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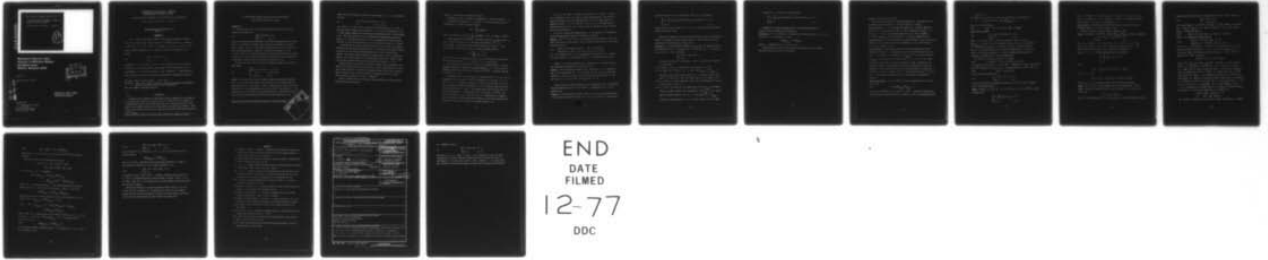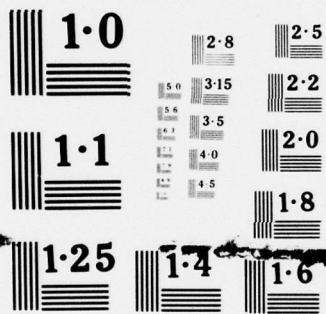JUL 77   H AMANN, M G CRANDALL   DAAG29-75-C-0024
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ON SOME EXISTENCE THEOREMS FOR SEMI-
LINEAR ELLIPTIC EQUATIONS

Herbert Amann & Michael G. Crandall

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ON SOME EXISTENCE THEOREMS FOR SEMI-LINEAR ELLIPTIC EQUATIONS

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ABSTRACT

Let A denote a strongly elliptic second order differential operator
and B be a first order boundary operator acting on functions u defined on
$\Omega \subseteq \mathbb{R}^N$. Let $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy $|f(x,\xi,\eta)| \leq C(|\xi|)(1 + |\eta|^2)$.
Under suitable assumptions, it is shown how the set of solutions of the
problem

$$(1) \qquad \begin{cases} Au = f(x,u,Du) & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

satisfying $\bar{u} \leq u \leq \hat{u}$ where $\bar{u}$ and $\hat{u}$ are, respectively, sub- and super-
solutions of (1) can be naturally identified with the fixed point set of a
self-mapping T of the order interval $[\bar{u},\hat{u}]$. Moreover, T has many desir-
able properties from which existence and multiplicity theorems are obtained.

AMS (MOS) Subject Classifications: 35J25, 35J60

Key Words: semi-linear elliptic equations, nonlinear boundary-value problem,
monotone iteration schemes, positive mappings

Work Unit Number 1 (Applied Analysis)

EXPLANATION

Boundary value problems for semilinear elliptic equations are considered.
If the nonlinear terms do not grow too fast as a function of the gradient of
the dependent variable and ordered upper and lower solutions are known, then
maximal and minimal solutions can be obtained by an iteration procedure. Other
results concerning the existence of additional solutions follow from topological
principles.

ON SOME EXISTENCE THEOREMS FOR SEMI-LINEAR ELLIPTIC EQUATIONS

Herbert Amann and Michael G. Crandall

Introduction

In a recent paper [3] one of the authors showed that the semi-linear elliptic boundary-value problem (BVP)

$$(1) \quad \begin{cases} Au = f(x,u,Du) & \text{in } \Omega \\ Bu = 0 & \text{on } \Omega \end{cases}$$

has one or several solutions provided that suitable sub- and supersolutions are known. Here A is a second order strongly elliptic differential operator on the bounded domain $\Omega \subseteq \mathbb{R}^N$, and B is a first order boundary operator of the sort which allows use of the maximum principle. Moreover, Du denotes the gradient of u and $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is required to satisfy certain assumptions detailed in Section 1. In this paper we obtain (slight) generalizations of these results by an argument which is considerably simpler and more direct than the presentation in [3].

The device used in [3] was to study the corresponding parabolic initial-boundary value problem

$$(2) \quad \begin{cases} \dfrac{\partial u}{\partial t} + Au = f(x,u,Du) & \text{in } \Omega \times (0,\infty) \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(\cdot,0) = u_0 & \text{on } \bar{\Omega} \ . \end{cases}$$

Roughly, it was shown that solving (2) yields an order preserving compact semigroup $\{S(t) : t \geq 0\}$ on the order interval $[\bar{u},\hat{u}]$ whenever $\bar{u},\hat{u}$ are ordered upper and lower solutions of (2). Rest points of $S(t)$ are solutions of (1), and one can obtain rest points of $S(t)$ by applying abstract results to $S(2^{-n})$ and letting $n \to \infty$. An essential point in this program is to associate solutions of (1) with fixed points of order preserving compact mappings with the same sub- and supersolutions as (1). A

simpler idea which achieves this same end is the following: Let $\lambda > 0$, $h \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ and solve

(3)
$$\begin{cases} u + \lambda(Au - f(x,u,h(Du))) = g & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega \end{cases},$$

for u when g in $[\bar{u},\hat{u}]$ is given. Denote the associated mapping by $u = T(g)$. In this paper (modulo technicalities) we show that $\lambda > 0$ and h can be chosen so that T is well-defined, compact, order preserving, and the fixed points of T are solutions of (1). This allows results for (1) to be deduced from standard abstract principles applied to T. The auxiliary problem (3) is in fact suggested by (2), because expressions like (3) arise upon approximating (2) with an implicit difference (in t) scheme.

