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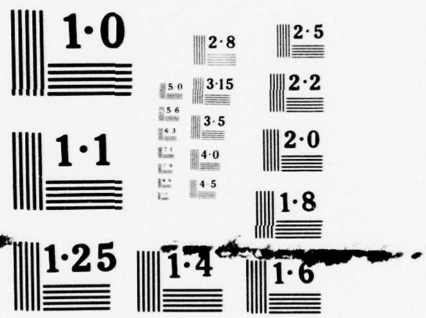

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PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS

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UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS

Paul H. Rabinowitz*

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ABSTRACT

The existence of periodic solutions of Hamiltonian systems of ordinary differential equations is proved in various settings. A case in which energy is prescribed is treated in Section 1. Both free and forced vibration problems, where the period is fixed, are studied in Section 2. The proofs involve finite dimensional approximation arguments, variational methods, and appropriate estimates.

SIGNIFICANCE AND EXPLANATION

Qualitative theorems for the existence of periodic solutions of Hamiltonian systems of ordinary differential equations are obtained in various settings. Cases are treated where either the period or the energy is prescribed and where there is explicit time dependence (forced vibration) or not (free vibrations).

AMS(MOS) Subject Classification - 34C15, 34C25

Key Words - Periodic solutions, Hamiltonian systems, ordinary differential equations, free vibrations, forced vibrations, variational methods.

Work Unit Number 1 - Applied Analysis

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Periodic Solutions of Hamiltonian Systems

Paul H. Rabinowitz*

Introduction

This paper concerns the existence of periodic solutions of the Hamiltonian system of ordinary differential equations:

$$\frac{dp}{dt} = -H_q \,, \qquad \frac{dq}{dt} = H_p \tag{0.1}$$

where $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and $p, q \in \mathbb{R}^n$. Letting $z = (p,q)$, (0.1) can be written more concisely as

$$\frac{dz}{dt} = \mathcal{J} H_z \tag{0.2}$$

where $\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

The search for periodic solutions of (0.2) will be carried out in two different but related settings. In §1 we look for solutions of (0.2) having prescribed energy while in §2 the period is fixed. To be more precise, the main result of §1, Theorem 1.1, states that if for some $b \neq 0$, $\mathfrak{m} \equiv H^{-1}(b)$

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is radially homeomorphic to S^{2n-1} and $(\zeta, H_z(\zeta))_{\mathbb{R}^{2n}} \neq 0$ for $\zeta \in \mathfrak{m}$, i.e. $\mathfrak{m}$ is appropriately star-shaped with respect to the origin, then (0.2) possesses a periodic solution on $\mathfrak{m}$. Note that the period is a priori unknown and one of the difficulties here is to determine it in the course of the solution.

There does not seem to have been much work on this sort of question in the literature. Seifert [1] showed if $Q(x, \frac{dx}{dt}) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{dx_i}{dt} \frac{dx_j}{dt}$ where Q is a positive definite quadratic form, $a_{ij}(x)$ and $U(x)$ are real analytic in $G \subset \mathbb{R}^n$, $U = E$ and $U_x \neq 0$ on ∂G, $U < E$ in G, and $\overline{G}$ is homeomorphic to the unit ball in $\mathbb{R}^n$, then the Lagrange equations corresponding to $Q - U$ have a periodic solution with energy E. See also Berger [2], Gordon [3], and Clark [4]. More recently considerable progress has been made on bifurcation questions for (0.2). In particular, A. Weinstein [5] showed if $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ and $H_{zz}(0)$ is positive definite, then for each small $b > 0$, $H^{-1}(b)$ contains at least n distinct periodic orbits (see also Moser [6], Chow and Mallet-Paret [7], and Fadell and Rabinowitz [8] for some generalizations and related results.) Since $H_{zz}(0)$ is positive definite, the hypotheses of Theorem 1.1 will be satisfied for small b. We suspect that $\mathfrak{m}$ contains at least n distinct periodic orbits.

In §2 we impose conditions on H near $z = 0$ and $z = \infty$ to obtain results on the existence of solutions of (0.2) having a prescribed period. To illustrate, suppose (i) $H(z) = o(|z|^2)$ at $z = 0$ while (ii) $0 \leq H(z)$ and $0 < H(z) \leq \theta(z, H_z(z))_{\mathbb{R}^{2n}}$ for $|z| > r$ where $\theta \in [0, \frac{1}{2})$. Then Theorem 2.1

states that for any $T > 0$, (0.2) possesses a nonconstant T-periodic solution. A caveat must be added here: We do not claim that T is the minimal period of the solution determined in Theorem 2.1. Indeed the conclusions of the theorem are unchanged if (i) is replaced by the assumption that $H_{zz}(0)$ is positive definite and for this case it is easy to give examples where there is an upper bound on any minimal period. We suspect however that under the hypotheses of Theorem 2.1 there is a nonconstant solution having any prescribed minimal period.

Some results have been obtained for bifurcation problems on the existence of solutions of (0.2) having a prescribed period by Chow and Mallet-Paret [7] and by Fadell and Rabinowitz [8].

The arguments employed here for the prescribed period case work equally well if H depends explicitly on t in a time periodic fashion. For this forced vibration case, which will also be treated in §2 , nontrivial solutions are obtained having the same period as $H(t,\cdot)$. Some stronger results of this nature in a more specialized setting have been obtained for $n = 1$ by Jacobowitz [9] and Hartman [10] .

As was pointed out to us by Jürgen Moser, Theorem 2.1 is related to the Poincaré-Birkhoff Theorem in the following way: Suppose $n = 1$, $H \in C^2(\mathbb{R}^2, \mathbb{R})$, and $H(z) > 0$ if $z \neq 0$. Consider the mapping $\zeta \rightarrow \varphi(T, \zeta)$ where $\varphi(T,\zeta)$ is the value at time T of the solution of (0.2) which is initially at ζ . Then by (i), for c sufficiently small, points on $H^{-1}(c)$ undergo a small twist while for c sufficiently large, points on $H^{-1}(c)$ undergo a large twist via (ii) . Hence by the Poincaré-Birkhoff

Theorem, $\varphi(T, \cdot)$ has a fixed point for some intermediate value of c and this provides a T-periodic solution of (0.2) .

To obtain our results, we employ methods from the calculus of variations. We try to find solutions of (0.2) as critical points of a suitable functional. For example, in §1 we consider the action integral subject to the constraint that an averaged Hamiltonian is prescribed. Since the action integral is not bounded from above or below on this manifold, it is rather subtle to obtain critical points for this problem. We do not know how to do this in any direct fashion and instead use an approximation procedure. Namely we minimax the action integral over appropriate subsets of a finite dimensional manifold, the subsets being chosen to exploit the symmetries inherent in this problem. For this purpose a cohomological index theory recently developed by E. Fadell and the author [8] is very helpful. Uniform bounds for the critical points of the approximating finite dimensional problem allow us to pass to a limit and find a solution of (0.2).

Similar arguments are used in §2 . There the topological arguments are simpler but an additional complication arises since we must avoid the trivial solution $z \equiv 0$ as well as any other constant solutions of (0.2). Much of the motivation for the techniques we use, especially those of §2 was provided by our recent paper [11] on free and forced vibrations for semilinear wave equations. Indeed the results of §2 can be considered to be the Hamiltonian analogues of [11] .

We thank Jürgen Moser who encouraged us to work on the problem of §1, Edward Fadell for many discussions on topological matters, and Charles Conley and Michael Crandall for some suggestions.

§1. The prescribed energy case.

In this section we will find a periodic solution of (0.2) when the energy, H, is prescribed. Let $(\cdot,\cdot)_{\mathbb{R}^j}$ denote the inner product in $\mathbb{R}^j$, $p, q \in \mathbb{R}^n$, and $z = (p, q)$. Our main result here is:

Theorem 1.1: Let $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$. Suppose

(H1) For some $b \neq 0$, $H^{-1}(b)$ is radially homeomorphic to S^{2n-1}, and

(H2) $(\zeta, H_z(\zeta))_{\mathbb{R}^{2n}} \neq 0$ for $\zeta \in H^{-1}(b)$.

Then the Hamiltonian system

$$(1.2) \qquad \frac{dz}{dt} = \mathcal{J} H_z$$

possesses a periodic solution on $H^{-1}(b)$.

As is clear from (H1) - (H2), we need only assume $H \in C^1$ near $H^{-1}(b)$.

The proof of Theorem 1.1 will be carried out in several steps. We begin with some simplifications and observations. First we further assume $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$. The C^1 case will be obtained by a limit argument later. On dividing H by b, we can assume $b = 1$. Next observe that if $\psi \in C(H^{-1}(1), S^{2n-1})$ is the homeomorphism of (H1), then $\psi \in C^2(H^{-1}(1), S^{2n-1})$ since $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$. Indeed if $\zeta \in H^{-1}(1)$, by (H2), $H_z(\zeta) \neq 0$ so by the implicit function theorem we can solve for one coordinate, say z_1, in terms of $z^* \equiv (z_2, \cdots, z_{2n})$ near ζ and $z_1 = \chi(z^*)$ is a C^2 function. Hence $\psi(z) = (\chi(z^*), z^*)(\chi(z^*))^2 + |z^*|^2)^{-\frac{1}{2}}$ is also a C^2 function. Now a new function $\overline{H}(z)$ is introduced as follows. Set $\overline{H}(0) = 0$. For $z \neq 0$, by (H1), there is a unique $\alpha = \alpha(z) > 0$ and $w = w(z) \in H^{-1}(1)$ such that $z = \alpha w$, namely $w = \psi^{-1}(z/|z|)$ and $\alpha = |z| \; |\psi^{-1}(z/|z|)|^{-1}$. Now define

$$\overline{H}(z) = \alpha(z)^2 , \qquad z \neq 0 . \tag{1.3}$$

It readily follows that $\overline{H}$ satisfies:

$$\left\{ \begin{array}{ll} \text{(i)} & \overline{H} \in C^2(\mathbb{R}^{2n}\setminus\{0\}, \mathbb{R}) \cap C^{1,\mathrm{Lip}}(\mathbb{R}^{2n}, \mathbb{R}) , \\ \text{(ii)} & H^{-1}(1) = \overline{H}^{-1}(1) , \\ \text{(iii)} & (\zeta, \overline{H}_z(\zeta))_{\mathbb{R}^{2n}} = 2\overline{H}(\zeta) \\ \text{(iv)} & \overline{H}(\zeta)|\zeta|^{-2} \text{ and } |\overline{H}_z(\zeta)| \; |\zeta|^{-1} \text{ are uniformly bounded .} \end{array} \right. \tag{1.4}$$

These properties will work to our advantage later.

