner, E., "On Bending of Elastic Plates," Quarterly Applied Mathematics, Vol. 5, No. 1, April 1947.

18. Novozhilov, V. V., Foundations of the Nonlinear Theory of Elasticity, New York, Graylock Press, 1953.

19. Tsao, C. H., "Strain-Displacement Relations in Large Displacement Theory of Shells," AIAA J., Vol. 2, No. 11, 1964, pp. 2060-2062.

20. Plantema, F. J., Sandwich Construction, John Wiley and Sons, New York, 1966.

21. Kraus, H., Thin Elastic Shells, New York, John Wiley and Sons, 1967.

22. Naghdi, P. M., "Foundations of Elastic Shell Theory," Progress in Solid Mechanics, I. N. Sneddon and R. Hill (editors), Amsterdam, North-Holland, 1963.

23. Timoshenko, S. and Woinowsky-Krieger, S., Theory of Plates and Shells, second edition, New York, McGraw-Hill, 1959.

DISTRIBUTION

	No. of Copies
Basic Distribution List	34
Defense Documentation Center Cameron Station Alexandria, VA 22314	12
Commander US Army Materiel Research and Development Command	
ATTN: DRCCG	1
DRCRD	1
5001 Eisenhower Avenue Alexandria, VA 22333	
DRCPM-RSE, Dr. Eldridge	20
DRSMI-LP, Mr. Voigt	1
DRDMI-T, Dr. Kobler	1
-TL, Mr. Lewis	1
-TLA, Mr. Pettey	1
-TBD	3
-TI (Reference Copy)	1
(Record Copy)	1