A related scheme was used by Chandra and Davis in [6] for the case of an ordinary differential equation which depends linearly on Du. We learned of the papers of Chandra and Bernfeld [7] and Chandra, Laksmikantham and Leela [8] after our own work was complete. These discuss ordinary differential equation cases in which f depends non-linearly on Du and the equation is an infinite system respectively. However, assumptions are made eliminating the need for the analog of our truncator h in (3).

Beyond the facts that the proofs given here are simpler and more constructive than those in [3], we expect the idea of the proof to be useful in other problems (e.g., non-linear boundary conditions). Moreover, we are able to obtain a sharp result, Theorem 3, which is not given in [3]. Finally, using Bony's maximum principle [5], we obtain some new technical generality at no cost in complexity.

Section 1 contains notations, definitions, and the statements of the main results. The proofs are given in Section 2.

Section 1. Preliminaries and Statements of Results.

Throughout this paper Ω denotes a bounded domain in $\mathbb{R}^N$ whose boundary Γ is a C^2 submanifold of dimension $N - 1$ such that Ω lies locally on one side of Γ. We denote by A the real differential operator

$$Au = - \sum_{j,k=1}^{N} a_{jk} D_j D_k u$$

where $a_{jk} \in C(\bar{\Omega})$, $a_{jk} = a_{kj}$ and $\sum_{j,k=1}^{N} a_{jk}(x) \xi^j \xi^k > 0$ for $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^N \setminus \{0\}$.

We also suppose that the boundary is the disjoint union of two _closed_ subsets Γ_0 and Γ_1 each of which is an $N - 1$ dimensional submanifold of $\mathbb{R}^N$. Let $\beta \in C^1(\Gamma_1; \mathbb{R}^N)$ be an outward pointing, nowhere tangent vector field on Γ_1 and $b_0 \in C^1(\Gamma_1; \mathbb{R})$ satisfy $b_0 \geq 0$. Then B denotes the boundary operator

$$Bu = \begin{cases} u & \text{on } \Gamma_0 , \\ \dfrac{\partial u}{\partial \beta} + b_0 u & \text{on } \Gamma_1 . \end{cases}$$

Thus B is the Dirichlet boundary operator on Γ_0 and the Neumann or regular oblique derivative boundary operator on Γ_1. Either Γ_0 or Γ_1 may be empty.

In the following all function spaces consist of real-valued functions. Moreover, throughout this paper p denotes a _real number satisfying_ $p > N$. Hence the Sobolev imbedding theorems imply that the usual Sobolev space $W^{k,p}(\Omega)$ is compactly imbedded in $C^{k-1}(\bar{\Omega})$ for $k = 1,2$.

The nonlinear term f in (1.1) will be a continuous function $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. A _solution_ of (1) is a function $u \in W^{2,p}(\Omega)$ satisfying $Au = f(x,u,Du)$ a.e. on Ω and $Bu = 0$ on Γ. One defines solutions of (3) and its variants below in the same way. The notion of solution is independent of $p > N$ by classical regularity results. If the coefficients of A, f and Γ are suitably smooth, then every solution of (1) is a classical solution. A supersolution of (1) is a function $u \in W^{2,p}(\Omega)$ such that $Au \geq f(x,u,Du)$ a.e. in Ω and $Bu \geq 0$ on Γ. A supersolution is _strict_ if it is not a solution. Similarly, one defines subsolutions and strict subsolutions by reversing the inequalities. (These notions evidently depend on p.)

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If $v, w : \Omega \to \mathbb{R}$ then $v \leq w$ means $v(x) \leq w(x)$ a.e. on Ω and $v < w$ means $v \leq w$ and moreover $v(x) < w(x)$ holds on a set of positive measure. The relation $\leq$ induces an ordering on each space considered below. Let $[v,w]$ denote the set of measurable functions $u : \Omega \to \mathbb{R}$ such that $v \leq u \leq w$. If X is a Banach space of real-valued functions on $\bar{\Omega}$ we set $[v,w]_X = [v,w] \cap X$ and regard $[v,w]_X$ as having the relative topology from X.