<u>Lemma 1.5</u>: For initial data $\omega \in H^{-1}(1)$, (1.2) and

$$\frac{d\zeta}{dt} = \mathcal{J}\overline{H}_z(\zeta) \tag{1.6}$$

have the same orbits. In particular they have the same periodic orbits.

Proof: By (1.4) (ii), $H^{-1}(1)$ is also a level set for $\overline{H}$. Hence $\overline{H}_z(z) = \beta(z)H_z(z)$ for $z \in H^{-1}(1)$ where $0 \neq \beta \in C^1(H^{-1}(1), \mathbb{R})$. Moreover since (1.2) and (1.6) are Hamiltonian systems, if $\omega \in H^{-1}(1)$, the corresponding solutions $z(t)$, $\zeta(t)$ of (1.2), (1.6) remain on $H^{-1}(1)$. It then follows that (1.2) and (1.6) have the same orbits although their parameterizations will be different in general. Indeed $\zeta(t) = z(r(t))$ where r satisfies

$$\frac{dr}{dt} = \beta(z(r(t))), \qquad r(0) = 0 \quad . \tag{1.7}$$

As a consequence of Lemma 1.5, to prove Theorem 1.1, it suffices to find a periodic solution of (1.6) on $H^{-1}(1)$. Stretching the time variable, $t \to \tau = 2\pi T^{-1}t \equiv \lambda^{-1}t$, (1.6) is replaced by

$$\dot{z} = \lambda \mathcal{J}\overline{H}_z \tag{1.8}$$

where $\cdot$ denotes $\frac{d}{d\tau}$ and the unknown period appears explicitly as a parameter via λ. Thus we have reduced the proof of Theorem 1.1 to determining a pair $(\lambda, z(\tau))$ satisfying (1.8) with $z(\tau)$ 2π-periodic and lying on $H^{-1}(1)$. This will be accomplished by a variational argument.

The corresponding variational problem will be formulated next. Let E denote the set of $2n$-tuples of 2π periodic functions $z(t) = (p(t), q(t))$ which have one square integrable derivative. The usual Hilbert space inner product will be employed in E, i.e.

$$(z, \zeta)_E = \int_0^{2\pi} [(\dot{z}(\tau), \dot{\zeta}(\tau))_{\mathbb{R}^{2n}} + (z(\tau), \zeta(\tau))_{\mathbb{R}^{2n}}]\, d\tau \ .$$

For $z = (p, q) \in E$, the action integral of E is defined as

$$A(z) = \int_0^{2\pi} (p, \dot{q})_{\mathbb{R}^n}\, d\tau \ . \tag{1.9}$$

Let

$$S = \{z \in E \mid \frac{1}{2\pi} \int_0^{2\pi} \overline{H}(z(\tau))d\tau = 1\} \ .$$

It is a straightforward exercise in the calculus of variations to verify that if z is a critical point of $A|_S$, then z satisfies (1.8) for some $\lambda \neq 0$, λ appearing as a Lagrange multiplier. Since (1.8) is a Hamiltonian system, $\overline{H}(z(t)) \equiv \overline{b}$, a constant. The definition of S then shows $\overline{b} = 1$ and we have our desired periodic solution.

Unfortunately, we know of no direct method to find critical points of $A|_S$, one difficulty being that A is neither bounded from above nor from below on S. However by replacing this variational problem by an approximating finite dimensional one, exploiting the symmetries present in A and S to obtain critical points of the new problem, and getting suitable bounds for these critical points, we can pass to a limit to get a solution of (1.8) on S.

To carry out this program, several preliminaries are needed. Let $e_1, \cdots, e_{2n}$ denote the usual basis in $\mathbb{R}^{2n}$, i.e. $e_1 = (1, 0, \cdots, 0), \cdots,$ $e_{2n} = (0, \cdots, 0, 1)$. Let

$$E_m = \{ \sum_{k=1}^{2n} (\sum_{j=0}^{m} a_{jk} \cos jt + b_{jk} \sin jt) e_k \,|\, a_{jk}, b_{jk} \in \mathbb{R}\}$$

i.e. $E_m = \mathrm{span}\{\cos jt\, e_k, \sin jt\, e_k \,|\, 0 \le j \le m\,,\ 1 \le k \le 2n\}$.

A convenient set of functions to introduce in E_m is

$$\begin{cases} \varphi_{jk} = \sin jt\, e_k - \cos jt\, e_{k+n} \\ \psi_{jk} = \cos jt\, e_k + \sin jt\, e_{k+n} \\ \theta_{jk} = \sin jt\, e_k + \cos jt\, e_{k+n} \\ \zeta_{jk} = \cos jt\, e_k - \sin jt\, e_{k+n} \end{cases}$$

for $0 \le j \le m$, $1 \le k \le n$. These functions form a basis for E_m so

$$E_m = \mathrm{span}\{\varphi_{jk}, \psi_{jk}, \theta_{jk}, \zeta_{jk} \,|\, 0 \le j \le m\,,\ 1 \le k \le n\} .$$

Note also that $\varphi_{0,k} = -\theta_{0,k}$, $\psi_{0,k} = \zeta_{0,k}$, and

$$(1.10) \qquad \begin{cases} A(\varphi_{0,k}) = 0 = A(\psi_{0,k}) \\ A(\varphi_{jk}) = j\pi = A(\psi_{jk}) \\ A(\theta_{jk}) = -j\pi = A(\zeta_{jk}) \end{cases}$$

For $z \in E$, let $P_1 z = (z_1, \cdots, z_n)$ and $P_2 z = (z_{n+1}, \cdots, z_{2n})$.

Lemma 1.11: If $\mathfrak{F}_m = \{\varphi_{jk}, \psi_{jk}, \theta_{jk}, \zeta_{jk} | 1 \le j \le m,\ 0 \le k \le n\}$, the functions in $\mathfrak{F}_m$ are both L^2 and A orthogonal, i.e. if $z, \zeta \in \mathfrak{F}_m$ and $z \not\equiv \zeta$, then

$$\int_0^{2\pi} (z, \zeta)_{\mathbb{R}^{2n}}\, d\tau = 0$$

and $\int_0^{2\pi} [(P_1 z, P_2 \dot{\zeta})_{\mathbb{R}^n} + (P_1 \zeta, P_2 \dot{z})_{\mathbb{R}^n}]\, d\tau = 0$.

Proof: This is an easy computation. (We identify $\varphi_{0,k}$ and $-\theta_{0,k}$, etc.)

It follows from Lemma 1.11 that $\mathfrak{F}_m$ is an orthogonal basis for E_m and for $z, \zeta \in \mathfrak{F}_m$, $z \not\equiv \zeta$,

$$A(z+\zeta) = A(z) + A(\zeta) \ . \tag{1.12}$$

Set

$$E^+ = \text{span}\,\{\varphi_{jk}, \psi_{jk} | j \in \mathbb{N}^+,\ 1 \le k \le n\}\ ,$$

$$E^- = \text{span}\,\{\theta_{jk}, \zeta_{jk} | j \in \mathbb{N}^+,\ 1 \le k \le n\}\ ,$$

$$E^0 = \text{span}\,\{\varphi_{0,k}, \psi_{0,k} | 1 \le k \le n\}\ .$$

Then by Lemma 1.11 and (1.10) and (1.12), E^+, E^-, and E^0 are L^2 orthogonal subspaces of E with $A > 0$ on $E^+\backslash\{0\}$, $A < 0$ on $E^-\backslash\{0\}$, and $A \equiv 0$ on E^0.

Next some invariance properties of our problem will be studied. If $z(\tau) \in \text{span}\{\varphi_{jk}, \psi_{jk}\}$, then $L_t z \equiv z(\tau+t) \in \text{span}\{\varphi_{jk}, \psi_{jk}\}$ for all $t \in [0,2\pi]$. This and similar observations imply that $E_m, E_m \cap E^{\pm}$, and $E_m \cap E^{o}$ are invariant under $\{L_t \mid t \in [0,2\pi]\}$. These translations induce an S^1 action on $E\backslash\{0\}$. Indeed we can write $z \in E$ as

$$z(\tau) = \sum_{j=-\infty}^{\infty} \gamma_j e^{ij\tau} \equiv \varphi(e^{i\tau})$$

where $\gamma_j \in \mathbb{C}^n$ and $\gamma_{-j} = \overline{\gamma}_j$. Then the S^1 action on E corresponding to the above translations is given by $(\rho\varphi)(e^{i\tau}) = \varphi(\rho e^{i\tau})$ for $\rho \in S^1$. We call mappings of E to E which commute with this action or real valued functions on E which are constant on orbits of the action equivariant mappings. Likewise a subset K of $E\backslash\{0\}$ is called invariant if $z \in K$ implies $L_t z \in K$ for all $t \in [0,2\pi]$. Since $A(z) = A(L_t z)$ for all $z \in E$, the action integral is an equivariant mapping. Similarly S and $S \cap E_m$ are invariant sets. Note that our S^1 action is not free. In fact E^{o} is a fixed point set for the S^1 action and there are also isotropy subgroups in S^1 of arbitrary order.

To exploit the effect of our S^1 action, an index theory is required. The category theory of Ljusternik-Schnirelmann on $(E\backslash E^{o})/S^1$ could be employed; however it has several technical disadvantages. Instead we will use a cohomological index theory developed recently in [8]. Let $\mathcal{E}$ denote the family of invariant subsets of $E\backslash\{0\}$. The L^2 orthogonal complement of a subspace $F \subset E$ will be denoted by $F^{\perp}$.

Lemma 1.13: There is an index theory, i.e. a mapping $i : \mathcal{E} \to \mathbb{N} \cup \{\infty\}$ having the following properties. For $K, \hat{K} \in \mathcal{E}$,

1° $i(K) < \infty$ if and only if $K \cap E^\circ = \phi$.

2° (Monotonicity) If there is an $f \in C(K, \hat{K})$ with f equivariant, then $i(K) \le i(\hat{K})$.

3° (Subadditivity) $i(K \cup \hat{K}) \le i(K) + i(\hat{K})$

4° (Continuity) If K is closed, there exists a closed neighborhood K^* of K such that $i(K^*) = i(K)$.

5° (Normalization) If $z \in E \backslash E^\circ$ and $S^1 z$ denotes the orbit of the action through z, $i(S^1 z) = 1$.

6° If F is a finite dimensional invariant subspace of $(E^\circ)^\perp$ and $\mathcal{S}$ is the unit sphere in E, $i(F \cap \mathcal{S}) = \frac{1}{2} \dim F$.

Proof: The proof of Lemma 1.13 can be found in §6-7 of [8]. Here we will only give the definition of $i(K)$ (which is denoted by $\text{Index}^*_{\mathbb{C}} K$ in [8]). Recall the Hopf fibration $S^{2n+1} \xrightarrow{p_n} \mathbb{C}P^n$ where $\mathbb{C}P^n$ denotes complex projective space. Set $S^\infty \equiv \bigcup_{n \in \mathbb{N}} S^{2n+1}$, $\mathbb{C}P^\infty \equiv \bigcup_{n \in \mathbb{N}} \mathbb{C}P^n$. The circle group S^1 acts freely on S^∞ in the usual fashion (i.e. coordinatewise on each S^{2n+1}) and hence acts freely on $S^\infty \times K$. Therefore we have a principle S^1-bundle $q_K : S^\infty \times K \to (S^\infty \times K)/S^1 \equiv S^\infty \times_{S^1} K$ and a classifying map

$$\begin{array}{ccc} S^\infty \times K & \xrightarrow{\hat{f}} & S^\infty \\ \downarrow q_K & & \downarrow q \\ (S^\infty \times K)/S^1 & \xrightarrow{f} & \mathbb{C}P^\infty \end{array}$$

induced by the projection $\hat{f}$. (The vertical maps are the usual maps into the orbit space). Then $i(K) \equiv \operatorname{index}^*_\alpha K + 1$ where

$$\operatorname{index}^*_\alpha K = \max\{k \in \mathbb{N} \mid f^*(\alpha^k) \neq 0\}$$

α being a generator of the rational cohomology group $H^2(\mathbb{C}P^\infty, \mathbb{Q}) = \mathbb{Q}$ and f^* is the induced cohomology homomorphism.

With the aid of this index theory we can determine the existence of critical points of A on $S \cap E_m \equiv S_m$. To do so, observe first that if

$$\Psi(z) = \frac{1}{2\pi} \int_0^{2\pi} \overline{H}(z(\tau))\, d\tau ,$$

and $\Psi'(z)\zeta$ denotes the Frechet derivative of Ψ at z acting on $\zeta \in E$, then by (iii) of (1.4),

$$\Psi'(z)z = \frac{1}{2\pi} \int_0^{2\pi} (\overline{H}_z(z), z)_{\mathbb{R}^{2n}}\, d\tau = 2\Psi(z) . \tag{1.14}$$

In particular if $z \in S_m$, $\Psi'(z)z = 2$. It now follows from (1.4) that S_m is a $C^{1,\text{Lip}}$ manifold in E_m which is radially homeomorphic to $\mathbf{S} \cap E_m$. Hence by 2° and 6° of Lemma 1.13, $i(S_m \cap (E^\circ)^\perp) = \frac{1}{2}\dim((E^\circ)^\perp \cap E_m) = 2mn$.

There are some obvious critical points of $A|_{S_m}$, namely the points at which A achieves its maximum and minimum on S_m. Unfortunately as we observed earlier, A is not bounded from above or from below on S so these critical points are not useful. However given the symmetries our problem possesses and a corresponding index theory, there is a standard way in which to attempt to find critical points of $A|_{S_m}$. Namely define

$$\gamma_j = \inf_{\substack{K \in S_m \\ i(K) \geq j}} \max_{z \in K} A(z) , \quad 1 \leq j \leq 2mn \tag{1.15}$$

and show that these numbers are critical values of $A|_{S_m}$. This approach will not succeed here without further qualification, the difficulty being the presence of the fixed point set E^0 of our S^1 action. Indeed observe that by 1^0 of Lemma 1.13, every neighborhood of $S \cap E^0$ contains sets of *infinite* index while $A(z) = 0$ if $z \in E^0$. Hence $\gamma_j \leq 0$. In general 0 will not be a critical value of $A|_{S_m}$. Therefore the only chance γ_j has to be a critical value is when $\gamma_j < 0$.

<u>Lemma 1.16</u>: For $1 \leq j \leq mn$, γ_j is a negative critical value of $A|_{S_m}$.

<u>Proof</u>: The proof of Lemma 1.16 relies on the following standard lemma. Let $r \in \mathbf{R}$, $\mathcal{C}_r = \{z \in S_m | A(z) \leq r\}$, and

$$\mathcal{K}_r = \{z \in S_m | A(z) = r \text{ and } A'(z) - \frac{(A'(z), \Psi'(z))_{L^2}}{\|\Psi'(z)\|^2_{L^2}} \Psi'(z) = 0\} .$$

Lemma 1.17: If $\overline{H}$ satisfies (1.4), for any $c \in \mathbf{R}$ and any neighborhood $\mathcal{O}$ of $\mathcal{K}_c$, there is an $\varepsilon > 0$ and $\eta \in C([0,1] \times S_m, S_m)$ such that

1° $\eta(s,\cdot)$ is equivariant for each $s \in [0,1]$,

2° $\eta(1, G_{c+\varepsilon} \backslash \mathcal{O}) \subset G_{c-\varepsilon}$,

3° If $\mathcal{K}_c = \phi$, $\eta(1, G_{c+\varepsilon}) \subset G_{c-\varepsilon}$.

Proof: The lemma without 1° is well known. (See e.g. [12] or [13]). For completeness we sketch the proof indicating in the process why 1° is also true. The function η is determined as the solution of the ordinary differential equation

$$
(1.18) \qquad \begin{cases} \dfrac{d\eta}{ds} = -A'(\eta) + \dfrac{(A'(\eta), \Psi'(\eta))_{L^2}}{\|\Psi'(\eta)\|^2_{L^2}} \Psi'(\eta) \equiv V(\eta) \\[2ex] \eta(0,z) = z, \quad z \in S_m \end{cases}
$$

In (1.18), A', Ψ' refer to the Frechet derivatives of these functions as elements of $E_m^* = E_m$. Since $\hat{H} \in C^{1,\mathrm{Lip}}$ and $\Psi'(z) \neq 0$ for $z \neq 0$, $V \in C^{0,\mathrm{Lip}}$ near S_m. Therefore there exists a unique continuously differentiable η satisfying (1.18) for small $|s|$. Since

$$
\frac{d}{ds} \Psi(\eta(s,z)) \equiv 0 ,
$$

the orbit $\eta(s,z)$ lies on S_m and therefore exists for all s. Next observe that

$$\text{(1.19)} \qquad \frac{d}{ds} A(\eta(s,z)) = (A', \frac{d\eta}{ds})_{L^2} = -\|A'\|^2_{L^2} + \frac{(A', \Psi')^2_{L^2}}{\|\Psi'\|^2_{L^2}} \le 0$$

by the Schwarz inequality and the right hand side of (1.19) vanishes if and only if $V(z) = 0$. Hence it is straightforward to verify 2° and 3°. To prove 1°, we need only show V is equivariant. To check this, observe that $A(L_t z) = A(z)$ for all $t \in [0, 2\pi]$ implies that

$$(A'(z), \zeta)_{L^2} = (A'(L_t z), L_t \zeta)_{L^2}$$

for all $\zeta \in E_m$. Choosing $\zeta = L_{-t}\xi$ and observing that $L_t = L^*_{-t}$ yields

$$(A'(z), L_{-t}\xi)_{L^2} = (A'(L_t z), \xi)_{L^2} = (L_t A'(z), \xi)_{L^2}$$

for all $\xi \in E_m$. Hence $A'(L_t z) = L_t A'(z)$ for all $t \in [0, 2\pi]$ so A' and similarly Ψ' are equivariant maps. The equivariance of the remaining terms follows in a more simple fashion.