If $f(x,\xi,\eta)$ grows at most quadratically in η, the existence of an ordered pair of sub- and supersolutions implies the existence of a solution:

Theorem 1. Let $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $\partial f/\partial \xi$ and $\partial f/\partial \eta$ exist and are continuous where (x,ξ,η) denotes a generic point of $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$. Assume moreover that:

$$(1.1) \quad \begin{cases} \text{There is an increasing function } c : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that} \\ |f(x,\xi,\eta)| \leq c(|\xi|)(1 + |\eta|^2) \quad \text{for} \quad (x,\xi,\eta) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N . \end{cases}$$

Let $\bar{v}$ be a subsolution and $\hat{v}$ be a supersolution of (1) such that $\bar{v} \leq \hat{v}$. Then (1) has a least solution $\bar{u}$ and a greatest solution $\hat{u}$ in the order interval $[\bar{v},\hat{v}]$.

Theorem 1 is a slight generalization of [3, Theorem (1.1)]. The next theorem is the corresponding generalization of [3, Theorem (1.6)].

Theorem 2. Let the hypotheses of Theorem 1 hold. Suppose that $\bar{v}_j$ is a subsolution and $\hat{v}_j$ is a supersolution for $j = 1,2$ such that $\bar{v}_1 < \hat{v}_1 < \bar{v}_2 < \hat{v}_2$. Assume moreover that $\hat{v}_1$ and $\bar{v}_2$ are strict. Then (1) has at least three solutions u_j such that $\bar{v}_1 \leq u_1 < u_3 < u_2 \leq \hat{v}_2$ and $u_j \in [\bar{v}_j, \hat{v}_j]$ for $j = 1,2$.

The next result is a sharp statement which was not obtained in [3]. It generalizes the corresponding result for the case in which f is independent of Du (cf. [2, Theorem (15.3)).

Theorem 3. Let the hypotheses of Theorem 1 hold. Let $\bar{v}$ be a strict subsolution and $\hat{v}$ be a strict supersolution such that $\bar{v} < \hat{v}$. If $u_1 = \bar{u}$ and $u_2 = \hat{u}$ are the least

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and greatest solutions of (1) in the interval $[\bar{v},\hat{v}]$, $u_1 \neq u_2$ and the BVP

$$
\begin{cases}
Ah - \displaystyle\sum_{j=1}^{N} f_{\eta_j}(x,u_i(x),Du_i(x))D_j h - f_\xi(x,u_i(x),Du_i(x))h = 0 & \text{in } \Omega \\[2mm]
Bh = 0 & \text{on } \Gamma
\end{cases}
$$

does not have a positive solution for $i = 1$ or $i = 2$, then (1) has at least three distinct solutions in $[\bar{v},\hat{v}]$.

The main new ingredient in our proof of the above results is the next proposition, which is of independent interest. Indeed, Theorems 1, 2 and 3 follow at once from this proposition and known abstract results ([2]).

<u>Proposition 1</u>. Let the hypotheses of Theorem 1 hold. Let $\bar{v}$ be a subsolution and $\hat{v}$ be a supersolution of (1). Then there exist $h \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ and $\lambda > 0$ such that

(a) For every $g \in [\bar{v},\hat{v}]$ the problem

$$
\begin{cases}
u + \lambda(Au - f(x,u,h(Du))) = g \\
Bu = 0
\end{cases}
$$

has exactly one solution u satisfying $u \in [\bar{v},\hat{v}]$. This solution is denoted by $u = T(g)$ below.

(b) A function $u \in [\bar{v},\hat{v}]$ is a solution of (1) if and only if $u = T(u)$.

(c) Let $C_B^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : Bu = 0 \text{ on } \Gamma\}$. Then $T : [\bar{v},\hat{v}]_{L^p(\Omega)} \to [\bar{v},\hat{v}]_{C_B^1(\Omega)}$ is continuous, compact and strongly increasing (i.e. $g \leq \hat{g}$ implies $T(g) \leq T(\hat{g})$ and if $g < \hat{g}$ then $T(\hat{g}) - T(g)$ lies in the interior of $\{u \in C_B^1(\bar{\Omega}) : u \geq 0\}$).

(d) If $w \in [\bar{v},\hat{v}]$ is a strict subsolution (resp., strict supersolution) of (1), then $w < T(w)$ (resp., $T(w) < w$).

(e) If $\bar{v}$ and $\hat{v}$ are strict sub- and supersolutions then $[\bar{v},\hat{v}]_{C_B^1(\bar{\Omega})}$ has nonempty interior in $C_B^1(\bar{\Omega})$. Moreover, as a self-mapping of $[\bar{v},\hat{v}]_{C_B^1(\bar{\Omega})}$ T has a strongly positive Fréchet derivative $T'(u)$ for u in the interior of $[\bar{v},\hat{v}]_{C_B^1(\Omega)}$.

Finally, for each fixed point u of T in $[\bar{v},\hat{v}]$, $T'(u)h = h$ for $h \in C_B^1(\Omega)$

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exactly when h is a solution of the linear BVP

$$\begin{cases} Ah - \sum_{j=1}^{N} f_{\eta_j}(x,u(x),Du(x))D_jh - f_\xi(x,u(x),Du(x))h = 0 & \text{in } \Omega \\ Bh = 0 & \text{on } \Gamma . \end{cases}$$

Proposition 1 is proved by maximum principle and continuation arguments in conjunction with the following a priori estimate:

<u>Proposition 2</u>.   Let f satisfy (1.1).   Then there is an increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that if u is a solution of (1) then

$$\|u\|_{W^{2,p}(\Omega)} \le \gamma(\|u\|_{C(\bar{\Omega})}) .$$

Moreover, γ depends only on A, B, Ω, N, p and c.

While Proposition 2 is more or less known, we do not know an explicit reference and give the proof in Section 2 for completeness.

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Section 2. Proofs of the Main Results

First we show how Theorems 1-3 follow from Proposition 1. Then Proposition 1 is established assuming Proposition 2. Finally, Proposition 2 is proved.

Proof of Theorem 1. If T is the mapping of Proposition 1, it follows immediately from (a), (b), (c) of Proposition 1 that $\hat{u}_n = T^n(\hat{v})$ decreases to an element $\hat{u}$ of $[\bar{v}, \hat{v}]$ as $n \to \infty$ and $\hat{u}$ is the maximal solution of (1) in $[\bar{v}, \hat{v}]$. Similarly, $\bar{u}_n = T^n(\bar{v})$ increases to the minimal solution $\bar{u}$. (Or see [2, Corollary (6.2)].)

Proof of Theorem 2. This result follows at once from Proposition 1 and [2, Theorem 14.2] applied to T. It should be noted that the assumption $\hat{v}_1 < \bar{v}_2$ can be weakened to $\bar{v}_1 < \hat{v}_2$ and $\bar{v}_2 \not\leq \hat{v}_1$ (cf. [4]).

Proof of Theorem 3. The assertions of Theorem 3 follow from Proposition 1 and [2, Theorem 14.4] applied to T provided that we can show the Frechét derivative $T'(u)$ (here T is regarded as a self-mapping of $[\bar{v}, \hat{v}]_{C_B^1(\bar{\Omega})}$ and u is in the interior of $[\bar{v}, \hat{v}]_{C_B^1(\bar{\Omega})}$) has a spectral radius different from 1 if u is u_1 or u_2. Since $T'(u_i)$ is a strongly positive compact endomorphism of $C_B^1(\bar{\Omega})$ by Proposition 1(e), the spectral radius is an eigenvalue and it is the only eigenvalue with a positive eigenvector. These assertions follow from the Krein-Rutman Theorem. Consequently, if $T'(u_i)h \neq h$ for every $h \in C_B^1(\bar{\Omega})$ with $h > 0$ then the spectral radius of $T'(u_i)$ is not 1. But, according to Proposition 1(e), this holds under the assumptions of Theorem 3.

Proof of Proposition 1. Let $\bar{v}, \hat{v} \in W^{2,p}(\Omega)$ be sub- and supersolutions of (1) with $\bar{v} \leq \hat{v}$. Let

$$(2.1) \qquad m = \max(\|\bar{v}\|_{C^1(\bar{\Omega})}, \|\hat{v}\|_{C^1(\bar{\Omega})}) + 1 .$$

Let $\mathcal{K} = \{h \in C^1(\mathbb{R}^N; \mathbb{R}^N): |h(\eta)| \leq 2|\eta|$ for $\eta \in \mathbb{R}^N\}$. It follows from Proposition 2 and the imbedding theorems that there is a constant M with the following properties:

(2.2) $\begin{cases} \text{(i)} \quad m \le M \\ \text{(ii)} \quad \text{If} \quad h \in \mathcal{H} \quad \text{and} \quad u \in [\bar{v}, \hat{v}] \quad \text{is a solution of} \quad Au = f(x,u,h(Du)) \quad \text{in} \\ \qquad \Omega \quad \text{and} \quad Bu = 0 \quad \text{on} \quad \Gamma, \quad \text{then} \quad \|u\|_{C^1(\bar{\Omega})} < M . \end{cases}$

Choose $h \in \mathcal{H}$ which satisfies

(2.3) $\qquad\qquad h(\eta) = \eta \quad \text{for} \quad |\eta| < M \quad \text{and} \quad h(\mathbb{R}^N) \quad \text{is bounded} ,$

and consider the problem

(2.4) $\qquad\qquad \begin{cases} u + \lambda(Au - f(x,u,h(Du))) = g \quad \text{in} \quad \Omega \\ Bu = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \Gamma \end{cases}$

where $\lambda > 0$ and $g \in [\bar{v}, \hat{v}]$. If (2.4) has a solution u and $u = g$, then $\|u\|_{C^1(\bar{\Omega})} < M$ by (2.2). By (2.3) we thus have $h(Du) = Du$ and u is a solution of (1). Conversely, any solution u of (1) with $u \in [\bar{v}, \hat{v}]$ is a solution of (2.4) with $g = u$. We next show that (2.4) in fact has a unique solution $u \in [\bar{v}, \hat{v}]$ for every $g \in [\bar{v}, \hat{v}]$ and the mapping $g \to u = T(g)$ so defined has the properties of Proposition 1 provided that λ is chosen suitably small.