<u>Proof of Lemma 1.16</u>: First we show $\gamma_j < 0$, $1 \le j \le mn$. Since $\gamma_j \le \gamma_{j+1}$, all we need verify is that $\gamma_{mn} < 0$ and to do this, it suffices to produce a set $\hat{K} \subset S_m$ with $i(\hat{K}) \ge mn$ and $\max_{\hat{K}} A < 0$. With the aid of 6° of Lemma 1.13, this is clearly the case for $\hat{K} = S_m \cap E^-$. Now to prove that γ_j is a critical value of $A|_{S_m}$ for $1 \le j \le mn$, assume the contrary. Then by Lemma 1.17, there is an $\varepsilon > 0$ and $\eta \in C([0,1] \times S_m, S_m)$ such that

$$(1.20) \qquad \eta(1, G_{\gamma_j+\varepsilon}) \subset G_{\gamma_j-\varepsilon} \ .$$

Choose $K \in G_{\gamma_j+\varepsilon}$ such that $i(K) \geq j$. Since by 1^o of Lemma 1.17, $\eta(1, \cdot)$ is equivariant, 2^o of Lemma 1.13 implies that $i(\eta(1,K)) \geq j$. Hence by the definition of γ_j ,

$$(1.21) \qquad \max_{\eta(1,K)} A \geq \gamma_j$$

while by (1.20) ,

$$(1.22) \qquad \max_{\eta(1,K)} A \leq \gamma_j - \varepsilon \ ,$$

a contradiction.

Remark 1.23: If $\gamma_j = \cdots = \gamma_{j+r} \equiv \gamma$, where $j+r \leq mn$, a standard argument using 2^o-4^o of Lemma 1.13 and 1^o-2^o of Lemma 1.17 shows $i(K_\gamma) \geq r+1$. However we will not need this below. We also observe that positive critical values of A can be obtained by working with $-A$ rather than A .

Set $c_m = \gamma_{mn}$. We will obtain upper and lower bounds for c_m independent of m . This in turn will enable us to get uniform estimates for any corresponding critical points z_m and periods λ_m and show that a subsequence of (λ_m, z_m) converges to a solution of (1.8). To aid us in this process we require:

Lemma 1.24: Let $K \subset E_m$ with $i(K) \geq mn$. Let F be an invariant subspace of E_m containing E^0 and satisfying $\dim F \geq 2mn + 2n + 2$. Then $K \cap F \neq \phi$.

Proof: Since F is invariant, so is $F^\perp \cap E_m$. Let P_m denote the L^2 orthogonal projector of E_m to $F^\perp \cap E_m$. If $K \cap F = \phi$, $P_m \in C(K, (F^\perp \cap E_m)\backslash\{0\})$ and P_m is equivariant. Hence by 2^0 of Lemma 1.13, $i(K) \leq i(P_m(K))$. Projecting $P_m(K)$ radially into $\mathbb{S} \cap F^\perp \cap E_m$ and applying 2^0 of Lemma 1.13 again yields $i(P_m(K)) \leq i(\mathbb{S} \cap F^\perp \cap E_m)$. Since $\dim E_m = 4mn + 2n$, $\dim F^\perp \cap E_m \leq 2mn - 2$. Consequently $i(\mathbb{S} \cap F^\perp \cap E_m) \leq mn - 1$ by 6^0 of Lemma 1.13 and this implies $i(K) \leq mn - 1$, a contradiction.

Now the bounds for c_m can be obtained. Let

$$\mu = \max_{|\zeta| = 1} \overline{H}(\zeta) \, ; \quad \nu = \min_{|\zeta| = 1} \overline{H}(\zeta) \, .$$

The definition of $\overline{H}$ then implies that

$$\nu |\zeta|^2 \leq \overline{H}(\zeta) \leq \mu |\zeta|^2$$

for all $\zeta \in \mathbb{R}^{2n}$. Consequently for $z \in E$,

$$\frac{\nu}{2\pi} \|z\|_{L^2}^2 \leq \Psi(z) \leq \frac{\mu}{2\pi} \|z\|_{L^2}^2$$

and in particular for $z \in S$,

$$(1.25) \qquad \frac{2\pi}{\mu} \leq \|z\|_{L^2}^2 \leq \frac{2\pi}{\nu} .$$

Lemma 1.26: $-\frac{\pi}{\nu} \leq c_m \leq -\frac{\pi}{\mu}$.

Proof: As was shown earlier,

$$(1.27) \qquad c_m \leq \max_{z \in E^- \cap S_m} A(z) .$$

Let B_r denote the ball of radius r in E under $\|\cdot\|_{L^2}$. Then by (1.25) , (1.27) , and the form of A ,

$$(1.28) \qquad c_m \leq \max_{z \in E^- \cap E_m \cap B_{(\frac{2\pi}{\mu})^{\frac{1}{2}}}} A(z) = -\frac{\pi}{\mu}$$

since the maximum is taken on when $z \in B_{(\frac{2\pi}{\mu})^{\frac{1}{2}}} \cap \operatorname{span}\{\theta_{1k}, \zeta_{1k} \mid k = 1, \cdots, n\}$. For the lower bound on c_m , choose any $K \in S \cap E_m$ such that $i(K) \geq mn$ and set $F = (E^+ \cap E_m) \oplus E^0 \oplus \operatorname{span}\{\theta_{11}, \zeta_{11}\}$. Then F is invariant and $\dim F = 2mn + 2n + 2$. By Lemma 1.24, $K \cap F \neq \phi$. Therefore if $\hat{z} \in K \cap F$ (and a fortiori $z \in S$) ,

$$\min_{z \in F \cap S} A(z) \leq A(\hat{z}) \leq \max_{z \in K} A(z) .$$

Since K was arbitrary,

$$\min_{F \cap S} A(z) \le c_m \ .$$

As in (1.27) - (1.28) ,

$$\min_{z \in F \cap S} A(z) \ge \min_{z \in F \cap B_{(\frac{2\pi}{\nu})^{\frac{1}{2}}}} A(z) = -\frac{\pi}{\nu} \ .$$

Next let $z_m = (p_m, q_m)$ denote a critical point of A/S_m corresponding to c_m. The Euler equation satisfied by z_m is

$$(1.29) \qquad 0 = \int_0^{2\pi} \{(p_m, \dot{q})_{\mathbb{R}^n} + (p, \dot{q}_m)_{\mathbb{R}^n} - \lambda_m[(\overline{H}_p(z_m), p)_{\mathbb{R}^n} + (\overline{H}_q(z_m), q)_{\mathbb{R}^n}]\} \, d\tau$$

for all $(p,q) \in E_m$ where λ_m is the Lagrange multiplier for our problem. Choosing $(p,q) = (p_m, q_m)$ and using (1.4) (iii) yields

$$(1.30) \qquad 2\int_0^{2\pi} (p_m, \dot{q}_m)_{\mathbb{R}^n} \, d\tau = 2\, c_m = \lambda_m \int_0^{2\pi} (z_m, \overline{H}_z(z_m))_{\mathbb{R}^{2n}} \, d\tau = 4\pi \lambda_m \ .$$

Combining (1.30) with Lemma 1.26, we have shown

Lemma 1.31: $-\frac{1}{2\nu} \le \lambda_m = \frac{c_m}{2\pi} \le -\frac{1}{2\mu}$.

Note in particular that λ_m is bounded away from 0 and $-\infty$ independently of m .

Lemma 1.32: There is a constant M independent of m such that

(1.33) $$\|z_m\|_E \le M .$$

Proof: We already have bounds for $\|z_m\|_{L^2}$ independent of m . Thus similar estimates are required for $\|\dot{z}_m\|_{L^2}$. Choosing $(p,q) = (\dot{q}_m, -\dot{p}_m)$ in (1.29) and integrating by parts gives

(1.34) $$\|\dot{z}_m\|_{L^2} \le |\lambda_m| \; \|\overline{H}_z(z_m)\|_{L^2} .$$

By (1.4) (iv),

$$|\overline{H}_z(\zeta)| \le a|\zeta|$$

for all $\zeta \in \mathbb{R}^{2n}$ where a depends on bounds in the C^1 norm for ψ and H restricted to $H^{-1}(1)$. Therefore,

(1.35) $$\|\dot{z}_m\|_{L^2} \le a|\lambda_m| \; \|z_m\|_{L^2}$$

so (1.33) follows from (1.25) and Lemma 1.31.

Completion of proof of Theorem 1.1: First we finish with the C^2 case. Lemmas 1.31, 1.32, and the Sobolev Imbedding Theorem imply that along some subsequence $\lambda_m \to \lambda \in (-\infty, 0)$ and z_m converges weakly in E and strongly in L^∞ to $z = (p, q) \in S$ satisfying

$$(1.36) \qquad 0 = \int_0^{2\pi} \{(p, \dot{\hat{q}})_{\mathbb{R}^n} + (\hat{p}, \dot{q})_{\mathbb{R}^n} - \lambda(\overline{H}_z(z), \hat{z})_{\mathbb{R}^{2n}}\} \, d\tau$$

for all $\hat{z} = (\hat{p}, \hat{q}) \in \bigcup_{m \in \mathbb{N}} E_m$. This implies (λ, z) satisfies (1.8) a.e. But the right hand side of (1.8) can be assumed to be continuous. Therefore $\dot{z}$ is also continuous and Theorem 1.1 is proved for this case.

Next suppose $H \in C^1$. Let $H_j(\zeta)$ be a sequence of C^2 functions which converge to H in the C^1 norm in a neighborhood of $H^{-1}(1)$. Then we can assume H_j satisfies (H1) - (H2) and by the result just proved, (1.8) possesses a solution $(\hat{\lambda}_j, \hat{z}_j)$ with $\overline{H}$ replaced by $\overline{H}_j$. Note in particular that $\hat{\lambda}_j$ satisfies the estimate of Lemma 1.31 with μ and ν replaced by

$$\mu_j = \max_{|\zeta| = 1} \overline{H}_j(\zeta) ; \qquad \nu_j = \min_{|\zeta| = 1} \overline{H}_j(\zeta) .$$

Since $\overline{H}_j(\zeta)$ converges uniformly to $\overline{H}(\zeta)$ for $\zeta \in S^{2n-1}$,

$$(1.37) \qquad -\frac{1}{\nu} \le \hat{\lambda}_j \le -\frac{1}{4\mu}$$

for all large j . Since $\hat{z}_j$ lies on $H_j^{-1}(1)$, (1.8) and the C^1 convergence of H_j to H near $H^{-1}(1)$ then provide uniform bounds for $\|\hat{z}_j\|_{C^1}$. Hence (1.8) , (1.37) and these bounds imply $(\hat{\lambda}_j, \hat{z}_j)$ converge along a subsequence to a solution (λ, z) of (1.8) with $z(\tau)$ lying on $H^{-1}(1)$. The proof is complete.