Set $k(x,\xi,\eta) = f(x,\xi,g(\eta))$ and let $\lambda > 0$ satisfy

(2.5) $\qquad\qquad 1 - \lambda k_\xi(x,\xi,\eta) > 0 \quad \text{for} \quad x \in \bar{\Omega}, \ |\xi| \le m, \ \eta \in \mathbb{R}^N .$

It is possible to choose such a λ since $h(\mathbb{R}^N)$ is bounded and $k_\xi(x,\xi,\eta) = f_\xi(x,\xi,h(\eta))$. As a final notational convenience we fix λ as above and define $G : C^1(\bar{\Omega}) \to C(\bar{\Omega})$ by

(2.6) $\qquad\qquad G(u)(x) = k(x,u(x),Du(x))$

so that (2.4) may be abbreviated to

(2.7) $\qquad\qquad \begin{cases} u + \lambda(Au - G(u)) = g \quad \text{in} \quad \Omega \\ Bu = 0 \qquad\qquad\qquad\qquad \text{on} \quad \Gamma . \end{cases}$

To prove the existence and uniqueness of solutions of (2.7) we will use the following form of the maximum principle:

Lemma 1. Suppose $a_j \in L^\infty(\Omega)$ for $j = 0,\ldots,N$ and $a_0 > 0$. Let $u \in W^{2,p}(\Omega)$ satisfy the inequalities

$$\begin{cases} Au + \sum_{j=1}^{N} a_j D_j u + a_0 u \ge 0 \quad \text{in} \quad \Omega \\ Bu \ge 0 \qquad\qquad\qquad\qquad\qquad \text{on} \quad \Gamma . \end{cases}$$

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Then $u \geq 0$. Moreover, if $u \neq 0$ then $u(x) > 0$ for every $x \in \Omega$. If $u \neq 0$ and $u(x_0) = 0$ for some $x_0 \in \Gamma$, then $(\partial u / \partial \alpha)(x_0) < 0$ where α is an arbitrary outward pointing vector at x_0 which is not tangential to Γ.

Proof of Lemma 1. The assertion follows from Bony's maximum principle [5] by means of standard arguments as given, for example, in [10].

Lemma 1 may be used to prove the following comparison result:

Lemma 2. Suppose that $u, v \in W^{2,p}(\Omega)$ satisfy $\|u\|_{C(\bar{\Omega})}$, $\|v\|_{C(\bar{\Omega})} \leq m$ and the inequalities

$$u + \lambda(Au - G(u)) \geq v + \lambda(Av - G(v)) \quad \text{in } \Omega$$

$$Bu \geq Bv \qquad\qquad \text{on } \Gamma .$$

Then $u \geq v$. Moreover, if $u \neq v$ then $u(x) > v(x)$ for $x \in \Omega$. If $u \neq v$ and $u(x_0) = v(x_0)$ for some $x_0 \in \Gamma$, then $(\partial u / \partial \alpha)(x_0) - (\partial v / \partial \alpha)(x_0)$ where α is an arbitrary outward pointing vector at x_0 which is not tangential to Γ.

Proof. Set $w = u - v$. Then the hypotheses imply the inequalities

$$Aw + \sum_{j=1}^{N} a_j D_j w + a_0 w \geq 0 \quad \text{in } \Omega$$

$$Bw \geq 0$$

where

$$a_j(x) = - \int_0^1 k_{\eta_j}(x, v(x) + \tau w(x), Dv(x) + \tau Dw(x)) d\tau$$

for $j = 1, \ldots, N$ and

$$a_0(x) = \lambda^{-1} \int_0^1 (1 - \lambda k_\xi(x, v(x) + \tau w(x), Dv(x) + \tau Dw(x)) d\tau .$$

We have $a_0 > 0$ by (2.5) and hence Lemma 1 implies the desired conclusions.

Lemma 3. Let $g \in [\bar{v}, \hat{v}]$. Then the problem (2.7) has a unique solution $u \in [\bar{v}, \hat{v}]$.

Proof. The uniqueness assertion follows at once from Lemma 2. Let $g \in [\bar{v}, \hat{v}]$ be fixed and set

$$\hat{\psi} = \hat{v} + \lambda(A\hat{v} - G(\hat{v})) - g .$$

Since $\hat{v}$ is a supersolution and $\hat{v} - g \geq 0$ we have $\hat{\psi} \geq 0$. From the L^p-theory of elliptic

boundary value problems and Lemma 1 it follows that for every $q \in L^p(\Omega)$ the problem

$$(2.8) \qquad \begin{cases} (A + 1)w = q & \text{in } \Omega \\ Bw = 0 & \text{on } \Gamma \end{cases}$$

has a unique solution $w = Kq \in W^{2,p}(\Omega)$. Moreover, $K : L^p(\Omega) \to W^{2,p}(\Omega)$ is continuous.

Set $\hat{z} = -K(A + 1)\hat{v}$ and $\hat{w} = \hat{z} + \hat{v}$. Then $\hat{w} \in W^{2,p}(\Omega)$, $(A + 1)\hat{w} = 0$ and $B\hat{w} = B\hat{v}$.