§2. The prescribed period case.

This section concerns the existence of solutions of (1.2) when the period is prescribed. The arguments given here are based strongly on those used in a recent paper of the author [11] dealing with periodic solutions of semilinear wave equations. As in [11], these methods can also be applied to the forced case where H depends explicitly on t in a periodic fashion. We begin with a free vibration problem in which $H_{zz}(0)$ vanishes. Then we indicate the modifications necessary to treat the analogous t dependent case. Next we investigate the situation where $H_{zz}(0)$ is a positive definite matrix. Lastly the special case $H(p,q) = Q(p) + V(q)$ will be studied.

The same notation as in §1 will be used below.

Theorem 2.1: Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies

(H3) $H(z) = o(|z|^2)$ at $z = 0$,

(H4) $H \geq 0$ and $0 < H(z) \leq \theta(z, H_z)_{\mathbb{R}^{2n}}$ for $|z| \geq \bar{r}$ where $\theta \in [0, \frac{1}{2})$.

Then for any $T > 0$, there exists a non-constant T-periodic solution of

(2.2) $$\frac{dz}{dt} = \mathcal{J} H_z$$

Remark 2.3: Observe that $z \equiv 0$ is a trivial periodic solution of (2.2) and (H4) permits the existence of other such trivial equilibrium solutions. Also note that if we write $z = r\zeta$ with $\zeta \in S^{2n-1}$, (H4) implies that

$$\frac{dH}{dr} = \frac{1}{r}(z, H_z)_{\mathbb{R}^{2n}} \geq \frac{1}{\theta r} H \tag{2.4}$$

for $r \geq \bar{r}$ and on integration (2.4) yields

$$H(z) \geq a_1 |z|^{\frac{1}{\theta}} - a_2 \tag{2.5}$$

for all $z \in \mathbb{R}^{2n}$ where $a_1, a_2 \geq 0$ are constants.

The proof of Theorem 2.1 is given as a sequence of lemmas. Again making the change of variables $\tau = 2\pi T^{-1} t \equiv \lambda^{-1} t$, it suffices to find a 2π periodic solution of

$$\dot{z} = \lambda \mathcal{J} H_z . \tag{2.6}$$

For $z \in E$, consider

$$\int_0^{2\pi} [(p, \dot{q})_{\mathbb{R}^n} - \lambda H(z)] \, d\tau . \tag{2.7}$$

A formal calculation shows critical points of (2.7) in E are solutions of (2.6). A direct treatment of this variational problem encounters the same obstacles as in §1 with the further complication that $z \equiv 0$ is a known critical point. As in §1 we use a finite dimensional approximation argument attempting to find critical points of (2.7) restricted to E_m and passing to a limit as earlier. Additional care must be taken to avoid the trivial solution $z \equiv 0$. Such a finite dimensional existence argument for (2.7) succeeds in determining approximate critical points but some difficulties in passing to a limit due to the fact that there are no restrictions on the growth of H at ∞ lead us to replace H by a modified function which grows at a prescribed rate (that we choose to be $|z|^4$) at ∞.

Let $K > \overline{r}$ and select $\chi \in C^{\infty}(\mathbb{R}^+, \mathbb{R}^+)$ such that $\chi(s) = 1$ if $s \le K$, $\chi(s) = 0$ if $s \ge K+1$, and $\chi'(s) < 0$ if $s \in (K, K+1)$. Set

$$H_K(z) = \chi(|z|)H(z) + \rho(K)(1-\chi(|z|))|z|^4 \,. \tag{2.8}$$

Lemma 2.9: H_K satisfies (H3) - (H4) with θ replaced by $\hat{\theta} = \max(\theta, \frac{1}{4})$ provided that

$$\rho(K) \ge (K+1)^{-4} \max_{K \le |z| \le K+1} H(z) \,. \tag{2.10}$$

Proof: (H3) is obvious. To verify (H4), note that

$$(z, H_{Kz})_{\mathbb{R}^{2n}} = \chi(z, H_z)_{\mathbb{R}^{2n}} + |z|\chi' H + \rho(K)[4(1-\chi)|z|^4 - \chi'|z|^5]$$

Thus for $\overline{r} \le |z| \le K$,

$$(z, H_{Kz})_{\mathbb{R}^{2n}} = (z, H_z)_{\mathbb{R}^{2n}} \ge \theta^{-1} H = \theta^{-1} H_K$$

while for $|z| \ge K+1$,

$$(z, H_{Kz})_{\mathbb{R}^{2n}} = 4\rho(K)|z|^4 = 4H_K .$$

Finally for $K \le |z| \le K+1$,

$$\begin{aligned}(z, H_{Kz})_{\mathbb{R}^{2n}} &\ge \theta^{-1}\chi H + 4\rho(K)(1-\chi)|z|^4 + |z|\chi'(H-\rho(K)|z|^4) \\ &\ge \min(\theta^{-1}, 4)H_K\end{aligned}$$

provided that (2.10) is satisfied. Hence (H4) holds.

For what follows, we assume $\rho(K)$ satisfies (2.10). Having introduced H_K, we replace (2.7) by

$$I(z) = \int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - \lambda H_K(z)]\, d\tau \tag{2.11}$$

and seek critical points of $I|_{E_m}$. This is a rather different situation than the constrained variational problem of §1. Critical points will be obtained with the aid of the following lemma. Here for
$s < j$, $\mathbb{R}^s = \{x \in \mathbb{R}^j | x = (x_1, \cdots, x_s, 0, \cdots, 0)\}$,
$(\mathbb{R}^s)^\perp = \{x \in \mathbb{R}^j | x = (0, \cdots, 0, x_{s+1}, \cdots, x_j)\}$ and $\hat{B}_r = \{x \in \mathbb{R}^j |\ |x| < r\}$.

<u>Lemma 2.12</u>: Let $J \in C^1(\mathbb{R}^j, \mathbb{R})$, $k < j$, and $I : \mathbb{R}^j \to \mathbb{R}$ with $J(x) \le I(x)$ for all $x \in \mathbb{R}^j$. Suppose

(I1) $I(x) \le 0$ for all $x \in \mathbb{R}^k$,

(I2) There is a constant $\rho > 0$ such that $J > 0$ in $(B_\rho \setminus \{0\}) \cap (\mathbb{R}^k)^\perp$

(I3) There is a constant $R > \rho$ such that $I \le 0$ in $\mathbb{R}^j \setminus B_R$.

Then J has a positive critical value characterized by

$$(2.13) \qquad b = \inf_{h \in \Gamma} \max_{x \in \overline{B_R} \cap \mathbb{R}^{k+1}} J(h(x))$$

where

$$\Gamma = \{h \in C(\overline{B_R} \cap \mathbb{R}^{k+1}, \mathbb{R}^j) | h(x) = x \text{ if } I(x) \le 0\} .$$

Proof: A proof can be found in [11] .

Identifying E_m with $\mathbb{R}^j$ where $j = 2n(2m+1)$, it is easy to verify that the hypotheses of Lemma 2.12 are satisfied by I as defined in (2.11) with $J = I$, $k = 2mn + 2n$, $\mathbb{R}^k = (E^o \oplus E^-) \cap E_m$, $(\mathbb{R}^k)^\perp = E^+ \cap E_m$, and $\mathbb{R}^{k+1} = (E^o \oplus E^- \oplus \text{span}\{\varphi_{11}\}) \cap E_m \equiv V_m$. Indeed (H4) implies (I1) and (I3) and (H3) implies (I2) . Thus (2.13) defines a critical value $b_m > 0$ of $I|_{E_m}$ where

$$\Gamma = \Gamma_m = \{h \in C(\overline{B_{R(m)}} \cap V_m, E_m) | h(z) = z \text{ if } I(z) \le 0\} .$$

The minimax characterization (2.13) will be used to obtain a uniform upper bound for b_m which will lead in turn to an estimate for z_m, a critical point of $I|_{E_m}$ corresponding to b_m .

Lemma 2.14: There are constants M_1, M_2 independent of m and K such that

$$b_m \leq M_1 \tag{2.15}$$

$$\int_0^{2\pi} (z_m, H_{Kz}(z_m))_{\mathbb{R}^{2n}}\, d\tau \leq M_2 \,. \tag{2.16}$$

Proof: Choose $h(z) = z \in \Gamma_m$. By (H4),

$$0 < b_m \leq \max_{z \in \bar{B}_R \cap V_m} I(z) \leq \max_{z \in V_m} I(z) \,. \tag{2.17}$$

Any function $z(\tau) \in V_m$ can be written as

$$z(\tau) = r(\zeta(\tau)\cos\omega + (2\pi)^{-\frac{1}{2}} \varphi_{11}(\tau)\sin\omega)$$

where $\zeta \in (E^- \oplus E^0) \cap E_m$ with $\|\zeta\|_{L^2} = 1$, $\omega \in [0, 2\pi]$, and $r = \|z\|_{L^2}$. Choosing $z \in V_m$ which maximizes the right hand side of (2.17) — such as z exists via (H4) — and substituting in (2.17) yields

$$\lambda \int_0^{2\pi} H_K(z)d\tau \leq \frac{1}{2}\|z\|^2_{L^2} \,. \tag{2.18}$$

Hency by (2.5) (for H_K), (2.18), and the Hölder inequality,

$$\frac{1}{2}\|z\|^2_{L^2} \geq a_1 \int_0^{2\pi} |z(\tau)|^{\frac{1}{\hat{\theta}}} d\tau - 2\pi a_2 \geq a_3 \|z\|^{\frac{1}{\hat{\theta}}}_{L^2} - 2\pi a_2 \tag{2.19}$$

where the constants a_2, a_3 are independent of K. Since $\hat{\theta}^{-1} > 2$, (2.19) implies a bound on $\|z\|_{L^2}$ independent of m and K, say $\|z\|_{L^2} \le \hat{r}$. Hence from (2.17) and the form of z,

$$b_m \le \frac{1}{2}\hat{r}^2 \equiv M_1 . \tag{2.20}$$

To obtain (2.16), observe that at z_m we have

$$I'(z_m)\zeta = 0 \tag{2.21}$$

for all $\zeta \in E_m$. Taking $\zeta = (\varphi, \psi)$ gives

$$0 = \int_0^{2\pi} [(p_m, \dot{\psi})_{\mathbb{R}^n} + (\varphi, \dot{q}_m)_{\mathbb{R}^n} - \lambda(H_{Kz}(z_m), \zeta)_{\mathbb{R}^{2n}}]\, d\tau \tag{2.22}$$

where $z_m = (p_m, q_m)$. Choosing $\zeta = z_m$ and forming $I(z_m) - \frac{1}{2} I'(z_m)z_m$ yields

$$\begin{aligned} b_m &= \lambda \int_0^{2\pi} [\tfrac{1}{2}(z_m, H_{Kz}(z_m))_{\mathbb{R}^{2n}} - H_K(z_m)]\, d\tau \\ &\ge \lambda \int_0^{2\pi} \{(\tfrac{1}{2} - \theta)\chi(|z_m|)(z_m, H_z(z_m))_{\mathbb{R}^{2n}} + \\ &\quad + \rho(K)(1-\chi(|z_m|))|z_m|^4 + \tfrac{1}{2}|z_m|\chi'(|z_m|)(H(z_m) - \rho(K)|z_m|^4)\}\, d\tau \\ &\quad - \lambda \int_{T_m} (H(z_m) + (\tfrac{1}{2} - \theta)(z_m, H_z(z_m))_{\mathbb{R}^{2n}})\, d\tau \end{aligned} \tag{2.23}$$

where $T_m = \{\tau \in [0,2\pi] \mid |z_m(\tau)| \le \overline{r}\}$. Hence using (2.10) we have

$$(2.24) \qquad b_m \ge \min((\tfrac{1}{2} - \theta), \tfrac{1}{4}) \int_0^{2\pi} (z_m, H_{Kz}(z_m))_{\mathbb{R}^{2n}} \, d\tau - a_4$$

where a_4 is independent of m and K so (2.24) implies (2.16).

The next step in the proof is to obtain bounds for $\dot{z}_m$.

Lemma 2.25: There is a constant M_3 depending on K but independent of m such that $\|\dot{z}_m\|_{L^2} \le M_3$.

Proof: Choosing $(\varphi, \psi) = (-\dot{p}_m, \dot{q}_m)$ in (2.22) and using the Schwarz inequality gives

$$(2.26) \qquad \|\dot{z}_m\|_{L^2} \le \lambda \|H_{Kz}(z_m)\|_{L^2} \le a_5(1 + \|z_m^3\|_{L^2})$$

via the definition of H_K where a_5 depends on K. The right hand side of (2.26) involves $\|z_m\|_{L^6}$ for which we do not yet have an upper bound. By (2.16) and the definition of H_K,

$$(2.27) \qquad \|z_m\|_{L^4} \le a_6$$

where a_6 depends on K. The Gagliardo-Nirenberg inequality [14] implies that

$$\|\zeta\|_{L^\infty} \le a_7 \|\zeta\|_E^{1/3} \|\zeta\|_{L^4}^{2/3} \tag{2.28}$$

for all $\zeta \in E$. Hence by (2.26) - (2.28),

$$\begin{aligned} \|z_m\|_{L^\infty} &\le a_7(\|z_m\|_{L^2}^2 + \|\dot{z}_m\|_{L^2}^2)^{1/6} \|z_m\|_{L^4}^{2/3} \\ &\le a_8(1 + \|z_m^3\|_{L^2}^2)^{1/6} \le a_8(1 + \|z_m\|_{L^\infty}^2 \|z_m\|_{L^4}^4)^{1/6} \\ &\le a_9(1 + \|z_m\|_{L^\infty}^2)^{1/6} \end{aligned} \tag{2.29}$$

Consequently (2.29) provides an upper bound for $\|z_m\|_{L^\infty}$ independent of m but depending on K. Returning to (2.26), we get the desired bound for $\|\dot{z}_m\|_{L^2}$.

By Lemma 2.25 and the Sobolev Imbedding Theorem, a subsequence of z_m converges weakly in E and strongly in L^∞ to $z_K = (p_K, q_K) \in E$ satisfying

$$0 = \int_0^{2\pi} [(p_K, \dot{\psi})_{\mathbb{R}^n} + (\varphi, \dot{q}_K)_{\mathbb{R}^n} - \lambda(H_{Kz}(z_K), \zeta)_{\mathbb{R}^{2n}}]\, d\tau \tag{2.30}$$

for all $\zeta = (\varphi, \psi) \in \bigcup_{m \in \mathbb{N}} E_m$. It follows as in §1 that z_K is a classical solution of

$$\dot{z} = \lambda \mathcal{J} H_{Kz}. \tag{2.31}$$

The following lemma enables us to obtain a solution of (2.6) from (2.31) .

Lemma 2.32 : There is a constant M_4 independent of K such that $\|z_K\|_{L^\infty} \le M_4$.

Proof : Suppose for convenience that $K \ge \bar{r}$. By Lemma 2.9 ,

$$(2.33) \qquad H_K(\zeta) \le \hat{\theta}(\zeta, H_{Kz}(\zeta))_{\mathbb{R}^{2n}} + M_5$$

for all $\zeta \in \mathbb{R}^{2n}$ where M_5 is independent of K . Choosing $\zeta = z_K$, integrating (2.33) with respect to τ and using (2.16) (which is valid for z_K) gives

$$(2.34) \qquad \int_0^{2\pi} H_K(z_K)\, d\tau \le \hat{\theta} M_2 + 2\pi M_5 .$$

Since z_K is a solution of the Hamiltonian system (2.31) , $H_K(z_K(\tau)) \equiv$ constant. Hence (2.34) implies

$$(2.35) \qquad H_K(z_K) \le \frac{\hat{\theta}}{2\pi} M_2 + M_5$$

and the lemma follows from (2.5) for H_K .

Lemma 2.32 implies that for $K \ge M_4$, $H_{Kz}(z_K) = H_z(z_K)$ and therefore z_K satisfies (2.6) . The next lemma completes the proof of Theorem 2.1.

Lemma 2.36: $z_K(\tau) \not\equiv$ constant.

Proof: We use a comparison argument. By (H3) - (H4) for each $\varepsilon > 0$, there is a positive constant A_ε depending on K such that

$$(2.37) \qquad H_K(z) \le \frac{\varepsilon}{2}|z|^2 + \frac{A_\varepsilon}{4}\sum_{i=1}^{n}(p_i^4 + q_i^4) \equiv \hat{H}_K(z)$$

for all $z \in \mathbb{R}^{2n}$ where $z = (p,q) \equiv (p_1,\cdots,p_n,q_1,\cdots,q_n)$.
For $z \in E$, define

$$J(z) = \int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - \lambda\hat{H}_K(z)]\, d\tau .$$

By (2.37), $J(z) \le I(z)$ for all $z \in E$. It is easy to verify that for ε sufficiently small (and independent of m) $J|_{E_m}$ satisfies the hypotheses of Lemma 2.12. Therefore by that lemma, $J|_{E_m}$ has a critical value $\hat{b}_m$ and by (2.13),

$$(2.38) \qquad 0 < \hat{b}_m \le b_m .$$

If $\hat{z}_m$ is a critical point of $J|_{E_m}$ corresponding to $\hat{b}_m$, then

$$(2.39) \qquad \hat{b}_m = J(\hat{z}_m) - \frac{1}{2}J'(\hat{z}_m)\hat{z}_m = \frac{\lambda}{4}A_\varepsilon\int_0^{2\pi}\sum_{i=1}^{n}(\hat{p}_i^4 + \hat{q}_i^4)\, d\tau .$$

Now suppose that $z_K(\tau)$ is a constant. Since z_K satisfies (2.31), $H_K(z_K) = 0$. Along some subsequence we have $z_m \xrightarrow{L^\infty} z_K$ and therefore $H_K(z_m) \xrightarrow{L^\infty} H_K(z_K)$. From (2.26) we see that $\dot{z}_m \xrightarrow{L^2} 0$. Observing that $b_m > 0$, the form of I implies that

$$b_m = I(z_m) \to -\lambda \int_0^{2\pi} H_K(z_K)\, d\tau = 0 \quad . \tag{2.40}$$

By (2.38), $\hat{b}_m \to 0$ along the same subsequence and by (2.39), $\hat{z}_m \xrightarrow{L^4} 0$. Lemmas 2.14 and 2.25 applied to the J problem show that $\{\hat{z}_m\}$ is bounded in E. Hence by (2.28), $\hat{z}_m \xrightarrow{L^\infty} 0$ along our subsequence. We claim this is impossible.