Define $H : C^1(\bar{\Omega}) \times \mathbb{R} \to C^1(\bar{\Omega})$ by

$$(2.9) \qquad H(u,\tau) = \lambda(u - \tau\hat{w}) + K[(1 - \lambda)u - \lambda G(u) - g - \tau(\hat{\psi} - \lambda\hat{w})] .$$

Clearly, H is continuously differentiable and

$$(2.10) \qquad \begin{cases} H(u,\tau) = K[u + \lambda(Au - G(u)) - g - \tau\hat{\psi}] \\ \text{provided } u - \tau\hat{w} \in W^{2,p}(\Omega) \text{ and } B(u - \tau\hat{w}) = 0 . \end{cases}$$

Hence $H(\hat{v},1) = 0$ and a solution u of $H(u,0) = 0$ is a solution of (2.7). We will show how to continue the solution $u = \hat{v}$, $\tau = 1$ of $H = 0$ to a solution at $\tau = 0$. Let $D_1 H(u,\tau)$ denote the Fréchet derivative of the mapping $u \to H(u,\tau)$. Clearly $D_1 H(u,\tau) : C^1(\bar{\Omega}) \to C^1(\bar{\Omega})$ is given by

$$D_1 H(u,\tau)h = \lambda h + K[(1 - \lambda)h - \lambda G'(u)h]$$

where $G'(u)h = \sum_{j=1}^{n} k_{\eta_j}(x,u,Du)D_j h + k_\xi(x,u,Du)h$. A function $h \in C^1(\bar{\Omega})$ satisfies $D_1 H(u,\tau)h = 0$ if and only if $h \in W^{2,p}(\Omega)$, $Bh = 0$ on Γ and $h + \lambda(A - G'(u))h = 0$ in Ω. Lemma 1 and (2.5) imply therefore that $D_1 H(u,\tau)$ is injective if $\|u\|_{C(\bar{\Omega})} \leq m$. Since K is bounded from $L^p(\Omega)$ to $W^{2,p}(\Omega)$, it is compact from $L^p(\Omega)$ to $C^1(\bar{\Omega})$ and it follows that $D_1 H(u,\tau)$ is an automorphism of $C^1(\bar{\Omega})$ for $(u,\tau) \in C^1(\bar{\Omega}) \times \mathbb{R}$ with $\|u\|_{C(\bar{\Omega})} \leq m$. Therefore, by the implicit function theorem, there is a neighborhood $V \times W$ of $(\hat{v},1)$ in $C^1(\bar{\Omega}) \times \mathbb{R}$ such that $H(u,\tau) = 0$ has a unique solution $u = u(\tau)$ for $\tau \in W$ with $u(\tau) \in V$. In particular, $u(1) = \hat{v}$. Since $\|\hat{v}\|_{C(\bar{\Omega})} < m$ and $\tau \to u(\tau)$ is continuous, we can assume $\|u(\tau)\| < m$ by choosing W sufficiently small. From $H(u(\tau),\tau) = 0$ and (2.10) we deduce that $u(\tau) \in W^{2,p}(\Omega)$ and

$$\begin{cases} u(\tau) + \lambda[Au(\tau) - G(u(\tau))] = g + \tau\hat{\psi} \\ Bu(\tau) = \tau B\hat{w} = \tau B\hat{v} \leq B\hat{v} \end{cases}$$

for $\tau \in W \cap [0,1]$. Since also $g + \tau\hat{\psi} \leq g + \hat{\psi} = \hat{v} + \lambda(A\hat{v} - G(\hat{v}))$ for such τ, Lemma 1

implies $u(\tau) \leq \hat{v}$. In a similar way we see that $u(\tau) \geq \bar{v}$ for $\tau \in W \cap [0,1]$. Using in addition (2.3), (2.6) and the definition of h, it follows that $G(u(\tau))$ remains bounded in $L^\infty(\Omega)$. Then (2.9) and $H(u(\tau),\tau) = 0$ shows that $\{u(\tau):\tau \in W \cap [0,1]\}$ is bounded in $W^{2,p}(\Omega)$ and hence precompact in $C^1(\bar{\Omega})$. A standard continuation argument now establishes the existence of a continuous mapping $u:[0,1] \to C^1(\bar{\Omega})$ such that $H(u(\tau),\tau) = 0$ and $\bar{v} \leq u(\tau) \leq \hat{v}$ for $\tau \in [0,1]$. Hence Lemma 3 is proved.