Indeed dropping the subscript m, we write $\hat{z} = \zeta + Z$ where $\zeta = (\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n) \in E^0$ and $z = (P_1, \cdots, P_n, Q_1, \cdots, Q_n) \in (E^+ \oplus E^-) \cap E_m$ Choosing $(\varphi, \psi) = \zeta$ in (2.22) shows

$$2\pi(\varepsilon|\zeta|^2 + A_\varepsilon \sum_{i=1}^{n} (\xi_i^4 + \eta_i^4)) = A_\varepsilon \sum_{i=1}^{n} \int_0^{2\pi} [(\xi_i^3 - \hat{p}_i^3)\xi_i + (\eta_i^3 - \hat{q}_i^3)\eta_i]\, d\tau \quad . \tag{2.41}$$

Simplifying gives

$$2\pi \sum_{i=1}^{n} (\xi_i^4 + \eta_i^4) \le - \sum_{i=1}^{n} \int_0^{2\pi} [(3\xi_i^2 P_i + 3\xi_i P_i^2 + P_i^3)\xi_i + (3\eta_i^2 Q_i + 3\eta_i Q_i^2 + Q_i^3)\eta_i]\, d\tau \, . \tag{2.42}$$

Consequently by the Hölder inequality and some crude estimates

$$(2.43) \qquad |\zeta| \le \alpha_1 \|Z\|_{L^4} \le \alpha_2 \|Z\|_{L^\infty} .$$

From (2.22) again with $(\varphi, \psi) = (\dot{Q}, -\dot{P})$, we get

$$(2.44) \qquad \|\dot{Z}\|^2_{L^2} \le \varepsilon \|Z\|^2_{L^2} + \alpha_3 \|\hat{z}\|^3_{L^\infty} \|Z\|_{L^2} \le 2\varepsilon \|Z\|^2_{L^2} + \alpha_4 \|\hat{z}\|^6_{L^\infty}$$

where we used Young's inequality and α_4 depends on ε. For any $w \in E^+ \oplus E^-$,

$$w(\tau) - w(t) = \int_t^\tau \dot{w}(s)ds .$$

Integrating with respect to t shows

$$2\pi w(\tau) = \int_0^{2\pi} (\int_t^\tau \dot{w}(s)ds) \, d\tau .$$

Hence

$$(2.45) \qquad \|w\|_{L^\infty} \le \|\dot{w}\|_{L^1} \le (2\pi)^{1/2} \|\dot{w}\|_{L^2}$$

Combining (2.43) - (2.45) yields

$$(2.46) \qquad \|Z\|^2_{L^\infty} \le 2(2\pi)^3 \varepsilon \|Z\|^2_{L^\infty} + \alpha_5 \|Z\|^6_{L^\infty} .$$

Letting $\varepsilon = (2\pi)^{-4}$, (2.46) implies $\{\|Z_m\|_{L^\infty}\}$ is bounded away from 0. ($Z_m \neq 0$ for then $J(\hat{z}_m) < 0$). Therefore $\hat{z}_m$ cannot converge to 0 in L^∞ contrary to what was shown above. The proof is complete.

Next we study the effect of weakening the hypotheses of Theorem 2.1. The ideas used here are similar to those in the proof of the above theorem so we will be brief.

Suppose first that $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$, $H(t,z)$ has period T in t, and (H3)-(H4) are satisfied. We then seek a T periodic solution of (2.2) or making our usual change of variables, a 2π periodic solution of (2.6). An examination of the proof of Theorem 2.1 shows that it carries over essentially verbatim to this case with the exception of Lemma 2.32. Thus we have a nonconstant 2π periodic solution z_K of (2.31) for each $K > 0$ and it remains to show that for K sufficiently large, z_K satisfies (2.6). We do not know if this is the case without imposing further conditions on H, the difficulty being that (2.35) is no longer valid. However suppose we assume

(H5) $$|H_z(t,\zeta)| \le \alpha(\zeta, H_z(\zeta))_{\mathbb{R}^{2n}} + \beta$$

for all $\zeta \in \mathbb{R}^{2n}$ for some constants $\alpha, \beta > 0$, i.e. the radial component of H_z cannot get too small compared to $|H_z|$. Then we can obtain the necessary pointwise bound for z_K. Indeed by (2.34), (2.5) and the Hölder inequality,

$$(2.47) \qquad \|z_K\|_{L^2} \le M_6$$

where M_6 is independent of K. Furthermore by (2.31), (H5), and (2.16)

$$(2.48) \qquad \|\dot{z}_K\|_{L^1} \le \lambda \|H_{Kz}(\cdot, z_K)\|_{L^1} \le M_7 ,$$

M_7 being independent of K. Letting $z_K = \zeta_K + Z_K$ where $\zeta_K \in E^0$ and $Z_K \in E^+ \oplus E^-$, (2.47) gives a bound for $|\zeta_K|$ independent of K while (2.45) and (2.48) provide a similar bound for $\|Z_K\|_{L^\infty}$. This gives the desired estimate for $\|z_K\|_{L^\infty}$ and we have shown

<u>Theorem 2.49</u>: *Let $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ with H having period T in t and satisfying (H3)-(H5). Then (2.2) possesses at least one nonconstant T periodic solution.*

<u>Remark 2.50</u>: Other hypotheses can be used instead of (H5) to get existence results in the t dependent case. For example, consider the perturbation situation where $H(t, z) = \tilde{H}(z) + G(t, z)$ with $\tilde{H}$ satisfying (H3)-(H4), $G \ge 0$ and satisfying (H3), and G is uniformly bounded in C^1 over $\mathbb{R} \times \mathbb{R}^{2n}$. Then (2.31) has a solution z_K as earlier and

$$\frac{d}{d\tau} H_K(t, z_K) = \frac{\partial G}{\partial \tau}(\tau, z_K) .$$

Therefore

$$H_K(\tau, z_K(\tau)) \le \frac{1}{2\pi} \| H_K(\cdot, z_K) \|_{L^1} + \| \frac{\partial G}{\partial \tau}(\tau, z_K(\tau)) \|_{L^\infty}$$

$$\le \hat{\theta} M_2 + \text{constant}$$

via (2.34). Hence (2.5) gives a K independent bound for $\|z_K\|_{L^\infty}$.

Next we study the effect of replacing (H3)-(H4) by

(H6) $H(z) = Q(z) + \hat{H}(z)$ where $\hat{H}$ satisfies (H3)-(H4) and $Q(z)$ is a positive definite quadratic form.

Theorem 2.51: If $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies (H6), then for any $T > 0$, (2.2) possesses a nonconstant T-periodic solution.

Proof: We will sketch the proof. Consider first the linear Hamiltonian system

(2.52) $$\dot{w} = \mathcal{J} Q_{zz} w .$$

Since Q is positive definite, (2.52) possesses $2n$ linearly independent periodic solutions. Indeed if μ is an eigenvalue of $\mathcal{J} Q_{zz}$ with a corresponding eigenvector ξ, then μ is purely imaginary and (2.52) has a solution of the form $w(t) = e^{\mu t}\xi$ with period $2\pi i \mu^{-1}$. Since (2.52) is a real system, $\overline{w}(t)$ is also a solution of (2.52). Let $\xi_1, \cdots, \xi_{2n}$ be linearly independent eigenvectors of $\mathcal{J} Q_{zz}$ with corresponding eigenvalues $\mu_1, \cdots, \mu_{2n}$. We can assume $\xi_{j+n} = \overline{\xi}_j$, $1 \le j \le n$. Set

$$f_{jk} = e^{ik\tau}\xi_j - e^{-ik\tau}\bar{\xi}_j\,, \quad j = 1,\cdots,2n, \quad k \in \mathbb{Z}\,.$$

Then $z = f_{jk}$ satisfies

$$\dot{z} = \frac{ik}{\mu}\mathcal{J}Q_{zz}z$$

and

$$\int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - \lambda Q(z)]\,d\tau = (\frac{ik}{\mu_j} - \lambda)\int_0^{2\pi} Q(z)d\tau\,.$$

We can assume that the functions $f_{jk}(\tau)$ are A orthogonal in the sense of Lemma 1.11. Note that $E_m = \text{span}\{f_{jk} | 1 \le j \le 2n,\ |k| \le m\}$.

Let

$$N^+ = \text{span}\{f_{jk} | \frac{ik}{\mu_j} > \lambda\}$$

$$N^- = \text{span}\{f_{jk} | \frac{ik}{\mu_j} < \lambda\}$$

$$N^o = \text{span}\{f_{jk} | \frac{ik}{\mu_j} = \lambda\}\,.$$

Now as usual to get a solution of (2.2) we try to find critical points of

$$I(z) = \int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - \lambda(Q(z) + \hat{H}_K(z))]\,d\tau$$

where $\hat{H}_K = \chi\hat{H} + (1-\chi)|z|^4$ and we have made our usual change of the time variable. Applying Lemma 2.12 to $I|_{E_m}$ with $\mathbb{R}^k \simeq (N^- \oplus N^0) \cap E_m$ and $\mathbb{R}^{k+1} \simeq N^- \oplus N^0 \oplus \text{span}\{f_{1\hat{k}}\}$ where $\hat{k}$ is the smallest positive value of k such that $\frac{ik}{\mu_j} > \lambda$ for some eigenvalue μ_j, we get a corresponding critical value b_m and critical point z_m as in the proof of Theorem 2.1. With minor modifications, the earlier proof then yields a solution z of (2.31). It remains to show $z(\tau) \not\equiv$ constant, i.e. we need an analogue of Lemma 2.36 for this problem. By the argument of Lemma 2.36 (modified by the inclusion of a Q term in J), $\hat{z}_m \xrightarrow{L^\infty} 0$ as earlier. Again we will show this is impossible. Two cases are considered:

Case i: $N^0 = \{0\}$.