According to Lemma 3 we now have defined a mapping $[\bar{v},\hat{v}] \ni g \to u = T(g)$ where u is the unique solution of (2.7) in $[\bar{v},\hat{v}]$. From (2.2) and (2.3) the fixed points of T are precisely the solutions of (1) which lie in the interval $[\bar{v},\hat{v}]$. Thus (a) and (b) of Proposition 1 hold. Moreover, as above, we see that $T([\bar{v},\hat{v}])$ is bounded in $W^{2,p}(\Omega)$ and hence precompact in $C_B^1(\bar{\Omega})$. To see that T is continuous as a mapping $T:[\bar{v},\hat{v}]_{L^p(\Omega)} \to [\bar{v},\hat{v}]_{C_B^1(\bar{\Omega})}$ we then need only check that it has closed graph, which follows at once from the uniqueness. The fact that T is strongly increasing follows at once from Lemma 2. Thus (c) of Proposition 1 holds. Next let w be a subsolution of (1) with $w \in [\bar{v},\hat{v}]$. The interval $[\bar{v},\hat{v}]$ may be replaced by $[w,\hat{v}]$ in the above proof (w satisfies the same assumptions as $\bar{v}$ and $\bar{v} \leq w \leq \hat{v}$ implies the choices of λ,m and h work for w in place of $\bar{v}$), so $T([w,\hat{v}]) \subseteq [w,\hat{v}]$ and so $w \leq Tw$. If w is not a solution, then $w \neq Tw$ so $w < Tw$. Similarly, supersolutions w of (1) satisfy $T(w) \leq w$ with strict inequality for strict supersolutions. This establishes (d). We turn now to (e). If $\bar{v}$ and $\hat{v}$ are strict we have $\hat{v} > \bar{v}$. By (c), there is an $\epsilon > 0$ such that if B_ϵ is the ball of radius ϵ about the origin in $C_B^1(\bar{\Omega})$ then $T(\hat{v}) - T(\bar{v}) + B_\epsilon \subseteq \{u \in C_B^1(\bar{\Omega}):u \geq 0\}$. But then

$$[\bar{v},\hat{v}] \supseteq [T\bar{v},T\hat{v}] \supset \frac{T(\bar{v}) + T(\hat{v})}{2} + B_\epsilon ,$$

which shows that $[\bar{v},\hat{v}]$ has nonempty interior. To see that T is continuously differentiable as a mapping from the interior of $[\bar{v},\hat{v}]_{C_B^1(\bar{\Omega})}$ (a set which clearly contains the fixed points of T if $\bar{v}$ and $\hat{v}$ are strict) to $C^1(\bar{\Omega})$, we need only recall that T was obtained by applying the implicit function theorem. Indeed, let

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us write $H(u,\tau,g)$ to indicate the dependence of H on g in (2.10). Since H is of class C^1, $H(T(g),0,g) = 0$ and $D_1 H(T(g),0,g)$ is an automorphism of $C^1(\bar{\Omega})$, $g \to T(g)$ is continuously differentiable. Calculating the derivative shows that $T'(g)h = w$ implies $w \in W^{2,p}(\Omega)$ and

$$(2.11) \quad \begin{cases} w + \lambda(Aw - G'(T(g))w) = h & \text{in } \Omega \\ Bw = 0 & \text{on } \Gamma . \end{cases}$$

Hence Lemma 1 implies that $T'(g)$ is a strongly positive linear operator. Finally, if $u = g$ is a fixed point of T and $T'(u)h = h$, (2.11) reduces to the final assertion of (e).

It remains to prove Proposition 2.

<u>Proof of Proposition 2</u>. The Proposition will follow at once from the next lemma.

<u>Lemma 3</u>. For every $b \in L^\infty(\Omega)$ there is exactly one solution $u \in W^{2,p}(\Omega)$ of the problem

$$(2.12) \quad \begin{cases} (A + 1)u = b(1 + |Du|^2) & \text{in } \Omega \\ Bu = 0 & \text{on } \Gamma . \end{cases}$$

Moreover, there is an increasing function $\gamma_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|u\|_{W^{2,p}(\Omega)} \leq \gamma_0(\|b\|_{L^\infty(\Omega)})$. The function γ_0 depends only on A, Ω, B, p and N.

Indeed, if u is a solution of (1), it is also a solution of $(A + 1)u = b(1 + |Du|^2)$, $Bu = 0$, where $b \doteq (f(x,u,Du) + u)/(1 + |Du|^2)$. Thus if we set $\gamma(r) = \gamma_0(c(r) + r)$, the assertions of Proposition 2 follow from Lemma 3.

<u>Proof of Lemma 3</u>. Let $\sigma_1, \sigma_2 \in [0,1]$ and assume $(A + 1)u_i = b(\sigma_i + |Du_i|^2)$, $Bu_i = 0$. Set $w = u_1 - u_2$ and $M = |\sigma_1 - \sigma_2| \|b\|_{L^\infty(\Omega)}$. Then it follows that

$$(A + 1)(M - w) - b \sum_{j=1}^{N} D_j(u_1 + u_2)D_j(M - w) = |\sigma_1 - \sigma_2| \|b\|_{L^\infty} - (\sigma_1 - \sigma_2)b \geq 0 \quad \text{in } \Omega ,$$

and

$$B(M - w) = M \quad \text{on } \Gamma_0 \quad \text{and} \quad B(M - w) = b_0 M \geq 0 \quad \text{on } \Gamma_1 .$$

Consequently, by Lemma 1, $w \leq M$. Similarly one shows that $w \geq -M$. Thus

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(2.13)
$$\|u_1 - u_2\|_{C(\bar{\Omega})} \le |\sigma_1 - \sigma_2| \|b\|_{L^\infty(\Omega)} \quad .$$

In particular, if $\sigma_1 = \sigma_2$ then $u_1 = u_2$ and solutions of (2.12) are unique as a consequence.

With the notation of the preceding step we also have

$$(A + 1)w = (\sigma_1 - \sigma_2)b + b(|Du_1|^2 - |Du_2|^2)$$

$$= (\sigma_1 - \sigma_2)b + b(|Du_1|^2 - |Dw - Du_1|^2)$$

in Ω and $Bw = 0$ on Γ. Consequently

$$\|(A + 1)w\|_{L^p(\Omega)} \le 2 \|b\|_{L^\infty(\Omega)} \| |Dw|^2 \|_{L^p(\Omega)}$$

$$+ \|b\|_{L^\infty(\Omega)} (\mu(\Omega)^{1/p} + 3 \| |Du_1|^2 \|_{L^p(\Omega)}) \quad .$$

where $\mu(\Omega)$ is the Lebesque measure of Ω. The Gagliardo-Nirenberg interpolation inequality (see, e.g., [9, Theorem I.10.1]) supplies a constant γ_1 such that

$$\| |Dw|^2 \|_{L^p(\Omega)} \le \gamma_1 \|w\|_{L^\infty(\Omega)} \|w\|_{W^{2,p}(\Omega)} \quad .$$

Combining this fact with the above inequality, (2.13) and the existence of a $\gamma_3 > 0$ such that $\|(A + 1)w\|_{L^p(\Omega)} \ge \gamma_3 \|w\|_{W^{2,p}(\Omega)}$, we find

(2.14)
$$\|u_1 - u_2\|_{W^{2,p}(\Omega)} \le |\sigma_1 - \sigma_2| \gamma_3^{-1} \gamma_1 \|b\|_{L^\infty(\Omega)} \|u_1 - u_2\|_{W^{2,p}(\Omega)}$$

$$+ \gamma_2 (\|b\|_{L^\infty(\Omega)}, \|u_1\|_{C^1(\bar{\Omega})})$$

where $\gamma_2 : \mathbb{R}_+^2 \to \mathbb{R}_+$ is nondecreasing in each argument. Choosing $\sigma_1 = 0$, $u_1 = 0$ and an integer $n > 0$ such that $n^{-1}\gamma_1^{-1}\gamma_3 \|b\|_{L^\infty(\Omega)} < 1/2$, (2.14) implies that if $\sigma_2 \in [0, n^{-1}]$, then

(2.15)
$$\|u_2\|_{W^{2,p}(\Omega)} \le 2\gamma_2 (\|b\|_{L^\infty(\Omega)}, 0) \quad .$$

All of the above estimates are equally valid if b is replaced by τb for $\tau \in [0,1]$. Next we attempt to solve

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$$(2.16) \quad \begin{cases} (A + 1)u = b(\frac{1}{n} + |Du|^2) & \text{in } \Omega \\ Bu = 0 & \text{on } \Gamma . \end{cases}$$

If we are able to do so, we then set $\sigma_2 = n^{-1}$, $u_2 = u$ in (2.14) and use (2.15) to obtain an estimate

$$\|u_1\|_{W^{2,P}(\Omega)} \leq \gamma_4 (\|b\|_{L^\infty(\Omega)})$$

for $\sigma_1 \in [n^{-1}, 2n^{-1}]$. Continuing in this way the result is proved in n steps. To show (2.16) has a solution, let $T(\tau, \varphi)$ denote the solution v of

$$(2.17) \quad \begin{cases} (A + 1)v = \tau b(\frac{1}{n} + |D\varphi|^2) & \text{in } \Omega \\ Bv = 0 . \end{cases}$$

As a mapping $T: [0,1] \times C^1(\bar{\Omega}) \to C^1(\bar{\Omega})$ T is compact, continuous and the solutions of $\varphi = T(\tau, \varphi)$ are uniformly bounded by the above estimates. Moreover, $T(0, \varphi) = 0$ for $\varphi \in C^1(\bar{\Omega})$. Hence $T(1, \cdot)$ has a fixed point by the Leray-Schauder fixed point principle, and the proof is complete.

The proof of Proposition 2 is based on methods developed by Tomi [11] (cf. also V. Wahl [12] and [3, Theorem 2.2]). We remark that our use of the Leray-Schauder theorem above could have been replaced by a continuation argument based on the implicit function theorem, thereby making the proof of Lemma 3 more constructive.

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Let A denote a strongly elliptic second order differential operator and B be a first order boundary operator acting on functions u defined on $\Omega \subseteq \mathbb{R}^N$. Let $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy $|f(x,\xi,\eta)| \leq c(|\xi|)(1 + |\eta|^2)$. Under suitable assumptions, it is shown how the set of solutions of the problem

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20. ABSTRACT (Cont'd.)

$$(1) \qquad \begin{cases} Au = f(x,u,Du) & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

satisfying $\bar{u} \leq u \leq \hat{u}$ where $\bar{u}$ and $\hat{u}$ are, respectively, sub- and super-solutions of (1) can be naturally identified with the fixed point set of a self-mapping T of the order interval $[\bar{u},\hat{u}]$. Moreover, T has many desirable properties from which existence and multiplicity theorems are obtained.