Then

$$\dot{w} = \lambda \mathcal{J} Q_{zz} w$$

has no nontrivial null vectors. Hence the same is true of

$$\dot{w} = \lambda \mathcal{J} Q_{zz} w + \varepsilon\lambda \mathcal{J} w$$

for all ε sufficiently small, say $\varepsilon < \hat{\varepsilon}$. Equation (2.21) for J' with $z = z_m$ can be written as:

$$\dot{\hat{z}}_m - \lambda \mathcal{J} Q_{zz} \hat{z}_m - \lambda\varepsilon \mathcal{J} \hat{z}_m = \lambda P_m \mathcal{J} G_z(\hat{z}_m) \tag{2.53}$$

where P_m denotes the L^2 orthogonal projector of E onto E_m and $G(z) = \frac{1}{4} A_\varepsilon \sum_{i=1}^{n} (p_i^4 + q_i^4)$. Since the left hand side of (2.53) is an isomorphism from e.g. E to L^2, by the Sobolev Imbedding Theorem,

(2.54) $$\|\hat{z}_m\|_{L^\infty} \le \alpha_1 \|\hat{z}_m\|_E \le \alpha_2 \|\hat{z}_m\|_{L^\infty}^3$$

with α_1, α_2 independent of ε provided that $\varepsilon \le \hat{\varepsilon}$. Consequently $\hat{z}_m$ is bounded away from 0 in L^∞ and this case is settled.

Case ii: $\dim N^0 > 0$.

Replace Q by $Q_\delta(z) = Q(z) + \delta|z|^2$ where $\delta > 0$. Let

$$I_\delta(z) = I(z) - \delta \int_0^{2\pi} |z|^2 \, d\tau$$

and define J_δ similarly where ε is small compared to δ. Thus

(2.55) $$J_\delta(z) \le I_\delta(z) \le I(z)$$

for all $z \in E$. Let

$$N_\delta^- = \operatorname{span}\{f_{jk} \mid \frac{ik}{\mu_j} < \lambda + \delta\}, \text{ etc.}$$

Then $N^o \oplus N^- \subset N_\delta^-$ and $N_\delta^o = \{0\}$ for small $\delta > 0$. If $I(z_m) \to 0$, then $J_\delta(\tilde{z}_m) \to 0$ where $\tilde{z}_m$ is a critical point of $J_\delta|_{E_m}$ satisfying $0 < J_\delta(\tilde{z}_m) \le I(z_m)$. But the argument of Case i implies $\{\|\tilde{z}_m\|_{L^\infty}\}$ is bounded away from 0. The proof is complete.

Remark 2.56: As was mentioned in the Introduction, Theorem 2.51 must be interpreted with care since it contains no information on minimal periods of solutions of (2.2). It is easy to give examples where any minimal period is bounded from above. E.g. if $H(z) = |z|^2 + f(|z|^2) \equiv g(|z|^2)$, the corresponding Hamiltonian system can be written in complex form as

$$\dot{\zeta} = 2ig'(|\zeta|^2)\zeta \tag{2.57}$$

where $\zeta = p + iq$. Hence $\zeta = \zeta_0 e^{2ig't}$ and if T is the minimal period of (2.57),

$$T = \frac{\pi}{g'(|\zeta|^2)} \le \pi$$

provided that $f' \ge 0$.

We conclude this section with a case of interest in mechanics which is not covered directly by the theory presented thus far. Suppose $H(p,q) = Q(\dot{q}) + V(q)$ where Q is a positive definite quadratic form and V is non-negative. For simplicity, we take $Q(p) = \frac{1}{2}|p|^2$. Then the corresponding Hamiltonian system is

(2.58) $$\dot{p} = -V_q, \ \dot{q} = p$$

or

(2.59) $$\ddot{q} = -V_q \ .$$

Treating the corresponding variational problem (for 2π periodic solutions) involves studying

(2.60) $$I(q) = \int_0^{2\pi} [\tfrac{1}{2}|\dot{q}|^2 - \lambda^2 V(q)] \, d\tau \ .$$

This functional can be dealt with in a more direct fashion than our previous cases.

<u>Theorem 2.61</u>: Suppose $V \in C^1(\mathbb{R}^n, \mathbb{R})$ and satisfies (H3)-(H4). Then for every $T > 0$, (2.59) possesses a non-constant T-periodic solution.

<u>Proof</u>: We will produce the solution as a critical point of I on $\tilde{E} = \{(p,q) \in E \mid p = 0\}$. No approximation argument is needed for this case. We use the following generalization of Lemma 2.12.

Below (PS) denotes the Palais-Smale condition which means every sequence u_m such that $I(u_m)$ is bounded and $I'(u_m) \to 0$ is precompact.

Lemma 2.62: Let X be a real Banach space, $I \in C^1(X, \mathbb{R})$, and satisfy (PS). Suppose further that $X = X_k \oplus \hat{X}$ where $\dim X_k = k$ and

(I4) $I|_{X_k} \leq 0$.

(I5) There are constants $\rho, \alpha > 0$ such that $I > 0$ in $(B_\rho \setminus \{0\}) \cap \hat{X}$ and $I \geq \alpha$ on $\partial B_\rho \cap \hat{X}$.

(I6) For each finite dimensional subspace $Y \subset X$, there is an $R = R(Y)$ such that $I \leq 0$ on $Y \setminus B_R$.

Then I has a positive critical value c in X characterized by

$$c = \inf_{h \in \Gamma} \max_{u \in \bar{B}_{R(X_{k+1})} \cap X_{k+1}} I(h(u))$$

where

$$\Gamma = \{h \in C(B_{R(X_{k+1})} \cap X_{k+1}, X) \,|\, h(u) = u \text{ if } I(u) \leq 0\}$$

and $X_{k+1} = X_k \oplus \operatorname{span}\{\varphi\}$ for any fixed $\varphi \in \hat{X} \setminus \{0\}$.

Proof: The proof of Lemma 2.62 can be found in [15].

Proof of Theorem 2.61: Take $X = \tilde{E}$, $X_k = E^0 \cap \tilde{E}$, and $\hat{X} = (E^0)^\perp \cap \tilde{E}$. Then it is easy to verify that $I \in C^1(X, \mathbb{R})$ and (I4)-(I6) are satisfied. (E.g. (6) follows with the aid of (2.5)). Assuming (PS) for the moment, Theorem 2.61 is immediate since any critical point of q of I in $\tilde{E}$ is a classical solution of (2.59) and since $c > 0$, q is non-constant.

To verify (PS), first define

$$\|q\|_\beta = (\|\dot{q}\|_{L^2}^2 + \beta\|q\|_{L^2}^2)^{\frac{1}{2}}$$

where $\beta > 0$. Thus $\|\cdot\|_\beta$ is equivalent to $\|\cdot\|_{\tilde{E}}$. Now suppose q_m is a sequence in $\tilde{E}$ with $|I(q_m)| \le K$ and $I'(q_m) \to 0$. Then for m large enough,

$$|I'(q_m)q_m| = |\,\|q_m\|_\beta^2 - \beta\|q_m\|_{L^2}^2 - \lambda^2 \int_0^{2\pi} (q_m, V_q(q_m))_{\mathbb{R}^n}\, d\tau| \le \|q_m\|_\beta \tag{2.63}$$

Therefore

$$\lambda^2 \int_0^{2\pi} (q_m, V_q(q_m))_{\mathbb{R}^n}\, d\tau \le \|q_m\|_\beta + \|q_m\|_\beta^2 . \tag{2.64}$$

For $y \in \mathbb{R}^n$ sufficiently large, by (H4) and (2.5),

$$|y|^2 \le \theta(y, V_q(y))_{\mathbb{R}^n} . \tag{2.65}$$

Hence by (H4) and (2.64)-(2.65), for some constant γ independent of m,

$$K \geq \frac{1}{2}\|q_m\|_\beta^2 - \frac{\beta}{2}\|q_m\|_{L^2}^2 - \lambda^2 \int_0^{2\pi} V(q_m)\, d\tau \tag{2.66}$$

$$\geq \frac{1}{2}\|q_m\|_\beta^2 - (\frac{\beta}{2} + \lambda^2)\theta\lambda^{-2}(\|q_m\|_\beta + \|q_m\|_\beta^2) - \gamma$$

Choosing $\beta \in (0, \lambda^2(1-2\theta))$, (2.66) gives a uniform bound for $\|q_m\|_{\hat{E}}$. Thus a subsequence of q_m converges weakly in $\tilde{E}$ and strongly in L^∞ to $\bar{q} \in \tilde{E}$. Moreover along this subsequence the component of q_m in $E^0 \cap \tilde{E}$, i.e. the mean value of q_m converges to the mean value of $\bar{q}$. Restricted to $(E^0)^\perp \cap \tilde{E}$, $I'(q)$ is of the form $q - \Phi(q)$ where Φ is compact. Hence $I'(q_m) \to 0$ implies $q_m \to \Phi(\bar{q})$ along our subsequence and (PS) is verified.

<u>Remark 2.67</u>: If V depends explicitly on t in a time periodic fashion, the above proof carries over unchanged and (H5) is not necessary here. Likewise if (H3)-(H4) is replaced by (H6) (with $Q(q) = \frac{1}{2}|q|^2$) we get the analogue of Theorem 2.51 directly via Lemma 2.62. Finally if $\frac{1}{2}|q|^2$ is replaced by a positive definite quadratic Q and (2.58) is appropriately changed, similar results obtain. However we will not carry out the details.

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The existence of periodic solutions of Hamiltonian systems of ordinary differential equations is proved in various settings. A case in which energy is prescribed is treated in Section 1. Both free and forced vibration problems, where the period is fixed, are studied in Section 2. The proofs involve finite dimensional approximation arguments, variational methods, and appropriate estimates.

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